Toric Nearly Kähler 6-manifolds

Andrei Moroianu
CNRS - Paris-Sud University

Progress and Open Problems 2019:
September 8-11, 2019, SCGP, Stony Brook
– joint work with Paul-Andi Nagy –

Nearly Kähler manifolds were originally introduced as the class $\mathcal{W}_1$ in the Gray-Hervella classification of almost Hermitian manifolds.

More precisely, an almost Hermitian manifold (M^{2n}, g, J) is called *nearly Kähler* (NK) if

$$(\nabla_X J)(X) = 0$$

for every vector field X on M, where ∇ denotes the Levi-Civita covariant derivative of g. A NK manifold is called *strict* if $(\nabla J)_p \neq 0$ for every $p \in M$.

Remark. In dimension $2n = 4$, NK = Kähler.

Examples:

- Kähler manifolds.

- twistor spaces over positive QK manifolds, endowed with the non-integrable almost complex structure and with the metric rescaled by a factor 2 on the fibres.

- naturally reductive 3-symmetric spaces G/H where G is compact, H is the invariant group of an automorphism σ of G of order 3, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, and $\mathfrak{p}$ has a scalar product such that for every $X, Y, Z \in \mathfrak{p}$:

$$\langle [X, Y]_{\mathfrak{p}}, Z \rangle + \langle [X, Z]_{\mathfrak{p}}, Y \rangle = 0.$$

  The almost complex structure is determined by the endomorphism J of $\mathfrak{p}$ satisfying

$$\sigma_* = -\frac{1}{2}\mathrm{Id}_{\mathfrak{p}} + \frac{\sqrt{3}}{2} J.$$

A product of NK manifolds is again NK. Conversely, the factors of the de Rham decomposition of a NK manifold are NK.

Theorem. (Nagy 2002): Every simply connected, complete, de Rham irreducible NK manifold is either one of the above examples, or a strict NK 6-manifold.

From now on, we restrict our attention to strict NK 6-manifolds. These are interesting for several reasons:

Properties of strict NK 6-manifolds:

- carry real Killing spinors $\rightsquigarrow$ positive Einstein; after rescaling the metric, one can normalize them to having scalar curvature 30 (like the round S^6).

- ∇J has constant norm $\rightsquigarrow$ SU(3)-structure

- carry a connection with parallel and skew-symmetric torsion

$$\tilde{\nabla}_X = \nabla_X - \frac{1}{2} J \circ \nabla_X J$$

- the Riemannian cone $(M \times \mathbb{R}^*_+, t^2 g + \mathrm{d}t^2)$ of a normalized NK 6-manifold (M, g, ω) has holonomy contained in G_2, defined by the positive 3-form

$$\varphi = \frac{1}{3}\mathrm{d}(t^3\omega) = \frac{1}{3}t^3\mathrm{d}\omega + t^2\mathrm{d}t \wedge \omega$$

Main problem: lack of examples.

3-symmetric spaces were classified by Gray. In dimension 6:

- $S^6 = \mathsf{G}_2/\mathsf{SU}(3)$
- $\mathsf{SU}(2) \times \mathsf{SU}(2) \times \mathsf{SU}(2)/_\Delta \sim \mathcal{S}^3 \times S^3$
- $\mathsf{Sp}(2)/\mathsf{U}(2) \sim \mathbb{C}\mathsf{P}^3$
- $\mathsf{SU}(3)/\mathsf{U}(1) \times \mathsf{U}(1) \sim F(1,2)$.

Theorem. (Butruille 2004) These are all homogeneous SNK 6-manifolds.

Foscolo and Haskins (2017): 2 new examples (of cohomogeneity 1) on S^6 and $S^3 \times S^3$, both with isometry group SU(2) $\times$ SU(2).

Deformations of SNK 6-manifolds were studied by –, Nagy, Semmelmann (2008, 2010, 2011).

The moduli space is isomorphic to the space of co-closed primitive $(1,1)$-forms which are eigenforms of the Laplace operator for the eigenvalue 12.

Using representation theory one can compute this space on the homogeneous examples. It vanishes except on $F(1,2)$ where it has dimension 8. However, these infinitesimal deformations are obstructed (Foscolo 2017).

SU(3)-structures on SNK 6-manifolds

Let M^6 be an oriented manifold. An SU(3)-structure on M is a triple (g, J, ψ), where

- g is a Riemannian metric,
- J is a compatible almost complex structure (i.e. $\omega := g(J\cdot, \cdot)$ is a 2-form),
- $\psi = \psi^+ + i\psi^-$ is a $(3,0)$ complex volume form satisfying

$$\psi^+ \wedge \psi^- = 4\mathrm{vol}_g = \frac{2}{3}\omega^3.$$

It is possible to characterize SU(3)-structures in terms of exterior forms only (Hitchin).

