TORSIONLESS THREE-DIMENSIONAL HETEROTIC SOLITONS WITH HARMONIC CURVATURE ARE RIGID

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Abstract. We prove the following rigidity result: every compact three-dimensional Heterotic soliton with vanishing torsion and harmonic curvature is rigid, namely, it is an isolated point in the moduli space.

1. Introduction

The primary objective of this note is to prove a rigidity result for the Heterotic soliton system with auxiliary vanishing torsion on a compact three-manifold. Inspired in Heterotic supergravity [2, 3], the general Heterotic system was introduced in [13], where its compact three-dimensional solutions with non-vanishing parallel torsion were classified in terms of hyperbolic three-manifolds and compact quotients of the Heisenberg group equipped with a left-invariant metric. The classification of compact Heterotic solitons with vanishing auxiliary torsion is notably more difficult and remains an open problem. For vanishing auxiliary torsion, the three-dimensional Heterotic soliton system for non-flat solitons reduces to the following system of equations:

$$\mathrm{Ric}^g + \nabla^g \mathrm{d}\phi - \frac{1}{2}e^{2\phi} g + \kappa\, \mathcal{R}^g \circ_g \mathcal{R}^g = 0\,, \tag{1.1}$$

$$\mathrm{d}^*_{\nabla^g}\mathcal{R}^g + \mathcal{R}^g(\mathrm{d}\phi) = 0\,, \tag{1.2}$$

$$\delta^g \mathrm{d}\phi + |\mathrm{d}\phi|^2_g - e^{2\phi} + \kappa\, |\mathcal{R}^g|^2_g = 0\,, \tag{1.3}$$

for couples (g, ϕ) consisting of a Riemannian metric g on a three-manifold M and a function $\phi \in C^\infty(M)$, the so-called *dilaton* of the system. Here ∇^g denotes the Levi-Civita connection of g, Ric^g is its Ricci tensor, $\mathcal{R}^g$ is its Riemann tensor, $\mathrm{d}^*_{\nabla^g}$ is the formal adjoint of the exterior covariant derivative d_{∇^g} associated to ∇^g, δ^g is the formal adjoint of the exterior derivative d, and where we have defined:

$$(\mathcal{R}^g \circ_g \mathcal{R}^g)(v_1, v_2) := \langle v_1 \lrcorner \mathcal{R}^g, v_2 \lrcorner \mathcal{R}^g \rangle_g = \frac{1}{2}\sum_{i,j,k=1}^{3} \mathcal{R}^g_{v_1,e_i}(e_j, e_k)\mathcal{R}^g_{v_2,e_i}(e_j, e_k)\,,$$

$$|\mathcal{R}^g|^2_g := \langle \mathcal{R}^g, \mathcal{R}^g\rangle_g = \frac{1}{2}\sum_{i,j=1}^{3}\langle \mathcal{R}^g_{e_i,e_j}, \mathcal{R}^g_{e_i,e_j}\rangle_g = \frac{1}{4}\sum_{i,j,k,l=1}^{3} \mathcal{R}^g_{e_i,e_j}(e_k, e_l)\, \mathcal{R}^g_{e_i,e_j}(e_k, e_l)$$

in terms of any local orthonormal frame e_i and vectors $v_1, v_2 \in \mathfrak{X}(M)$. The symbol $\langle\cdot,\cdot\rangle_g$ denotes the *determinant* metric associated to g. Its associated norm is denoted by $|\cdot|^2_g$. The system of equations (1.1), (1.2), and (1.3) is the object of study of this note.

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Equation (1.1) is the *Einstein equation* of the system, whereas (1.2) is a Yang-Mills type of equation for the Levi-Civita connection. On the other hand, Equation (1.3) is typically called the *dilaton equation* of the system. Solutions (g, ϕ) to equations (1.1), (1.2), and (1.3) are called *Heterotic solitons with vanishing torsion* or *torsionless Heterotic solitons.*

Currently, all known three-dimensional torsionless compact Heterotic solitons are Einstein with constant dilaton. If non-flat, such a Heterotic soliton is hyperbolic of scalar curvature $-24\kappa^{-1}$. We refer to these solitons simply as *hyperbolic*, since the hyperbolicity of the metric forces a compact torsionless Heterotic soliton to have constant dilaton. It remains a major open problem to determine whether compact, torsionless Heterotic solitons with non-constant dilaton actually exist. A natural strategy for constructing a non-flat, torsionless Heterotic soliton with a non-constant dilaton is to deform a hyperbolic one. The main theorem of this note proves that this is not possible, not even if we allow the curvature of g to be harmonic instead of strictly hyperbolic.

Theorem 1.1. *Let (g, ϕ) be a torsionless, non-flat, three-dimensional compact Heterotic soliton with vanishing torsion and harmonic curvature. Then, the vector space of essential deformations of (g, ϕ) is zero-dimensional.*

The notion of *essential deformation* that we consider is the natural generalization of Koisos's [10]. That is, an essential deformation is an infinitesimal deformation orthogonal to the infinitesimal deformations generated by the action of diffeomorphisms. By the previous theorem, torsionless Heterotic solitons with harmonic curvature are infinitesimally rigid. In turn, due to the existence of a slice and a local Kuranishi model, which follow from the general theory of Diez and Rudolph [6, 7], this implies that they are also isolated points in the moduli space of torsionless Heterotic solitons. Alternatively, we can think of Theorem 4.6 as a natural generalization of the Mostow rigidity theorem to the Heterotic soliton system.

Theorem 1.1 is proved in two steps. First, in Theorem 4.6 we prove that three-dimensional *hyperbolic* compact Heterotic solitons are infinitesimally rigid, which, combined with Corollary 4.2, proves rigidity. Then, in Theorem 5.2 we prove that non-flat three-dimensional compact Heterotic solitons with harmonic curvature are, in fact, hyperbolic, and consequently also rigid.

