Lectures on Kähler Geometry

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Contents

Introduction

These notes, based on a graduate course I gave at Hamburg University in 2003, are intended to students having basic knowledges of differential geometry. I assume, in particular, that the reader is familiar with following topics:

- differential manifolds, tensors, Lie groups;
- principal fibre bundles, vector bundles, connexions, holonomy groups;
- Riemannian metrics, de Rham decomposition theorem, Levi–Civita connexion, Killing vector fields.

This background material is well covered in the classical literature, and can be found for instance in [**8**], Ch. 1-4.

The main purpose of these notes is to provide a quick and accessible introduction to different aspects of Kähler geometry. They should hopefully be useful for graduate students in mathematics and theoretical physics. The text is self–contained with a few notable exceptions – the Newlander–Nirenberg theorem, the Hodge theorem, the Calabi conjecture, the Hirzebruch–Riemann–Roch formula, the Cheeger–Gromoll theorem and the Kodaira embedding theorem. I considered that including the proofs of these results would have add too much technicality in the presentation and would have doubled the volume of the text without bringing essentially new insights to our objects of interest.

The text is organized as follows. In the first part I quickly introduce complex manifolds, and in Part 2 I define Kähler manifolds from the point of view of Riemannian geometry. Most of the remaining material concerns compact manifolds. In Part 3 I review Hodge and Dolbeault theories, and give a simple way of deriving the famous Kähler identities. Part 4 is devoted to the Calabi conjecture and in Part 5 I obtain several vanishing results using Weitzenböck techniques. Finally, in Part 6, different aspects of Calabi–Yau manifolds are studied using techniques from algebraic geometry.

Most of the sections end up with a series of exercises whose levels of difficulty range from low to medium.

Part 1

Complex geometry

1. Complex structures and holomorphic maps

1.1. Preliminaries. Kähler manifolds may be considered as special Riemannian manifolds. Besides the Riemannian structure, they also have compatible symplectic and complex structures. Here are a few examples of Kähler manifolds:

- $(\mathbb{C}^m, \langle\ ,\ \rangle)$, where $\langle\ ,\ \rangle$ denotes the Hermitian metric $ds^2 = \mathrm{Re}(\sum dz_i d\bar{z}_i)$;
- any oriented 2–dimensional Riemannian manifold;
- the complex projective space $(\mathbb{C}\mathrm{P}^m, FS)$ endowed with the Fubini-Study metric;
- every projective manifold, that is, submanifold of $\mathbb{C}\mathrm{P}^m$ defined by homogeneous polynomials in $\mathbb{C}^{m+1}$.

We give here a short definition, which will be detailed later.

Definition 1.1. *A Kähler structure on a Riemannian manifold (M^n, g) is given by a 2–form Ω and a field of endomorphisms of the tangent bundle J satisfying the following*

- *algebraic conditions*
  a) *J is an almost complex structure: $J^2 = -Id$.*
  b) $g(X,Y) = g(JX,JY)\ \ \forall\ X,Y \in TM$.
  c) $\Omega(X,Y) = g(JX,Y)$.
- *analytic conditions*
  d) *the 2–form Ω is symplectic: $d\Omega = 0$.*
  e) *J is integrable in the sense that its Nijenhuis tensor vanishes (see (4) below).*

Condition a) requires the real dimension of M to be even. Obviously, given the metric and one of the tensors J and Ω, we can immediately recover the other one by the formula c).

Kähler structures were introduced by Erich Kähler in his article [**7**] with the following motivation. Given any Hermitian metric h on a complex manifold, we can express the fundamental two–form Ω in local holomorphic coordinates as follows:

$$\Omega = i \sum h_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta,$$

where

$$h_{\alpha\bar{\beta}} := h\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right).$$

He then noticed that the condition $d\Omega = 0$ is equivalent to the local existence of some function u such that

$$h_{\alpha\bar{\beta}} = \frac{\partial^2 u}{\partial z_\alpha \partial \bar{z}_\beta}.$$

In other words, the whole metric tensor is defined by a unique function! This remarkable (*bemerkenswert*) property of the metric allows one to obtain simple explicit expressions for the Christoffel symbols and the Ricci and curvature tensors, and "a long list of miracles occur then". The function u is called *Kähler potential.*

There is another remarkable property of Kähler metrics, which, curiously, Kähler himself did not seem to have noticed. Recall that every point x in a Riemannian manifold has a local coordinate system x_i such that the metric osculates to the Euclidean metric to the order 2 at x. These special coordinate systems are the *normal coordinates* around each point. Now, on a complex manifold with Hermitian metric, the existence of normal *holomorphic* coordinates around each point is equivalent to the metric being Kähler!

Kähler manifolds have found many applications in various domains like Differential Geometry, Complex Analysis, Algebraic Geometry or Theoretical Physics. To illustrate their importance let us make the following remark. With two exceptions (the so–called Joyce manifolds in dimensions 7 and 8), the only known compact examples of manifolds satisfying Einstein's equations

$$R_{\alpha\beta} = 0$$

(Ricci–flat in modern language) are constructed on Kähler manifolds. Generic Ricci–flat Kähler manifolds, also called *Calabi–Yau manifolds*, will be studied later on in these notes.

1.2. Holomorphic functions. A function $F = f + ig : U \subset \mathbb{C} \to \mathbb{C}$ is called holomorphic if it satisfies the Cauchy–Riemann equations:

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} \quad \text{and} \quad \frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} = 0.$$

Let j denote the endomorphism of $\mathbb{R}^2$ corresponding to the multiplication by i on $\mathbb{C}$ via the identification of $\mathbb{R}^2$ with $\mathbb{C}$ given by $z = x + iy \mapsto (x, y)$. The endomorphism j can be expressed in the canonical base as

$$j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The differential of F (viewed as real function $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$) is of course the linear map

$$F_* = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}.$$

Then it is easy to check that the Cauchy–Riemann relations are equivalent to the commutation relation $jF_* = F_* j$.

Similarly, we identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ via

$$(z_1, \ldots, z_m) = (x_1 + iy_1, \ldots, x_m + iy_m) \mapsto (x_1, \ldots, x_m, y_1, \ldots, y_m),$$

and denote by j_m the endomorphism of $\mathbb{R}^{2m}$ corresponding to the multiplication by i on $\mathbb{C}^m$:

$$j_m = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}.$$

A function $F : U \subset \mathbb{C}^n \to \mathbb{C}^m$ is then holomorphic if and only if the differential F_* of F as real map $F : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ satisfies $j_m F_* = F_* j_n$.

1.3. Complex manifolds. A complex manifold of complex dimension m is a topological space M with an open covering $\mathcal{U}$ such that for every point $x \in M$ there exists $U \in \mathcal{U}$ containing x and a homeomorphism $\phi_U : U \to \tilde{U} \subset \mathbb{C}^m$, such that for every intersecting $U, V \in \mathcal{U}$, the map between open sets of $\mathbb{C}^m$

$$\phi_{UV} := \phi_U \circ \phi_V^{-1}$$

is holomorphic. A pair (U, ϕ_U) is called a chart and the collection of all charts is called a *holomorphic structure.*

Important example. The complex projective space $\mathbb{C}\mathrm{P}^m$ can be defined as the set of complex lines of $\mathbb{C}^{m+1}$ (a line is a vector subspace of dimension one). If we define the equivalence relation $\sim$ on $\mathbb{C}^{m+1} - \{0\}$ by

$$(z_0, \dots, z_m) \sim (\alpha z_0, \dots, \alpha z_m), \qquad \forall\, \alpha \in \mathbb{C}^*,$$

then $\mathbb{C}\mathrm{P}^m = \mathbb{C}^{m+1} - \{0\}/\sim$. The equivalence class of $(z_0, \dots z_m)$ will be denoted by $[z_0 : \dots : z_m]$. Consider the open cover U_i, $i = 0, \dots, m$ of $\mathbb{C}\mathrm{P}^m$ defined by

$$U_i := \{[z_0 : \dots : z_m] \mid z_i \neq 0\}$$

and the maps $\phi_i : U_i \to \mathbb{C}^m$,

$$\phi_i([z_0 : \dots : z_m]) = \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i} \dots, \frac{z_m}{z_i}\right).$$

It is then an easy exercise to compute

$$\phi_i \circ \phi_j^{-1}(w_1, \dots, w_m) = \left(\frac{w_1}{w_i}, \dots, \frac{w_{i-1}}{w_i}, \frac{w_{i+1}}{w_i}, \dots, \frac{w_j}{w_i}, \frac{1}{w_i}, \frac{w_{j+1}}{w_i}, \dots, \frac{w_m}{w_i}\right),$$

which is obviously holomorphic on its domain of definition.

A function $F : M \to \mathbb{C}$ is called holomorphic if $F \circ \phi_U^{-1}$ is holomorphic for every $U \in \mathcal{U}$. This property is local. To check it in the neighborhood of a point x it is enough to check it for a single $U \in \mathcal{U}$ containing x.

The most important object on a complex manifold from the differential geometric point of view is the almost complex structure J, which is a field of endomorphisms of the tangent bundle defined as follows. For every $X \in T_xM$, choose $U \in \mathcal{U}$ containing x and define

$$J_U(X) = (\phi_U)_*^{-1} \circ j_n \circ (\phi_U)_*(X).$$

If we take some other $V \in \mathcal{U}$ containing x, then $\phi_{VU} = \phi_V \circ \phi_U^{-1}$ is holomorphic, and $\phi_V = \phi_{VU} \circ \phi_U$, so

$$\begin{aligned} J_V(X) &= (\phi_V)_*^{-1} \circ j_n \circ (\phi_V)_*(X) = (\phi_V)_*^{-1} \circ j_n \circ (\phi_{VU})_* \circ (\phi_U)_*(X) \\ &= (\phi_V)_*^{-1} \circ (\phi_{VU})_* \circ j_n \circ (\phi_U)_*(X) = (\phi_U)_*^{-1} \circ j_n \circ (\phi_U)_*(X) \\ &= J_U(X), \end{aligned}$$

thus showing that J_U does not depend on U and their collection is a well–defined tensor J on M. This tensor clearly satisfies $J^2 = -Id$.

Definition 1.2. *A (1,1)–tensor J on a differential manifold M satisfying $J^2 = -Id$ is called an almost complex structure. The pair (M, J) is then referred to as almost complex manifold.*

A complex manifold is thus in a canonical way an almost complex manifold. The converse is only true under some integrability condition (see Theorem 1.4 below).

1.4. The complexified tangent bundle. Let (M, J) be an almost complex manifold. We would like to diagonalize the endomorphism J. In order to do so, we have to complexify the tangent space. Define

$$TM^{\mathbb{C}} := TM \otimes_{\mathbb{R}} \mathbb{C}.$$

We extend all real endomorphisms and differential operators from TM to $TM^{\mathbb{C}}$ by $\mathbb{C}$–linearity. Let $T^{1,0}M$ (resp. $T^{0,1}M$) denote the eigenbundle of $TM^{\mathbb{C}}$ corresponding to the eigenvalue i (resp. $-i$) of J. The following algebraic lemma is an easy exercise.

Lemma 1.3. *One has*

$$T^{1,0}M = \{X - iJX \mid X \in TM\} \ , \qquad T^{0,1}M = \{X + iJX \mid X \in TM\}.$$

and $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$.

The famous Newlander–Nirenberg theorem can be stated as follows:

Theorem 1.4. *Let (M, J) be an almost complex manifold. The almost complex structure J comes from a holomorphic structure if and only if the distribution $T^{0,1}M$ is integrable.*

Proof. We will only prove here the "only if" part. The interested reader can find the proof of the hard part for example in [**5**].

Suppose that J comes from a holomorphic structure on M. Consider a local chart (U, ϕ_U) and let $z_\alpha = x_\alpha + iy_\alpha$ be the α–th component of ϕ_U. If $\{e_1, \ldots, e_{2m}\}$ denotes the standard basis of $\mathbb{R}^{2m}$, we have by definition:

$$\frac{\partial}{\partial x_\alpha} = (\phi_U)_*^{-1}(e_\alpha) \qquad \text{and} \qquad \frac{\partial}{\partial y_\alpha} = (\phi_U)_*^{-1}(e_{m+\alpha}).$$

Moreover, $j_m(e_\alpha) = e_{m+\alpha}$, so we obtain directly from the definition

$$J\left(\frac{\partial}{\partial x_\alpha}\right) = \frac{\partial}{\partial y_\alpha}. \tag{1}$$

We now make the following notations

$$\frac{\partial}{\partial z_\alpha} := \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - i\frac{\partial}{\partial y_\alpha}\right), \qquad \frac{\partial}{\partial \bar{z}_\alpha} := \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} + i\frac{\partial}{\partial y_\alpha}\right).$$

From (1) we obtain immediately that $\frac{\partial}{\partial z_\alpha}$ and $\frac{\partial}{\partial \bar{z}_\alpha}$ are local sections of $T^{1,0}M$ and $T^{0,1}M$ respectively. They form moreover a local basis in each point of U. Let now Z and W be two local sections of $T^{0,1}M$ and write $Z = \sum Z_\alpha \frac{\partial}{\partial \bar{z}_\alpha}$, $W = \sum W_\alpha \frac{\partial}{\partial \bar{z}_\alpha}$. A direct calculation then gives

$$[Z, W] = \sum_{\alpha,\beta=1}^{n} Z_\alpha \frac{\partial W_\beta}{\partial \bar{z}_\alpha} \frac{\partial}{\partial \bar{z}_\beta} - \sum_{\alpha,\beta=1}^{n} W_\alpha \frac{\partial Z_\beta}{\partial \bar{z}_\alpha} \frac{\partial}{\partial \bar{z}_\beta},$$

which is clearly a local section of $T^{0,1}M$. □

An almost complex structure arising from a holomorphic structure is called a *complex structure*.

REMARK. The existence of local coordinates satisfying (1) is actually the key point of the hard part of the theorem. Once we have such coordinates, it is easy to show that the transition functions are holomorphic: suppose that u_α, v_α is another local system of coordinates, satisfying

$$\frac{\partial}{\partial v_\alpha} = J \frac{\partial}{\partial u_\alpha}.$$

We then have

$$\frac{\partial}{\partial x_\alpha} = \sum_{\beta=1}^{m} \frac{\partial u_\beta}{\partial x_\alpha} \frac{\partial}{\partial u_\beta} + \sum_{\beta=1}^{m} \frac{\partial v_\beta}{\partial x_\alpha} \frac{\partial}{\partial v_\beta} \tag{2}$$

and

$$\frac{\partial}{\partial y_\alpha} = \sum_{\beta=1}^{m} \frac{\partial u_\beta}{\partial y_\alpha} \frac{\partial}{\partial u_\beta} + \sum_{\beta=1}^{m} \frac{\partial v_\beta}{\partial y_\alpha} \frac{\partial}{\partial v_\beta}. \tag{3}$$

Applying J to (2) and comparing to (3) yields

$$\frac{\partial u_\beta}{\partial x_\alpha} = \frac{\partial v_\beta}{\partial y_\alpha} \qquad \text{and} \qquad \frac{\partial u_\beta}{\partial y_\alpha} = -\frac{\partial v_\beta}{\partial x_\alpha},$$

thus showing that the transition functions are holomorphic.

1.5. Exercises.

(1) Prove Lemma 1.3.

(2) Let $A + iB \in \mathrm{Gl}_m(\mathbb{C})$. Compute the product

$$\begin{pmatrix} I_m & 0 \\ -iI_m & I_m \end{pmatrix} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} I_m & 0 \\ iI_m & I_m \end{pmatrix}$$

and use this computation to prove that for every invertible matrix $A + iB \in \mathrm{Gl}_m(\mathbb{C})$, the determinant of the real $2m \times 2m$ matrix

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

is strictly positive.

(3) Show that every almost complex manifold is orientable.

(4) Let $\alpha > 1$ be some real number. Let Γ be the subgroup of $\mathrm{Gl}_m(\mathbb{C})$ generated by αI_m. Show that Γ acts freely and properly discontinuously on $\mathbb{C}^m - \{0\}$. Use this to prove that $S^1 \times S^{2m-1}$ carries a complex structure.

2. Holomorphic forms and vector fields

2.1. Decomposition of the (complexified) exterior bundle. Let (M,J) be an almost complex manifold. We now turn our attention to exterior forms and introduce the complexified exterior bundle $\Lambda^*_{\mathbb{C}}M = \Lambda^*M \otimes_{\mathbb{R}} \mathbb{C}$. The sections of $\Lambda^*_{\mathbb{C}}M$ can be viewed as complex–valued forms or as formal sums $\omega + i\tau$, where ω and τ are usual real forms on M.

We define the following two sub–bundles of $\Lambda^1_{\mathbb{C}}M$:

$$\Lambda^{1,0}M := \{\xi \in \Lambda^1_{\mathbb{C}}M \mid \xi(Z) = 0 \ \forall\ Z \in T^{0,1}M\}$$

and

$$\Lambda^{0,1}M := \{\xi \in \Lambda^1_{\mathbb{C}}M \mid \xi(Z) = 0 \ \forall\ Z \in T^{1,0}M\}.$$

The sections of these sub–bundles are called forms of type (1,0) or forms of type (0,1) respectively. Similarly to Lemma 1.3 we have

Lemma 2.1. *One has*

$$\Lambda^{1,0}M = \{\omega - i\omega \circ J \mid \omega \in \Lambda^1 M\}\ , \qquad \Lambda^{0,1}M = \{\omega + i\omega \circ J \mid \omega \in \Lambda^1 M\}.$$

and $\Lambda^1_{\mathbb{C}}M = \Lambda^{1,0}M \oplus \Lambda^{0,1}M$.

Let us denote the k–th exterior power of $\Lambda^{1,0}$ (resp. $\Lambda^{0,1}$) by $\Lambda^{k,0}$ (resp. $\Lambda^{0,k}$) and let $\Lambda^{p,q}$ denote the tensor product $\Lambda^{p,0} \otimes \Lambda^{0,q}$. The exterior power of a direct sum of vector spaces can be described as follows

$$\Lambda^k(E \oplus F) \simeq \oplus_{i=0}^k \Lambda^i E \otimes \Lambda^{k-i} F.$$

Using Lemma 2.1 we then get

$$\Lambda^k_{\mathbb{C}}M \simeq \oplus_{p+q=k} \Lambda^{p,q}M.$$

Sections of $\Lambda^{p,q}M$ are called forms of type (p,q). It is easy to check that a complex–valued k–form ω is a section of $\Lambda^{k,0}M$ if and only if $Z \lrcorner\, \omega = 0$ for all $Z \in T^{0,1}M$. More generally, a k–form is a section of $\Lambda^{p,q}M$ if and only if it vanishes whenever applied to $p+1$ vectors from $T^{1,0}M$ or to $q+1$ vectors from $T^{0,1}M$.

If J is a complex structure, we can describe these spaces in terms of a local coordinate system. Let $z_\alpha = x_\alpha + iy_\alpha$ be the α–th coordinate of some ϕ_U. Extending the exterior derivative on functions by $\mathbb{C}$–linearity we get complex–valued forms $dz_\alpha = dx_\alpha + idy_\alpha$ and $d\bar{z}_\alpha = dx_\alpha - idy_\alpha$. Then $\{dz_1, \ldots, dz_m\}$ and $\{d\bar{z}_1, \ldots, d\bar{z}_m\}$ are local basis for $\Lambda^{1,0}M$ and $\Lambda^{0,1}M$ respectively, and a local basis for $\Lambda^{p,q}M$ is given by

$$\{dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q},\ i_1 < \ldots < i_p,\ j_1 < \ldots < j_q\}.$$

To every almost complex structure J one can associate a (2,1)–tensor N^J called the Nijenhuis tensor, satisfying

$$N^J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY], \qquad \forall\ X,Y \in \mathcal{C}^\infty(TM). \tag{4}$$

Proposition 2.2. *Let J be an almost complex structure on M^{2m}. The following statements are equivalent:*

(a) *J is a complex structure.*
(b) *$T^{0,1}M$ is integrable.*
(c) *$d\mathcal{C}^\infty(\Lambda^{1,0}M) \subset \mathcal{C}^\infty(\Lambda^{2,0}M \oplus \Lambda^{1,1}M)$.*
(d) *$d\mathcal{C}^\infty(\Lambda^{p,q}M) \subset \mathcal{C}^\infty(\Lambda^{p+1,q}M \oplus \Lambda^{p,q+1}M)\ \forall\ 0 \le p, q \le m$.*
(e) *$N^J = 0$.*

Proof. (a) $\iff$ (b) is given by Theorem 1.4.

(b) $\iff$ (c) Let ω be a section of $\Lambda^{1,0}M$. The $\Lambda^{0,2}M$–component of $d\omega$ vanishes if and only if $d\omega(Z, W) = 0\ \forall\ Z, W \in T^{0,1}M$. Extend Z and W to local sections of $T^{0,1}M$ and write

$$d\omega(Z, W) = Z(\omega(W)) - W(\omega(Z)) - \omega([Z, W]) = -\omega([Z, W]).$$

Thus

$$\begin{aligned} & d\omega(Z, W) = 0 \quad \forall\ Z, W \in T^{0,1}M,\ \forall\ \omega \in \Lambda^{1,0}M \\ \iff\ & \omega([Z, W]) = 0 \quad \forall\ Z, W \in T^{0,1}M,\ \forall\ \omega \in \Lambda^{1,0}M \\ \iff\ & [Z, W] \in T^{0,1}M \quad \forall\ Z, W \in T^{0,1}M. \end{aligned}$$

(c) $\iff$ (d) One implication is obvious. Suppose now that (c) holds. By conjugation we get immediately $d\mathcal{C}^\infty(\Lambda^{0,1}M) \subset \mathcal{C}^\infty(\Lambda^{0,2}M \oplus \Lambda^{1,1}M)$. It is then enough to apply Leibniz' rule to any section of $\Lambda^{p,q}M$, locally written as a sum of decomposable elements $\omega_1 \wedge \ldots \wedge \omega_p \wedge \bar\tau_1 \wedge \ldots \wedge \bar\tau_q$, where $\omega_i \in \mathcal{C}^\infty(\Lambda^{1,0}M)$ and $\bar\tau_i \in \mathcal{C}^\infty(\Lambda^{0,1}M)$.

(b) $\iff$ (e) Let $X, Y \in \mathcal{C}^\infty(TM)$ be local vector fields and let Z denote the bracket $Z := [X + iJX, Y + iJY]$. An easy calculation gives $Z - iJZ = N^J(X, Y) - iJN^J(X, Y)$. Thus $Z \in T^{0,1}M \iff N^J(X, Y) = 0$, which proves that $T^{0,1}M$ is integrable if and only if $N^J \equiv 0$ ☐

2.2. Holomorphic objects on complex manifolds. In this section (M, J) will denote a complex manifold of complex dimension m. We start with the following characterization of holomorphic functions.

Lemma 2.3. *Let $f : M \to \mathbb{C}$ be a smooth complex–valued function on M. The following assertions are equivalent:*

(1) *f is holomorphic.*
(2) *$Z(f) = 0\ \ \forall\ Z \in T^{0,1}M$.*
(3) *df is a form of type $(1, 0)$.*

Proof. (2) $\iff$ (3). $df \in \Lambda^{1,0}M \iff df(Z) = 0\ \forall\ Z \in T^{0,1}M \iff Z(f) = 0\ \ \forall\ Z \in T^{0,1}M$.

(1) $\iff$ (3). The function f is holomorphic if and only if $f \circ \phi_U^{-1}$ is holomorphic for every holomorphic chart (U, ϕ_U), which is equivalent to $f_* \circ (\phi_U)_*^{-1} \circ j_m = if_* \circ (\phi_U)_*^{-1}$, that is, $f_* \circ J = if_*$. This last equation just means that for every real vector X, $df(JX) = idf(X)$, hence $idf(X + iJX) = 0\ \ \forall\ X \in TM$, which is equivalent to $df \in \Lambda^{1,0}M$. ☐

Using Proposition 2.2, for every fixed (p,q) we define the differential operators $\partial : \mathcal{C}^\infty(\Lambda^{p,q}M) \to \mathcal{C}^\infty(\Lambda^{p+1,q}M)$ and $\bar\partial : \mathcal{C}^\infty(\Lambda^{p,q}M) \to \mathcal{C}^\infty(\Lambda^{p,q+1}M)$ by $d = \partial + \bar\partial$.