Lemma 1 *A pair* $(\omega, \psi^+) \in C^\infty(\Lambda^2 M \times \Lambda^3 M)$ *defines an* SU(3)*-structure on* M *provided that:*

- $\omega^3 \neq 0$ *(i.e.* ω *is non-degenerate).*

- $\omega \wedge \psi^+ = 0$*.*

- *If* $K \in \mathrm{End(TM)} \otimes \Lambda^6\mathrm{M}$ *is defined by*

  $K(X) := (X \lrcorner \psi^+) \wedge \psi^+ \in \Lambda^5 M \simeq \mathrm{TM} \otimes \Lambda^6 \mathrm{M}$,

  then $\mathrm{tr} K^2 = -\frac{1}{6}(\omega^3)^2 \in (\Lambda^6 M)^{\otimes 2}$

- $\omega(X, K(X))/\omega^3 > 0$ *for every* $X \neq 0$*.*

<u>"Proof":</u> Define $J := 6K/\omega^3$, $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$, $\psi^- := -\psi^+(J\cdot,\cdot,\cdot)$.

A normalized SNK structure (g, J, ω) on M^6

$\rightsquigarrow$ SU(3)-structure $(g, J, \omega, \psi^+, \psi^-)$ where

$$\psi^+ := \nabla\omega, \qquad \psi^- := -\psi^+(J\cdot, \cdot, \cdot).$$

This satisfies the exterior differential system

$$\begin{cases} \mathrm{d}\omega = 3\psi^+ \\ \mathrm{d}\psi^- = -2\omega^2. \end{cases}$$

Conversely, an SU(3)-structure satisfying this system is a normalized SNK structure (Hitchin).

This is similar to the case of G_2 structures, where a stable 3-form is parallel if and only if it is harmonic.

Toric NK 6-manifolds

An *infinitesimal automorphism* of a normalized SNK 6-manifold $(M, g, J, \omega, \psi^{\pm})$ is a vector field ξ whose flow preserves the whole structure (enough to have $\mathcal{L}_\xi \omega = 0 = \mathcal{L}_\xi \psi^+$).

Lemma. $\mathrm{rk}(\mathfrak{aut}(M, g, J)) \leq 3$.

If equality holds, (M, g, J) is called toric. The only homogeneous example is $S^3 \times S^3$.

Assume that (M, g, J) is toric and let ξ_1, ξ_2, ξ_3 be a basis of a Cartan subalgebra of $\mathfrak{aut}(M, g, J)$.

Lemma. The vector fields

$$\xi_1,\ \xi_2,\ \xi_3,\ J\xi_1,\ J\xi_2,\ J\xi_3$$

are linearly independent on a dense open subset M_0 of M.

$\rightsquigarrow$ dual basis $\{\theta^1, \theta^2, \theta^3, \gamma^1, \gamma^2, \gamma^3\}$ of $\Lambda^1 M_0$.

Define the functions

$$\mu_{ij} := \omega(\xi_i, \xi_j), \qquad \varepsilon := \psi^-(\xi_1, \xi_2, \xi_3).$$

The Cartan formula and $\begin{cases} \mathrm{d}\omega = 3\psi^+ \\ \mathrm{d}\psi^- = -2\omega^2 \end{cases}$ $\rightsquigarrow$

$$\begin{aligned} \mathrm{d}\mu_{ij} &= \mathrm{d}(\xi_j \lrcorner \xi_i \lrcorner \omega) = -\xi_j \lrcorner \mathrm{d}(\xi_i \lrcorner \omega) \\ &= \xi_j \lrcorner \xi_i \lrcorner \mathrm{d}\omega = -3\xi_i \lrcorner \xi_j \lrcorner \psi^+. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{d}\varepsilon &= \mathrm{d}(\xi_3 \lrcorner \xi_2 \lrcorner \xi_1 \lrcorner \psi^-) = -\xi_3 \lrcorner \xi_2 \lrcorner \xi_1 \lrcorner \mathrm{d}\psi^- \\ &= 2\xi_3 \lrcorner \xi_2 \lrcorner \xi_1 \lrcorner \omega^2. \end{aligned}$$

Remarks:

1. $\psi^+(\xi_1, \xi_2, \xi_3) = 0$ on M.

2. ε does not vanish on M_0.