2. Preliminaries

Let M be an oriented three-dimensional manifold equipped with a Riemannian metric g. Throughout the paper, when there is no risk of confusion, we will identify 1-forms with vector fields and 2-forms with skew-symmetric endomorphisms using the metric. In our conventions, the Riemann tensor $\mathcal{R}^g$ of g is defined by:

$$\mathcal{R}^g_{v_1,v_2} v_3 = \nabla^g_{v_1}\nabla^g_{v_2} v_3 - \nabla^g_{v_2}\nabla^g_{v_1} v_3 - \nabla^g_{[v_1,v_2]} v_3\,, \qquad v_1, v_2, v_3 \in \mathfrak{X}(M)\,.$$

We will understand the curvature tensor $\mathcal{R}^g$ as a section of $\Omega^2(M)\otimes\Omega^2(M)$, upon identifying skew-symmetric endomorphisms with two-forms, using the underlying Riemannian metric g. The Ricci tensor Ric^g is defined in our conventions as follows:

$$\mathrm{Ric}^g(v_1, v_2) = \sum_i^3 \mathcal{R}^g_{e_i v_1}(v_2, e_i)\,, \qquad v_1, v_2 \in \mathfrak{X}(M)\,,$$

where $\{e_1, e_2, e_3\}$ is an orthonormal frame . The scalar curvature is given by $s_g = \mathrm{tr}_g \mathrm{Ric}^g$.

For further reference, we in introduce the Kulkarni-Nomizu product $⊘$ acting on symmetric tensors $A, B \in \Gamma(T^*M \otimes T^*M)$ as follows:

$$(A ⊘ B)(v_1, v_2, v_3, v_4) = A(v_1, v_3)B(v_2, v_4) + A(v_2, v_4)B(v_1, v_3) \\ - A(v_1, v_4)B(v_2, v_3) - A(v_2, v_3)B(v_1, v_4)\,,$$

for every $v_1, v_2, v_3, v_4 \in \mathfrak{X}(M)$. In dimension three, we can write the Riemannian tensor in terms of the Ricci tensor and the scalar curvature:

$$\mathcal{R}^g = -g ⊘ \mathrm{Ric}^g + \frac{1}{4} s_g\, g ⊘ g\,. \tag{2.1}$$

Alternatively, we can write the Riemann tensor as:

$$\mathcal{R}^g_{v_1,v_2} = \frac{1}{2} s_g\, v_1 \wedge v_2 + v_2 \wedge \mathrm{Ric}^g(v_1) + \mathrm{Ric}^g(v_2) \wedge v_1\,, \qquad v_1, v_2 \in \mathfrak{X}(M)\,. \tag{2.2}$$

Using the previous formula, it is easy to show that the contraction $\mathcal{R}^g \circ_g \mathcal{R}^g$ is given by:

$$\mathcal{R}^g \circ_g \mathcal{R}^g = -\mathrm{Ric}^g \circ_g \mathrm{Ric}^g + s_g \mathrm{Ric}^g + \left(|\mathrm{Ric}^g|^2_g - \frac{1}{2} s_g^2\right) g\,, \tag{2.3}$$

where we have defined:

$$\mathrm{Ric}^g \circ_g \mathrm{Ric}^g(v_1, v_2) := g(\mathrm{Ric}^g(v_1), \mathrm{Ric}^g(v_2))\,, \qquad v_1, v_2 \in \mathfrak{X}(M)\,.$$

In particular, the norm of $\mathcal{R}^g$ is given by:

$$|\mathcal{R}^g|^2_g = \frac{1}{2} \mathrm{tr}_g(\mathcal{R}^g \circ_g \mathcal{R}^g) = |\mathrm{Ric}^g|^2_g - \frac{1}{4} s_g^2\,. \tag{2.4}$$

We also recall the standard contracted second Bianchi identity

$$\mathrm{d}^*_{\nabla^g} \mathcal{R}^g(v_1, v_2, v_3) = d_{\nabla^g} \mathrm{Ric}^g(v_2, v_3, v_1)\,, \qquad v_1, v_2, v_3 \in \mathfrak{X}(M)\,. \tag{2.5}$$

Using these formulas, the three-dimensional Heterotic soliton system given in Equations (1.1), (1.2), and (1.3) can be further expanded and simplified.

Proposition 2.1. *A pair (g, ϕ) is a non-flat Heterotic soliton with vanishing torsion on a three-dimensional manifold M if and only if:*

$$-\kappa \mathrm{Ric}^g \circ_g \mathrm{Ric}^g + (1 + \kappa s_g)\mathrm{Ric}^g + \frac{1}{3}\left(-s_g - \frac{3\kappa}{4} s_g^2 - |\mathrm{d}\phi|^2_g + e^{2\phi}\right) g + \nabla^g \mathrm{d}\phi = 0\,, \tag{2.6}$$

$$\mathrm{d}^*_{\nabla^g}\left(-g ⊘ \mathrm{Ric}^g + \frac{1}{4} s_g\, g ⊘ g\right) + \left(-g ⊘ \mathrm{Ric}^g + \frac{1}{4} s_g\, g ⊘ g\right)(\mathrm{d}\phi) = 0\,, \tag{2.7}$$

$$s_g - 3\delta^g \mathrm{d}\phi - 2|\mathrm{d}\phi|^2_g + \frac{1}{2} e^{2\phi} = 0\,. \tag{2.8}$$

Proof. First, subtracting twice the dilaton equation (1.3) from the trace of the Einstein equation (1.1) yields (2.8).

On the other hand, by the general formula (2.1), the Yang-Mills equation (1.2) is equivalent to (2.7).

Now, using (2.3), the Einstein equation (1.1) can be equivalently written

$$-\kappa \mathrm{Ric}^g \circ_g \mathrm{Ric}^g + (1 + \kappa s_g)\mathrm{Ric}^g + \left(\kappa\, |\mathrm{Ric}^g|^2_g - \frac{\kappa}{2} s_g^2 - \frac{1}{2} e^{2\phi}\right) g + \nabla^g \mathrm{d}\phi = 0\,. \tag{2.9}$$

Taking the trace of (2.9) shows

$$\kappa\,|\mathrm{Ric}^g|^2_g = \frac{1}{2}\left(-s_g + \frac{\kappa}{2}s_g^2 + \frac{3}{2}e^{2\phi} + \delta^g \mathrm{d}\phi\right) = \frac{1}{2}\left(-\frac{2}{3}s_g + \frac{\kappa}{2}s_g^2 + \frac{5}{3}e^{2\phi} - \frac{2}{3}|\mathrm{d}\phi|^2_g\right),$$

where in the second equality we have used (2.8) in order to express $\delta^g \mathrm{d}\phi$. Substituting back into (2.9) yields

$$-\kappa \mathrm{Ric}^g \circ_g \mathrm{Ric}^g + (1+\kappa s_g)\mathrm{Ric}^g + \frac{1}{3}\left(-s_g - \frac{3\kappa}{4}s_g^2 - |\mathrm{d}\phi|^2_g + e^{2\phi}\right) g + \nabla^g \mathrm{d}\phi = 0\,,$$

which is (2.6). Doing the computations backwards proves the converse. □

In the following, we will consider the two formulations of the three-dimensional Heterotic soliton presented (and, occasionally, a combination of them) as most convenient.