Lemma 2.4. *The following identities hold:*

$$\partial^2 = 0, \qquad \bar\partial^2 = 0, \qquad \partial\bar\partial + \bar\partial\partial = 0.$$

Proof. We have $0 = d^2 = (\partial + \bar\partial)^2 = \partial^2 + \bar\partial^2 + (\partial\bar\partial + \bar\partial\partial)$, and the three operators in the last term take values in different sub–bundles. □

Definition 2.5. *A vector field Z in $\mathcal{C}^\infty(T^{1,0}M)$ is called holomorphic if $Z(f)$ is holomorphic for every locally defined holomorphic function f. A p–form ω of type $(p,0)$ is called holomorphic if $\bar\partial\omega = 0$.*

Definition 2.6. *A real vector field X is called real holomorphic if $X - iJX$ is a holomorphic vector field.*

Lemma 2.7. *Let X be a real vector field on a complex manifold (M, J). The following assertions are equivalent:*

- *X is real holomorphic*
- *$\mathcal{L}_X J = 0$*
- *The flow of X consists of holomorphic transformations of M.*

Although not explicitly stated, the reader might have guessed that a map $f : (M, J_1) \to (N, J_2)$ between two complex manifolds is called holomorphic if its differential commutes with the complex structures at each point: $f_* \circ J_1 = J_2 \circ f_*$.

Proof. The equivalence of the last two assertions is tautological. In order to prove the equivalence of the first two assertions, we first notice that a complex vector field Z is of type (0,1) if and only if $Z(f) = 0$ for every locally defined holomorphic function f. Suppose that X is real holomorphic and let Y be an arbitrary vector field and f a local holomorphic function. As $(X + iJX)f = 0$, we have $(X - iJX)f = 2X(f)$. By definition $X(f)$ is then holomorphic so by Lemma 2.3 we get $(Y + iJY)(X(f)) = 0$ and $(Y + iJY)(f) = 0$. This implies $[Y + iJY, X](f) = 0$. This holds for every holomorphic f so $[Y + iJY, X]$ has to be of type (0,1), that is $[JY, X] = J[Y, X]$. Hence $\mathcal{L}_X J(Y) = \mathcal{L}_X(JY) - J(\mathcal{L}_X Y) = [X, JY] - J[X, Y] = 0$ for all vector fields Y, *i.e.* $\mathcal{L}_X J = 0$. The converse is similar and left to the reader. □

We close this section with the following important result:

Proposition 2.8. (The local $i\partial\bar\partial$–Lemma). *Let $\omega \in \Lambda^{1,1}M \cap \Lambda^2 M$ be a real 2–form of type (1,1) on a complex manifold M. Then ω is closed if and only if every point $x \in M$ has an open neighborhood U such that $\omega|_U = i\partial\bar\partial u$ for some real function u on U.*

Proof. One implication is clear from Lemma 2.4:

$$d(i\partial\bar\partial) = i(\partial + \bar\partial)\partial\bar\partial = i(\partial^2\bar\partial - \partial\bar\partial^2) = 0.$$

The other implication is more delicate and needs the following counterpart of the Poincaré Lemma (see [**2**] p. 25 for a proof):

LEMMA 2.9. $\bar\partial$–Poincaré Lemma. *A $\bar\partial$–closed (0,1)–form is locally $\bar\partial$–exact.*

Let ω be a closed real form of type (1,1). From the Poincaré Lemma, there exists locally a 1–form τ with $d\tau = \omega$. Let $\tau = \tau^{1,0} + \tau^{0,1}$ be the decomposition of τ in forms of type (1,0) and (0,1). Clearly, $\tau^{1,0} = \overline{\tau^{0,1}}$. Comparing types in the equality

$$\omega = d\tau = \bar\partial\tau^{0,1} + (\partial\tau^{0,1} + \bar\partial\tau^{1,0}) + \partial\tau^{1,0},$$

we get $\bar\partial\tau^{0,1} = 0$ and $\omega = (\partial\tau^{0,1} + \bar\partial\tau^{1,0})$. The $\bar\partial$–Poincaré Lemma yields a local function f such that $\tau^{0,1} = \bar\partial f$. By conjugation we get $\tau^{1,0} = \partial\bar f$, hence $\omega = (\partial\tau^{0,1} + \bar\partial\tau^{1,0}) = \partial\bar\partial f + \bar\partial\partial\bar f = i\partial\bar\partial(2\mathrm{Im}(f))$, and the Proposition follows, with $u := 2\mathrm{Im}(f)$. $\square$

2.3. Exercises.

(1) Prove Lemma 2.1.

(2) Prove that the object defined by formula (4) is indeed a tensor.

(3) Show that a almost complex structure on a real 2–dimensional manifold is always integrable.

(4) Show that $\{dz_\alpha\}$ and $\{\frac{\partial}{\partial z_\alpha}\}$ are dual basis of $\Lambda^{1,0}M$ and $T^{1,0}M$ at each point of the local coordinate system.

(5) Show that a 2–form ω is of type (1,1) if and only if $\omega(X, Y) = \omega(JX, JY)$, $\forall\, X, Y \in TM$.

(6) Let M be a complex manifold with local holomorphic coordinates $\{z_\alpha\}$.
- Prove that a local vector field of type (1,0) $Z = \sum Z_\alpha \frac{\partial}{\partial z_\alpha}$ is holomorphic if and only if Z_α are holomorphic functions.
- Prove that a local form of type (1,0) $\omega = \sum \omega_\alpha dz_\alpha$ is holomorphic if and only if ω_α are holomorphic functions.

(7) If X is a real holomorphic vector field on a complex manifold, prove that JX has the same property.

(8) Prove the converse in Lemma 2.7.

(9) Show that in every local coordinate system one has

$$\partial f = \sum_{\alpha=1}^{n} \frac{\partial f}{\partial z_\alpha} dz_\alpha \qquad \text{and} \qquad \bar{\partial} f = \sum_{\alpha=1}^{n} \frac{\partial f}{\partial \bar{z}_\alpha} d\bar{z}_\alpha.$$

(10) Let N be a manifold, and let T be a complex sub–bundle of $\Lambda^1_{\mathbb{C}} N$ such that $T \oplus \Lambda^1 N = \Lambda^1_{\mathbb{C}} N$. Show that there exists a unique almost complex structure J on N such that $T = \Lambda^{1,0} N$ with respect to J.

3. Complex and holomorphic vector bundles

3.1. Holomorphic vector bundles. Let M be a complex manifold and let $\pi : E \to M$ be a complex vector bundle over M (*i.e.* each fiber $\pi^{-1}(x)$ is a complex vector space). E is called *holomorphic vector bundle* if there exists a trivialization with holomorphic transition functions. More precisely, there exists an open cover $\mathcal{U}$ of M and for each $U \in \mathcal{U}$ a diffeomorphism $\psi_U : \pi^{-1}(U) \to U \times \mathbb{C}^k$ such that

- the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}U \arrow[r, "\psi_U"] \arrow[d, "\pi"'] & U \times \mathbb{C}^k \arrow[dl, "\mathrm{pr}_U"] \\ U & \end{tikzcd}$$

- for every intersecting U and V one has $\psi_U \circ \psi_V^{-1}(x, v) = (x, g_{UV}(x)v)$, where $g_{UV} : U \cap V \to \mathrm{Gl}_k(\mathbb{C}) \subset \mathbb{C}^{k^2}$ are holomorphic functions.

EXAMPLES. 1. The tangent bundle of a complex manifold M^{2m} is holomorphic. To see this, take a holomorphic atlas (U, ϕ_U) on M and define $\psi_U : TM|_U \to U \times \mathbb{C}^m$ by $\psi_U(X_x) = (x, (\phi_U)_*(X))$. The transition functions $g_{UV} = (\phi_U)_* \circ (\phi_V)_*^{-1}$ are then clearly holomorphic.

2. The cotangent bundle, and more generally the bundles $\Lambda^{p,0}M$ are holomorphic. Indeed, using again a holomorphic atlas of the manifold one can trivialize locally $\Lambda^{p,0}M$ and the chain rule

$$dz_{\alpha_1} \wedge \ldots \wedge dz_{\alpha_p} = \sum_{\beta_1,..,\beta_p} \frac{\partial z_{\alpha_1}}{\partial w_{\beta_1}} \cdots \frac{\partial z_{\alpha_p}}{\partial w_{\beta_p}} dw_{\beta_1} \wedge \ldots \wedge dw_{\beta_p}$$

shows that the transition functions are holomorphic.

For every holomorphic bundle E one defines the bundles $\Lambda^{p,q}E := \Lambda^{p,q}M \otimes E$ of E–valued forms on M of type (p, q) and the $\bar\partial$–operator $\bar\partial : \mathcal{C}^\infty(\Lambda^{p,q}E) \to \mathcal{C}^\infty(\Lambda^{p,q+1}E)$ in the following way. If a section σ of $\Lambda^{p,q}(E)$ is given by $\sigma = (\omega_1, \ldots, \omega_k)$ in some local trivialization (where ω_i are local (p, q)–forms), we define $\bar\partial\sigma := (\bar\partial\omega_1, \ldots, \bar\partial\omega_k)$. Suppose that σ is written $\sigma = (\tau_1, \ldots, \tau_k)$ in some other trivialization of E. Then one has $\tau_j = \sum g_{jk}\omega_k$ for some holomorphic functions g_{jk}, thus $\bar\partial\tau_j = \sum g_{jk}\bar\partial\omega_k$, showing that $\bar\partial\sigma$ does not depend on the chosen trivialization. By construction one has $\bar\partial^2 = 0$ and $\bar\partial$ satisfies the Leibniz rule:

$$\bar\partial(\omega \wedge \sigma) = (\bar\partial\omega) \wedge \sigma + (-1)^{p+q}\omega \wedge (\bar\partial\sigma), \qquad \forall\ \omega \in \mathcal{C}^\infty(\Lambda^{p,q}M),\ \sigma \in \mathcal{C}^\infty(\Lambda^{r,s}E).$$

Notice that the bundles $\Lambda^{p,q}M$ *are not* in general holomorphic bundles for $q \neq 0$.

3.2. Holomorphic structures. An operator $\bar\partial : \mathcal{C}^\infty(\Lambda^{p,q}E) \to \mathcal{C}^\infty(\Lambda^{p,q+1}E)$ on a complex vector bundle E satisfying the Leibniz rule is called a *pseudo–holomorphic structure*. If, moreover, $\bar\partial^2 = 0$, then $\bar\partial$ is called a *holomorphic structure*.

A section σ in a pseudo–holomorphic vector bundle $(E,\bar\partial)$ is called holomorphic if $\bar\partial\sigma = 0$.

LEMMA 3.1. *A pseudo–holomorphic vector bundle $(E,\bar\partial)$ of rank k is holomorphic if and only if each $x \in M$ has an open neighborhood U and k holomorphic sections σ_i of E over U such that $\{\sigma_i(x)\}$ form a basis of E_x (and hence on some neighborhood of x).*

PROOF. If E is holomorphic, one can define for every local holomorphic trivialization (U,ψ_U) a local basis of holomorphic sections by $\sigma_i(x) := \psi_U^{-1}(x,e_i)$, $\forall\ x \in U$. Conversely, every local basis of holomorphic sections defines a local trivialization of E, and if $\{\sigma_i\}$ and $\{\tilde\sigma_i\}$ are two such holomorphic basis, we can write $\sigma_i = \sum g_{ij}\tilde\sigma_j$, which immediately yields (applying $\bar\partial$ and using Leibniz' rule) that $\bar\partial g_{ij} = 0$, hence the transition functions are holomorphic. □

THEOREM 3.2. *A complex vector bundle E is holomorphic if and only if it has a holomorphic structure $\bar\partial$.*

PROOF. The "only if" part follows directly from the discussion above. Suppose, conversely, that E is a complex bundle over M of rank k with holomorphic structure $\bar\partial$ satisfying Leibniz' rule and $\bar\partial^2 = 0$. In order to show that E is holomorphic, it is enough to show, using Lemma 3.1, that one can trivialize E around each $x \in M$ by holomorphic sections. Let $\{\sigma_1,\dots,\sigma_k\}$ be local sections of E which form a basis of E over some open set U containing x. We define local (0,1)–forms τ_{ij} on U by the formula

$$\bar\partial\sigma_i = \sum_{j=1}^k \tau_{ij}\otimes\sigma_j.$$

The condition $\bar\partial^2 = 0$, together with Leibniz' rule, yields

$$0 = \bar\partial^2\sigma_i = \sum_{j=1}^k \bar\partial\tau_{ij}\otimes\sigma_j - \sum_{j,l=1}^k \tau_{il}\wedge\tau_{lj}\otimes\sigma_j,$$

hence

$$\bar\partial\tau_{ij} = \sum_{l=1}^k \tau_{il}\wedge\tau_{lj},\quad \forall\ 1\le i,j\le k. \tag{5}$$

From now on we will use the summation convention on repeating indexes. Suppose one can find a map $f : U' \to \mathrm{Gl}_k(\mathbb{C})$, $f = (f_{ij})$ such that

$$0 = \bar\partial f_{ij} + f_{il}\tau_{lj},\quad \forall\ 1\le i,j\le k, \tag{6}$$

for some open subset U' of U containing x. It is then easy to check that the local sections s_j of E over U' defined by $s_j := f_{jl}\sigma_l$ are holomorphic:

$$\bar\partial s_j = \bar\partial f_{jl}\otimes\sigma_l + f_{jr}\tau_{rl}\otimes\sigma_l = 0.$$

The theorem thus follows from the next lemma. □

LEMMA 3.3. *Suppose that* $\tau := (\tau_{ij})$ *is a* $\mathfrak{gl}_k(\mathbb{C})$*–valued (0,1)–form on* U *satisfying* $\bar\partial\tau = \tau\wedge\tau$*, or equivalently (5). Then for every* $x \in U$ *there exists some open subset* U' *of* U *containing* x *and a map* $f : U' \to \mathrm{Gl}_k(\mathbb{C})$*,* $f = (f_{ij})$ *such that* $\bar\partial f + f\tau = 0$*, or equivalently such that (6) holds.*

PROOF. The main idea is to define an almost complex structure locally on $U \times \mathbb{C}^k$ using τ, to show that its integrability is equivalent to (5), and finally to obtain f as the matrix of some frame defined by τ in terms of holomorphic coordinates given by the theorem of Newlander–Nirenberg.

We denote by N the product $U \times \mathbb{C}^k$. We may suppose that U is an open subset of $\mathbb{C}^m$ with holomorphic coordinates z_α and denote by w_i the coordinates in $\mathbb{C}^k$.

It is an easy exercise to check that any complement T of Λ^1N in the complexified bundle $\Lambda^1_{\mathbb{C}}N$ of some manifold N^{2n}, with $iT = T$, defines an almost complex structure on N, such that T becomes the space of (1,0)–forms on N.

Consider the sub–bundle T of $\Lambda^1N\otimes\mathbb{C}$ generated by the 1–forms

$$\{dz_\alpha,\ dw_i - \tau_{il}w_l \mid 1\le\alpha\le m,\ 1\le i\le k\}.$$

We claim that the almost complex structure induced on N by T is integrable. By Proposition 2.2, we have to show that $d\mathcal{C}^\infty(T)\subset\mathcal{C}^\infty(T\wedge\Lambda^1_{\mathbb{C}}N)$. It is enough to check this on the local basis defining T. Clearly $d(dz_\alpha) = 0$ and from (5) we get

$$\begin{aligned} d(dw_i - \tau_{il}w_l) &= -\partial\tau_{il}w_l - \bar\partial\tau_{il}w_l + \tau_{il}\wedge dw_l \\ &= -\partial\tau_{il}w_l - \tau_{is}\wedge\tau_{sl}w_l + \tau_{is}\wedge dw_s \\ &= -\partial\tau_{il}w_l + \tau_{is}\wedge(dw_s - \tau_{sl}w_l), \end{aligned}$$

which clearly is a section of $\mathcal{C}^\infty(T\wedge\Lambda^1_{\mathbb{C}}N)$. We now use the Newlander–Nirenberg theorem and complete the family $\{z_\alpha\}$ to a local holomorphic coordinate system $\{z_\alpha, u_l\}$ on some smaller neighborhood U' of x. Since du_l are sections of T, we can find functions F_{li} and $F_{l\alpha}$, $1\le i,l\le k$, $1\le\alpha\le m$ such that

$$du_l = F_{li}(dw_i - \tau_{ik}w_k) + F_{l\alpha}dz_\alpha.$$

We apply the exterior derivative to this system and get

$$0 = dF_{li}\wedge(dw_i - \tau_{ik}w_k) + F_{li}(-d\tau_{ik}w_k + \tau_{ik}\wedge dw_k) + dF_{l\alpha}\wedge dz_\alpha.$$

We evaluate this last equality for $w_i = 0$, and get

$$0 = dF_{lk}(z,0)\wedge dw_k + F_{li}(z,0)\tau_{ik}\wedge dw_k + dF_{l\alpha}\wedge dz_\alpha. \tag{7}$$

If we denote $f_{lk}(z) := F_{lk}(z,0)$, then the $\Lambda^{0,1}U'$–part of $dF_{lk}(z,0)$ is just $\bar\partial f_{lk}$. Therefore, the vanishing of the $\Lambda^{0,1}U'\otimes\Lambda^{1,0}\mathbb{C}^k$–components of (7) just reads

$$0 = \bar\partial f_{lk} + f_{li}\tau_{ik}.$$

□

3.3. The canonical bundle of $\mathbb{C}\mathrm{P}^m$. For a complex manifold (M^{2m}, J), the complex line bundle $K_M := \Lambda^{m,0}M$ is called the *canonical bundle* of M. We already noticed that K_M has a holomorphic structure.

On the complex projective space there is some distinguished holomorphic line bundle called the *tautological line bundle*. It is defined as the complex line bundle $\pi : L \to \mathbb{C}\mathrm{P}^m$ whose fiber $L_{[z]}$ over some point $[z] \in \mathbb{C}\mathrm{P}^m$ is the complex line $< z >$ in $\mathbb{C}^{m+1}$.

We consider the canonical holomorphic charts (U_α, ϕ_α) on $\mathbb{C}\mathrm{P}^m$ and the local trivializations $\psi_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times \mathbb{C}$ of L defined by $\psi_\alpha([z], w) = ([z], w_\alpha)$. It is an easy exercise to compute the transition functions:

$$\psi_\alpha \circ \psi_\beta^{-1}([z], \lambda) = ([z], g_{\alpha\beta}([z])\lambda), \text{ where } g_{\alpha\beta}([z]) = \frac{z_\alpha}{z_\beta},$$

which are clearly holomorphic. The aim of this subsection is to prove the following

Proposition 3.4. *The canonical bundle of $\mathbb{C}\mathrm{P}^m$ is isomorphic to the $m+1^{st}$ power of the tautological bundle.*

Proof. A trivialization for $p : \Lambda^{m,0}\mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^m$ is given by $(\phi_\alpha^*)^{-1} : p^{-1}U_\alpha \to U_\alpha \times \Lambda^{m,0}\mathbb{C}^m$, so the transition functions are $h_{\alpha\beta} := (\phi_\alpha^*)^{-1} \circ (\phi_\beta^*)$. Let now $\omega := dw_1 \wedge \ldots \wedge dw_m$ be the canonical generator of $\Lambda^{m,0}\mathbb{C}^m$. We introduce holomorphic coordinates on $U_\alpha \cap U_\beta$: $a_i := \frac{z_i}{z_\alpha}$ for $i \in \{0, \ldots m\} - \{\alpha\}$ and $b_i := \frac{z_i}{z_\beta}$ for $i \in \{0, \ldots m\} - \{\beta\}$. Then

$$\phi_\alpha^*(\omega) = da_0 \wedge \ldots \wedge da_{\alpha-1} \wedge da_{\alpha+1} \wedge \ldots \wedge da_m$$

and

$$\phi_\beta^*(\omega) = db_0 \wedge \ldots \wedge db_{\beta-1} \wedge db_{\beta+1} \wedge \ldots \wedge db_m.$$

Therefore we can write

$$db_0 \wedge \ldots \wedge db_{\beta-1} \wedge db_{\beta+1} \wedge \ldots \wedge db_m = h_{\alpha\beta} da_0 \wedge \ldots \wedge da_{\alpha-1} \wedge da_{\alpha+1} \wedge \ldots \wedge da_m. \tag{8}$$

On the other hand, for every $i \neq \alpha, \beta$ we have $a_i = b_i a_\beta$ and $a_\beta b_\alpha = 1$. This shows that $da_i = a_\beta db_i + b_i da_\beta$ for $i \neq \alpha, \beta$ and $da_\beta = -\frac{1}{b_\alpha^2} db_\alpha = -a_\beta^2 db_\alpha$, and an easy algebraic computation then yields

$$da_0 \wedge \ldots \wedge da_{\alpha-1} \wedge da_{\alpha+1} \wedge \ldots \wedge da_m = (-1)^{\alpha-\beta} a_\beta^{m+1} db_0 \wedge \ldots \wedge db_{\beta-1} \wedge db_{\beta+1} \wedge \ldots \wedge db_m.$$

Using (8) we thus see that the transition functions are given by

$$h_{\alpha\beta} = (-1)^{\alpha-\beta} a_\beta^{-m-1} = (-1)^{\alpha-\beta} \left(\frac{z_\alpha}{z_\beta}\right)^{m+1}.$$

Finally, denoting $c_\alpha := (-1)^\alpha$ we have $c_\alpha h_{\alpha\beta} c_\beta^{-1} = g_{\alpha\beta}^{m+1}$, which proves that

$$K_{\mathbb{C}\mathrm{P}^m} \simeq L^{m+1}.$$

$\square$

3.4. Exercises.

(1) Prove that any holomorphic function on a compact manifold $f : M \to \mathbb{C}$ is constant. *Hint:* use the maximum principle.

(2) Let $E \to M$ be a rank k complex vector bundle whose transition functions with respect to some open cover $\{U_\alpha\}$ of M are $g_{\alpha\beta}$. Show that a section $\sigma : M \to E$ of E can be identified with a collection $\{\sigma_\alpha\}$ of smooth maps $\sigma_\alpha : U_\alpha \to \mathbb{C}^k$ satisfying $\sigma_\alpha = g_{\alpha\beta}\sigma_\beta$ on $U_\alpha \cap U_\beta$.

(3) Let $\pi E \to M$ be a complex vector bundle over a complex manifold M. Prove that E has a holomorphic structure if and only if there exists a complex structure on E as manifold, such that the projection π is a holomorphic map.

(4) *The tautological line bundle.* Let L be the complex line bundle $\pi : L \to \mathbb{C}\mathrm{P}^m$ whose fiber $L_{[z]}$ over some point $[z] \in \mathbb{C}\mathrm{P}^m$ is the complex line $< z >$ in $\mathbb{C}^{m+1}$. Prove that E is a holomorphic line bundle. *Hint:* Use the local trivializations $\psi_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times \mathbb{C}$ defined by $\psi_\alpha([z], w) = ([z], w_\alpha)$.

(5) Show that the tautological line bundle L has no non–trivial holomorphic sections.

(6) *The hyperplane line bundle.* Let $H := L^*$ be the dual of L. Thus, the fiber of H over some point $[z] \in \mathbb{C}\mathrm{P}^m$ is the set of linear maps $< z > \to \mathbb{C}$. Find local trivializations for H with holomorphic transition functions. Find the dimension of the space of holomorphic sections of H.

Part 2

Hermitian and Kähler structures

4. Hermitian bundles

4.1. Connections on complex vector bundles. Let M be a differentiable manifold (not necessarily complex) and let $E \to M$ be a complex vector bundle over M.