It follows that the map $\mu : M \to \Lambda^2\mathbb{R}^3 \cong \mathfrak{so}(3)$ defined by

$$\mu := \begin{pmatrix} 0 & \mu_{12} & \mu_{13} \\ \mu_{21} & 0 & \mu_{23} \\ \mu_{31} & \mu_{32} & 0 \end{pmatrix}$$

is the multi-moment map of the strong geometry (M, ψ^+) defined by Madsen and Swann (and studied further by Dixon in the particular case where $M = S^3 \times S^3$).

Similarly, the function ε is the multi-moment map associated to the stable closed 4-form $d\psi^-$.

Consider the symmetric 3×3 matrix

$$C := (C_{ij}) = (g(\xi_i, \xi_j)).$$

In terms of the basis $\{\theta^1, \theta^2, \theta^3, \gamma^1, \gamma^2, \gamma^3\}$ of $\Lambda^1 M_0$ we can write

$$\psi^+ = \varepsilon(\gamma^{123} - \theta^{12} \wedge \gamma^3 - \theta^{31} \wedge \gamma^2 - \theta^{23} \wedge \gamma^1),$$

$$\psi^- = \varepsilon(\theta^{123} - \gamma^{12} \wedge \theta^3 - \gamma^{31} \wedge \theta^2 - \gamma^{23} \wedge \theta^1),$$

where $\gamma^{123} = \gamma^1 \wedge \gamma^2 \wedge \gamma^3$ etc. Similarly,

$$\omega = \sum_{1 \le i < j \le 3} \mu_{ij}(\theta^{ij} + \gamma^{ij}) + \sum_{i,j=1}^{3} C_{ij} \theta^i \wedge \gamma^j$$

The normalization condition

$$\psi^+ \wedge \psi^- = \frac{2}{3} \omega^3$$

translates into

$$\det(C) = \varepsilon^2 + \sum_{i,j=1}^{3} C_{ij} y_i y_j,$$

where

$$y_1 := \mu_{23}, \ y_2 := \mu_{31}, \ y_3 := \mu_{12}.$$

The previous formula $\mathrm{d}\mu_{ij} = -3\xi_i \lrcorner \xi_j \lrcorner \psi^+$can be restated as

$$\mathrm{d}y_i = -3\varepsilon\gamma^i, \qquad i = 1, 2, 3.$$

Similarly, $\mathrm{d}\varepsilon = 2\xi_3 \lrcorner \xi_2 \lrcorner \xi_1 \lrcorner \omega^2$ is equivalent to

$$\mathrm{d}\varepsilon = 4 \sum_{i,j=1}^{3} C_{ij} y_i \gamma^j.$$

Remark also that $\xi_j \lrcorner \mathrm{d}\theta^i = 0 \quad \rightsquigarrow \quad$ explicit expression of $\mathrm{d}\theta^i$ in terms of γ_j, y_j, ε and C.

Let $U := M_0/T^3$ be the set of orbits of the T^3-action generated by the vector fields ξ_i.

All invariant functions and basic forms descend to $U \quad \rightsquigarrow \quad y_i$, ε, γ^i, C_{ij}, etc. Since ε does not vanish on $M_0 \quad \rightsquigarrow \quad \{y_i\}$ define a local coordinate system on U.

Key point: The system $\begin{cases} \mathrm{d}\omega = 3\psi^+ \\ \mathrm{d}\psi^- = -2\omega^2 \end{cases}$

$\rightsquigarrow \quad \exists\varphi$ on U such that $\mathrm{Hess}(\varphi) = C$ in the coordinates $\{y_i\}$.

Let us introduce the operator ∂_r of radial differentiation, acting on functions on U by

$$\partial_r f := \sum_{i=1}^{3} y_i \frac{\partial f}{\partial y_i}.$$

Claim: The function φ can be chosen in such a way that

$$\varepsilon^2 = \frac{8}{3}(\varphi - \partial_r \varphi).$$

<u>Proof:</u> It is enough to show that the exterior derivatives of the two terms coincide. Since

$$\frac{\partial(\partial_r \varphi)}{\partial y_j} = \sum_{i=1}^{3} \frac{\partial^2 \varphi}{\partial y_i \partial y_j} y_i + \frac{\partial \varphi}{\partial y_j},$$

we get:

$$\begin{aligned} \mathrm{d}(\partial_r \varphi - \varphi) &= \sum_{i,j=1}^{3} C_{ij} y_i \mathrm{d} y_j = -3 \sum_{i,j=1}^{3} C_{ij} y_i \varepsilon \gamma^j \\ &= -\frac{3}{4} \varepsilon \mathrm{d}\varepsilon = -\frac{3}{8} \mathrm{d}(\varepsilon^2). \end{aligned}$$