Corollary 2.2. *Let (g,ϕ) be a three-dimensional Heterotic soliton with vanishing torsion. Then:*

$$\mathrm{Ric}^g(\mathrm{d}\phi) = \frac{1}{2}\mathrm{d}s_g. \tag{2.10}$$

Proof. Let $\{e_1, e_2, e_3\}$ be a local orthonormal frame. The Yang–Mills equation reads

$$-\nabla^g_{e_j}\mathcal{R}^g(e_j, v_1, v_2, v_3) + \mathcal{R}^g(\mathrm{d}\phi, v_1, v_2, v_3) = 0, \qquad v_1, v_2, v_3 \in \mathfrak{X}(M)\,,$$

with the summation convention over repeating subscripts. Setting $v_1 = v_2 = e_i$ and summing over i gives

$$-\nabla^g_{e_j}\left(\mathcal{R}^g(e_j, e_i, e_i, v_3)\right) + \mathcal{R}^g(\mathrm{d}\phi, e_i, e_i, v_3) = 0\,.$$

Since $\mathcal{R}^g(e_i,\cdot,\cdot,e_i) = \mathrm{Ric}^g$ by definition, using the symmetries of $\mathcal{R}^g$ we obtain

$$-\nabla^g_{e_j}\mathrm{Ric}^g(e_j, v_3) + \mathrm{Ric}^g(\mathrm{d}\phi, v_3) = 0\,.$$

The contracted second Bianchi identity reads $\nabla^g_{e_j}\mathrm{Ric}^g(e_j, v_3) = \frac{1}{2}\,\mathrm{d}s_g(v_3)$, which together with the previous equation gives (2.10). □

Remark 2.3. Viewing the Riemann tensor $\mathcal{R}^g$ as a section of $\Omega^2(M)\otimes\Omega^2(M)$, the Yang–Mills equation becomes an equation in $\Omega(M)\otimes\Omega^2(M)$. Via the Hodge star operator, we obtain an isomorphism $\Omega^1(M)\otimes\Omega^2(M)\cong\Omega^1(M)\otimes\Omega^1(M)$, which decomposes into trace, skew-symmetric, and traceless symmetric parts. Identity (2.10) is precisely the trace component of this decomposition.

We proceed to characterize traceless Heterotic solitons with constant dilaton.

Proposition 2.4. *A non-flat compact Heterotic soliton with vanishing torsion (g,ϕ) has constant dilaton if and only if it is hyperbolic, in which case $\kappa s_g = -24$, and $\kappa e^{2\phi} = 48$.*

Proof. First, assume that ϕ is constant. Then the system (2.6)–(2.8) reduces to:

$$-\kappa \mathrm{Ric}^g \circ_g \mathrm{Ric}^g + (1+\kappa s_g)\mathrm{Ric}^g - \Big(\frac{\kappa}{4}s_g^2 + s_g\Big)g = 0, \tag{2.11}$$

$$d_{\nabla^g}\mathrm{Ric}^g = 0, \tag{2.12}$$

$$s_g + \frac{1}{2}e^{2\phi} = 0\,, \tag{2.13}$$

where in the first equation we used (2.13) in order to eliminate $e^{2\phi}$ and in the second equation we used (2.5). Equation (2.13) shows that s_g is a negative constant, and Equation (2.12) says that Ric^g is a Codazzi tensor, which has constant eigenvalues thanks to

(2.11). By [4, §16.12], (M,g) is either Einstein or locally a product $\mathbb{R}\times\Sigma$ where Σ is a surface of constant curvature. In either case, the eigenvalues are globally constant, and, by Equation (2.11), each eigenvalue λ satisfies the quadratic equation

$$\kappa\lambda^2-(1+\kappa s_g)\lambda+\left(\frac{\kappa}{4}s_g^2+s_g\right)=0\,. \tag{2.14}$$

We now examine the two possible geometric structures.

If (M,g) is locally a product $\mathbb{R}\times\Sigma$, then Ric^g has eigenvalues $(0,\mu,\mu)$ with $\mu\neq 0$ and $s_g=2\mu$. Setting $\lambda=0$ in (2.14) gives $\frac{\kappa}{4}s_g^2+s_g=0$, whence $s_g=-\frac{4}{\kappa}$ (since $s_g\neq 0$). Thus $\mu=\frac{1}{2}s_g=-\frac{2}{\kappa}$. Now the eigenvalue μ must also satisfy (2.14); but substituting $\mu=-\frac{2}{\kappa}$ and $s_g=-\frac{4}{\kappa}$ into the left-hand side of (2.14) gives

$$\kappa\cdot\frac{4}{\kappa^2}-\left(1-\kappa\cdot\frac{4}{\kappa}\right)\left(-\frac{2}{\kappa}\right)+\left(\frac{\kappa}{4}\cdot\frac{16}{\kappa^2}-\frac{4}{\kappa}\right)=\frac{4}{\kappa}-\frac{6}{\kappa}+\frac{4}{\kappa}-\frac{4}{\kappa}=-\frac{2}{\kappa}\neq 0\,,$$

a contradiction. Hence, this case cannot occur.

If (M,g) is Einstein, then $\mathrm{Ric}^g=\frac{1}{3}s_g g$. Substituting $\lambda=\frac{1}{3}s_g$ into (2.14) yields

$$\frac{1}{3}s_g\left(\frac{\kappa}{12}s_g+2\right)=0\,.$$

Since $s_g\neq 0$ (otherwise the soliton would be flat), we obtain $s_g=-\frac{24}{\kappa}$. Consequently $e^{2\phi}=-2s_g=\frac{48}{\kappa}$, and $\mathrm{Ric}^g=-\frac{8}{\kappa}\,g$, i.e., (g,ϕ) is hyperbolic. □

3. Linearization of certain curvature operators

In this section, we study the linearization of certain curvature operators that appear in the three-dimensional Heterotic soliton system, as a preliminary step toward the linearization of the Heterotic soliton system in Section 4.2. Let $h\in\Gamma(T^*M\otimes T^*M)$. For a given Riemannian metric g, the differential of a given curvature operator along h at g will be denoted by $\mathrm{d}_g(-)(h)$. Since $\owedge$ is linear and symmetric, the variation of the Riemann tensor is simply:

$$\mathrm{d}_g\mathcal{R}^g(h)=-h\owedge\mathrm{Ric}^g-g\owedge\mathrm{d}_g\mathrm{Ric}^g(h)+\frac{1}{4}\mathrm{d}_g s_g(h)\,g\owedge g+\frac{1}{2}s_g\,g\owedge h. \tag{3.1}$$