DEFINITION 4.1. *A* ($\mathbb{C}$–linear) connection *on a E is a $\mathbb{C}$–linear differential operator $\nabla : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(\Lambda^1(E))$ satisfying the Leibniz rule*

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma, \quad \forall\ f \in \mathcal{C}^\infty(M).$$

One can extend any connection to the bundles of E–valued p–forms on M by

$$\nabla(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^p \omega \wedge \nabla\sigma,$$

where the wedge product has to be understood as

$$\omega \wedge \nabla\sigma := \sum_{i=1}^{n} \omega \wedge e_i^* \otimes \nabla_{e_i}\sigma$$

for any local basis $\{e_i\}$ of TM with dual basis $\{e_i^*\}$.
The *curvature* of ∇ is the $\mathrm{End}(E)$–valued 2–form R^∇ defined by

$$R^\nabla(\sigma) := \nabla(\nabla\sigma)) \quad \forall\ \sigma \in \mathcal{C}^\infty(E).$$

To check that this is indeed tensorial, we can write:

$$\nabla^2(f\sigma) = \nabla(df \otimes \sigma + f\nabla\sigma) = d^2 f \otimes \sigma - df \wedge \nabla\sigma + df \wedge \nabla\sigma + f\nabla^2\sigma = f\nabla^2\sigma.$$

More explicitly, if $\{\sigma_1, \ldots, \sigma_k\}$ are local sections of E which form a basis of each fiber over some open set U, we define the connection forms $\omega_{ij} \in \Lambda^1(U)$ (relative to the choice of the base) by

$$\nabla\sigma_i = \omega_{ij} \otimes \sigma_j.$$

We define the two–forms R_{ij}^∇ by

$$R^\nabla(\sigma_i) = R_{ij}^\nabla \otimes \sigma_j,$$

and compute

$$R_{ij}^\nabla \otimes \sigma_j = R^\nabla(\sigma_i) = \nabla(\omega_{ij} \otimes \sigma_j) = (d\omega_{ij}) \otimes \sigma_j - \omega_{ik} \wedge \omega_{kj} \otimes \sigma_j,$$

showing that

$$R_{ij}^\nabla = d\omega_{ij} - \omega_{ik} \wedge \omega_{kj}. \tag{9}$$

4.2. Hermitian structures and connections. Let $E \to M$ be a complex rank k bundle over some manifold M. We do not assume for the moment that M has an almost complex structure.

DEFINITION 4.2. *A* Hermitian structure H *on E is a smooth field of Hermitian products on the fibers of E, that is, for every $x \in M$, $H : E_x \times E_x \to \mathbb{C}$ satisfies*

- *$H(u, v)$ is $\mathbb{C}$–linear in u for every $v \in E_x$.*
- *$H(u, v) = \overline{H(v, u)} \quad \forall\ u, v \in E_x$.*

- *$H(u,u) > 0 \quad \forall\, u \neq 0$.*
- *$H(u,v)$ is a smooth function on M for every smooth sections u,v of E.*

It is clear from the above conditions that H is $\mathbb{C}$–anti–linear in the second variable. The third condition shows that H is non–degenerate. In fact, it is quite useful to think to H as to a $\mathbb{C}$–anti–linear isomorphism $H : E \to E^*$.

Every rank k complex vector bundle E admits Hermitian structures. To see this, just take a trivialization (U_i, ψ_i) of E and a partition of the unity f_i subordinate to the open cover $\{U_i\}$ of M. For every $x \in U_i$, let $(H_i)_x$ denote the pull–back of the Hermitian metric on $\mathbb{C}^k$ by the $\mathbb{C}$–linear map $\psi_i|_{E_x}$. Then $H := \sum f_i H_i$ is a well–defined Hermitian structure on E.

Suppose now that M is a complex manifold. Consider the projections $\pi^{1,0} : \Lambda^1(E) \to \Lambda^{1,0}(E)$ and $\pi^{0,1} : \Lambda^1(E) \to \Lambda^{0,1}(E)$. For every connection ∇ on E, one can consider its (1,0) and (0,1)–components $\nabla^{1,0} := \pi^{1,0} \circ \nabla$ and $\nabla^{0,1} := \pi^{0,1} \circ \nabla$. From Proposition 2.2, we can extend these operators to $\nabla^{1,0} : \mathcal{C}^\infty(\Lambda^{p,q}(E)) \to \mathcal{C}^\infty(\Lambda^{p+1,q}(E))$ and $\nabla^{0,1} : \mathcal{C}^\infty(\Lambda^{p,q}(E)) \to \mathcal{C}^\infty(\Lambda^{p,q+1}(E))$ satisfying the Leibniz rule:

$$\nabla^{1,0}(\omega \otimes \sigma) = \partial\omega \otimes \sigma + (-1)^{p+q}\omega \wedge \nabla^{1,0}\sigma, \qquad \nabla^{0,1}(\omega \otimes \sigma) = \bar\partial\omega \otimes \sigma + (-1)^{p+q}\omega \wedge \nabla^{0,1}\sigma,$$

for all $\omega \in \mathcal{C}^\infty(\Lambda^{p,q}M)$, $\sigma \in \mathcal{C}^\infty(E)$. Of course, $\nabla^{0,1}$ is a pseudo–holomorphic structure on E for every connection ∇.

For every section σ of E one can write

$$R^\nabla(\sigma) = \nabla^2\sigma = (\nabla^{1,0} + \nabla^{0,1})^2(\sigma) = (\nabla^{1,0})^2(\sigma) + (\nabla^{0,1})^2(\sigma) + (\nabla^{1,0}\nabla^{0,1} + \nabla^{0,1}\nabla^{1,0})(\sigma),$$

so the $\Lambda^{0,2}$ part of the curvature is given by

$$(R^\nabla)^{0,2} = (\nabla^{0,1})^2.$$

Theorem 3.2 shows that if the $\Lambda^{0,2}$–part of the curvature of some connection D on E vanishes, then E is a holomorphic bundle with holomorphic structure $\bar\partial := \nabla^{0,1}$. The converse is also true: simply choose an arbitrary Hermitian metric on E and apply Theorem 4.3 below.

We say that ∇ is a H–connection if H, viewed as a field of $\mathbb{C}$–valued real bilinear forms on E, is parallel with respect to ∇. We can now state the main result of this section:

THEOREM 4.3. *For every Hermitian structure H in a holomorphic bundle E with holomorphic structure $\bar\partial$, there exists a unique H–connection ∇ (called the* Chern connection*) such that $\nabla^{0,1} = \bar\partial$.*

PROOF. Let us first remark that the dual vector bundle E^* is also holomorphic, with holomorphic structure denoted by $\bar\partial$, and that any connection ∇ on E induces canonically a connection, also denoted by ∇, on E^* by the formula

$$(\nabla_X\sigma^*)(\sigma) := X(\sigma^*(\sigma)) - \sigma^*(\nabla_X\sigma), \quad \forall\, X \in TM,\ \sigma \in \mathcal{C}^\infty(E),\ \sigma^* \in \mathcal{C}^\infty(E^*). \tag{10}$$

Note also that $\nabla^{0,1} = \bar\partial$ on E just means that $\nabla\sigma \in \mathcal{C}^\infty(\Lambda^{1,0}(E))$ for every holomorphic section σ of E. From (10), if this property holds on E, then it holds on E^*, too.

After these preliminaries, suppose that ∇ is a H–connection with $\nabla^{0,1} = \bar\partial$. The $\mathbb{C}$–anti–linear isomorphism $H : E \to E^*$ is then parallel, so for every section σ of E and every real vector X on M we get

$$\nabla_X(H(\sigma)) = \nabla_X(H)(\sigma) + H(\nabla_X \sigma) = H(\nabla_X \sigma).$$

By the $\mathbb{C}$–anti–linearity of H, for every complex vector $Z \in TM^{\mathbb{C}}$ we have

$$\nabla_Z(H(\sigma)) = H(\nabla_{\bar Z}\sigma).$$

For $Z \in T^{1,0}M$, this shows that

$$\nabla^{1,0}(\sigma) = H^{-1} \circ \nabla^{0,1}(H(\sigma)) = H^{-1} \circ \bar\partial(H(\sigma)), \tag{11}$$

hence $\nabla = \bar\partial + H^{-1} \circ \bar\partial \circ H$, which proves the existence and uniqueness of ∇.

□

Remark. The (0,2)–component of the curvature of the Chern connection vanishes. Indeed,

$$R^{0,2}(\sigma) = \nabla^{0,1}(\nabla^{0,1}(\sigma)) = \bar\partial^2(\sigma) = 0.$$

Its (2,0)–component actually vanishes, too, since by (11),

$$\nabla^{1,0}(\nabla^{1,0}(\sigma)) = \nabla^{1,0}(H^{-1} \circ \bar\partial(H(\sigma)) = H^{-1} \circ \bar\partial^2(H(\sigma)) = 0.$$

4.3. Exercises.

(1) Let $E \to M$ be a complex vector bundle and denote by E^* and $\bar E$ its dual and its conjugate. (Recall that for every $x \in M$, the fibre of E^* over x is just the dual of E_x and the fibre $\bar E_x$ of $\bar E$ is equal to E_x as a set, but has the conjugate complex structure, in the sense that the action of some complex number z on $\bar E_x$ is the same as the action of $\bar z$ on E_x). If $g_{\alpha\beta}$ denote the transition functions of E with respect to some open cover $\{U_\alpha\}$ of M, find the transition functions of E^* and $\bar E$ with respect to the same open cover.

(2) Show that a Hermitian structure on a complex vector bundle E defines an isomorphism between E^* and $\bar E$ as complex vector bundles.

(3) Let $E \to M$ be a rank k complex vector bundle. Viewing local trivializations of E as local basis of sections of E, show that if the transition functions of E with respect to some local trivialization take values in the unitary group $\mathrm{U}_k \subset \mathrm{Gl}_k(\mathbb{C})$ then there exists a canonically defined Hermitian structure on E.

(4) Prove the naturality of the Chern connection with respect to direct sums and tensor products of holomorphic vector bundles.

5. Hermitian and Kähler metrics

5.1. Hermitian metrics. We start with the following

DEFINITION 5.1. *A* Hermitian metric *on an almost complex manifold (M, J) is a Riemannian metric h such that $h(X, Y) = h(JX, JY)$, $\forall\ X, Y \in TM$. The* fundamental form *of a Hermitian metric is defined by $\Omega(X, Y) := h(JX, Y)$.*

The extension (also denoted by h) of the Hermitian metric to $TM^{\mathbb{C}}$ by $\mathbb{C}$–linearity satisfies

$$\begin{cases} h(\bar{Z}, \bar{W}) = \overline{h(Z, W)}, \quad \forall\ Z, W \in TM^{\mathbb{C}}. \\ h(Z, \bar{Z}) > 0 \quad \forall\ Z \in TM^{\mathbb{C}} - \{0\}. \\ h(Z, W) = 0, \quad \forall\ Z, W \in T^{1,0}M \text{ and } \forall\ Z, W \in T^{0,1}M. \end{cases} \tag{12}$$

Conversely, each symmetric tensor on $TM^{\mathbb{C}}$ with these properties defines a Hermitian metric by restriction to TM (exercise).

REMARK. The tangent bundle of an almost complex manifold is in particular a complex vector bundle. If h is a Hermitian metric on M, then $H(X, Y) := h(X, Y) - ih(JX, Y) = (h - i\Omega)(X, Y)$ defines a Hermitian structure on the complex vector bundle (TM, J), as defined in the previous section. Conversely, any Hermitian structure H on TM as complex vector bundle defines a Hermitian metric h on M by $h := \mathrm{Re}(H)$.

REMARK. Every almost complex manifold admits Hermitian metrics. Simply choose an arbitrary Riemannian metric g and define $h(X, Y) := g(X, Y) + g(JX, JY)$.

Let z_α be holomorphic coordinates on a complex Hermitian manifold (M^{2m}, J, h) and denote by $h_{\alpha\bar{\beta}}$ the coefficients of the metric tensor in these local coordinates:

$$h_{\alpha\bar{\beta}} := h\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right).$$

LEMMA 5.2. *The fundamental form is given by*

$$\Omega = i \sum_{\alpha,\beta=1}^{m} h_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta.$$

The proof is left as an exercise.

5.2. Kähler metrics. Suppose that the fundamental form Ω of a complex Hermitian manifold is closed. From the $i\partial\bar{\partial}$–Lemma we get locally a real function u such that $\Omega = i\partial\bar{\partial}u$, which in local coordinates reads

$$h_{\alpha\bar{\beta}} = \frac{\partial^2 u}{\partial z_\alpha \partial \bar{z}_\beta}.$$

This particularly simple expression for the metric tensor in terms of one single real function deserves the following

Definition 5.3. *A Hermitian metric h on an almost complex manifold (M, J) is called a* Kähler metric *if J is a complex structure and the fundamental form Ω is closed:*

$$h \text{ is Kähler} \iff \begin{cases} N^J = 0 \\ d\Omega = 0 \end{cases}$$

A local real function u satisfying $\Omega = i\partial\bar{\partial}u$ is called a local Kähler potential *of the metric h.*

Our aim (as Riemannian geometers) is to express the Kähler condition in terms of the covariant derivative of the Levi–Civita connection of h. We start with doing so for the Nijenhuis tensor.

Lemma 5.4. *Let h be a Hermitian metric on an almost complex manifold (M, J), with Levi–Civita covariant derivative ∇. Then J is integrable if and only if*

$$(\nabla_{JX}J)Y = J(\nabla_X J)Y, \qquad \forall\ X, Y \in TM. \tag{13}$$

Proof. Let us fix a point $x \in M$ and extend X and Y to vector fields on M parallel with respect to ∇ at x. Then we can write

$$\begin{aligned} N^J(X,Y) &= [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] \\ &= J(\nabla_X J)Y - J(\nabla_Y J)X - (\nabla_{JX}J)Y + (\nabla_{JY}J)X \\ &= (J(\nabla_X J)Y - (\nabla_{JX}J)Y) - (J(\nabla_Y J)X - (\nabla_{JY}J)X), \end{aligned}$$

thus proving that (13) implies $N^J = 0$. Conversely, suppose that $N^J = 0$ and denote by $A(X,Y,Z) = h(J(\nabla_X J)Y - (\nabla_{JX}J)Y), Z)$. The previous equation just reads $A(X,Y,Z) = A(Y,X,Z)$. On the other hand, A is skew–symmetric in the last two variables, since J and $\nabla_X J$ are anti–commuting skew–symmetric operators. Thus $A(X,Y,Z) = -A(X,Z,Y)$, so by circular permutations we get

$$A(X,Y,Z) = -A(Y,Z,X) = A(Z,X,Y) = -A(X,Y,Z),$$

which implies (13).

□

Theorem 5.5. *A Hermitian metric h on an almost complex manifold is Kähler if and only if J is parallel with respect to the Levi–Civita connection of h.*

Proof. One direction is obvious, since if J is parallel, then N^J clearly vanishes, and as $\Omega = h(J\cdot,\cdot)$, we also have $\nabla\Omega = 0$, so in particular $d\Omega = 0$. Conversely, suppose that h is Kähler and denote by B the tensor $B(X,Y,Z) := h((\nabla_X J)Y, Z)$. As J and $\nabla_X J$ anti–commute we have

$$B(X,Y,JZ) = B(X,JY,Z).$$

From (13) we get

$$B(X,Y,JZ) + B(JX,Y,Z) = 0.$$

Combining these two relations also yields

$$B(X,JY,Z) + B(JX,Y,Z) = 0.$$

We now use $d\Omega = 0$ twice, first on X, Y, JZ, then on X, JY, Z and get:

$$B(X, Y, JZ) + B(Y, JZ, X) + B(JZ, X, Y) = 0,$$
$$B(X, JY, Z) + B(JY, Z, X) + B(Z, X, JY) = 0.$$

Adding these two relations and using the previous properties of B yields $2B(X, Y, JZ) = 0$, that is, J is parallel. □

5.3. Characterization of Kähler metrics. We will now prove the analytic characterization of Kähler metrics described in the first section.

THEOREM 5.6. *Let h be a Hermitian metric on a complex manifold (M^{2m}, J). Then h is Kähler if and only if around each point of M there exist holomorphic coordinates in which h osculates to the standard Hermitian metric up to the order 2.*

PROOF. Suppose that we can find holomorphic local coordinates $z_\alpha = x_\alpha + iy_\alpha$ around $x \in M$ such that $h_{\alpha\bar\beta} = \frac{1}{2}\delta_{\alpha\beta} + r_{\alpha\beta}$, with $r_{\alpha\beta}(x) = 0$

$$\frac{\partial r_{\alpha\beta}}{\partial x_\gamma}(x) = \frac{\partial r_{\alpha\beta}}{\partial y_\gamma}(x) = 0$$

at x. Then

$$d\Omega = i \sum_{\alpha,\beta,\gamma=1}^{m} \left(\frac{\partial h_{\alpha\bar\beta}}{\partial x_\gamma} dx_\gamma + \frac{\partial h_{\alpha\bar\beta}}{\partial y_\gamma} dy_\gamma \right) \wedge dz_\alpha \wedge d\bar z_\beta$$

clearly vanishes at x. As x was arbitrary, this means $d\Omega = 0$.

Conversely, if the metric is Kähler, for every $x \in M$ we take an orthonormal basis of T_xM of the form $\{e_1, \ldots, e_m, Je_1, \ldots, Je_m\}$ and choose a local holomorphic coordinate system $(z_\alpha = x_\alpha + iy_\alpha)$ around x such that

$$e_\alpha = \frac{\partial}{\partial x_\alpha}(x) \qquad \text{and} \qquad Je_\alpha = \frac{\partial}{\partial y_\alpha}(x).$$

The fundamental 2–form Ω can be written as

$$\Omega = i \sum_{\alpha,\beta} \left(\frac{1}{2}\delta_{\alpha\beta} + \sum_\gamma a_{\alpha\beta\gamma} z_\gamma + \sum_\gamma a_{\alpha\beta\bar\gamma} \bar z_\gamma + o(|z|) \right) dz_\alpha \wedge d\bar z_\beta,$$

where $o(|z|)$ denotes generically a function whose 1–jet vanishes at x. The condition $h_{\alpha\bar\beta} = \overline{h_{\beta\bar\alpha}}$ together with Lemma 5.2 implies

$$a_{\alpha\beta\bar\gamma} = \overline{\alpha_{\beta\alpha\gamma}}, \tag{14}$$

and from $d\Omega = 0$ we find

$$a_{\alpha\beta\gamma} = a_{\gamma\beta\alpha}. \tag{15}$$

We now look for a local coordinate change in which the fundamental form has vanishing first order terms. We put

$$z_\alpha = w_\alpha + \frac{1}{2} \sum_{\beta,\gamma} b_{\alpha\beta\gamma} w_\beta w_\gamma,$$

where $b_{\alpha\beta\gamma}$ are complex numbers satisfying $b_{\alpha\beta\gamma} = b_{\alpha\gamma\beta}$. This coordinate change is well–defined locally thanks to the holomorphic version of the local inversion theorem. We then have

$$dz_\alpha = dw_\alpha + \sum_{\beta,\gamma} b_{\alpha\beta\gamma} w_\beta dw_\gamma,$$

whence (using Einstein's summation convention)

$$\begin{aligned}
\Omega &= i\left(\frac{1}{2}\delta_{\alpha\beta} + a_{\alpha\beta\gamma} z_\gamma + a_{\alpha\beta\bar\gamma}\bar z_\gamma + o(|z|)\right) dz_\alpha \wedge d\bar z_\beta \\
&= i\left(\frac{1}{2}\delta_{\alpha\beta} + a_{\alpha\beta\gamma} w_\gamma + a_{\alpha\beta\bar\gamma}\bar w_\gamma + o(|z|)\right)(dw_\alpha + b_{\alpha\varepsilon\tau} w_\varepsilon dw_\tau) \wedge (d\bar w_\beta + \overline{b_{\beta\varepsilon\tau}}\bar w_\varepsilon d\bar w_\tau) \\
&= i\left(\frac{1}{2}\delta_{\alpha\beta} + a_{\alpha\beta\gamma} w_\gamma + a_{\alpha\beta\bar\gamma}\bar w_\gamma + b_{\beta\gamma\alpha} w_\gamma + \overline{b_{\alpha\gamma\beta}}\bar w_\gamma + o(|z|)\right) dw_\alpha \wedge d\bar w_\beta.
\end{aligned}$$

If we choose $b_{\beta\gamma\alpha} = -a_{\alpha\beta\gamma}$, (which is possible because of (15) which ensures that $a_{\alpha\beta\gamma}$ is symmetric in α and γ), then from (14) we get

$$\overline{b_{\alpha\gamma\beta}} = -\overline{a_{\beta\alpha\gamma}} = -a_{\alpha\beta\bar\gamma},$$

showing that

$$\Omega = i\left(\frac{1}{2}\delta_{\alpha\beta} + o(|z|)\right) dw_\alpha \wedge d\bar w_\beta.$$

□

5.4. Comparison of the Levi–Civita and Chern connections. Our next aim is to express the $\bar\partial$–operator on a Hermitian manifold in terms of the Levi–Civita connection. In order to do so, we have to remember that TM is identified with a complex vector bundle via the complex structure J. In other words, a product iX for some $X \in TM$ is identified with JX.

Lemma 5.7. *For every section Y of the complex vector bundle TM, the $\bar\partial$–operator, as TM–valued (0,1)–form is given by*

$$\bar\partial^\nabla Y(X) = \frac{1}{2}(\nabla_X Y + J\nabla_{JX} Y - J(\nabla_Y J)X),$$

where ∇ denotes the Levi–Civita connection of any Hermitian metric h on M.

Proof. We first recall that $\bar\partial f(X) = \frac{1}{2}(X + iJX)(f)$, so

$$\begin{aligned}
\bar\partial^\nabla(fY)(X) &= f\frac{1}{2}(\nabla_X Y + J\nabla_{JX} Y - J(\nabla_Y J)X) + \frac{1}{2}(X(f)Y + JX(f)JY) \\
&= f\bar\partial^\nabla Y(X) + \bar\partial f(X)Y,
\end{aligned}$$

which shows that the above defined operator $\bar\partial^\nabla$ satisfies the Leibniz rule. Moreover, a vector field Y is a holomorphic section of TM if and only if it is real holomorphic. By Lemma 2.7, this is equivalent to $\mathcal{L}_Y J = 0$, which means that for every vector field $X \in \mathcal{C}^\infty(TM)$ one has

$$\begin{aligned} 0 &= (\mathcal{L}_Y J)X = \mathcal{L}_Y(JX) - J\mathcal{L}_Y X = [Y, JX] - J[Y, X] \\ &= \nabla_Y JX - \nabla_{JX} Y - J\nabla_Y X + J\nabla_X Y \\ &= (\nabla_Y J)X - \nabla_{JX} Y + J\nabla_X Y = J(\nabla_X Y + J\nabla_{JX} Y - J(\nabla_Y J)X), \end{aligned}$$

thus showing that $\bar\partial^\nabla Y$ vanishes for every holomorphic section Y. This proves that $\bar\partial^\nabla = \bar\partial$. □

A Hermitian manifold (M, h, J) two natural linear connections: the Levi–Civita connection ∇ and the Chern connection $\bar\nabla$ on TM as Hermitian vector bundle.

Proposition 5.8. *The Chern connection coincides with the Levi–Civita connection if and only if h is Kähler.*

Proof. Let $H := h - i\Omega$ denote the Hermitian structure of TM. By definition, J is parallel with respect to the Chern connection, which is a complex connection. Thus, if $\nabla = \bar\nabla$ then J is ∇–parallel, so h is Kähler by Theorem 5.5. Conversely, suppose that h is Kähler. Then the Levi–Civita connection is a well–defined complex connection in TM since $\nabla J = 0$, by Theorem 5.5 again. Moreover, it is a H–connection since $\nabla h = 0$ and $\nabla\Omega = 0$. Finally, the condition $\nabla^{0,1} = \bar\partial$ follows from Lemma 5.7, as $\nabla^{0,1}_X = \frac{1}{2}(\nabla_X + i\nabla_{JX}) = \frac{1}{2}(\nabla_X + J\nabla_{JX})$.