On the other hand,

$$\partial_r^2\varphi = \partial_r(\sum_{i=1}^{3} y_i\frac{\partial f}{\partial y_i}) = \sum_{i,j=1}^{3} C_{ij}y_iy_j + \partial_r\varphi.$$

Summing up, the previous relation

$$\mathrm{det}(C) = \varepsilon^2 + \sum_{i,j=1}^{3} C_{ij}y_iy_j$$

becomes:

$$\boxed{\mathrm{det}(\mathrm{Hess}(\varphi)) = \tfrac{8}{3}\varphi - \tfrac{11}{3}\partial_r\varphi + \partial_r^2\varphi.}$$

This Monge-Ampère equation is enough to recover (locally) the full structure of the toric SNK manifold provided some positivity constraints hold.

The inverse construction

We will show that a solution φ of

$$\det(\mathrm{Hess}(\varphi)) = \frac{8}{3}\varphi - \frac{11}{3}\partial_r\varphi + \partial_r^2\varphi$$

on some open set $U \subset \mathbb{R}^3$ defines a toric SNK structure on $U_0 \times \mathbb{T}^3$, where U_0 is some open subset of U.

Let y_1, y_2, y_3 be the standard coordinates on U and let μ be the 3×3 skew-symmetric matrix

$$\mu := \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix}.$$

Define the 6×6 symmetric matrix

$$D := \begin{pmatrix} \mathrm{Hess}(\varphi) & -\mu \\ \mu & \mathrm{Hess}(\varphi) \end{pmatrix}.$$

Let $U_0 \subset U$ denote the open subset

$$U_0 := \{x \in U \mid \varphi(x) - \partial_r\varphi(x) > 0 \text{ and } D > 0\}.$$

Note that the matrix D is positive definite if and only if $C := \mathrm{Hess}(\varphi) > 0$ and $\langle \mu a, b\rangle^2 < \langle Ca, a\rangle\langle Cb, b\rangle$ for all $(a, b) \in (\mathbb{R}^3 \times \mathbb{R}^3) \setminus (0, 0)$.

On U_0 we define a positive function ε by

$$\varepsilon^2 = \frac{8}{3}(\varphi - \partial_r\varphi),$$

and 1-forms γ^i by $\mathrm{d}y_i = -3\varepsilon\gamma^i$.

We pull-back ε, y_i, and γ_i to $U_0 \times \mathbb{T}^3$ and define θ_i on $U_0 \times \mathbb{T}^3$ as connection forms whose curvature is given by the explicit expression of $\mathrm{d}\theta_i$ in the direct construction in terms of C, ε, y_i, and γ_i.

It remains to check that ω and $\psi^\pm$ defined by the previous expressions form indeed an SU(3)-structure on $U_0 \times \mathbb{T}^3$.

Example

Let $K := \mathrm{SU}_2$ with Lie algebra $\mathfrak{k} = \mathfrak{su}_2$ and $G := K \times K \times K$ with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k} \oplus \mathfrak{k}$. We consider the 6-dimensional manifold $M = G/K$, where K is diagonally embedded in G. The tangent space of M at $o = eK$ can be identified with

$$\mathfrak{p} = \{(X, Y, Z) \in \mathfrak{k} \oplus \mathfrak{k} \oplus \mathfrak{k} \,|\, X + Y + Z = 0\}.$$

The Killing form B on $\mathfrak{su}_2$ induces a scalar product on $\mathfrak{g}$ by

$$|(X, Y, Z)|^2 := B(X, X) + B(Y, Y) + B(Z, Z)$$

which defines a 3-symmetric nearly Kähler metric g on $M = S^3 \times S^3$.

The G-automorphism σ of order 3 defined by $\sigma(a_1, a_2, a_3) = (a_2, a_3, a_1)$ induces a canonical almost complex structure on the 3-symmetric space M by the relation

$$\sigma = \frac{-\mathrm{Id} + \sqrt{3}J}{2} \qquad \text{on } \mathfrak{p}.$$

$$J(X,Y,Z) = \tfrac{2}{\sqrt{3}}(Y,Z,X) + \tfrac{1}{\sqrt{3}}(X,Y,Z).$$

Let ξ be a unit vector in $\mathfrak{su}_2$ with respect to B. The right-invariant vector fields on G generated by the elements

$$\tilde{\xi}_1 = (\xi, 0, 0), \quad \tilde{\xi}_2 = (0, \xi, 0), \quad \tilde{\xi}_3 = (0, 0, \xi)$$

of $\mathfrak{g}$, define three commuting Killing vector fields ξ_1, ξ_2, ξ_3 on M.