Standard computations give the following identities (see, e.g., [4, §1.174] or [5]):

$$\mathrm{d}_g\mathrm{Ric}^g(h)=\frac{1}{2}\Delta_L^g h-\nabla^{S,g}\nabla^{g*}h-\frac{1}{2}\nabla^g\mathrm{d}\,\mathrm{tr}_g(h)\,, \tag{3.2}$$

$$\mathrm{d}_g s_g(h)=\delta^g\mathrm{d}(\mathrm{tr}_g h)+\delta^g\nabla^{g*}h-g(h,\mathrm{Ric}^g)\,, \tag{3.3}$$

where ∇^{g*} is the formal adjoint of ∇^g, $\nabla^{S,g}$ is the symmetrization of ∇^g and

$$\Delta_L^g h=\nabla^{g*}\nabla^g h+h\circ_g\mathrm{Ric}^g+\mathrm{Ric}^g\circ_g h-2\mathcal{R}_0(h)$$

is the Lichnerowicz Laplacian, with $\mathcal{R}_0$ defined through

$$\mathcal{R}_0(h)(v_1,v_2)=g(\mathcal{R}(\cdot,v_1,v_2,\cdot),h)\,.$$

In the following, assume that the background metric g is Einstein. These assumptions give the following standard relations:

$$\mathcal{R}^g=-\frac{1}{12}s_g\,g\owedge g\,,\qquad \mathrm{Ric}^g=\frac{1}{3}s_g\,g\,. \tag{3.4}$$

Lemma 3.1. *Let (M,g) be a three-dimensional Einstein manifold. Then, the linearizations of $\mathcal{R}^g \circ_g \mathcal{R}^g$ and $|\mathcal{R}|^2_g$ in the direction of $h \in \Gamma(T^*M \otimes T^*M)$ are given by:*

$$\mathrm{d}_g(\mathcal{R}^g \circ_g \mathcal{R}^g)(h) = \frac{1}{3} s_g\, \mathrm{d}_g\mathrm{Ric}^g(h) - \frac{1}{18} s_g^2\, h\,, \tag{3.5}$$

$$\mathrm{d}_g(|\mathcal{R}|^2_g)(h) = \frac{1}{6} s_g\, \mathrm{d}_g s_g(h)\,, \tag{3.6}$$

where $\mathrm{d}_g\mathrm{Ric}^g(h)$ and $\mathrm{d}_g s_g(h)$ are the linearizations of the Ricci tensor and the scalar curvature.

Proof. Set $3\lambda = s_g$. From (2.3), we compute:

$$\begin{aligned}\mathrm{d}_g(\mathcal{R}^g \circ_g \mathcal{R}^g)(h) =& -\mathrm{d}_g(\mathrm{Ric}^g \circ_g \mathrm{Ric}^g)(h) + \mathrm{d}_g(s_g\mathrm{Ric}^g)(h)\\ &+ \mathrm{d}_g|\mathrm{Ric}^g|^2_g(h) g - s_g \mathrm{d}_g s_g(h) g + (|\mathrm{Ric}^g|^2_g - \frac{1}{2}s_g^2)h\,. \end{aligned} \tag{3.7}$$

We evaluate term-wise on the Einstein background, and we consider the linearization of $\mathrm{Ric}^g \in \Gamma(T^*M \otimes T^*M)$ as a symmetric two-form on M, rather than as an endomorphism of TM. Hence, for the first term we have:

$$\mathrm{d}_g(\mathrm{Ric}^g \circ_g \mathrm{Ric}^g)(h) = -\lambda^2 h + 2\lambda\, \mathrm{d}_g\mathrm{Ric}^g(h)\,.$$

where we have taken into account that the *composition* $(-)\circ_g(-)$ of symmetric two-forms involves the (inverse) metric and must therefore be linearized, yielding the first term on the right-hand side of the previous equation. More precisely, we can write:

$$(\mathrm{Ric}^g \circ_g \mathrm{Ric}^g)(v_1, v_2) = g^*(\mathrm{Ric}^g(v_1), \mathrm{Ric}^g(v_2))\,, \qquad v_1, v_2 \in \mathfrak{X}(M)\,,$$

where g^* denotes the metric induced by g on T^*M. Linearizing the *inverse* metric g^* and evaluating on the given Einstein metric, yields the term $-\lambda^2 h$ in the equation above. For the second term, we have:

$$\mathrm{d}_g(s_g\mathrm{Ric}^g)(h) = \lambda\, \mathrm{d}_g s_g(h)\, g + 3\lambda\, \mathrm{d}_g\mathrm{Ric}^g(h)\,.$$

For the third term, we compute:

$$\begin{aligned}\mathrm{d}_g|\mathrm{Ric}^g|^2_g(h) &= 2g^*(\mathrm{Ric}^g, \mathrm{d}_g\mathrm{Ric}^g(h)) - 2g^*(h, \mathrm{Ric}^g \circ_g \mathrm{Ric}^g)\\ &= 2\lambda\, \mathrm{tr}_g[\mathrm{d}_g\mathrm{Ric}^g(h)] - 2\lambda^2\, \mathrm{tr}_g h\,,\end{aligned}$$

where we have taken into account that the norm square of Ric^g involves twice the induced metric g^*. Using this equation, the second line in Equation (3.7) simplifies as follows:

$$\begin{aligned}&\mathrm{d}_g|\mathrm{Ric}^g|^2_g(h) g - s_g \mathrm{d}_g s_g(h) g + (|\mathrm{Ric}^g|^2_g - \frac{1}{2}s_g^2)h =\\ &= \left(2\lambda\, \mathrm{tr}_g[\mathrm{d}_g\mathrm{Ric}^g(h)] - 2\lambda^2\, \mathrm{tr}_g h - 3\lambda\, \mathrm{d}_g s_g(h)\right) g - \frac{3}{2}\lambda^2 h\,.\end{aligned}$$