□

5.5. Exercises.

(1) Prove that every Hermitian metric on a 2–dimensional almost complex manifold is Kähler.

(2) Prove that the fundamental form of a Hermitian metric is a (1,1)–form.

(3) If $h_{\alpha\bar\beta}$ denote the coefficients of a Hermitian metric tensor in some local holomorphic coordinate system, show that $h_{\alpha\bar\beta} = \overline{h_{\beta\bar\alpha}}$.

(4) Show that the extension of a Hermitian metric h by $\mathbb{C}$–linearity is a symmetric bilinear tensor satisfying

$$\begin{cases} h(\bar Z, \bar W) = \overline{h(Z, W)}, \quad \forall\ Z, W \in TM^{\mathbb{C}}. \\ h(Z, \bar Z) > 0 \quad \forall\ Z \in TM^{\mathbb{C}} - \{0\}. \\ h(Z, W) = 0, \quad \forall\ Z, W \in T^{1,0}M \text{ and } \forall\ Z, W \in T^{0,1}M. \end{cases}$$

and conversely, any symmetric complex bilinear tensor satisfying this system arises from a Hermitian metric.

6. The curvature tensor of Kähler manifolds

6.1. The curvature tensor. Let (M^n, g) be a Riemannian manifold with Levi–Civita connection ∇ and denote by R its curvature tensor:

$$R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \quad \forall\ X,Y,Z \in \mathcal{C}^\infty(TM).$$

It is sometimes convenient to identify the curvature tensor with the following tensor:

$$R(X,Y,Z,T) := h(R(X,Y)Z,T), \quad \forall\ X,Y,Z,T \in TM.$$

The symmetries of the curvature operator then read:

- $R(X,Y,Z,T) = -R(X,Y,T,Z)$;
- $R(X,Y,Z,T) = R(Z,T,X,Y)$;
- $R(X,Y,Z,T) + R(Y,Z,X,T) + R(Z,X,Y,T) = 0$ (first Bianchi identity);
- $(\nabla_X R)(Y,Z,T,W) + (\nabla_Y R)(Z,X,T,W) + (\nabla_Z R)(X,Y,T,W) = 0$ (second Bianchi identity).

The *Ricci tensor* of M is defined by

$$\mathrm{Ric}(X,Y) := \mathrm{Tr}\{V \mapsto R(V,X)Y\},$$

or equivalently

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{2m} R(e_i, X, Y, e_i),$$

where e_i is a local orthonormal basis of TM. We recall that the Ricci tensor of every Riemannian manifold is symmetric, as can be easily seen from the symmetries of the curvature. A Riemannian manifold (M^n, g) is called *Einstein* if the Ricci tensor Ric is proportional to the metric tensor g at each point of M

$$\mathrm{Ric}(X,Y) = \lambda g(X,Y) \qquad \forall\ X,Y \in T_xM.$$

If $n \geq 3$, it is easy to check that λ (which a priori depends on x) has to be constant on M (see [**8**]).

Suppose now that (M^{2m}, h, J) is a Kähler manifold. Since J is ∇–parallel, the curvature tensor has more symmetries:

$$R(X,Y)JZ = JR(X,Y)Z, \quad \forall\ X,Y,Z \in \mathcal{C}^\infty(TM). \tag{16}$$

This immediately implies

$$R(X,Y,JZ,JT) = R(X,Y,Z,T) = R(JX,JY,Z,T),$$

hence

$$\mathrm{Ric}(JX,JY) = \sum_{i=1}^{2m} R(e_i, JX, JY, e_i) = \sum_{i=1}^{2m} R(Je_i, X, Y, Je_i) = \mathrm{Ric}(X,Y),$$

since for every orthonormal basis $\{e_i\}$, the set $\{Je_i\}$ is also an orthonormal basis.

This last equation justifies the following

DEFINITION 6.1. *The* Ricci form ρ *of a Kähler manifold is defined by*

$$\rho(X,Y) := \mathrm{Ric}(JX,Y), \quad \forall\ X,Y \in TM.$$

The Ricci form is one of the most important objects on a Kähler manifold. Among its properties which will be proved in the next few sections we mention:

- the Ricci form ρ is closed;
- the cohomology class of ρ is (up to some real multiple) equal to the Chern class of the canonical bundle of M;
- in local coordinates, ρ can be expressed as $\rho = -i\partial\bar\partial \log\det(h_{\alpha\bar\beta})$, where $\det(h_{\alpha\bar\beta})$ denotes the determinant of the matrix $(h_{\alpha\bar\beta})$ expressing the Hermitian metric.

For the moment being we use the Bianchi identities for the curvature tensor to prove the following

PROPOSITION 6.2. *(i) The Ricci tensor of a Kähler manifold satisfies*

$$\mathrm{Ric}(X,Y) = \frac{1}{2}\mathrm{Tr}(R(X,JY)\circ J).$$

(ii) The Ricci form is closed.

PROOF. Let (e_i) be a local orthonormal basis of TM. (i) Using the first Bianchi identity we get

$$\begin{aligned}
\mathrm{Ric}(X,Y) &= \sum_i R(e_i,X,Y,e_i) = \sum_i R(e_i,X,JY,Je_i)\\
&= \sum_i(-R(X,JY,e_i,Je_i) - R(JY,e_i,X,Je_i))\\
&= \sum_i(R(X,JY,Je_i,e_i) + R(Y,Je_i,X,Je_i))\\
&= \mathrm{Tr}(R(X,JY)\circ J) - \mathrm{Ric}(X,Y).
\end{aligned}$$

(ii) From (i) we can write $2\rho(X,Y) = \mathrm{Tr}(R(X,Y)\circ J)$. Therefore,

$$\begin{aligned}
2d\rho(X,Y,Z) &= 2((\nabla_X\rho)(Y,Z) + (\nabla_Y\rho)(Z,X) + (\nabla_Z\rho)(X,Y))\\
&= \mathrm{Tr}((\nabla_X R)(Y,Z)\circ J + (\nabla_Y R)(Z,X)\circ J + (\nabla_Z R)(X,Y)\circ J) = 0
\end{aligned}$$

from the second Bianchi identity. □

6.2. Kähler metrics in local coordinates. Let (M^{2m},h,J) be a Kähler manifold with Levi–Civita covariant derivative ∇ and let (z_α) be a system of local holomorphic coordinates. We introduce the following local basis of the complexified tangent space:

$$Z_\alpha := \frac{\partial}{\partial z_\alpha}, \qquad Z_{\bar\alpha} := \frac{\partial}{\partial \bar z_\alpha}, \quad 1 \le \alpha \le m,$$

and we let subscripts $A, B, C, \dots$ run over the set $\{1, \dots, m, \bar{1}, \dots, \bar{m}\}$. We denote the components of the Kähler metric in these coordinates by

$$h_{AB} := h(Z_A, Z_B).$$

Of course, since the metric is Hermitian we have

$$h_{\alpha\beta} = h_{\bar{\alpha}\bar{\beta}} = 0, \quad h_{\bar{\beta}\alpha} = h_{\alpha\bar{\beta}} = \overline{h_{\beta\bar{\alpha}}}. \tag{17}$$

Let $h^{\alpha\bar{\beta}}$ denote the coefficients of the inverse matrix of $(h_{\alpha\bar{\beta}})$. The Christoffel symbols are defined by

$$\nabla_{Z_A} Z_B = \Gamma_{AB}^C Z_C.$$

Using the convention $\bar{\bar{\alpha}} = \alpha$, we get by conjugation

$$\overline{\Gamma_{AB}^C} = \Gamma_{\bar{A}\bar{B}}^{\bar{C}}.$$

Since ∇ is torsion–free we have

$$\Gamma_{AB}^C = \Gamma_{BA}^C,$$

and since $T^{1,0}$ is ∇–parallel we must have

$$\Gamma_{A\bar{\beta}}^{\gamma} = 0.$$

These relations show that the only non–vanishing Christoffel symbols are

$$\Gamma_{\alpha\beta}^{\gamma} \qquad \text{and} \qquad \Gamma_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}.$$

Now, in order to compute these coefficients we notice that $\Gamma_{\alpha\bar{\delta}}^C = 0$ implies

$$\nabla_{Z_\alpha} Z_{\bar{\delta}} = 0, \tag{18}$$

hence

$$\frac{\partial h_{\beta\bar{\delta}}}{\partial z_\alpha} = h(\nabla_{Z_\alpha} Z_\beta, Z_{\bar{\delta}}) = \Gamma_{\alpha\beta}^{\gamma} h_{\gamma\bar{\delta}}.$$

This proves the formulas

$$\Gamma_{\alpha\beta}^{\gamma} h_{\gamma\bar{\delta}} = \frac{\partial h_{\beta\bar{\delta}}}{\partial z_\alpha} \qquad \text{and} \qquad \Gamma_{\alpha\beta}^{\gamma} = h^{\gamma\bar{\delta}} \frac{\partial h_{\beta\bar{\delta}}}{\partial z_\alpha}. \tag{19}$$

The curvature tensor can be viewed either as (3,1)– or as (4,0)–tensor. The corresponding coefficients are defined by

$$R(Z_A, Z_B) Z_C = R_{ABC}^D Z_D \qquad \text{and} \qquad R_{ABCD} = R(Z_A, Z_B, Z_C, Z_D) = h_{DE} R_{ABC}^E.$$

From the fact that $T^{1,0}M$ is parallel we immediately get $R_{AB\bar{\delta}}^{\gamma} = R_{AB\delta}^{\bar{\gamma}} = 0$, hence $R_{AB\gamma\delta} = R_{AB\bar{\gamma}\bar{\delta}} = 0$. Using the curvature symmetries we finally see that the only non–vanishing components of R are

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}}, \ R_{\alpha\bar{\beta}\bar{\gamma}\delta}, \ R_{\bar{\alpha}\beta\gamma\bar{\delta}}, \ R_{\bar{\alpha}\beta\bar{\gamma}\delta}$$

and

$$R_{\alpha\bar{\beta}\gamma}^{\delta}, \ R_{\alpha\bar{\beta}\bar{\gamma}}^{\bar{\delta}}, \ R_{\bar{\alpha}\beta\gamma}^{\delta}, \ R_{\bar{\alpha}\beta\bar{\gamma}}^{\bar{\delta}}.$$

From (18) and (19) we obtain

$$R^{\delta}_{\alpha\bar{\beta}\gamma} Z_\delta = R(Z_\alpha, Z_{\bar{\beta}})Z_\gamma = -\nabla_{Z_{\bar{\beta}}}(\nabla_{Z_\alpha} Z_\gamma) = -\nabla_{Z_{\bar{\beta}}}(\Gamma^{\delta}_{\alpha\gamma} Z_\delta) = -\frac{\partial \Gamma^{\delta}_{\alpha\gamma}}{\partial \bar{z}_\beta} Z_\delta,$$

therefore

$$R^{\delta}_{\alpha\bar{\beta}\gamma} = -\frac{\partial \Gamma^{\delta}_{\alpha\gamma}}{\partial \bar{z}_\beta}. \tag{20}$$

Using this formula we can compute the components of the Ricci tensor:

$$\mathrm{Ric}_{\gamma\bar{\beta}} = \mathrm{Ric}_{\bar{\beta}\gamma} = R^{A}_{A\bar{\beta}\gamma} = R^{\alpha}_{\alpha\bar{\beta}\gamma} = -\frac{\partial \Gamma^{\alpha}_{\alpha\gamma}}{\partial \bar{z}_\beta}.$$

Let us denote by d the determinant of the matrix $(h_{\alpha\bar{\beta}})$. Using the Lemma 6.3 below and (19) we get

$$\Gamma^{\alpha}_{\alpha\gamma} = \Gamma^{\alpha}_{\gamma\alpha} = h^{\alpha\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z_\gamma} = \frac{1}{d}\frac{\partial d}{\partial z_\gamma} = \frac{\partial \log d}{\partial z_\gamma}.$$

This proves the following simple expressions for the Ricci tensor

$$\mathrm{Ric}_{\alpha\bar{\beta}} = -\frac{\partial^2 \log d}{\partial z_\alpha \partial \bar{z}_\beta},$$

and for the Ricci form

$$\rho = -i\partial\bar{\partial}\log d. \tag{21}$$

Lemma 6.3. *Let $(h_{ij}) = (h_{ij}(t))$ be the coefficients of a map $h : \mathbb{R} \to \mathrm{Gl}_m(\mathbb{C})$ with $h^{ij} := (h_{ij})^{-1}$ and let $d(t)$ denote the determinant of (h_{ij}). Then the following formula holds*

$$d'(t) = d \sum_{i,j=1}^{m} h'_{ij}(t) h^{ji}(t).$$

Proof. Recall the definition of the determinant

$$d = \sum_{\sigma \in S_m} \varepsilon(\sigma) h_{1\sigma_1} \ldots h_{m\sigma_m}.$$

If we denote

$$\tilde{h}^{ji} := \frac{1}{d} \sum_{\sigma \in S_m,\ \sigma_i = j} \varepsilon(\sigma) h_{1\sigma_1} \ldots h_{i-1\sigma_{i-1}} h_{i+1\sigma_{i+1}} \ldots h_{m\sigma_m},$$

then we obtain easily

$$\sum_{j=1}^{m} h_{ij}\tilde{h}^{ji} = \frac{1}{d}\sum_{j=1}^{m} \sum_{\sigma \in S_m,\ \sigma_i = j} \varepsilon(\sigma) h_{1\sigma_1} \ldots h_{m\sigma_m} = \frac{1}{d}\sum_{\sigma \in S_m} \varepsilon(\sigma) h_{1\sigma_1} \ldots h_{m\sigma_m} = \frac{1}{d} d = 1,$$

and

$$\sum_{j=1}^{m} h_{kj}\tilde{h}^{ji} = \frac{1}{d}\sum_{j=1}^{m} \sum_{\sigma \in S_m,\ \sigma_i = j} \varepsilon(\sigma) h_{1\sigma_1} \ldots h_{i-1\sigma_{i-1}} h_{k\sigma_i} h_{i+1\sigma_{i+1}} \ldots h_{m\sigma_m} = 0$$

for $k \neq i$ since in the last sum each term corresponding to a permutation σ is the opposite of the term corresponding to the permutation $(ik) \circ \sigma$, where (ik) denotes the transposition of i and k. This shows that $\tilde{h}^{ji} = h^{ji}$ are the coefficients of the inverse matrix of h. We now get

$$\begin{aligned} d'(t) &= \sum_{\sigma \in S_m} \sum_{i=1}^{m} \varepsilon(\sigma) h'_{i\sigma_i}(t) h_{1\sigma_1} \dots h_{i-1\sigma_{i-1}} h_{i+1\sigma_{i+1}} \dots h_{m\sigma_m} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{\sigma \in S_m,\ \sigma_i = j} \varepsilon(\sigma) h'_{ij}(t) h_{1\sigma_1} \dots h_{i-1\sigma_{i-1}} h_{i+1\sigma_{i+1}} \dots h_{m\sigma_m} \\ &= d \sum_{i,j=1}^{m} h'_{ij}(t) \tilde{h}^{ji} = d \sum_{i,j=1}^{m} h'_{ij}(t) h^{ji}(t). \end{aligned}$$

$\square$

6.3. Exercises.

(1) Let $S := \mathrm{Tr}(\mathrm{Ric})$ denote the scalar curvature of a Kähler manifold M with Ricci form ρ. Using the second Bianchi identity, prove the formula:

$$\delta\rho = -\frac{1}{2} J dS.$$

In particular, the Ricci form of M is harmonic if and only if the scalar curvature S is constant.

(2) Prove that the curvature of a Kähler manifold, viewed as a symmetric endomorphism of the space of complex 2–forms, maps $\Lambda^{0,2}$ and $\Lambda^{2,0}$ to 0. Compute the image of the fundamental form through this endomorphism.

(3) Let h be a Hermitian metric on some complex manifold M^{2m} and let $z_\alpha = x_\alpha + iy_\alpha$ be a local system of holomorphic local coordinates on M. Using (17) show that the determinant of the complex $m \times m$ matrix $(h_{\alpha\bar{\beta}})$ is a positive real number whose square is equal to the determinant of the real $2m \times 2m$ matrix h_{ij} representing the metric in the local coordinate system (x_i, y_i).

(4) Let h and h' be two Kähler metrics on some complex manifold (M, J) having the same (Riemannian) volume form. Prove that the Ricci tensors of h and h' are equal.

7. Examples of Kähler metrics

7.1. The flat metric on $\mathbb{C}^m$. Its coefficients in holomorphic coordinates are

$$h_{\alpha\bar\beta} = h\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar z_\beta}\right) = \frac{1}{4} h\left(\frac{\partial}{\partial x_\alpha} - i\frac{\partial}{\partial y_\alpha}, \frac{\partial}{\partial x_\beta} + i\frac{\partial}{\partial y_\beta}\right) = \frac{1}{2}\delta_{\alpha\beta},$$

so by Lemma 5.2 the fundamental form is

$$\Omega = i\frac{1}{2}\sum_{\alpha=1}^{m} dz_\alpha \wedge d\bar z_\alpha = \frac{i}{2}\partial\bar\partial |z|^2.$$

Thus $u(z) = \frac{1}{2}|z|^2$ is a Kähler potential for the canonical Hermitian metric on $\mathbb{C}^m$.

7.2. The Fubini–Study metric on the complex projective space $\mathbb{C}\mathrm{P}^m$. Consider the holomorphic atlas (U_j, ϕ_j) on $\mathbb{C}\mathrm{P}^m$ described in the first section. Let $\pi : \mathbb{C}^{m+1} - \{0\} \to \mathbb{C}\mathrm{P}^m$ be the canonical projection

$$\pi(z_0, \ldots, z_m) = [z_0 : \ldots : z_m].$$

This map is clearly surjective. It is moreover a principal $\mathbb{C}^*$–fibration, with local trivializations $\psi_j : \pi^{-1}U_j \to U_j \times \mathbb{C}^*$ given by

$$\psi_j(z) = ([z], z_j),$$

and satisfying $\psi_j \circ \psi_k^{-1}([z], \alpha) = ([z], \frac{z_j}{z_k}\alpha)$.

Consider the functions $u : \mathbb{C}^m \to \mathbb{R}$ and $v : \mathbb{C}^{m+1} - \{0\} \to \mathbb{R}$ defined by $u(w) = \log(1 + |w|^2)$ and $v(z) = \log(|z|^2)$. For every $j \in \{0, \ldots, m\}$, we define $f_j = \phi_j \circ \pi$.

$$\begin{tikzcd}
\mathbb{C}^{m+1} - \{0\} \arrow[r, "\pi"] \arrow[d, "f_j"'] & U_j \subset \mathbb{C}\mathrm{P}^m \arrow[dl, "\phi_j"] \\
\mathbb{C}^m &
\end{tikzcd}$$

The map f_j is clearly holomorphic and a direct calculation yields $u \circ f_j(z) = v(z) - \log|z_j|^2$. As $\partial\bar\partial \log|z_j|^2 = 0$, this shows that $(f_j)^*(\partial\bar\partial u) = \partial\bar\partial v$ for every j. We thus can define a 2–form Ω on $\mathbb{C}\mathrm{P}^m$ by

$$\Omega|_{U_j} := i(\phi_j)^*(\partial\bar\partial u),$$

which satisfies

$$\pi^*(\Omega) = i\partial\bar\partial v. \tag{22}$$

Clearly Ω is a closed real $(1,1)$–form, so the tensor h defined by

$$h(X, Y) := \Omega(X, JY), \quad \forall\, X, Y \in T\mathbb{C}\mathrm{P}^m$$

is symmetric and Hermitian. The next lemma proves that h defines a Kähler metric on $\mathbb{C}\mathrm{P}^m$.

Lemma 7.1. *The tensor h is positive definite on $\mathbb{C}\mathrm{P}^m$.*

Proof. Let us fix some local holomorphic chart $\phi_j : U_j \to \mathbb{C}^m$. Clearly, $h = (\phi_j)^*(\hat{h})$, where $\hat{h}$ is the symmetric tensor on $\mathbb{C}^m$ defined by $\hat{h}(X,Y) := i\partial\bar{\partial}u(X,JY), \quad \forall\, X,Y \in T\mathbb{C}^m$. We have to prove that $\hat{h}$ is positive definite. Now, since U_m is a group of holomorphic transformations of $\mathbb{C}^m$ preserving u, it also preserves $\hat{h}$. Moreover, it is transitive on the unit sphere of $\mathbb{C}^m$, so it is enough to prove that $\hat{h}$ is positive definite at a point $p = (r, 0, \dots, 0) \in \mathbb{C}^m$ for some positive real number r. We have

$$\partial\bar{\partial}\log(1+|z|^2) = \partial\Big(\frac{1}{1+|z|^2}\Big(\sum_{i=1}^m z_i d\bar{z}_i\Big)\Big) = \frac{1}{1+|z|^2}\sum_{i=1}^m dz_i\wedge d\bar{z}_i - \frac{1}{(1+|z|^2)^2}\Big(\sum_{i=1}^m \bar{z}_i dz_i\Big)\wedge\Big(\sum_{i=1}^m z_i d\bar{z}_i\Big).$$

At p this 2–form simplifies to

$$\frac{1}{(1+r^2)^2}\left(dz_1\wedge d\bar{z}_1 + (1+r^2)\sum_{i=2}^m dz_i\wedge d\bar{z}_i\right),$$

which shows that

$$\hat{h}_p(X,Y) = \frac{2}{(1+r^2)^2}\mathrm{Re}(X_1\bar{Y}_1 + (1+r^2)\sum_{i=2}^m X_i\bar{Y}_i)$$

hence $\hat{h}$ is positive definite. □

The Kähler metric on $\mathbb{C}\mathrm{P}^m$ constructed in this way is called the *Fubini–Study metric* and is usually denoted by h_{FS}.

7.3. Geometrical properties of the Fubini–Study metric. The Fubini–Study metric was defined via its fundamental 2–form, which was expressed by local Kähler potentials. We provide here a more geometrical description of this metric, showing that it is the projection to $\mathbb{C}\mathrm{P}^m$ of some symmetric tensor field of $\mathbb{C}^{m+1} - \{0\}$.

Lemma 7.2. *For every $z \in \mathbb{C}^{m+1} - \{0\}$, the canonical projection $\pi : \mathbb{C}^{m+1} - \{0\} \to \mathbb{C}\mathrm{P}^m$ is a submersion, and the kernel of its differential $\pi_{*z} : T_z(\mathbb{C}^{m+1} - \{0\}) \to T_{\pi(z)}\mathbb{C}\mathrm{P}^m$ is the complex line generated by z.*

Proof. Let $z \in \mathbb{C}^{m+1}$ with $z_j \neq 0$. The composition $f_j := \phi_j \circ \pi$ is given by

$$f_j(z_0, \dots, z_m) = \frac{1}{z_j}(z_0, \dots, z_{j-1}, z_{j+1}, \dots, z_m).$$

We put $j = 0$ for simplicity and denote $f = f_0$. Its differential at z applied to some tangent vector v is

$$f_{*z}(v) = \frac{1}{z_0}(v_1, \dots, v_m) - \frac{v_0}{z_0^2}(z_1, \dots, z_m).$$

Thus $v \in \ker(\pi_*)_z \iff v \in \ker(f_*)_z \iff v = \frac{v_0}{z_0}z$. This shows that $\ker((\pi_*)_z)$ is the complex line generated by z, and for dimensional reasons $(\pi_*)_z$ has to be surjective. □

Consider the complex orthogonal $z^\perp$ of z in $\mathbb{C}^{m+1}$ with respect to the canonical Hermitian metric, *i.e.* the set

$$z^\perp := \{y \in \mathbb{C}^{m+1} \mid \sum_{j=0}^{m} z_j \bar{y}_j = 0\}.$$

This defines a codimension 1 complex distribution D in $\mathbb{C}^{m+1} - \{0\}$ with $D_z := z^\perp$. Let $X \mapsto X^\perp$ denote the orthogonal projection onto $z^\perp$ in $T_z(\mathbb{C}^{m+1} - \{0\})$ and define a bilinear symmetric tensor $\tilde{h}$ on $\mathbb{C}^{m+1} - \{0\}$ by

$$\tilde{h}(X, Y) := \frac{2}{|z|^2} \langle X^\perp, Y^\perp \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical Hermitian product.