Let us compute $g(\xi_1, J\xi_2)$ at some point $aK \in M$, where $a = (a_1, a_2, a_3)$ is some element of G. By the definition of J we have

$$g(\xi_1, J\xi_2)_{aK} = \frac{1}{\sqrt{3}} B(a_1^{-1}\xi a_1, a_2^{-1}\xi a_2).$$

We introduce the functions $y_1, y_2, y_3 : G \to \mathbb{R}$ defined by

$$y_i(a_1, a_2, a_3) = \frac{1}{\sqrt{3}} B(a_j^{-1}\xi a_j, a_k^{-1}\xi a_k),$$

for every permutation (i, j, k) of $(1, 2, 3)$.

A similar computation yields

$$C_{ij} := g(\xi_i, \xi_j)_{aK} = 2\delta_{ij} + \frac{1}{\sqrt{3}} y_k(a).$$

The function φ in the coordinates y_i such that Hess$(\varphi) = C$ is determined by

$$\varphi(y_1, y_2, y_3) = y_1^2 + y_2^2 + y_3^2 + \frac{1}{\sqrt{3}} y_1 y_2 y_3 + h,$$

up to some affine function h in the coordinates y_i. On the other hand, since

$$\det(C) = -\frac{2}{3}(y_1^2 + y_2^2 + y_3^2) + \frac{2}{3\sqrt{3}} y_1 y_2 y_3 + 8,$$

the above function φ satisfies the Monge–Ampère equation

$$\det(\text{Hess}(\varphi)) = \frac{8}{3}\varphi - \frac{11}{3}\partial_r \varphi + \partial_r^2 \varphi$$

for $h = 3$.

Radial solutions

We search here radial solutions to the Monge–Ampère equation on (some open subset of) $\mathbb{R}^3$ with coordinates y_1, y_2, y_3.

Write $\varphi(y_1, y_2, y_3) := x(\frac{r^2}{2})$ where x is a function of one real variable and $r^2 = y_1^2 + y_2^2 + y_3^2$. A direct computation yields

$$\begin{aligned}\mathrm{Hess}(\varphi) &= \begin{pmatrix} y_1^2 x'' + x' & y_1 y_2 x'' & y_1 y_3 x'' \\ y_1 y_2 x'' & y_2^2 x'' + x' & y_2 y_3 x'' \\ y_1 y_3 x'' & y_2 y_3 x'' & y_3^2 x'' + x' \end{pmatrix} \\ &= x' \mathrm{Id} + x''(\frac{r^2}{2}) V \cdot {}^t V\end{aligned}$$

where $V := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$. In particular,

$$\begin{aligned}\det \mathrm{Hess}(\varphi) &= (x')^2 x'' r^2 + (x')^3 \\ \partial_r \varphi &= r^2 x', \;\; \partial_r^2 \varphi = r^4 x'' + 2 r^2 x',\end{aligned}$$

whence after making the substitution $t := \frac{r^2}{2}$ we get:

Proposition 1 *Radial solutions to the Monge-Ampère equation are given by solutions of the second order* ODE

$$x'' = F(t, x, x')$$

where $F(t,p,q) := \frac{8p-(10tq+3q^3)}{6(q^2t-2t^2)}$.

To decide which solutions of this equation yield genuine Riemannian metrics in dimension six, we observe that

Proposition 2 *For any radial solution* $\varphi = x(\frac{r^2}{2})$*, the set*

$$U_0 := \{x \in U \mid \varphi(x) - \partial_r\varphi(x) > 0 \text{ and } D > 0\}.$$

defined above is given by

$$U_0 = \{t > 0 \mid x(t) > 2tx'(t) > 2t\sqrt{2t}\}.$$

Remark 1 *The solutions of the above ODE of the form* $x = kt^l$ *with* $k, l \in \mathbb{R}$ *are* $x_{1,2} = \pm\frac{2\sqrt{2}}{9}t^{\frac{3}{2}}$ *and* $x_3 = kt^{\frac{1}{2}}$*, corresponding to*

$$\varphi_{1,2} = \pm\frac{r^3}{9}, \qquad \varphi_3 = \frac{k}{\sqrt{2}}r.$$

However, they do not satisfy the positivity requirements from Proposition 2.

Admissible solutions can be obtained by solving the Cauchy problem with initial data

$$(t_0, x(t_0), x'(t_0)) \in \mathcal{S}$$

where

$$\mathcal{S} := \{(t, p, q) \in \mathbb{R}^3 : t > 0,\ p > 2tq > 2t\sqrt{2t}\}.$$