Altogether, we obtain:

$$\mathrm{d}_g(\mathcal{R}^g \circ_g \mathcal{R}^g)(h) = \lambda\, \mathrm{d}_g\mathrm{Ric}^g(h) - \frac{1}{2}\lambda^2 h + \left[2\lambda \mathrm{tr}_g(\mathrm{d}_g\mathrm{Ric}^g(h)) - 2\lambda^2 \mathrm{tr}_g h - 2\lambda\, \mathrm{d}_g s_g(h)\right]\, g\,,$$

but, since the linearized scalar curvature satisfies

$$\mathrm{d}_g s_g(h) = -g(h, \mathrm{Ric}^g) + \mathrm{tr}_g(\mathrm{d}_g\mathrm{Ric}^g(h)) = -\lambda\, \mathrm{tr}_g h + \mathrm{tr}_g(\mathrm{d}_g\mathrm{Ric}^g(h))\,,$$

the expression in brackets above cancels identically. Thus:

$$\mathrm{d}_g(\mathcal{R}^g \circ_g \mathcal{R}^g)(h) = \lambda \mathrm{d}_g\mathrm{Ric}^g(h) - \frac{1}{2}\lambda^2 h\,.$$

Substituting back λ, (3.5) follows. Taking the trace of (3.5) yields

$$\begin{aligned}\mathrm{tr}_g[\mathrm{d}_g(\mathcal{R}^g \circ_g \mathcal{R}^g)(h)] &= \lambda \mathrm{tr}_g[\mathrm{d}_g\mathrm{Ric}^g(h)] - \frac{1}{2}\lambda^2\mathrm{tr}_g h \\ &= \lambda\, \mathrm{d}_g s_g(h) + \lambda g(h, \mathrm{Ric}^g(h)) - \frac{1}{2}\lambda^2\mathrm{tr}_g h = \lambda\, \mathrm{d}_g s_g(h) + \frac{1}{2}\lambda^2\mathrm{tr}_g h\,.\end{aligned}$$

On the other hand, linearizing the relation $\mathrm{d}_g|\mathcal{R}^g|^2_g(h) = \frac{1}{2}\mathrm{d}_g\mathrm{tr}_g(\mathcal{R}^g \circ_g \mathcal{R}^g)$ gives

$$\begin{aligned}\mathrm{d}_g|\mathcal{R}^g|^2_g(h) &= -\frac{1}{2}g(h, \mathcal{R}^g \circ_g \mathcal{R}^g) + \frac{1}{2}\mathrm{tr}_g[\mathrm{d}_g(\mathcal{R}^g \circ_g \mathcal{R}^g)(h)] \\ &= -\frac{1}{4}\lambda^2\mathrm{tr}_g(h) + \frac{1}{2}\mathrm{tr}_g[\mathrm{d}_g(\mathcal{R}^g \circ_g \mathcal{R}^g)(h)]\,,\end{aligned}$$

where we have used $\mathcal{R}^g \circ_g \mathcal{R}^g = \frac{1}{2}\lambda^2 g$ by (2.3). Combining both expression produces

$$\mathrm{d}_g|\mathcal{R}^g|^2_g(h) = \frac{1}{2}\lambda\, \mathrm{d}_g s_g(h)\,.$$

Substituting back λ, (3.6) follows, completing the proof. □

Remark 3.2. A useful sanity check is linearizing $|\mathcal{R}^g|^2_g$ directly from its alternative expression (2.4):

$$\begin{aligned}\mathrm{d}_g|\mathcal{R}^g|^2_g(h) &= \mathrm{d}_g\left(|\mathrm{Ric}^g|^2_g - \frac{1}{4}s_g^2\right) \\ &= -2g^*(h, \mathrm{Ric}^g \circ_g \mathrm{Ric}^g) + 2g^*(\mathrm{Ric}^g, \mathrm{d}_g\mathrm{Ric}^g(h)) - \frac{1}{2}s_g\, \mathrm{d}_g s_g(h) \\ &= -\frac{2}{9}s_g^2\,\mathrm{tr}_g h + \frac{2}{3}s_g\,\mathrm{tr}_g[\mathrm{d}_g\mathrm{Ric}^g(h)] - \frac{1}{2}s_g\,\mathrm{d}_g s_g(h) = \frac{1}{6}s_g\mathrm{d}_g s_g(h)\,,\end{aligned}$$

where we have used $\mathrm{d}_g s_g(h) = -g(h, \mathrm{Ric}^g) + \mathrm{tr}_g[\mathrm{d}_g\mathrm{Ric}^g(h)]$ and (3.4). This is precisely Equation (3.6).

4. Rigidity of hyperbolic Heterotic solitons

In this section, we study the local structure of the moduli of the torsionless Heterotic soliton system around a hyperbolic soliton, showing that they are isolated points in moduli space and therefore rigid.

4.1. **Existence of a slice.** Let $\mathrm{Conf}(M)$ denote the configuration space of the torsionless Heterotic system, that is:

$$\mathrm{Conf}(M) = \mathrm{Met}(M) \times C^\infty(M)\,.$$

which we consider as a tame Fréchet manifold [9]. The diffeomorphism group $\mathrm{Diff}(M)$ acts smoothly on the configuration space by pullback:

$$\Psi\colon \mathrm{Conf}(M) \times \mathrm{Diff}(M) \to \mathrm{Conf}(M)\,, \qquad ((g,\phi), u) \mapsto (u^*g, \phi\circ u)\,.$$

For a fixed soliton (g, ϕ) we introduce the orbit map:

$$\Psi_{g,\phi}\colon \mathrm{Diff}(M) \to \mathrm{Conf}(M), \qquad u \mapsto (u^*g, \phi \circ u)\,.$$

Its differential at the identity $e \in \mathrm{Diff}(M)$ can be shown to be:

$$\mathrm{d}_e\Psi_{g,\phi}\colon \mathfrak{X}(M) \to T_{g,\phi}\mathrm{Conf}(M), \qquad v \mapsto (\mathcal{L}_v g,\, \mathrm{d}\phi(v))\,.$$

Thus, its formal L^2-adjoint becomes:

$$\mathrm{d}_e\Psi^*_{g,\phi}\colon T_{(g,\phi)}\mathrm{Conf}(M) \to \mathfrak{X}(M), \qquad (h, \xi) \mapsto 2\nabla^{g*}h + \xi\mathrm{d}\phi\,.$$

The symbol of $\mathrm{d}_e\Psi_{g,\phi}$ is injective, and hence it is overdetermined elliptic. Then, from the standard elliptic theory on closed manifolds (see for example [12]), the following L^2-orthogonal decomposition follows:

$$T_{g,\phi}\mathrm{Conf}(M) = \mathrm{Im}(\mathrm{d}_e\Psi_{g,\phi}) \oplus \mathrm{Ker}(\mathrm{d}_e\Psi^*_{g,\phi}),$$

where both factors are closed subspaces.