Lemma 7.3. *The (1,1)–form $\varphi(X, Y) := \tilde{h}(JX, Y)$ associated to the tensor $\tilde{h}$ satisfies $\varphi = i\partial\bar{\partial}\log(|z|^2)$ on $\mathbb{C}^{m+1} - \{0\}$.*

Proof. It is enough to prove this relation at a point $p = (r, 0, \ldots, 0) \in \mathbb{C}^{m+1} - \{0\}$ for some positive real number r because both members are U_{m+1}–invariant and U_{m+1} is transitive on spheres. We have

$$\partial\bar{\partial}\log(|z|^2) = \partial\Big(\frac{1}{|z|^2}\Big(\sum_{i=0}^{m} z_i d\bar{z}_i\Big)\Big) = \frac{1}{|z|^2}\sum_{i=0}^{m} dz_i \wedge d\bar{z}_i - \frac{1}{|z|^4}\Big(\sum_{i=0}^{m} \bar{z}_i dz_i\Big) \wedge \Big(\sum_{i=0}^{m} z_i d\bar{z}_i\Big).$$

At p this 2–form simplifies to

$$\frac{1}{r^2}\sum_{i=1}^{m} dz_i \wedge d\bar{z}_i.$$

On the other hand, we have at p

$$-i\varphi\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right) = -i\tilde{h}\left(i\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right) = \tilde{h}\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right)$$

which vanishes if $\alpha = 0$ or $\beta = 0$ and equals $\frac{1}{r^2}\delta_{\alpha\beta}$ otherwise. Thus

$$-i\varphi = \frac{1}{r^2}\sum_{i=1}^{m} dz_i \wedge d\bar{z}_i.$$

$\square$

Using this lemma and (22) we see that $\pi^* h = \tilde{h}$, showing that the Fubini–Study metric h_{FS} on $\mathbb{CP}^m$ is given by the projection of the above defined semi–positive symmetric tensor field $\tilde{h}$.

Proposition 7.4. *The group U_{m+1} acts transitively by holomorphic isometries on $(\mathbb{CP}^m, h_{FS})$*

Proof. For every $A \in \mathrm{U}_{m+1}$, $z \in \mathbb{C}^{m+1} - \{0\}$ and $\alpha \in \mathbb{C}^*$, we have $A(\alpha z) = \alpha A(z)$, showing that the canonical action of U_{m+1} on $\mathbb{C}^{m+1} - \{0\}$ descends to an action on $\mathbb{CP}^m$. For every $A \in \mathrm{U}_{m+1}$, let $\tilde{A}$ be the corresponding transformation of $\mathbb{CP}^m$. Looking at its expression in the canonical holomorphic charts, it is easy to check that every $\tilde{A}$ acts *holomorphically* on $\mathbb{CP}^m$.

In order to check that $\tilde{A}$ preserves the Fubini–Study metric, we first use (22) and the relation $v \circ A(z) = \log |Az|^2 = \log |z|^2 = v(z)$ to get

$$\pi^*(\tilde{A}^*(\Omega)) = A^*(i\partial\bar{\partial}v) = i\partial\bar{\partial}A^*v = i\partial\bar{\partial}v = \pi^*\Omega.$$

Lemma 7.2 shows that π_* is onto, so π^* is injective on exterior forms, hence $\tilde{A}^*(\Omega) = \Omega$. As $\tilde{A}$ also preserves the complex structure, too, this clearly implies that $\tilde{A}$ is an isometry. □

We will now use our computations in local coordinates from the previous section in order to show that the Fubini–Study metric is Einstein. Since there exists a transitive isometric action on $\mathbb{C}\mathrm{P}^m$, it is enough to check this at some point, say $p := [1 : 0 : \ldots : 0] \in \mathbb{C}\mathrm{P}^m$. From Lemma 7.1 we see that the fundamental form is given in the local chart ϕ_0 by

$$\Omega = \frac{i}{1+|z|^2}\sum_{i=1}^m dz_i \wedge d\bar{z}_i - \frac{i}{(1+|z|^2)^2}(\sum_{i=1}^m \bar{z}_i dz_i) \wedge (\sum_{i=1}^m z_i d\bar{z}_i).$$

Lemma 7.5. *Let dv denote the volume form on $\mathbb{C}^m$*

$$dv := dx_1 \wedge dy_1 \wedge \ldots \wedge dx_m \wedge dy_m = \frac{i}{2}dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge \frac{i}{2}dz_m \wedge d\bar{z}_m.$$

Then the fundamental 2–form Ω satisfies

$$\Omega \wedge \ldots \wedge \Omega = \frac{2^m m!}{(1+|z|^2)^{m+1}}dv.$$

Proof. Both terms are clearly invariant by the action of U_m on $\mathbb{C}^m$, which is transitive on spheres, so it is enough to prove the equality at points of the form $z = (r, 0, \ldots, 0)$, where it is actually obvious. □

Now, for every Hermitian metric h on $\mathbb{C}^m$ with fundamental form φ, the determinant d of the matrix $(h_{\alpha\bar{\beta}})$ satisfies

$$\frac{1}{m!}\varphi^m = d2^m dv.$$

Applying this to our situation and using the lemma above yields

$$d = \det(h_{\alpha\bar{\beta}}) = \frac{1}{(1+|z|^2)^{m+1}},$$

whence $\log d = -(m+1)\log(1+|z|^2)$, so from the local formula (21) for the Ricci form we get

$$\rho = -i\partial\bar{\partial}\log d = (m+1)i\partial\bar{\partial}\log(1+|z|^2) = (m+1)\Omega,$$

thus proving that the Fubini–Study metric on $\mathbb{C}\mathrm{P}^m$ is an Einstein metric, with Einstein constant $m+1$.

7.4. Exercises.

(1) A submersion $f : (M, g) \to (N, h)$ between Riemannian manifolds is called *Riemannian submersion* if for every $x \in M$, the restriction of $(f_*)_x$ to the g–orthogonal of the tangent space to the fiber $f^{-1}(f(x))$ is an isometry onto $T_{f(x)}N$. Prove that the restriction of the canonical projection π to S^{2m+1} defines a Riemannian submersion onto $(\mathbb{C}\mathrm{P}^m, \frac{1}{2}h_{FS})$.

(2) Show that $(\mathbb{C}\mathrm{P}^1, h_{FS})$ is isometric to the round sphere of radius $1/\sqrt{2}$, $S^2(1/\sqrt{2}) \subset \mathbb{R}^3$. *Hint:* Use the fact that a simply–connected manifold with constant positive sectional curvature K is isometric to the sphere of radius $1/\sqrt{K}$.

(3) Show that for every Hermitian tensor h on $\mathbb{C}^m$ with fundamental form φ, the determinant d of the matrix $(h_{\alpha\bar{\beta}})$ satisfies

$$\varphi^m = d2^m m! dv.$$

Part 3

The Laplace operator

8. Natural operators on Riemannian and Kähler manifolds

8.1. The formal adjoint of a linear differential operator. Let (M^n, g) be an oriented Riemannian manifold (not necessarily compact) with volume form dv and let E and F be Hermitian vector bundles over M with Hermitian structures denoted by $\langle \cdot, \cdot \rangle_E$ and $\langle \cdot, \cdot \rangle_F$.

DEFINITION 8.1. *Let $P : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(F)$ and $Q : \mathcal{C}^\infty(F) \to \mathcal{C}^\infty(E)$ be linear differential operators. The operator Q is called a* formal adjoint *of P if*

$$\int_M \langle P\alpha, \beta \rangle_F dv = \int_M \langle \alpha, Q\beta \rangle_E dv,$$

for every compactly supported smooth sections $\alpha \in \mathcal{C}_0^\infty(E)$ and $\beta \in \mathcal{C}_0^\infty(F)$.

LEMMA 8.2. *There exists at most one formal adjoint for every linear differential operator.*

PROOF. Suppose that $P : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(F)$ has two formal adjoints, denoted Q and Q'. Then their difference $R := Q - Q'$ satisfies

$$\int_M \langle \alpha, R\beta \rangle_E dv = 0 \qquad \forall\ \alpha \in \mathcal{C}_0^\infty(E),\ \forall\ \beta \in \mathcal{C}_0^\infty(F).$$

Suppose that there exists some $\sigma \in \mathcal{C}^\infty(F)$ and some $x \in M$ such that $R(\sigma)_x \neq 0$. Take a positive function f on M such that $f \equiv 1$ on some open set U containing x and $f = 0$ outside a compact set. Since R is a differential operator, the value of $R(\sigma)$ at x only depends on the germ of σ at x, so in particular $R(f\sigma)$ has compact support and $R(f\sigma)_x = R(\sigma)_x \neq 0$. Applying the formula above to the compactly supported sections $\alpha := R(f\sigma)$ and $\beta := f\sigma$ of E and F we get

$$0 = \int_M \langle \alpha, R\beta \rangle_E dv = \int_M |R(f\sigma)|^2 dv,$$

thus showing that the smooth positive function $|R(f\sigma)|^2$ has to vanish identically on M, contradicting the fact that its value at x is non–zero. □

The formal adjoint of an operator P is usually denoted by P^*. From the above lemma it is immediate to check that P is the formal adjoint of P^* and that $Q^* \circ P^*$ is the formal adjoint of $P \circ Q$. The lemma below gives a useful method to compute the formal adjoint:

LEMMA 8.3. *Let $P : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(F)$ and $Q : \mathcal{C}^\infty(F) \to \mathcal{C}^\infty(E)$ be linear differential operators. If there exists a section $\omega \in \mathcal{C}^\infty(E^* \otimes F^* \otimes \Lambda^{n-1}M)$ such that*

$$(\langle P\alpha, \beta \rangle_F - \langle \alpha, Q\beta \rangle_E) dv = d(\omega(\alpha, \beta)), \quad \forall\ \alpha \in \mathcal{C}^\infty(E),\ \beta \in \mathcal{C}^\infty(F), \tag{23}$$

then Q is the formal adjoint of P.

PROOF. The $n-1$–form $\omega(\alpha, \beta)$ has compact support for every compactly supported sections α and β. By Stokes' theorem we see that the integral over M of its exterior derivative vanishes. □

8.2. The Laplace operator on Riemannian manifolds. We start with an oriented Riemannian manifold (M^n, g) with volume form dv. We denote generically by $\{e_1, \ldots, e_n\}$ a local orthonormal frame on M parallel in a point and identify vectors and 1–forms via the metric g. In this way we can write for instance $dv = e_1 \wedge \ldots \wedge e_n$.

There is a natural embedding φ of $\Lambda^k M$ in $TM^{\otimes k}$ given by

$$\varphi(\omega)(X_1, \ldots, X_k) := \omega(X_1, \ldots, X_k),$$

which in the above local basis reads

$$\varphi(e_1 \wedge \ldots \wedge e_k) = \sum_{\sigma \in S_k} \varepsilon(\sigma) e_{\sigma_1} \otimes \ldots \otimes e_{\sigma_k}.$$

The Riemannian product g induces a Riemannian product on all tensor bundles. We consider the following weighted tensor product on $\Lambda^k M$:

$$\langle \omega, \tau \rangle := \frac{1}{k!} g(\varphi(\omega), \varphi(\tau)),$$

which can also be characterized by the fact that the basis

$$\{e_{i_1} \wedge \ldots \wedge e_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n\}$$

is orthonormal. With respect to this scalar product, the interior and exterior products are adjoint operators:

$$\langle X \lrcorner \omega, \tau \rangle = \langle \omega, X \wedge \tau \rangle, \qquad \forall\, X \in TM,\ \omega \in \Lambda^k M,\ \tau \in \Lambda^{k-1} M. \tag{24}$$

We define the Hodge *–operator $* : \Lambda^k M \to \Lambda^{n-k} M$ by

$$\omega \wedge *\tau := \langle \omega, \tau \rangle dv, \qquad \forall \omega, \tau \in \Lambda^k M.$$

It is well–known and easy to check on the local basis above that the following relations are satisfied:

$$*1 = dv, \quad *dv = 1, \tag{25}$$

$$\langle *\omega, *\tau \rangle = \langle \omega, \tau \rangle, \tag{26}$$

$$*^2 = (-1)^{k(n-k)} \quad \text{on } \Lambda^k M. \tag{27}$$

The exterior derivative $d : \mathcal{C}^\infty(\Lambda^k M) \to \mathcal{C}^\infty(\Lambda^{k+1} M)$

$$d = \sum_i e_i \wedge \nabla_{e_i}$$

has a formal adjoint $\delta : \mathcal{C}^\infty(\Lambda^{k+1} M) \to \mathcal{C}^\infty(\Lambda^k M)$ satisfying

$$\delta = -(-1)^{nk} * d* = -\sum_i e_i \lrcorner \nabla_{e_i}.$$

To see this, let $\alpha \in \Omega^p$ and $\beta \in \Omega^{p+1}$ be smooth forms. Then we have

$$\begin{aligned}
\langle d\alpha, \beta \rangle dv &= d\alpha \wedge *\beta = d(\alpha \wedge *\beta) - (-1)^p \alpha \wedge d * \beta \\
&= d(\alpha \wedge *\beta) - (-1)^{p+p(n-p)} \alpha \wedge * * d * \beta = d(\alpha \wedge *\beta) - (-1)^{np} \langle \alpha, *d * \beta \rangle dv,
\end{aligned}$$

so Lemma 8.3 shows that $d^* = (-1)^{np+1} * d*$ on $p+1$–forms.

Using the Hodge *–operator we get the following useful reformulation of Lemma 8.3: if there exists a section $\tau \in \mathcal{C}^\infty(E^* \otimes F^* \otimes \Lambda^1 M)$ such that

$$\langle P\alpha, \beta\rangle_F - \langle \alpha, Q\beta\rangle_E = \delta(\tau(\alpha,\beta)), \quad \forall\ \alpha \in \mathcal{C}^\infty(E),\ \beta \in \mathcal{C}^\infty(F), \tag{28}$$

then Q is the formal adjoint of P.

The Laplace operator $\Delta : \mathcal{C}^\infty(\Lambda^k M) \to \mathcal{C}^\infty(\Lambda^k M)$ is defined by

$$\Delta := d\delta + \delta d,$$

and is clearly formally self–adjoint.

8.3. The Laplace operator on Kähler manifolds. After these preliminaries, let now (M^{2m}, h, J) be an almost Hermitian manifold with fundamental form Ω. We define the following algebraic (real) operators acting on differential forms:

$$L : \Lambda^k M \to \Lambda^{k+2} M, \qquad L(\omega) := \Omega \wedge \omega = \frac{1}{2}\sum_i e_i \wedge Je_i \wedge \omega,$$

with adjoint Λ satisfying

$$\Lambda : \Lambda^{k+2} M \to \Lambda^k M, \qquad \Lambda(\omega) := \frac{1}{2}\sum_i Je_i \lrcorner\, e_i \lrcorner\, \omega.$$

These natural operators can be extended to complex–valued forms by $\mathbb{C}$–linearity.

Lemma 8.4. *The following relations hold:*

(1) *The Hodge *–operator maps (p,q)–forms to $(m-q, m-p)$–forms.*
(2) $[X \lrcorner\,, \Lambda] = 0$ *and* $[X \lrcorner\,, L] = JX \wedge$.

The proof is straightforward.

Let us now assume that M is Kähler. We define the twisted differential $d^c : \mathcal{C}^\infty(\Lambda^k M) \to \mathcal{C}^\infty(\Lambda^{k+1} M)$ by

$$d^c(\omega) := \sum_i Je_i \wedge \nabla_{e_i}\omega$$

whose formal adjoint is $\delta^c : \mathcal{C}^\infty(\Lambda^{k+1} M) \to \mathcal{C}^\infty(\Lambda^k M)$

$$\delta^c := - * d^c * = -\sum_i Je_i \lrcorner\, \nabla_{e_i}.$$

Lemma 8.5. *On a Kähler manifold, the following relations hold:*

$$[L, \delta] = d^c, \qquad [L, d] = 0 \tag{29}$$

and

$$[\Lambda, d] = -\delta^c, \qquad [\Lambda, \delta] = 0. \tag{30}$$

Proof. Using Lemma 8.4 (2) and the fact that J and Ω are parallel we get

$$[L,\delta] = -[L, e_i \lrcorner \nabla_{e_i}] = -[L, e_i \lrcorner]\nabla_{e_i} = Je_i \wedge \nabla_{e_i} = d^c.$$

The second relation in (29) just expresses the fact that the Kähler form is closed. The two relations in (30) follow by the uniqueness of the formal adjoint. ☐

Corresponding to the decomposition $d = \partial + \bar\partial$ we have the decomposition $\delta = \partial^* + \bar\partial^*$, where

$$\partial^* : \mathcal{C}^\infty(\Lambda^{p,q}M) \to \mathcal{C}^\infty(\Lambda^{p-1,q}M), \qquad \partial^* := - * \bar\partial *$$

and

$$\bar\partial^* : \mathcal{C}^\infty(\Lambda^{p,q}M) \to \mathcal{C}^\infty(\Lambda^{p,q-1}M), \qquad \bar\partial^* := - * \partial * .$$

Notice that ∂^* and $\bar\partial^*$ are formal adjoints of ∂ and $\bar\partial$ with respect to the Hermitian product H on complex forms given by

$$H(\omega,\tau) := \langle \omega, \bar\tau \rangle. \tag{31}$$

We define the Laplace operators

$$\Delta^\partial := \partial\partial^* + \partial^*\partial \qquad \text{and} \qquad \Delta^{\bar\partial} := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial.$$

One of the most important features of Kähler metrics is that these new Laplace operators are essentially the same as the usual one:

Theorem 8.6. *On any Kähler manifold one has* $\Delta = 2\Delta^\partial = 2\Delta^{\bar\partial}$.

Proof. Our identification of TM and T^*M via the metric maps $(1,0)$–vectors to $(0,1)$–forms and vice–versa. From the fact that $\Lambda^{p,q}M$ are preserved by the covariant derivative follows easily

$$\partial = \sum_j \frac{1}{2}(e_j + iJe_j) \wedge \nabla_{e_j} \qquad \text{and} \qquad \bar\partial = \sum_j \frac{1}{2}(e_j - iJe_j) \wedge \nabla_{e_j}.$$

From the definition of d^c we then get

$$d^c = i(\bar\partial - \partial), \tag{32}$$

and by adjunction

$$\delta^c = i(\partial^* - \bar\partial^*). \tag{33}$$

Applying (29) to a (p,q)–form and projecting onto $\Lambda^{p\pm1,q}M$ and $\Lambda^{p,q\pm1}M$ then yields

$$[L,\partial^*] = i\bar\partial, \qquad [L,\bar\partial^*] = -i\partial, \qquad [L,\partial] = 0, \qquad [L,\bar\partial] = 0, \tag{34}$$

and similarly from (30) we obtain

$$[\Lambda,\partial] = i\bar\partial^*, \qquad [\Lambda,\bar\partial] = -i\partial^*, \qquad [\Lambda,\partial^*] = 0, \qquad [\Lambda,\bar\partial^*] = 0. \tag{35}$$

Now, the relation $\bar\partial^2 = 0$ together with (35) gives

$$-i(\bar\partial\partial^* + \partial^*\bar\partial) = \bar\partial[\Lambda,\bar\partial] + [\Lambda,\bar\partial]\bar\partial = \bar\partial\Lambda\bar\partial - \bar\partial\Lambda\bar\partial = 0$$

and similarly

$$\partial\bar\partial^* + \bar\partial^*\partial = 0.$$

Thus,

$$\begin{aligned}\Delta &= (\partial+\bar\partial)(\partial^*+\bar\partial^*)+(\partial^*+\bar\partial^*)(\partial+\bar\partial)\\ &= (\partial\partial^*+\partial^*\partial)+(\bar\partial\bar\partial^*+\bar\partial^*\bar\partial)+(\bar\partial\partial^*+\partial^*\bar\partial)+(\partial\bar\partial^*+\bar\partial^*\partial)\\ &= \Delta^\partial+\Delta^{\bar\partial}.\end{aligned}$$

It remains to show the equality $\Delta^\partial=\Delta^{\bar\partial}$, which is an easy consequence of (35):

$$\begin{aligned}-i\Delta^\partial &= -i(\partial\partial^*+\partial^*\partial)=\partial[\Lambda,\bar\partial]+[\Lambda,\bar\partial]\partial=\partial\Lambda\bar\partial-\partial\bar\partial\Lambda+\Lambda\bar\partial\partial-\bar\partial\Lambda\partial\\ &= \partial\Lambda\bar\partial+\bar\partial\partial\Lambda-\Lambda\partial\bar\partial-\bar\partial\Lambda\partial=[\partial,\Lambda]\bar\partial+\bar\partial[\partial,\Lambda]=-i\bar\partial^*\bar\partial-i\bar\partial\bar\partial^*=-i\Delta^{\bar\partial}.\end{aligned}$$

□

8.4. Exercises.

(1) Consider the extension of J as derivation

$$J:\Lambda^kM\to\Lambda^kM,\qquad J(\omega):=\sum_i Je_i\wedge e_i\lrcorner\,\omega.$$

Show that the following relations hold:

- J is skew–Hermitian.
- $J(\alpha\wedge\beta)=J(\alpha)\wedge\beta+\alpha\wedge J(\beta)$ for all forms $\alpha\in\Omega^pM$ and $\beta\in\Omega^kM$.
- The restriction of J to $\Lambda^{p,q}M$ equals to the scalar multiplication by $i(q-p)$.
- $[J,\Lambda]=0$ and $[J,L]=0$.

(2) Let ω be a k–form on a n–dimensional Riemannian manifold M. Prove that $e_i\wedge(e_i\lrcorner\,\omega)=k\omega$ and $e_i\lrcorner\,(e_i\wedge\omega)=(n-k)\omega$.

(3) Show that $0=dd^c+d^cd=d\delta^c+\delta^cd=\delta\delta^c+\delta^c\delta=\delta d^c+d^c\delta$ on every Kähler manifold.

(4) Prove that $[J,d]=d^c$ and $[J,d^c]=-d$ on Kähler manifolds.

(5) Show that the Laplace operator commutes with L, Λ and J on Kähler manifolds.