As a direct consequence of the general theory developed by Diez and Rudolph [6], this action admits a slice [6, Definition 2.2]. To prove this, we first observe that, by a celebrated result of [8], the action of the diffeomorphism group on the Riemannian manifolds of a compact manifold admits a slice $S_g \subset \mathrm{Met}(M)$ through every metric $g \in \mathrm{Met}(M)$. The stabilizer of this action is the isometry group $\mathrm{Iso}(M, g)$ of g, which is well-known to be compact by the compactness of M. Hence, by [6, Theorem 3.15] the action of $\mathrm{Iso}(M, g)$ on $C^\infty(M)$ admits a smooth slice, which used in combination with [6, Proposition 3.29] proves the existence of a smooth slice for the action of $\mathrm{Diff}(M)$ on $\mathrm{Conf}(M)$ in the tame Fréchet category. In particular, we obtain the following result.

Proposition 4.1. *Let $(g, \phi) \in \mathrm{Conf}(M)$. There exists a smooth submanifold $S_{g,\phi} \subset \mathrm{Conf}(M)$ and an open neighbourhood $U_{g,\phi} \subset \mathrm{Conf}(M)$ of (g, ϕ) homeomorphic to $S_{g,\phi}/\mathcal{I}_{g,\phi}$, where $\mathcal{I}_{g,\phi}$ denotes the stabilizer of (g, ϕ) in $\mathrm{Diff}(M)$. If $\mathcal{I}_{g,\phi}$ is trivial, then $U_{g,\phi}$ is a smooth tame Fréchet manifold modeled on the tame Fréchet vector space $\mathrm{Ker}(\mathrm{d}_e\Psi^*_{g,\phi}) \subset T_{g,\phi}\mathrm{Conf}(M)$.*

Using the equations of the three-dimensional Heterotic system, we introduce the following smooth maps of Fréchet manifolds:

$$\begin{aligned}
\mathcal{E}_{\mathrm{E}} &: \mathrm{Conf}(M) \to \Gamma(T^*M \otimes T^*M)\,, && (g, \phi) \mapsto \mathrm{Ric}^g + \nabla^g \mathrm{d}\phi - \frac{1}{2}e^{2\phi} g + \kappa \mathcal{R}^g \circ_g \mathcal{R}^g\,,\\
\mathcal{E}_{\mathrm{YM}} &: \mathrm{Conf}(M) \to \Gamma(T^*M \otimes \Omega^2(M))\,, && (g, \phi) \mapsto \mathrm{d}^*_{\nabla^g}\mathcal{R}^g + \mathcal{R}^g(\mathrm{d}\phi)\,,\\
\mathcal{E}_{\mathrm{D}} &: \mathrm{Conf}(M) \to C^\infty(M)\,, && (g, \phi) \mapsto \delta^g \mathrm{d}\phi + |\mathrm{d}\phi|^2_g - e^{2\phi} + \kappa|\mathcal{R}^g|^2_g\,.
\end{aligned}$$

We set:

$$\mathcal{E} = (\mathcal{E}_{\mathrm{E}}, \mathcal{E}_{\mathrm{YM}}, \mathcal{E}_{\mathrm{D}})\colon \mathrm{Conf}(M) \to \Gamma(T^*M \otimes T^*M) \times \Gamma(T^*M \otimes \Omega^2(M)) \times C^\infty(M)\,.$$

The map $\mathcal{E}$ is diffeomorphism-equivariant, that is:

$$\mathcal{E}(u^*g, \phi \circ u) = (u^*\mathcal{E}_{\mathrm{E}}(g, \phi),\, u^*\mathcal{E}_{\mathrm{YM}}(g, \phi),\, \mathcal{E}_{\mathrm{D}}(g, \phi) \circ u)\,.$$

Consequently, the moduli space of three-dimensional Heterotic solitons with vanishing torsion is defined as:

$$\mathfrak{M}(M) := \frac{\mathcal{E}^{-1}(0)}{\mathrm{Diff}(M)}\,,$$

equipped with the quotient topology induced by the Fréchet topology on $\mathrm{Conf}(M)$. Due to the existence of a slice $S_{g,\phi}$ through every $(g, \phi) \in \mathrm{Conf}(M)$, if $(g, \phi) \in \mathcal{E}^{-1}(0)$ then $[g, \phi] \in \mathfrak{M}(M)$ has an open neighbourhood $U_{[g,\phi]}$ homeomorphic to $S_{g,\phi} \cap \mathcal{E}^{-1}(0)$:

$$U_{[g,\phi]} \simeq S_{g,\phi} \cap \mathcal{E}^{-1}(0)\,,$$

after possibly shrinking $S_{g,\phi}$. The *candidate* tangent space of $S_{g,\phi} \cap \mathcal{E}^{-1}(0)$ is precisely what we call, following Koiso [10], the vector space of *essential deformations* $\mathbb{E}_{(g,\phi)}$ of (g, ϕ), that is:

$$\mathbb{E}_{(g,\phi)} := \mathrm{Ker}(\mathrm{d}_{(g,\phi)}\mathcal{E}) \cap \mathrm{Ker}(\mathrm{d}_e\Psi^*_{g,\phi}).$$

This, together with the existence of a local Kuranishi model by [6, Theorem 5.3], provides us with the rigidity criteria that we will apply in the next subsection.

Corollary 4.2. *Let (g, ϕ) be such that $\mathbb{E}_{(g,\phi)} = 0$. Then, there exists an open neighbourhood of $[g, \phi] \in \mathfrak{M}(M)$ that is homeomorphic to a point. In particular, $[g, \phi]$ is an isolated point in moduli space.*

4.2. **Essential deformations.** Let (g, ϕ) be a three-dimensional torsionless Heterotic soliton with constant dilaton. By Proposition 2.4, it is hyperbolic Einstein:

$$\mathrm{Ric}^g = \frac{1}{3} s_g g\,, \qquad \kappa s_g = -24\,, \qquad \kappa e^{2\phi} = 48\,. \tag{4.1}$$

The existence of a slice implies that the tangent space to the moduli space at $[g, \phi]$ is contained in the intersection of the kernel of the linearized equations with the slice.

Let $(h, \xi) \in \mathbb{E}_{(g,\phi)}$. We proceed by linearizing the scalar identity:

$$s_g + |\mathrm{d}\phi|^2_g - \frac{5}{2} e^{2\phi} + 3\kappa |\mathcal{R}^g|^2_g = 0\,, \tag{4.2}$$

which follows from adding the dilaton equation (1.3) to the trace of the Einstein equation (1.1). Using that $\mathrm{d}\phi = 0$ on the given hyperbolic background (g, ϕ) together with Equation (3.6) yields:

$$0 = \mathrm{d}_g s_g(h) - 5 e^{2\phi} \xi + 3\kappa\, \mathrm{d}_g(|\mathcal{R}^g|^2_g)(h) = (1 + \frac{\kappa}{2} s_g) \mathrm{d}_g s_g(h) - 5 e^{2\phi} \xi\,. \tag{4.3}$$

Taking the exterior derivative of this equation, we obtain:

$$(1 + \frac{\kappa}{2} s_g) \mathrm{d}[\mathrm{d}_g s_g(h)] = 5 e^{2\phi}\, \mathrm{d}\xi\,.$$

On the other hand, linearizing Equation (2.10) at (g, ϕ) gives:

$$\mathrm{d}[\mathrm{d}_g s_g(h)] = \frac{2}{3} s_g\, \mathrm{d}\xi\,,$$

which substituted back into the previous equation yields:

$$0 = ((1 + \frac{\kappa}{2} s_g) \frac{2}{3} s_g - 5 e^{2\phi}) \mathrm{d}\xi\,,$$

Substituting now the background values $\kappa s_g = -24$, $\kappa e^{2\phi} = 48$ in this equation, we obtain:

$$-\frac{64}{\kappa}\, \mathrm{d}\xi = 0\,,$$

and hence ξ is constant. We prove next that $\xi = 0$.

Proposition 4.3. *Let (g, ϕ) be a hyperbolic Heterotic soliton with vanishing torsion. For every $(h, \xi) \in \mathbb{E}_{(g,\phi)}$ we have $\xi = 0$ and $\mathrm{d}_g s_g(h) = 0$.*

Proof. Since ξ is constant, the linearization of the dilaton equation (1.3) at (g, ϕ) reads:

$$-2 e^{2\phi} \xi + \kappa\, \mathrm{d}_g(|\mathcal{R}^g|^2_g)(h) = 0.$$

Using Lemma 3.1 together with the background values $\kappa s_g = -24$, $\kappa e^{2\phi} = 48$, this equation reduces to:

$$-\frac{96}{\kappa}\, \xi + \kappa \cdot \frac{1}{6} \cdot \frac{-24}{\kappa} \mathrm{d}_g s_g(h) = 0\,,$$

whence $\kappa\, \mathrm{d}_g s_g(h) = -24\,\xi$. On the other hand, linearizing the trace of the Einstein equation (1.1) yields:

$$\mathrm{d}_g s_g(h) - 3e^{2\phi}\xi + 2\kappa \cdot \frac{1}{6} s_g\, \mathrm{d}_g s_g(h) = 0.$$

Substituting the same background values and the expression for $\mathrm{d}_g s_g(h)$ gives

$$0 = -7\left(-\frac{24}{\kappa}\,\xi\right) - \frac{144}{\kappa}\,\xi = \frac{24}{\kappa}\,\xi\,,$$

hence $\xi = 0$ and consequently $\mathrm{d}_g s_g(h) = 0$. □

Next, we show that essential deformations are automatically transverse-traceless.

Lemma 4.4. *Let (g, ϕ) be a three-dimensional hyperbolic Heterotic soliton. For every $(h, \xi) \in \mathbb{E}_{(g,\phi)}$ we have $\mathrm{tr}_g h = 0$.*

Proof. Since $\nabla^{g*}h = 0$ by the slice condition, the linearization (3.3) of the scalar curvature on an Einstein manifold simplifies to

$$\mathrm{d}_g s_g(h) = \delta^g \mathrm{d}\,(\mathrm{tr}_g h) - \frac{1}{3} s_g\, \mathrm{tr}_g h,$$

Proposition 4.3 gives $\mathrm{d}_g s_g(h) = 0$, hence

$$\delta^g \mathrm{d}\,(\mathrm{tr}_g h) - \frac{1}{3} s_g\, \mathrm{tr}_g h = 0\,.$$

Multiplying by $\mathrm{tr}_g h$ and integrating over M yields

$$0 = \int_M \left(\mathrm{tr}_g h\, \delta^g \mathrm{d}\,(\mathrm{tr}_g h) - \frac{1}{3} s_g\, (\mathrm{tr}_g h)^2\right) \nu_g = \int_M \left(|\mathrm{d}(\mathrm{tr}_g h)|^2_g - \frac{1}{3} s_g\, (\mathrm{tr}_g h)^2\right) \nu_g\,.$$

Since $s_g < 0$, the two terms in the integrand are non-negative, thus both are necessarily zero pointwise. In particular, $-s_g\,(\mathrm{tr}_g h)^2 = 0$, which implies $\mathrm{tr}_g h = 0$. □

We now characterize h as an infinitesimal Einstein deformation.

Lemma 4.5. *Let (g, ϕ) be a three-dimensional hyperbolic Heterotic soliton. For every $(h, \xi) \in \mathbb{E}_{(g,\phi)}$, h is an infinitesimal Einstein deformation.*

Proof. From Proposition 4.3 and Lemma 4.4 we know $\xi = 0$, $\mathrm{tr}_g h = 0$ and $\nabla^{g*}h = 0$. Then, linearizing (1.1) and using (3.5) yields

$$\begin{aligned} 0 &= \mathrm{d}_g \mathrm{Ric}^g(h) - \frac{1}{2} e^{2\phi} h + \kappa\, \mathrm{d}_g(\mathcal{R}^g \circ_g \mathcal{R}^g) \\ &= (1 + \frac{\kappa}{3} s_g)\, \mathrm{d}_g \mathrm{Ric}^g(h) - (\frac{1}{2} e^{2\phi} + \frac{\kappa}{18} s_g^2)\, h\,. \end{aligned}$$