9. Hodge and Dolbeault theory

9.1. Hodge theory. In this section we assume that (M^n, g) is a compact oriented Riemannian manifold. From now on we denote the space of smooth complex–valued k–forms by $\Omega^k M := \mathcal{C}^\infty(\Lambda^k M \otimes \mathbb{C})$ and by $\mathcal{Z}^k(M)$ the space of *closed* complex k–forms on M. Since the exterior derivative satisfies $d^2 = 0$, one clearly has $d\Omega^{k-1}M \subset \mathcal{Z}^k M$. We define the De Rham cohomology groups by

$$H^k_{DR}(M, \mathbb{C}) := \frac{\mathcal{Z}^k M}{d\Omega^{k-1}M}.$$

The De Rham theorem says that the k^{th} cohomology group of M with complex coefficients is naturally isomorphic to the k^{th} De Rham cohomology group:

$$H^k(M, \mathbb{C}) \simeq H^k_{DR}(M, \mathbb{C}).$$

We now denote by $\mathcal{H}^k(M, \mathbb{C})$ the space of complex *harmonic k–forms* on M, *i.e.* forms in the kernel of the Laplace operator:

$$\mathcal{H}^k(M, \mathbb{C}) := \{\omega \in \Omega^k M \mid \Delta\omega = 0\}.$$

Lemma 9.1. *A form is harmonic if and only if it is closed and δ–closed.*

Proof. One direction is clear. Suppose conversely that ω is harmonic. Since M is compact and d and δ are formally adjoint operators we get

$$0 = \int_M H(\Delta\omega, \omega) dv = \int_M H(d\delta\omega + \delta d\omega, \omega) dv = \int_M |\delta\omega|^2 + |d\omega|^2 dv,$$

showing that $d\omega = 0$ and $\delta\omega = 0$. □

Theorem 9.2. (Hodge decomposition theorem). *The space of k–forms decomposes as a direct sum*

$$\Omega^k M = \mathcal{H}^k(M, \mathbb{C}) \oplus \delta\Omega^{k+1}M \oplus d\Omega^{k-1}M.$$

Proof. Using Lemma 9.1 it is immediate to check that the three spaces above are orthogonal with respect to the global Hermitian product on $\Omega^k M$ given by

$$(\omega, \tau) := \int_M H(\omega, \tau) dv.$$

The hard part of the theorem is to show that the direct sum of these three summands is the whole space $\Omega^k M$. A proof can be found in [**2**], pp. 84–100. □

The Hodge decomposition theorem shows that every k–form ω on M can be uniquely written as

$$\omega = d\omega' + \delta\omega'' + \omega^H,$$

where $\omega' \in \Omega^{k-1}M$, $\omega'' \in \Omega^{k+1}M$ and $\omega^H \in \mathcal{H}^k(M, \mathbb{C})$. If ω is closed, we can write

$$0 = (d\omega, \omega'') = (d\delta\omega'', \omega'') = \int_M |\delta\omega''|^2 dv,$$

showing that the second term in the Hodge decomposition of ω vanishes.

Proposition 9.3. (Hodge isomorphism). *The natural map* $f : \mathcal{H}^k(M,\mathbb{C}) \to H^k_{DR}(M,\mathbb{C})$ *given by* $\omega \mapsto [\omega]$ *is an isomorphism.*

Proof. First, f is well–defined because every harmonic form is closed (Lemma 9.1). The kernel of f is zero since the spaces of harmonic forms and exact forms are orthogonal, so in particular their intersection is $\{0\}$. Finally, for every De Rham cohomology class c, there exists a closed form ω such that $[\omega] = c$. We have seen that the Hodge decomposition of ω is $\omega = d\omega' + \omega^H$, showing that

$$f(\omega^H) = [\omega^H] = [d\omega' + \omega^H] = [\omega] = c,$$

hence f is surjective. □

The complex dimension $b_k(M) := \dim_{\mathbb{C}}(H^k_{DR}(M,\mathbb{C}))$ is called the k^{th} Betti number of M and is a topological invariant in view of De Rham's theorem.

Proposition 9.4. (Poincaré duality). *The spaces* $\mathcal{H}^k(M,\mathbb{C})$ *and* $\mathcal{H}^{n-k}(M,\mathbb{C})$ *are isomorphic. In particular* $b_k(M) = b_{n-k}(M)$ *for every compact* n*–dimensional manifold* M*.*

Proof. The isomorphism is simply given by the Hodge *–operator which sends harmonic k–forms to harmonic $n-k$–forms. □

We close this section with the following interesting application of Theorem 9.2.

Proposition 9.5. *Every Killing vector field on a compact Kähler manifold is real holomorphic.*

Proof. Let X be a Killing vector field, that is $\mathcal{L}_X g = 0$. We compute the Lie derivative of the fundamental 2–form with respect to X using Cartan's formula:

$$\mathcal{L}_X\Omega = d(X \lrcorner \Omega) + X \lrcorner d\Omega = d(X \lrcorner \Omega),$$

so $\mathcal{L}_X\Omega$ is exact. On the other hand, since the flow of X is isometric, it commutes with the Hodge *–operator, thus $\mathcal{L}_X \circ * = * \circ \mathcal{L}_X$. As we clearly have $d \circ \mathcal{L}_X = \mathcal{L}_X \circ d$, too, we see that $\mathcal{L}_X \circ \delta = \delta \circ \mathcal{L}_X$, whence

$$d(\mathcal{L}_X\Omega) = \mathcal{L}_X(d\Omega) = 0$$

and

$$\delta(\mathcal{L}_X\Omega) = \mathcal{L}_X(\delta\Omega) = 0,$$

because Ω is coclosed, being parallel. Thus $\mathcal{L}_X\Omega$ is harmonic and exact, so it has to vanish by the easy part of Theorem 9.2. This shows that the flow of X preserves the metric and the fundamental 2–form, it thus preserves the complex structure J, hence X is real holomorphic. □

9.2. Dolbeault theory. Let (M^{2m}, h, J) be a compact Hermitian manifold. We consider the Dolbeault operator $\bar\partial$ acting on the spaces of (p,q)–forms $\Omega^{p,q}M := \mathcal{C}^\infty(\Lambda^{p,q}M) \subset \Omega^{p+q}M$. Let $\mathcal{Z}^{p,q}M$ denote the space of $\bar\partial$–closed (p,q)–forms. Since $\bar\partial^2 = 0$, we see that $\bar\partial\Omega^{p,q-1}M \subset \mathcal{Z}^{p,q}M$. We define the Dolbeault cohomology groups

$$H^{p,q}M := \frac{\mathcal{Z}^{p,q}M}{\bar\partial\Omega^{p,q-1}M}.$$

In contrast to De Rham cohomology, the Dolbeault cohomology is no longer a topological invariant of the manifold, since it strongly depends on the complex structure J.

We define the space $\mathcal{H}^{p,q}M$ of *$\bar\partial$–harmonic (p,q)–forms* on M by

$$\mathcal{H}^{p,q}M := \{\omega \in \Omega^{p,q}M \mid \Delta^{\bar\partial}\omega = 0\}.$$

As before we have

LEMMA 9.6. *A form $\omega \in \Omega^{p,q}M$ is $\bar\partial$–harmonic if and only if $\bar\partial\omega = 0$ and $\bar\partial^*\omega = 0$.*

The proof is very similar to that of Lemma 9.1 and is left as an exercise.

THEOREM 9.7. (Dolbeault decomposition theorem). *The space of (p,q)–forms decomposes as a direct sum*

$$\Omega^{p,q}M = \mathcal{H}^{p,q}M \oplus \bar\partial^*\Omega^{p,q+1}M \oplus \bar\partial\Omega^{p,q-1}M.$$

PROOF. Lemma 9.6 shows that the three spaces above are orthogonal with respect to the global Hermitian product

$$(\cdot,\cdot) := \int_M H(\cdot,\cdot)dv$$

on $\Omega^{p,q}M$, and a proof for the hard part, which consists in showing that the direct sum of the three summands is the whole space $\Omega^{p,q}M$, can be found in [**2**], pp. 84–100. □

This shows that every (p,q)–form ω on M can be uniquely written as

$$\omega = \bar\partial\omega' + \bar\partial^*\omega'' + \omega^H,$$

where $\omega' \in \Omega^{p,q-1}M$, $\omega'' \in \Omega^{p,q+1}M$ and $\omega^H \in \mathcal{H}^{p,q}M$. This is called the Dolbeault decomposition of ω. As before,the second summand in the Dolbeault decomposition of ω vanishes if and only if $\bar\partial\omega = 0$. Specializing for $q = 0$ yields

PROPOSITION 9.8. *A $(p,0)$–form on a compact Hermitian manifold is holomorphic if and only if it is $\bar\partial$–harmonic.*

COROLLARY 9.9. (Dolbeault isomorphism). *The map $f : \mathcal{H}^{p,q}M \to H^{p,q}M$ given by $\omega \mapsto [\omega]$ is an isomorphism.*

The proof is completely similar to the proof of the Hodge isomorphism.

We denote by $h^{p,q}$ the complex dimension of $H^{p,q}M$. These are the *Hodge numbers* associated to the complex structure J of M.

Proposition 9.10. (Serre duality). *The spaces $\mathcal{H}^{p,q}M$ and $\mathcal{H}^{m-p,m-q}M$ are isomorphic. In particular $h^{p,q} = h^{m-p,m-q}$.*

Proof. Consider the composition of the Hodge $*$–operator with the complex conjugation

$$\bar{*} : \Omega^{p,q}M \to \Omega^{m-p,m-q}M, \qquad \bar{*}(\omega) := *\bar{\omega}.$$

We have

$$\begin{aligned}
\bar{*}\Delta^{\bar\partial}(\omega) &= *\overline{(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\omega} = *(\partial\partial^* + \partial^*\partial)\bar\omega \\
&= -*(\partial * \bar\partial * + * \bar\partial * \partial)\bar\omega = \bar\partial^*\bar\partial(\bar{*}\omega) - *^2\bar\partial * \partial\bar\omega \\
&= \bar\partial^*\bar\partial(\bar{*}\omega) - \bar\partial * \partial *^2 \bar\omega = \bar\partial^*\bar\partial(\bar{*}\omega) + \bar\partial\bar\partial^*(\bar{*}\omega) = \Delta^{\bar\partial}(\bar{*}\omega).
\end{aligned}$$

This shows that $\bar{*}$ is a ($\mathbb{C}$–anti–linear) isomorphism from $\mathcal{H}^{p,q}M$ to $\mathcal{H}^{m-p,m-q}M$. $\square$

If M is Kähler much more can be said about Hodge and Betti numbers, due to Theorem 8.6. Firstly, the fact that $\Delta = 2\Delta^{\bar\partial}$ shows that $\mathcal{H}^{p,q}M \subset \mathcal{H}^{p+q}M$. Secondly, since $\Delta^{\bar\partial}$ leaves the spaces $\Omega^{p,q}M$ invariant, we deduce that Δ has the same property, thus proving that the components of a harmonic form in its type decomposition are all harmonic. This shows that

$$\mathcal{H}^kM = \oplus_{p+q=k}\mathcal{H}^{p,q}M.$$

Moreover, as $\Delta^{\bar\partial}$ is a real operator, it commutes with the complex conjugation (in the general case we only have that $\overline{\Delta^{\bar\partial}\alpha} = \Delta^{\partial}\bar\alpha$) so the complex conjugation defines an isomorphism between the spaces $\mathcal{H}^{p,q}M$ and $\mathcal{H}^{q,p}M$. Consider now the fundamental form $\Omega \in \Omega^{1,1}M$. Since Ω^m is a non–zero multiple of the volume form, we deduce that all exterior powers $\Omega^p \in \Omega^{p,p}M$ are non–zero. Moreover, they are all harmonic since Ω is parallel so Ω^p is parallel, too, and a parallel form is automatically harmonic. We thus have proved the

Proposition 9.11. *In addition to Poincaré and Serre dualities, the following relations hold between Betti and Hodge numbers on compact Kähler manifolds:*

$$b_k = \sum_{p+q=k} h^{p,q}, \qquad h^{p,q} = h^{q,p}, \qquad h^{p,p} \geq 1 \quad \forall\ 0 \leq p \leq m. \tag{36}$$

In particular (36) shows that all Betti numbers of odd order are even and all Betti numbers of even order are non–zero.

9.3. Exercises.

(1) Prove that the complex manifold $S^1 \times S^{2k+1}$ carries no Kähler metric for $k \geq 1$.

(2) Let V be an Euclidean vector space, identified with V^* via the metric. Prove that the Lie algebra extension of an endomorphism A of V to Λ^kV is given by the formula

$$A(\omega) := A(e_i) \wedge e_i \lrcorner\, \omega,$$

for every orthonormal basis $\{e_i\}$ of V.

(3) *The global $i\partial\bar{\partial}$–Lemma.* Let φ be an exact real $(1,1)$–form on a compact Kähler manifold M. Prove that φ is $i\partial\bar{\partial}$–exact, in the sense that there exists a real function u such that $\varphi = i\partial\bar{\partial}u$.

(4) Show that there exists no global Kähler potential on a compact Kähler manifold.

(5) Let (M, h) be a compact Kähler manifold whose second Betti number is equal to 1. Show that if the scalar curvature of M is constant, then the metric h is Einstein.

Part 4

Prescribing the Ricci tensor on Kähler manifolds

10. Chern classes

10.1. Chern-Weil theory. The comprehensive theory of Chern classes can be found in [**8**], Ch.12. We will outline here the definition and properties of the first Chern class, which is the only one needed in the sequel. The following proposition can be taken as a definition

Proposition 10.1. *To every complex vector bundle E over a smooth manifold M one can associate a cohomology class $c_1(E) \in H^2(M, \mathbb{Z})$ called the* first Chern class *of E satisfying the following axioms:*

- (naturality) *For every smooth map $f : M \to N$ and complex vector bundle E over N, one has $f^*(c_1(E)) = c_1(f^*E)$, where the left term denotes the pull–back in cohomology and f^*E is the pull–back bundle defined by $f^*E_x = E_{f(x)}$ $\forall\, x \in M$.*
- (Whitney sum formula) *For every bundles E, F over M one has $c_1(E \oplus F) = c_1(E) + c_1(F)$, where $E \oplus F$ is the Whitney sum defined as the pull–back of the bundle $E \times F \to M \times M$ by the diagonal inclusion of M in $M \times M$.*
- (normalization) *The first Chern class of the tautological bundle of $\mathbb{C}\mathrm{P}^1$ is equal to -1 in $H^2(\mathbb{C}\mathrm{P}^1, \mathbb{Z}) \simeq \mathbb{Z}$, which means that the integral over $\mathbb{C}\mathrm{P}^1$ of any representative of this class equals -1.*

Let $E \to M$ be a complex vector bundle. We will now explain the *Chern–Weil theory* allows one to express the images in real cohomology of the Chern classes of E using the curvature of an arbitrary connection ∇ on E. Recall the formula (9) for the curvature of ∇ in terms of ∇:

$$R^\nabla(\sigma_i) =: R^\nabla_{ij}\sigma_j = (d\omega_{ij} - \omega_{ik} \wedge \omega_{kj})\sigma_j, \tag{37}$$

where $\{\sigma_1, \ldots, \sigma_k\}$ are local sections of E which form a basis of each fiber over some open set U and the connection forms $\omega_{ij} \in \Lambda^1(U)$ (relative to the choice of this basis) are defined by

$$\nabla\sigma_i = \omega_{ij} \otimes \sigma_j.$$

Notice that although the coefficients R^∇_{ij} of R^∇ depend on the local basis of sections (σ_i), its trace is a well–defined (complex–valued) 2–form on M independent of the chosen basis, and can be computed as $\mathrm{Tr}(R^\nabla) = \sum R^\nabla_{ii}$ in the local basis (σ_i). To compute this explicitly we use the following summation trick:

$$\sum_{i,k} \omega_{ik} \wedge \omega_{ki} = \sum_{k,i} \omega_{ki} \wedge \omega_{ik} = -\sum_{i,k} \omega_{ik} \wedge \omega_{ki},$$

where the first equality is given by interchanging the summation indices and the second by the fact that the wedge product is skew–symmetric on 1–forms. From (37) we thus get

$$\mathrm{Tr}(R^\nabla) = d(\sum \omega_{ii}), \tag{38}$$

where of course the trace of the connection form $\omega = (\omega_{ij})$ *does depend* on the local basis (σ_i). This shows that $\mathrm{Tr}(R^\nabla)$ is closed, being locally exact.

If ∇ and $\tilde{\nabla}$ are connections on E, the Leibniz rule shows that their difference $A := \tilde{\nabla} - \nabla$ is a zero–order operator, more precisely a smooth section in $\Lambda^1(M) \otimes \mathrm{End}(E)$. Thus $\mathrm{Tr}(A)$ is a well–defined 1–form on M and (38) readily implies

$$\mathrm{Tr}(R^{\tilde{\nabla}}) = \mathrm{Tr}(R^{\nabla}) + d\mathrm{Tr}(A), \tag{39}$$

We thus have proved the following

LEMMA 10.2. *The 2–form $\mathrm{Tr}(R^{\nabla})$ is closed and its cohomology class $[\mathrm{Tr}(R^{\nabla})] \in H^2(M, \mathbb{C})$ does not depend on ∇.*

It is actually easy to see that $[\mathrm{Tr}(R^{\nabla})]$ is a purely imaginary class, in the sense that it has a representative which is a purely imaginary 2–form. Indeed, let us choose an arbitrary Hermitian structure h on E and take ∇ such that h is ∇–parallel. If we start with a local basis $\{\sigma_i\}$ adapted to h, then we have

$$\begin{aligned} 0 &= \nabla(\delta_{ij}) = \nabla(h(\sigma_i, \sigma_j)) = h(\nabla\sigma_i, \sigma_j) + h(\sigma_i, \nabla\sigma_j) \\ &= \omega_{ij} + \overline{\omega_{ji}}. \end{aligned}$$

From (37) we get

$$\begin{aligned} \overline{R^{\nabla}_{ij}} &= d\overline{\omega_{ij}} - \overline{\omega_{ik}} \wedge \overline{\omega_{kj}} = -\omega_{ji} - \omega_{ki} \wedge \omega_{jk} \\ &= -\omega_{ji} + \omega_{jk} \wedge \omega_{ki} = -R^{\nabla}_{ji}, \end{aligned}$$

showing that the trace of R^{∇} is a purely imaginary 2–form.

THEOREM 10.3. *Let ∇ be a connection on a complex bundle E over M. The real cohomology class*

$$c_1(\nabla) := \left[\frac{i}{2\pi}\mathrm{Tr}(R^{\nabla})\right]$$

is equal to the image of $c_1(E)$ in $H^2(M, \mathbb{R})$.

PROOF. We have to check that $c_1(\nabla)$ satisfies the three conditions in Proposition 10.1. The naturality is straightforward. Recall that if $f : M \to N$ is smooth and $\pi : E \to N$ is a rank k vector bundle, then

$$f^*(E) := \{(x, v) \mid x \in M,\ v \in E,\ f(x) = \pi(v)\}.$$

If $\{\sigma_i\}$ is a local basis of sections of E, then

$$f^*\sigma_i : M \to f^*(E), \qquad x \mapsto (x, \sigma_i(f(x)))$$

is a basis of local sections of f^*E. The formula

$$f^*\nabla(f^*\sigma) := f^*(\nabla\sigma)$$

defines a connection on f^*E (one has to check the classical formulas for basis changes in order to prove that $f^*\nabla$ is well–defined), and with respect to this basis we obviously have

$$R^{f^*\nabla}_{ij} = f^*(R^{\nabla}_{ij}),$$

whence $c_1(f^*\nabla) = f^*(c_1(\nabla))$.

The Whitney sum formula is also easy to check. If E and F are complex bundles over M with connections ∇ and $\tilde{\nabla}$ then one can define a connection $\nabla \oplus \tilde{\nabla}$ on $E \oplus F$ by

$$(\nabla \oplus \tilde{\nabla})(\sigma \oplus \tilde{\sigma})(X) := \nabla\sigma(X) \oplus \tilde{\nabla}\tilde{\sigma}(X).$$

If $\{\sigma_i\}$, $\{\tilde{\sigma}_j\}$ are local basis of sections of E and F then $\{\sigma_i \oplus 0, 0 \oplus \tilde{\sigma}_j\}$ is a local basis for $E \oplus F$ and the curvature of $\nabla \oplus \tilde{\nabla}$ in this basis is a block matrix having R^{∇} and $R^{\tilde{\nabla}}$ on the principal diagonal. Its trace is thus the sum of the traces of R^{∇} and $R^{\tilde{\nabla}}$.

We finally check the normalization property. Let $L \to \mathbb{C}\mathrm{P}^1$ be the tautological bundle. For any section $\sigma : \mathbb{C}\mathrm{P}^1 \to L$ of L we denote by $\sigma_0 : U_0 \to \mathbb{C}$ and $\sigma_1 : U_1 \to \mathbb{C}$ the expressions of σ in the standard local trivializations of L, given by $\psi_i : \pi^{-1}U_i \to U_i \times \mathbb{C}$, $\psi_i(w) = (\pi(w), w_i)$.

The Hermitian product on $\mathbb{C}^2$ induces a Hermitian structure h on L. Let ∇ be the Chern connection on L associated to h. We choose a local holomorphic section σ and denote its square norm by u. If ω is the connection form of ∇ with respect to the section σ, $\nabla\sigma = \omega \otimes \sigma$, then we can write:

$$X(u) = X(h(\sigma,\sigma)) = h(\nabla_X\sigma,\sigma) + h(\sigma,\nabla_X\sigma) = \omega(X)u + \bar{\omega}(X)u, \quad \forall X \in T\mathbb{C}\mathrm{P}^1.$$

This just means $\omega + \bar{\omega} = d\log u$. On the other hand, since σ is holomorphic and $\nabla^{0,1} = \bar{\partial}$, we see that ω is a $(1,0)$–form. Thus $\omega = \partial\log u$. From (38) we get

$$R^{\nabla} = d\omega = d\partial\log u = \bar{\partial}\partial\log u. \tag{40}$$

We thus have to check the following condition:

$$\frac{i}{2\pi}\int_{\mathbb{C}\mathrm{P}^1} \bar{\partial}\partial\log u = -1.$$

It is clearly enough to compute this integral over $U_0 := \mathbb{C}\mathrm{P}^1 - \{[0:1]\}$. We denote by $z := \phi_0 = \frac{z_1}{z_0}$ the holomorphic coordinate on U_0. We now take a particular local holomorphic section σ such that $\sigma_0(z) = 1$. From the definition of σ_0 (as the image of σ through the trivialization ψ_0 of L), we deduce that $\sigma(z)$ is the unique vector lying on the complex line generated by (z_0, z_1) in $\mathbb{C}^2$, whose first coordinate is 1, *i.e.* $\sigma(z) = (1,z)$. This shows that $u = |(1,z)|^2 = 1 + |z|^2$. In polar coordinates $z = r\cos\theta + ir\sin\theta$ one can readily compute

$$\bar{\partial}\partial f = \frac{i}{2}\left(r\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial f}{\partial r}\right) dr \wedge d\theta.$$

Applying this formula to $f := \log(1 + r^2)$ we finally get

$$\begin{aligned}
\frac{i}{2\pi}\int_{\mathbb{C}\mathrm{P}^1} \bar{\partial}\partial\log u &= \frac{i}{2\pi}\int_{[0,\infty)\times[0,2\pi]} \frac{i}{2}\left(r\frac{\partial^2 f}{\partial r^2} + \frac{\partial f}{\partial r}\right) dr \wedge d\theta \\
&= -\frac{1}{2}\int_0^{\infty} d\left(r\frac{\partial f}{\partial r}\right) = \frac{1}{2}\lim_{r\to\infty} r\frac{\partial f}{\partial r} \\
&= -\lim_{r\to\infty} \frac{r}{2}\frac{2r}{1+r^2} = -1.
\end{aligned}$$

□

If M is an almost complex manifold, we define the first Chern class of M – denoted by $c_1(M)$ – to be the first Chern class of the tangent bundle TM, viewed as complex vector bundle:

$$c_1(M) := c_1(TM).$$

In the next sections we will see that a representative of the first Chern class of a Kähler manifold is $\frac{1}{2\pi}\rho$, where ρ denotes the Ricci form.

10.2. Properties of the first Chern class. Let M be a complex manifold and let E, F be two complex vector bundles over M.