Now, taking (4.1) into account, we obtain

$$0 = -7\mathrm{d}_g \mathrm{Ric}^g(h) - (\frac{24}{\kappa} + \frac{24^2}{18\kappa})\, h = -7\mathrm{d}_g \mathrm{Ric}^g(h) - \frac{56}{\kappa}\, h = -7\mathrm{d}_g \mathrm{Ric}^g(h) + \frac{7}{3} s_g h\,,$$

whence

$$\mathrm{d}_g \mathrm{Ric}^g(h) = \frac{1}{3} s_g h\,. \tag{4.4}$$

After all these substitutions, using (3.5) and (4.1), the linearized Einstein equation (1.1) reads after simplification:

$$\mathrm{d}_g \mathrm{Ric}^g(h) = \frac{1}{3} s_g h\,.$$

For a transverse-traceless tensor on an Einstein manifold, (3.2) gives

$$\mathrm{d}_g\mathrm{Ric}^g(h) = \frac{1}{2}\Delta_L h = \frac{1}{2}\left(\nabla^{g*}\nabla^g h + \frac{2}{3}s_g h - 2\mathcal{R}_0(h)\right) , \tag{4.5}$$

where we have used $h \circ_g \mathrm{Ric}^g + \mathrm{Ric}^g \circ_g h = \frac{2}{3}s_g h$. From (4.4) and (4.5) we thus obtain

$$\nabla^{g*}\nabla^g h - 2\mathcal{R}_0(h) = 0 . \tag{4.6}$$

By [1] (see also [4, §12.30]), (4.6) is equivalent to h being an infinitesimal Einstein deformation. □

We are now ready to prove our first main result.

Theorem 4.6. *Let (g, ϕ) be a non-flat three-dimensional compact Heterotic soliton with vanishing torsion and constant dilaton. Then, the vector space $\mathbb{E}_{(g,\phi)}$ of essential deformations of (g, ϕ) is zero-dimensional.*

Proof. By (2.4), the metric g is hyperbolic. Let $(h, \xi) \in \mathbb{E}_{(g,\phi)}$ be an essential deformation of the hyperbolic soliton (g, ϕ). By Proposition 4.3, we have $\xi = 0$. Furthermore, by Lemma 4.5, h is an essential infinitesimal Einstein deformation of the hyperbolic metric g. Then, by a classical result of Koiso [11] (see also [4, §12.73]), it follows that $h = 0$, and thus $\mathbb{E}_{(g,\phi)} = 0$. □

5. Heterotic solitons with harmonic curvature

In this section, we prove that if the Riemann curvature tensor of a three-dimensional compact Heterotic soliton with vanishing torsion is divergence-free, then the dilaton is necessarily constant. Combined with Proposition 2.4, this shows that the only non-flat solutions in this class are the hyperbolic ones and adds a layer of difficulty to the construction of compact Heterotic solitons with non-constant dilaton.

Definition 5.1. A Riemannian manifold (M, g) has *harmonic curvature* if its Riemann tensor is divergence-free, i.e. $\mathrm{d}^*_{\nabla^g}\mathcal{R}^g = 0$.

In three dimensions, the harmonicity of the Riemann tensor is equivalent to the Ricci tensor being Codazzi (2.5), which, in turn, is equivalent to (M, g) being locally conformally flat with constant scalar curvature. We leave open the existence of locally conformally flat torsionless Heterotic solitons with non-constant dilaton, and therefore non-constant scalar curvature.

Theorem 5.2. *Let M be a closed three-dimensional manifold and (g, ϕ) a non-flat Heterotic soliton with vanishing torsion. If g has harmonic curvature, then (g, ϕ) is hyperbolic.*

Proof. Assume that (g, ϕ) satisfies the system (1.1), (1.2), and (1.3) on a compact oriented three-manifold M. By hypothesis $\mathrm{d}^*_{\nabla^g}\mathcal{R}^g = 0$, whence the scalar curvature s_g is constant. Then, from Corollary 2.2 we have:

$$\mathrm{Ric}^g(\mathrm{d}\phi) = 0 . \tag{5.1}$$

Let $U := \{p \in M \mid \mathrm{d}\phi_p \neq 0\}$ be the open subset of M where the gradient of ϕ is non-vanishing. Assume U is non-empty. On U define the unit 1-form:

$$u := \frac{1}{|\mathrm{d}\phi|_g}\mathrm{d}\phi .$$

From (5.1), Ric^g has eigenvalue 0 in the direction of $\mathrm{d}\phi$ on U.

On the other hand, since the harmonicity of $\mathcal{R}^g$ is equivalent to $\mathrm{d}^{\nabla^g}\mathrm{Ric}^g = 0$, the Yang–Mills equation (1.2) reduces to $\mathcal{R}^g(\mathrm{d}\phi) = 0$, or, equivalently, taking the interior product with an arbitrary tangent field $v_1 \in \mathfrak{X}(M)$ and using (2.2),

$$\mathrm{d}\phi \wedge \left(\frac{1}{2}s_g v_1 - \mathrm{Ric}^g(v_1)\right) = 0\,. \tag{5.2}$$

Therefore, Ric^g has eigenvalue $\frac{s_g}{2}$ in the directions orthogonal to $\mathrm{d}\phi$ on U.

Moreover, (5.1) and (5.2) together show that we can write the Ricci tensor on U as

$$\mathrm{Ric}^g = \frac{1}{2}s_g\left(g - u\otimes u\right)\,. \tag{5.3}$$

From (5.3) we calculate $\mathrm{Ric}^g \circ_g \mathrm{Ric}^g = \frac{s_g^2}{4}(g - u\otimes u)$ and

$$|\mathrm{Ric}^g|_g^2 = \frac{1}{2}s_g^2\,. \tag{5.4}$$

Substituting (2.4) and (5.4) in the scalar identity (4.2), we obtain at each point of U:

$$|\mathrm{d}\phi|_g^2 - \frac{5}{2}e^{2\phi} + s_g + \frac{3\kappa}{4}s_g^2 = 0\,.$$

Define the function $f := |\mathrm{d}\phi|_g^2 - \frac{5}{2}e^{2\phi}$ on M. The equality above shows that

$$f = -s_g - \frac{3\kappa}{4}s_g^2$$

at each point of U, and, since s_g is constant on M, f is constant on U. However, f is, by construction, also locally constant on the interior of the complement of U in M. But the union of U with the interior of its complement is dense on M, thus it follows that the above function is locally constant on M, hence constant since M is connected. Therefore, ϕ takes the same value at a maximum and minimum, i.e. ϕ is constant. Then, by Proposition 2.4 the soliton is hyperbolic. □

Combining this result with Corollary 4.2 and Theorem 4.6 we obtain Theorem 1.1, as presented in the introduction.

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