PROPOSITION 10.4. (i) *$c_1(E) = c_1(\Lambda^k E)$, where k denotes the rank of E.*

(ii) *$c_1(E \otimes F) = rk(F)c_1(E) + rk(E)c_1(F)$.*

(iii) *$c_1(E^*) = -c_1(E)$, where E^* denotes the dual of E.*

PROOF. (i) Consider any connection ∇ in E, inducing a connection $\tilde\nabla$ on $\Lambda^k E$. If $\sigma_1, \ldots, \sigma_k$ denotes a local basis of sections of E, then $\sigma := \sigma_1 \wedge \ldots \wedge \sigma_k$ is a local non–vanishing section of $\Lambda^k E$. Let $\omega := (\omega_{ij})$ and $\tilde\omega$ be the connection forms of ∇ and $\tilde\nabla$ relative to these local basis:

$$\nabla\sigma_i = \omega_{ij} \otimes \sigma_j \qquad \text{and} \qquad \tilde\nabla\sigma = \tilde\omega \otimes \sigma.$$

We then compute

$$\begin{aligned}
\tilde\nabla\sigma &= \tilde\nabla(\sigma_1 \wedge \ldots \wedge \sigma_k)\\
&= \sum_i \sigma_1 \wedge \ldots \wedge \sigma_{i-1} \wedge (\sum_j \omega_{ij} \otimes \sigma_j) \wedge \sigma_{i+1} \wedge \ldots \wedge \sigma_k\\
&= \sum_{i=j} \omega_{ij} \otimes \sigma,
\end{aligned}$$

which proves that $\tilde\omega = \mathrm{Tr}(\omega)$. From (37) we then get

$$\mathrm{Tr}(R^{\tilde\nabla}) = R^{\tilde\nabla} = d\tilde\omega - \tilde\omega \wedge \tilde\omega = d\tilde\omega$$

and

$$\mathrm{Tr}(R^{\nabla}) = \sum_{i=j}(d\omega_{ij} - \omega_{ik} \wedge \omega_{kj}) = \sum_{i=j} d\omega_{ij} = d\mathrm{Tr}(\omega) = d\tilde\omega,$$

thus proving that $c_1(E) = c_1(\Lambda^k E)$.

(ii) Let us denote by e and f the ranks of E and F. Because of the canonical isomorphism $\Lambda^{ef}(E \otimes F) \cong (\Lambda^e E)^{\otimes f} \otimes (\Lambda^f F)^{\otimes e}$, it is enough to check this relation for line bundles E and F. Any connections ∇^E and ∇^F on E and F respectively induce a connection ∇ on $E \otimes F$ defined by

$$\nabla(\sigma^E \otimes \sigma^F) := (\nabla^E\sigma^E) \otimes \sigma^F + \sigma^E \otimes (\nabla^F\sigma^F).$$

The corresponding connection forms are then related by

$$\omega = \omega^E + \omega^F$$

so clearly

$$R^{\nabla} = d\omega = d(\omega^E + \omega^F) = R^{\nabla^E} + R^{\nabla^F}.$$

(iii) Again, since $(\Lambda^k E)^*$ is isomorphic to $\Lambda^k(E^*)$, we can suppose that E is a line bundle. But in this case the canonical isomorphism $E \otimes E^* \simeq \mathbb{C}$ (where $\mathbb{C}$ denotes the trivial line bundle) shows that $0 = c_1(\mathbb{C}) = c_1(E \otimes E^*) = c_1(E) + c_1(E^*)$. $\square$

10.3. Exercises.

(1) Consider the change of variables $z = r\cos\theta + ir\sin\theta$. Show that for every function $f : U \subset \mathbb{C} \to \mathbb{C}$ the following formula holds:

$$\bar{\partial}\partial f = \frac{i}{2}\left(r\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial f}{\partial r}\right) dr \wedge d\theta.$$

(2) Show that the first Chern class of a trivial bundle vanishes.

(3) Show that if E is a complex line bundle, there is a canonical isomorphism $E \otimes E^* \simeq \mathbb{C}$.

(4) Let ∇ be any connection on a complex bundle E and let ∇^* be the induced connection on the dual E^* of E defined by

$$(\nabla^*_X \sigma^*)(\sigma) := X(\sigma^*(\sigma)) - \sigma^*(\nabla_X \sigma).$$

Show that

$$R^{\nabla^*}(X, Y) = (R^{\nabla}(X, Y))^*,$$

where $A^* \in \mathrm{End}(E^*)$ denotes the adjoint of A, defined by $A^*(\sigma^*)(\sigma) := -\sigma^*(A(\sigma))$.

11. The Ricci form of Kähler manifolds

11.1. Kähler metrics as geometric $U(m)$–structures. We start by a short review on G–structures which will help us to characterize Kähler and Ricci–flat Kähler metrics. Let M be an n–dimensional manifold and let G be any closed subgroup of $\mathrm{Gl}_n(\mathbb{R})$.

DEFINITION 11.1. *A* topological *G–structure on M is a reduction of the principal frame bundle $\mathrm{Gl}(M)$ to G. A* geometrical *G–structure is given by a topological G–structure $G(M)$ together with a torsion–free connection on $G(M)$.*

Let us give some examples. An *orientation* on M is a $\mathrm{Gl}_n^+(\mathbb{R})$–structure. An almost complex structure is a $\mathrm{Gl}_m(\mathbb{C})$–structure, for $n = 2m$. A Riemannian metric is a O_n–structure. In general, if the group G can be defined as the group preserving an element of some Gl_n representation $\rho : \mathrm{Gl}_n(\mathbb{R}) \to \mathrm{End}(V)$, then a G–structure is simply a section σ in the associated vector bundle $\mathrm{Gl}(M) \times_\rho V$ with the same algebraic properties as v_0 in the sense that for every $x \in M$ there exists $u \in \mathrm{Gl}(M)$ with $\sigma(x) = [u, v_0]$. To see this, let G be given by

$$G := \{g \in \mathrm{Gl}_n(\mathbb{R}) \mid \rho(g)(v_0) = v_0\}.$$

If $G(M)$ is a G–structure, we define a section in $\mathrm{Gl}(M) \times_\rho V$ by $\sigma(x) := [u, v_0]$ where u is an arbitrary element of the fiber $G(M)_x$. This definition clearly does not depend on u. Conversely, the set $\{u \in \mathrm{Gl}(M) \mid \sigma = [u, v_0]\}$ defines a reduction of the structure group of $\mathrm{Gl}(M)$ to G. In this setting, the G–structure is geometrical if and only if there exists a torsion–free connection on M with respect to which σ is parallel.

PROPOSITION 11.2. *The U_m–structure defined by an almost complex structure J together with a Hermitian metric h on a manifold M is geometrical if and only if the metric is Kähler.*

PROOF. The point here is that if G is a closed subgroup of O_n then there exists *at most* one torsion–free connection on any G–structure (by the uniqueness of the Levi–Civita connection). As $\mathrm{U}_m = \mathrm{O}_{2m} \cap \mathrm{Gl}_m(\mathbb{C})$, the U_m structure is geometrical if and only if the tensor defining it (namely J) is parallel with respect to the Levi–Civita connection, which by Theorem 5.5 just means that h is Kähler. □

11.2. The Ricci form as curvature form on the canonical bundle. We now turn back to our main objects of interest. Let (M^{2m}, h, J) be a Kähler manifold with Ricci form ρ and canonical bundle $K := \Lambda^{m,0}M$. As before, we will interpret the tangent bundle TM as a complex (actually holomorphic) Hermitian vector bundle over M, where the multiplication by i corresponds to the tensor J and the Hermitian structure is $h - i\Omega$. From Proposition 5.8 we know that the Levi–Civita connection ∇ on M coincides with the Chern connection on TM.

LEMMA 11.3. *The curvature $R^\nabla \in \mathcal{C}^\infty(\Lambda^2 M \otimes \mathrm{End}(TM))$ of the Chern connection and the curvature tensor R of the Levi–Civita connection are related by*

$$R^\nabla(X, Y)\xi = R(X, Y)\xi,$$

where X, Y are vector fields on M and ξ is a section of TM.

Proof. The proof is tautological, provided we make explicit the definition of R^∇. Let $\{e_i\}$ denote a local basis of vector fields on M and let $\{e_i^*\}$ denote the dual local basis of $\Lambda^1 M$. Then

$$R^\nabla \xi = \nabla^2 \xi = \nabla(e_i^* \otimes \nabla_{e_i}\xi) = de_i^* \otimes \nabla_{e_i}\xi - e_i^* \wedge e_j^* \otimes \nabla_{e_j}\nabla_{e_i}\xi.$$

Denoting $X_i := e_i^*(X)$ and $Y_i := e_i^*(Y)$ we then obtain

$$\begin{aligned} R^\nabla(X,Y)\xi &= de_i^*(X,Y)\nabla_{e_i}\xi - (e_i^* \wedge e_j^*)(X,Y)\nabla_{e_j}\nabla_{e_i}\xi \\ &= (X(Y_i) - Y(X_i) - e_i^*([X,Y]))\nabla_{e_i}\xi - (X_iY_j - X_jY_i)\nabla_{e_j}\nabla_{e_i}\xi \\ &= -\nabla_{[X,Y]}\xi + (X(Y_i) - Y(X_i))\nabla_{e_i}\xi - X_i\nabla_Y\nabla_{e_i}\xi + Y_i\nabla_X\nabla_{e_i}\xi \\ &= -\nabla_{[X,Y]}\xi - \nabla_Y\nabla_X\xi + \nabla_X\nabla_Y\xi = R(X,Y)\xi. \end{aligned}$$

□

We are now ready to prove the following characterization of the Ricci form ρ on Kähler manifolds:

Proposition 11.4. *The curvature of the Chern connection of the canonical line bundle K is equal to $i\rho$ acting by scalar multiplication on K.*

Proof. We fix some notations: let r and r^* be the curvatures of the Chern connections of $K := \Lambda^{m,0}M$ and $K^* := \Lambda^{0,m}M$. They are related by $r = -r^*$ (exercise). Moreover, the connection induced on $\Lambda^m(TM)$ with the induced Hermitian structure by the Chern connection on TM is clearly the Chern connection of $\Lambda^m(TM)$. It is easy to check that $\Lambda^m(TM)$ is isomorphic to K^*, so from the proof of Proposition 10.4 and from Lemma 11.3 we get

$$r^*(X,Y) = \mathrm{Tr}(R^\nabla(X,Y)) = \mathrm{Tr}(R(X,Y)).$$

Since we will now use both complex and real traces, we will make this explicit by a superscript. By Proposition 6.2 we then obtain

$$\begin{aligned} i\rho(X,Y) &= i\mathrm{Ric}(JX,Y) = \frac{i}{2}\mathrm{Tr}^{\mathbb{R}}(R(X,Y)\circ J) \\ &= \frac{i}{2}(2i\mathrm{Tr}^{\mathbb{C}}(R(X,Y)) = -\mathrm{Tr}^{\mathbb{C}}(R(X,Y)) \\ &= -r^*(X,Y) = r(X,Y), \end{aligned}$$

where we used the fact that

$$\mathrm{Tr}^{\mathbb{R}}(A^{\mathbb{R}}\circ J) = 2i\mathrm{Tr}^{\mathbb{C}}(A)$$

for every skew–hermitian endomorphism A. □

11.3. Ricci–flat Kähler manifolds. Let (M^{2m}, h, J) be a Kähler manifold with canonical bundle K (endowed with the Hermitian structure induced from the Kähler metric on TM) and Ricci form ρ. We suppose, for simplicity, that M is simply connected. Then the previous results can be summarized as follows:

Theorem 11.5. *The five statements below are equivalent:*

(1) *M is Ricci–flat.*
(2) *The Chern connection of the canonical bundle K is flat.*

(3) *There exists a parallel complex volume form, that is, a parallel section of* $\Lambda^{m,0}M$.
(4) M *has a geometrical* SU_m*–structure.*
(5) *The Riemannian holonomy of* M *is a subgroup of* SU_m.

In the non–simply connected case, the last 3 statements are only local.

Proof. (1) $\Longleftrightarrow$ (2) is a direct consequence of Proposition 11.4.

(2) $\Longleftrightarrow$ (3) follows from the general principle that a connection on a line bundle is flat if and only if there exists a parallel section (globally defined if $\pi_1(M) = 0$, and locally defined otherwise).

(3) $\Longleftrightarrow$ (4) The special unitary group SU_m can be defined as the stabilizer of a vector in the canonical representation of U_m onto $\Lambda^{m,0}\mathbb{C}$. Thus, there exists a parallel section in $\Lambda^{m,0}M$ if and only if the geometrical U_m–structure defined by the Kähler metric can be further reduced to a geometrical SU_m–structure.

(4) $\Longrightarrow$ (5) If $G(M)$ is a G–structure, the holonomy of a connection in $G(M)$ is contained in G. Now, if M has a geometrical SU_m–structure, the torsion–free connection defining it is just the Levi–Civita connection, therefore the Riemannian holonomy group is a subgroup of SU_m.

(5) $\Longrightarrow$ (4) The reduction theorem ([**8**], Ch. 2, Thm. 7.1) shows that for every fixed frame u, the holonomy bundle (that is, the set of frames obtained from u by parallel transport) is a $\mathrm{Hol}_u(M)$–principal bundle, and the Levi–Civita connection can be restricted to it. Thus, if the Riemannian holonomy $\mathrm{Hol}(M)$ of M is a subgroup of SU_m, we get a geometrical SU_m–structure simply by extending the holonomy bundle to SU_m.

□

Notice that by Theorem 10.3 and Proposition 11.4, for a given Kähler manifold (M, h, J), the vanishing of the first Chern class of (M, J) is a necessary condition for the existence of a Ricci–flat Kähler metric on M compatible with J. The converse statement is also true if M is compact, and will be treated in the next section.

11.4. Exercises.

(1) Let G be a closed subgroup of $\mathrm{Gl}_n(\mathbb{R})$ containing SO_n. Show that every G–structure is geometrical.

(2) Let M^n be a connected differentiable manifold. Prove that M is orientable if and only if its frame bundle $\mathrm{Gl}_n(M)$ is not connected.

(3) Show that a U_m–structure on M defines an almost complex structure together with a Hermitian metric.

(4) Show that a geometrical $\mathrm{Gl}_m(\mathbb{C})$–structure is the same as an integrable almost complex structure. *Hint:* Start with a torsion free connection ∇ and consider the connection $\tilde{\nabla}$

defined by $\tilde{\nabla}_X Y := \nabla_X Y - A_X Y$, where $A_X Y = \frac{1}{4}(2J(\nabla_X J)Y + (\nabla_{JY} J)X + J(\nabla_Y J)X)$. Use the proof of Lemma 5.4 to check that A is symmetric if and only if J is integrable.

(5) Let A be a skew–hermitian endomorphism of $\mathbb{C}^m$ and let $A^{\mathbb{R}}$ be the corresponding real endomorphism of $\mathbb{R}^{2m}$. Show that

$$\mathrm{Tr}^{\mathbb{R}}(A^{\mathbb{R}} \circ J) = 2i\mathrm{Tr}^{\mathbb{C}}(A).$$

(6) The special unitary group SU_m is usually defined as the subgroup of $\mathrm{U}_m \subset \mathrm{Gl}_m(\mathbb{C})$ consisting of complex unitary matrices of determinant 1. Prove that SU_m is equal to the stabilizer in U_m of the form $dz_1 \wedge \ldots \wedge dz_m$.

(7) Let (L, h) be a complex line bundle with Hermitian structure over some smooth manifold M. Prove that the space of Hermitian connections is an affine space over the real vector space $\mathcal{C}^\infty(\Lambda^1 M)$. Equivalently, there is a free transitive group action of $\mathcal{C}^\infty(\Lambda^1 M)$ on the space of Hermitian connections on L.

(8) If L is a complex line bundle over M, show that every real closed 2–form in the cohomology class $c_1(L) \in H^2(M, \mathbb{R})$ is $\frac{i}{2\pi}$ times the curvature of some connection on L.

12. The Calabi conjecture

12.1. An overview. We have seen that the first Chern class of any compact Kähler manifold is represented by $\frac{1}{2\pi}\rho$. Conversely, we have the following famous result due to Calabi and Yau

THEOREM 12.1. *Let M^m be a compact Kähler manifold with fundamental form φ and Ricci form ρ. Then for every closed real $(1,1)$–form ρ_1 in the cohomology class $2\pi c_1(M)$, there exists a unique Kähler metric with fundamental form φ_1 in the same cohomology class as φ, whose Ricci form is exactly ρ_1.*

Before giving an outline of the proof, we state some corollaries.

COROLLARY 12.2. *If the first Chern class of a compact Kähler manifold vanishes, then M carries a Ricci–flat Kähler metric.*

COROLLARY 12.3. *If the first Chern class of a compact Kähler manifold is positive, then M is simply connected.*

PROOF. By the Calabi theorem M has a Kähler metric with positive Ricci curvature, so the result follows from Theorem 15.6 below. $\square$

The first step in the proof of Theorem 12.1 is to reformulate the problem in order to reduce it to a so–called *Monge–Ampère equation.* We denote by $\mathcal{H}$ the set of Kähler metrics in the same cohomology class as φ. The global $i\partial\bar\partial$–Lemma shows that

$$\mathcal{H} = \left\{ u \in \mathcal{C}^\infty(M) \mid \varphi + i\partial\bar\partial u > 0,\ \int_M u\varphi^m = 0 \right\} \tag{41}$$

(this last condition is needed since u is only defined up to a constant).

Now, if g and g_1 are Kähler metrics with Kähler forms φ and φ_1 in the same cohomology class, we denote by $dv := \frac{1}{m!}\varphi^m$ and $dv_1 := \frac{1}{m!}\varphi_1^m$ their volume forms and consider the real function f defined by $e^f\,dv = dv_1$. Since $[\varphi] = [\varphi_1]$ we also have $[\varphi^m] = [\varphi_1^m]$, that is

$$\int_M e^f\,dv = \int_M dv. \tag{42}$$

Let ρ and ρ_1 denote the corresponding Ricci forms. Since $i\rho$ is the curvature of the canonical bundle K_M, for every local holomorphic section ω of K_M we have

$$i\rho = \bar\partial\partial \log g(\omega,\bar\omega) \qquad \text{and} \qquad i\rho_1 = \bar\partial\partial \log g_1(\omega,\bar\omega). \tag{43}$$

It is easy to check that the Hodge operator acts on $\Lambda^{m,0}$ simply by scalar multiplication with $\varepsilon := i^m(-1)^{\frac{m(m+1)}{2}}$. We thus have

$$\varepsilon\omega\wedge\bar\omega = \omega\wedge *\bar\omega = g(\omega,\bar\omega)dv \tag{44}$$

and similarly

$$\varepsilon\omega\wedge\bar\omega = g_1(\omega,\bar\omega)dv_1 = e^f g_1(\omega,\bar\omega)dv. \tag{45}$$

From (43)–(45) we get

$$i\rho_1 - i\rho = \partial\bar{\partial}f. \tag{46}$$

This shows that the Ricci form of the modified Kähler metric $\varphi_1 = \varphi + i\partial\bar{\partial}u$ can be computed by the formula

$$\rho_1 = \rho - i\partial\bar{\partial}f, \qquad \text{where } f = \log\frac{(\varphi + i\partial\bar{\partial}u)^m}{\varphi^m}. \tag{47}$$

Now given closed real $(1,1)$–form ρ_1 in the cohomology class $2\pi c_1(M)$, the global $i\partial\bar{\partial}$–Lemma shows that there exists some real function f such that $\rho_1 = \rho - i\partial\bar{\partial}f$. Moreover, f is unique if we impose the normalization condition (42). We denote by $\mathcal{H}'$ the space of smooth functions on M satisfying this condition. The Calabi conjecture is then equivalent to the following

THEOREM 12.4. *The mapping* $Cal : \mathcal{H} \to \mathcal{H}'$ *defined by*

$$Cal(u) = \log\frac{(\varphi + i\partial\bar{\partial}u)^m}{\varphi^m}$$

is a diffeomorphism.

We first show that Cal is injective. It is clearly enough to show that $Cal(u) = 0$ and $u \in \mathcal{H}$ implies $u = 0$. If $Cal(u) = 0$ we have $\varphi_1^m = \varphi^m$, and since 2–forms commute we obtain

$$0 = \varphi_1^m - \varphi^m = i\partial\bar{\partial}u \wedge \sum_{k=0}^{m-1} \varphi_1^k \wedge \varphi^{m-k-1}.$$

Using the formula $2i\partial\bar{\partial} = dd^c$ and the fact that φ and φ_1 are closed forms we get after multiplication by u

$$\begin{aligned} 0 &= 2iu\partial\bar{\partial}u \wedge \sum_{k=0}^{m-1} \varphi_1^k \wedge \varphi^{m-k-1} = udd^cu \wedge \sum_{k=0}^{m-1} \varphi_1^k \wedge \varphi^{m-k-1} \\ &= d\left(ud^cu \wedge \sum_{k=0}^{m-1} \varphi_1^k \wedge \varphi^{m-k-1}\right) - du \wedge d^cu \wedge \sum_{k=0}^{m-1} \varphi_1^k \wedge \varphi^{m-k-1}. \end{aligned}$$

Integrating over M and using Stokes' theorem yields

$$0 = \sum_{k=0}^{m-1} \int_M du \wedge Jdu \wedge \varphi_1^k \wedge \varphi^{m-k-1}. \tag{48}$$

Now, since φ_1 defines a Kähler metric, there exists a local basis $\{e_1, Je_1, \ldots, e_m, Je_m\}$ orthonormal with respect to g such that

$$\varphi = \sum_{j=1}^{m} e_j \wedge Je_j \qquad \text{and} \qquad \varphi_1 = \sum_{j=1}^{m} a_j e_j \wedge Je_j,$$

where a_j are strictly positive local functions. This shows easily that for every k

$$\varphi_1^k \wedge \varphi^{m-k-1} = *(\sum_{j=1}^{m} b_j^k e_j \wedge Je_j), \quad b_j^k > 0.$$

In fact one can compute explicitly

$$b_j^k = k!(m-k-1)! \sum_{\substack{j_1 \neq j, \ldots, j_k \neq j \\ j_1 < \ldots < j_k}} a_{j_1} \ldots a_{j_k}.$$

This shows that the integrand in (48) is strictly positive unless $du = 0$. Thus u is a constant, so $u = 0$ because the integral of $u\,dv$ over M vanishes. Therefore Cal is injective.

To prove that it is a local diffeomorphism, we compute its differential at some $u \in \mathcal{H}$. By changing the reference metric if necessary, we may suppose without loss of generality that $u = 0$. For $v \in T_0\mathcal{H}$ we compute

$$\begin{aligned} Cal_*(v) &= \frac{d}{dt}|_{t=0}(Cal(tv)) = \frac{d}{dt}|_{t=0}\left(\frac{(\varphi + i\partial\bar{\partial}tv)^m}{\varphi^m}\right) \\ &= m\frac{i\partial\bar{\partial}v \wedge \varphi^{m-1}}{\varphi^m} = \Lambda(i\partial\bar{\partial}v) = -\bar{\partial}^*\bar{\partial}v = -\Delta^{\bar{\partial}}v. \end{aligned}$$

From the general elliptic theory we know that the Laplace operator is a bijection of the space of functions with zero integral over M. Thus Cal_* is bijective, so the Inverse Function Theorem shows that Cal is a local diffeomorphism.

The surjectivity of Cal, which is the hard part of the theorem, follows from *a priori* estimates, which show that Cal is proper. We refer the reader to [**6**] for details.

12.2. Exercises.

(1) Show that $*\omega = i^{m(m+2)}\omega$ for all $\omega \in \Lambda^{m,0}M$ on a Hermitian manifold M of complex dimension m.

(2) Prove that the mapping
$$u \mapsto \varphi + i\partial\bar{\partial}u$$
is indeed a bijection from the set defined in (41) to the set of Kähler metrics in the cohomology class $[\varphi]$.

(3) Prove that the total volume of a Kähler metric on a compact manifold only depends on the cohomology class of its fundamental form.

(4) Show that $*(\varphi^{m-1}) = (m-1)!\varphi$ on every Hermitian manifold M of complex dimension m. Using this, prove that if (M, φ) is Kähler, then $\rho \wedge \varphi^{m-1} = (m-1)!S$, where ρ denotes the Ricci form and S is the scalar curvature of the Kähler metric defined by φ.

(5) Let (M^m, J) be a compact complex manifold. Show that the integral over M of the scalar curvature of a Kähler metric only depends on the cohomology class of its fundamental form φ. More precisely one has

$$\int_M S dv = 2\pi m c_1(M) \cup [\varphi]^{m-1}.$$

13. Kähler–Einstein metrics

13.1. The Aubin–Yau theorem. We turn our attention to compact Kähler manifolds (M, g) satisfying the Einstein condition

$$\mathrm{Ric} = \lambda g, \qquad \lambda \in \mathbb{R}.$$

We will exclude the case $\lambda = 0$ which was treated above. If we rescale the metric by a positive constant, the curvature tensor does not change, so neither does the Ricci tensor, which was defined as a trace. This shows that we may suppose that $\lambda = \varepsilon = \pm 1$. The Kähler–Einstein condition reads

$$\rho = \varepsilon\varphi, \qquad \varepsilon = \pm 1.$$

As the first Chern class of M is represented by $\frac{\rho}{2\pi}$, we see that a necessary condition for the existence of a Kähler–Einstein manifold on a given compact Kähler manifold is that its first Chern class is *definite* (positive or negative). In the negative case, this condition turns out to be also sufficient:

THEOREM 13.1. (Aubin, Yau) *A compact Kähler manifold with negative first Chern class admits a unique Kähler–Einstein metric with Einstein constant* $\varepsilon = -1$.

We will treat simultaneously the two cases $\varepsilon = \pm 1$, in order to emphasize the difficulties that show up in the case $\varepsilon = 1$.

As before, we first reformulate the problem. Let $(M^{2m}, g, J, \varphi, \rho)$ be a compact Kähler manifold with definite first Chern class $c_1(M)$. By definition, there exists a positive closed $(1,1)$–form representing the cohomology class $2\pi\varepsilon c_1(M)$. We can suppose without loss of generality that this form is equal to φ (otherwise we just change the initial Kähler metric). Then $[\varphi] = 2\pi\varepsilon c_1(M) = [\varepsilon\rho]$, so the global $i\partial\bar\partial$–Lemma shows that there exists some function f with

$$\rho = \varepsilon\varphi + i\partial\bar\partial f. \tag{49}$$

We are looking for a new Kähler metric g_1 with fundamental form φ_1 and Ricci form ρ_1 such that $\rho_1 = \varepsilon\varphi_1$. Suppose we have such a metric. From our choice for φ we have

$$[2\pi\varphi] = \varepsilon c_1(M) = [2\pi\varepsilon\rho_1] = [2\pi\varphi_1].$$

From this equation and the global $i\partial\bar\partial$–Lemma it is clear that there exists a unique function $u \in \mathcal{H}$ such that $\varphi_1 = \varphi + i\partial\bar\partial u$. Now the previously obtained formula (47) for the Ricci form of the new metric reads

$$\rho_1 = \rho - i\partial\bar\partial \log \frac{(\varphi + i\partial\bar\partial u)^m}{\varphi^m}. \tag{50}$$

Using (49) and (50), the Kähler–Einstein condition for g_1 becomes

$$\varepsilon\varphi + i\partial\bar\partial f - i\partial\bar\partial \log \frac{(\varphi + i\partial\bar\partial u)^m}{\varphi^m} = \varepsilon\varphi_1, \tag{51}$$

which is equivalent to

$$\log \frac{(\varphi + i\partial\bar{\partial}u)^m}{\varphi^m} + \varepsilon u = f + const. \tag{52}$$

Conversely, if $u \in \mathcal{C}^\infty_+(M)$ satisfies this equation, then the Kähler metric $\varphi_1 := \varphi + i\partial\bar{\partial}u$ is Kähler–Einstein (we denote by $\mathcal{C}^\infty_+(M)$ the space of all smooth functions u on M such that $\varphi + i\partial\bar{\partial}u > 0$). The Aubin–Yau theorem is therefore equivalent to the fact that the mapping

$$Cal^\varepsilon : \mathcal{C}^\infty_+(M) \to \mathcal{C}^\infty(M) \qquad Cal^\varepsilon(u) := Cal(u) + \varepsilon u$$

is a diffeomorphism.

The injectivity of Cal^- can be proved as follows. Suppose that $Cal^-(u_1) = Cal^-(u_2)$ and denote by $\varphi_1 := \varphi + i\partial\bar{\partial}u_1$ and $\varphi_2 := \varphi + i\partial\bar{\partial}u_2$. Then

$$\log \frac{\varphi_1^m}{\varphi^m} - u_1 = \log \frac{\varphi_2^m}{\varphi^m} - u_2,$$

hence, denoting the difference $u_2 - u_1$ by u:

$$\log \frac{(\varphi_1 + i\partial\bar{\partial}u)^m}{\varphi_1^m} = u. \tag{53}$$

At a point where u attains its maximum, the $(1,1)$–form $i\partial\bar{\partial}u$ is negative semi–definite, since we can write (for any vector X parallel at that point)

$$\begin{aligned} i\partial\bar{\partial}u(X, JX) &= \frac{1}{2}(dd^c u)(X, JX) = \frac{1}{2}(X(d^c u(JX)) - JX(d^c u(X))) \\ &= \frac{1}{2}(H^u(X,X) + H^u(JX,JX)) \le 0, \end{aligned}$$

since the Hessian H^u of u is of course negative semi–definite at a point where u reaches its maximum. Taking into account (53) we see that $u \le 0$ at each of its maximum points, so $u \le 0$ on M. Similarly, $u \ge 0$ at each minimum points, so finally $u = 0$ on M, thus proving the injectivity of Cal^-.

We have already computed the differential of Cal at $u = 0$ applied to some $v \in T_0\mathcal{C}^\infty_+(M)$:

$$Cal_*(v) = -\Delta^{\bar{\partial}} v.$$

Consequently

$$Cal^-_*(v) = -v - \Delta^{\bar{\partial}} v$$

is a bijection of $\mathcal{C}^\infty(M)$ since the self–adjoint elliptic operator $v \mapsto \frac{1}{2}\Delta v + v$ has obviously no kernel and its index is zero.

As before, the surjectivity of Cal^- is harder to prove and requires non–trivial analysis (see [**6**]).

13.2. Holomorphic vector fields on compact Kähler–Einstein manifolds. Let M^{2m} be a compact Kähler manifold. We start by showing the following

LEMMA 13.2. *Let ξ be a holomorphic (real) vector field with dual 1–form also denoted by ξ. Then ξ can be decomposed in a unique manner as*

$$\xi = df + d^c h + \xi^H,$$

where f and h are functions with vanishing integral and ξ^H is the harmonic part of ξ in the usual Hodge decomposition.

PROOF. Since ξ is holomorphic we have $\mathcal{L}_\xi J = 0$, so $[\xi, JX] = J[\xi, X]$ for every vector field X. Thus $\nabla_{JX}\xi = J\nabla_X \xi$, so taking the scalar product with some JY and skew–symmetrising yields $d\xi(JX, JY) = d\xi(X, Y)$, *i.e.* $d\xi$ is of type $(1, 1)$. The global dd^c–Lemma shows that $d\xi = dd^c h$ for some function h. The form $\xi - d^c h$ is closed, so the Hodge decomposition theorem shows that

$$\xi - d^c h = df + \xi_0$$

for some function f and some harmonic 1–form ξ_0. Comparing this formula with the Hodge decomposition for ξ and using the fact that harmonic 1–forms are L^2–orthogonal to $d\mathcal{C}^\infty(M)$, $d^c\mathcal{C}^\infty(M)$ and $\delta\Omega^2(M)$ shows that ξ_0 equals ξ^H, the harmonic part of ξ. Finally, the uniqueness of f and h follows easily from the normalization condition, together with the fact that $d\mathcal{C}^\infty(M)$ and $d^c\mathcal{C}^\infty(M)$ are L^2–orthogonal. $\square$

Next, we have the following characterization of real holomorphic and Killing vector fields on compact Kähler–Einstein manifolds with positive scalar curvature.

LEMMA 13.3. *A vector field ξ (resp. ζ) on a compact Kähler–Einstein manifold M^{2m} with positive scalar curvature S is Killing (resp holomorphic) if and only if $\xi = Jdh$ (resp. $\zeta = df + d^c h$) where h (resp. f and h) are eigenfunctions of the Laplace operator corresponding to the eigenvalue $\frac{S}{m}$.*

PROOF. The Ricci tensor of M satisfies $\mathrm{Ric}(X) = \frac{S}{2m}X$ for every vector X. Let ξ be a vector field on M. If we view as usual TM as a holomorphic vector bundle, then the Weitzenböck formula (see (65) below) yields

$$2\bar{\partial}^*\bar{\partial}\xi = \nabla^*\nabla\xi + i\rho\xi = \nabla^*\nabla\xi - \mathrm{Ric}(\xi) = \nabla^*\nabla\xi - \frac{S}{2m}\xi. \tag{54}$$

The Bochner formula (Exercise 3 in the next section) reads

$$\Delta\xi = \nabla^*\nabla\xi + \mathrm{Ric}(\xi) = \nabla^*\nabla\xi + \frac{S}{2m}\xi. \tag{55}$$

Since $S > 0$, this shows that there are no harmonic 1–forms on M.

Suppose that ζ is holomorphic. From Lemma 13.2 we then can write ζ as a sum $\zeta = df + d^c h$, where f and h have vanishing integrals over M. Now, subtracting (54) from (55) yields $\Delta\zeta = \frac{S}{m}\zeta$, so $\Delta(df + d^c h) = d(\frac{S}{m}f) + d^c(\frac{S}{m}h)$, and since Δ commutes with d and d^c, and the images of d and d^c are L^2–orthogonal, this yields $\Delta f = \frac{S}{m}f + c_1$ and $\Delta h = \frac{S}{m}h + c_2$. Finally the constants have to vanish because of the normalization condition.

If ξ is Killing, then ξ is holomorphic by Proposition 9.5. The codifferential of every Killing vector field vanishes, and moreover δ anti–commutes with d^c. Thus $0 = \delta\xi = \delta df$, showing that $df = 0$, so $\xi = d^c h$ with $\Delta h = \frac{S}{m}h$.

Conversely, suppose that $\xi = df + d^c h$ and f and h are eigenfunctions of the Laplace operator corresponding to the eigenvalue $\frac{S}{m}$. Then $\Delta\xi = \frac{S}{m}\xi$, so from (55) we get

$$\frac{S}{m}\xi = \nabla^*\nabla\xi + \frac{S}{2m}\xi.$$

Then (54) shows that ξ is holomorphic.

If moreover $df = 0$, we have

$$\mathcal{L}_\xi\varphi = d(\xi \lrcorner \varphi) + \xi \lrcorner d\varphi = d(J\xi) = -ddh = 0,$$

where φ is the fundamental form of M. Together with $\mathcal{L}_\xi J = 0$, this shows that $\mathcal{L}_\xi g = 0$, so ξ is Killing. $\square$

We are now ready to prove the following result of Matsushima:

Theorem 13.4. *The Lie algebra $\mathfrak{g}(M)$ of Killing vector fields on a compact Kähler–Einstein manifold M with positive scalar curvature is a real form of the Lie algebra $\mathfrak{h}(M)$ of (real) holomorphic vector fields on M. In particular $\mathfrak{h}(M)$ is reductive, i.e. it is the direct sum of its center and a semi–simple Lie algebra.*

Proof. Let $F : \mathfrak{g}(M) \otimes \mathbb{C} \to \mathfrak{h}(M)$ be the linear map given by $F(\xi + i\zeta) := \xi + J\zeta$. Since J maps holomorphic vector fields to holomorphic vector fields, F is well–defined. The two lemmas above clearly show that F is a vector space isomorphism. Moreover, F is a Lie algebra morphism because Killing vector fields are holomorphic, so

$$\begin{aligned} F([\xi + i\zeta, \xi_1 + i\zeta_1]) &= [\xi, \xi_1] - [\zeta, \zeta_1] + J([\xi, \zeta_1] + [\zeta, \xi_1]) = [\xi, \xi_1] + J^2[\zeta, \zeta_1] + [J\xi, \zeta_1] + [J\zeta, \xi_1] \\ &= [\xi, \xi_1] + [J\zeta, J\zeta_1] + [J\xi, \zeta_1] + [J\zeta, \xi_1] = [F(\xi + i\zeta), F(\xi_1 + i\zeta_1)]. \end{aligned}$$

The last statement follows from the fact that the isometry group of M is compact, and every Lie algebra of compact type, as well as its complexification, is reductive. $\square$

There exist compact Kähler manifolds with positive first Chern class whose Lie algebra of holomorphic vector fields is not reductive. Therefore such a manifold carries no Kähler–Einstein metric, thus showing that Theorem 13.1 cannot hold in the positive case.

Part 5

Vanishing results

14. Weitzenböck techniques

14.1. The Weitzenböck formula. The aim of the next 3 sections is to derive vanishing results under certain positivity assumptions on the curvature using Weitzenböck techniques.

The general principle is the following: let $(E,h)\to M$ be some holomorphic Hermitian bundle over a compact Kähler manifold (M^{2m},g,J), with holomorphic structure $\bar\partial:\mathcal{C}^\infty(\Lambda^{p,q}M\otimes E)\to\mathcal{C}^\infty(\Lambda^{p,q+1}M\otimes E)$ and Chern connection $\nabla:\mathcal{C}^\infty(\Lambda^{p,q}M\otimes E)\to\mathcal{C}^\infty(\Lambda^1_{\mathbb{C}}M\otimes\Lambda^{p,q}M\otimes E)$. If $\bar\partial^*$ and ∇^* are the formal adjoints of $\bar\partial$ and ∇, it turns out that the difference of the differential operators of order two $\nabla^*\nabla$ and $2(\bar\partial^*\bar\partial+\bar\partial\bar\partial^*)$ is a zero–order operator, depending only on the curvature of the Chern connection:

$$2(\bar\partial^*\bar\partial+\bar\partial\bar\partial^*)=\nabla^*\nabla+\mathcal{R}, \tag{56}$$

where $\mathcal{R}$ is a section in $\mathrm{End}(\Lambda^{p,q}M\otimes E)$. If $\mathcal{R}$ is a *positive operator* on $\Lambda^{p,0}M\otimes E$, then every holomorphic section of $\Lambda^{p,0}M\otimes E$ is ∇–parallel, and if $\mathcal{R}$ is *strictly positive* on $\Lambda^{p,0}M\otimes E$, then this holomorphic bundle has no holomorphic section. This follows by applying (56) to some holomorphic section σ of $\Lambda^{p,0}M\otimes E$, taking the scalar product with σ and integrating over M, using the fact that $\bar\partial^*$ vanishes identically on $\Lambda^{p,0}M\otimes E$.

We start with the following technical lemma:

LEMMA 14.1. *If $\{e_j\}$ is a local orthonormal basis in TM (identified via the metric g with an orthonormal basis of Λ^1M), and ∇ denotes the Chern connection of E, as well as its prolongation to $\Lambda^{p,q}M\otimes E$ using the Levi–Civita connection on the left–hand side of this tensor product, then $\bar\partial$, $\bar\partial^*$, ∇^* and $\nabla^*\nabla$ are given locally by*

$$\bar\partial:\mathcal{C}^\infty(\Lambda^{p,q}M\otimes E)\to\mathcal{C}^\infty(\Lambda^{p,q+1}M\otimes E)\qquad \bar\partial\sigma=\frac{1}{2}(e_j-iJe_j)\wedge\nabla_{e_j}(\sigma), \tag{57}$$

$$\bar\partial^*:\mathcal{C}^\infty(\Lambda^{p,q}M\otimes E)\to\mathcal{C}^\infty(\Lambda^{p,q-1}M\otimes E)\qquad \bar\partial^*\sigma=-\frac{1}{2}(e_j+iJe_j)\lrcorner\nabla_{e_j}(\sigma), \tag{58}$$

$$\nabla^*:\mathcal{C}^\infty(\Lambda^1_{\mathbb{C}}M\otimes\Lambda^{p,q}M\otimes E)\to\mathcal{C}^\infty(\Lambda^{p,q}M\otimes E)\qquad \nabla^*(\omega\otimes\sigma)=(\delta\omega)\sigma-\nabla_\omega\sigma, \tag{59}$$

$$\nabla^*\nabla:\mathcal{C}^\infty(\Lambda^{p,q}M\otimes E)\to\mathcal{C}^\infty(\Lambda^{p,q}M\otimes E)\qquad \nabla^*\nabla\sigma=\nabla_{\nabla_{e_j}e_j}\sigma-\nabla_{e_j}\nabla_{e_j}\sigma. \tag{60}$$

PROOF. If (E,h^E) and (F,h^F) are Hermitian bundles, their tensor product inherits a natural Hermitian structure given by

$$h^{E\otimes F}(\sigma^E\otimes\sigma^F,s^E\otimes s^F):=h^E(\sigma^E,s^E)h^F(\sigma^F,s^F).$$

The Hermitian structure, on $\Lambda^{p,q}M\otimes E$ with respect to which one defines the adjoint operators above is obtained in this way from the Hermitian structure h of E and the Hermitian structure H of $\Lambda^{p,q}M$ given by (31). We will use the same symbol H for this Hermitian structure, by a slight abuse of notation.

The relation (57) is more or less tautological, using the definition of $\bar\partial$ and the fact that e_j-iJe_j is a $(1,0)$–vector, identified via the metric g with a $(0,1)$–form. Of course, the wedge product there only concerns the $\Lambda^{p,q}M$–part of σ.

For $\sigma \in \mathcal{C}^\infty(\Lambda^{p,q}M \otimes E)$ and $s \in \mathcal{C}^\infty(\Lambda^{p,q-1}M \otimes E)$ we define the 1–form α by

$$\alpha(X) := \frac{1}{2} H((X + iJX) \lrcorner \sigma, s).$$

By choosing the local basis $\{e_j\}$ parallel in a point for simplicity, then we get at that point:

$$\begin{aligned} -\delta\alpha &= e_j(\alpha(e_j)) = \frac{1}{2} H((e_j + iJe_j) \lrcorner \nabla_{e_j}\sigma, s) + \frac{1}{2} H((e_j + iJe_j) \lrcorner \sigma, \nabla_{e_j} s) \\ &= \frac{1}{2} H((e_j + iJe_j) \lrcorner \nabla_{e_j}\sigma, s) + \frac{1}{2} H(\sigma, (e_j - iJe_j) \wedge \nabla_{e_j} s) \\ &= \frac{1}{2} H((e_j + iJe_j) \lrcorner \nabla_{e_j}\sigma, s) + H(\sigma, \bar\partial s), \end{aligned}$$

thus showing that the operator $-\frac{1}{2}(e_j + iJe_j) \lrcorner \nabla_{e_j}$ is the formal adjoint of $\bar\partial$.

The proof of (59) is similar: for $\omega \otimes \sigma \in \mathcal{C}^\infty(\Lambda^1_{\mathbb{C}}M \otimes \Lambda^{p,q}M \otimes E)$ and $s \in \mathcal{C}^\infty(\Lambda^{p,q}M \otimes E)$ we define the 1–form α by

$$\alpha(X) := H((\omega(X))\sigma, s)$$

and compute

$$\begin{aligned} -\delta\alpha &= e_j(\alpha(e_j)) = -H((\delta\omega)\sigma, s) + H((\omega(e_j))\nabla_{e_j}\sigma, s) + H((\omega(e_j))\sigma, \nabla_{e_j} s) \\ &= H(\nabla_\omega \sigma - (\delta\omega)\sigma, s) + H(\omega \otimes \sigma, \nabla s), \end{aligned}$$

whence $\nabla^*(\omega \otimes \sigma) = (\delta\omega)\sigma - \nabla_\omega\sigma$.

Finally, we apply (59) to some section $\nabla\sigma = e_j \otimes \nabla_{e_j}\sigma$ of $\Lambda^1_{\mathbb{C}}M \otimes \Lambda^{p,q}M \otimes E$ and get

$$\begin{aligned} \nabla^*\nabla\sigma &= (\delta e_j)\nabla_{e_j}\sigma - \nabla_{e_j}\nabla_{e_j}\sigma = -g(e_k, \nabla_{e_k} e_j)\nabla_{e_j}\sigma - \nabla_{e_j}\nabla_{e_j}\sigma \\ &= g(\nabla_{e_k} e_k, e_j)\nabla_{e_j}\sigma - \nabla_{e_j}\nabla_{e_j}\sigma = \nabla_{\nabla_{e_j} e_j}\sigma - \nabla_{e_j}\nabla_{e_j}\sigma. \end{aligned}$$

□

We are now ready for the main result of this section

THEOREM 14.2. *Let $(E, h) \to (M^{2m}, g, J)$ be a holomorphic Hermitian bundle over a Kähler manifold M. For vectors $X, Y \in TM$, let $\tilde R(X, Y) \in \mathrm{End}(\Lambda^{p,q}M \otimes E)$ be the curvature operator of the tensor product connection on $\Lambda^{p,q}M \otimes E$ induced by the Levi–Civita connection on $\Lambda^{p,q}M$ and the Chern connection on E. Then the following formula holds*

$$2(\bar\partial^*\bar\partial + \bar\partial\bar\partial^*) = \nabla^*\nabla + \mathcal{R}, \tag{61}$$

where $\mathcal{R}$ is the section of $\mathrm{End}(\Lambda^{p,q}M \otimes E)$ *defined by*

$$\mathcal{R}(\sigma) := \frac{i}{2}\tilde R(Je_j, e_j)\sigma - \frac{1}{2}(e_j - iJe_j) \wedge (e_k + iJe_k) \lrcorner (\tilde R(e_j, e_k)\sigma). \tag{62}$$

Proof. The proof is a simple computation in a local orthonormal frame parallel at a point, using Lemma 14.1, (57),(58) and (60):

$$\begin{aligned}
2\bar\partial^*\bar\partial &= -\frac{1}{2}\Big((e_k + iJe_k) \lrcorner \nabla_{e_k}((e_j - iJe_j)\wedge \nabla_{e_j})\Big)\\
&= -\frac{1}{2}\Big((e_k + iJe_k) \lrcorner ((e_j - iJe_j)\wedge \nabla_{e_k}\nabla_{e_j})\Big)\\
&= -(g(e_k,e_j) + ig(Je_k,e_j))\nabla_{e_k}\nabla_{e_j} + \frac{1}{2}\Big((e_j - iJe_j)\wedge (e_k + iJe_k) \lrcorner \nabla_{e_k}\nabla_{e_j}\Big)\\
&= \nabla^*\nabla - ig(Je_k,e_j)\nabla_{e_k}\nabla_{e_j} + \frac{1}{2}\Big((e_j - iJe_j)\wedge (e_k + iJe_k) \lrcorner \nabla_{e_j}\nabla_{e_k}\Big)\\
&\quad + \frac{1}{2}\Big((e_j - iJe_j)\wedge (e_k + iJe_k) \lrcorner \tilde R(e_k,e_j)\Big)\\
&= \nabla^*\nabla - \frac{i}{2}g(Je_k,e_j)\tilde R(e_k,e_j) + \frac{1}{2}\Big((e_j - iJe_j)\wedge \nabla_{e_j}((e_k + iJe_k) \lrcorner \nabla_{e_k})\Big)\\
&\quad + \frac{1}{2}\Big((e_j - iJe_j)\wedge (e_k + iJe_k) \lrcorner \tilde R(e_k,e_j)\Big)\\
&= \nabla^*\nabla - 2\bar\partial\bar\partial^* + \mathcal{R}.
\end{aligned}$$

□

14.2. Vanishing results on Kähler manifolds. Most of the applications will concern the case $q=0$. The expression of the curvature term becomes then particularly simple, since the last term in (62) automatically vanishes. Let $\rho^{(p)}$ denote the action of the Ricci–form of M on $\Lambda^{p,0}$ given by

$$\rho^{(p)}\omega := \rho(e_j)\wedge e_j \lrcorner \omega.$$

It is easy to check that this action preserves the space $\Lambda^{p,0}$.

Proposition 14.3. *If $q=0$, for every section $\omega\otimes\xi$ of $\Lambda^{p,0}M\otimes E$ we have*

$$2\bar\partial^*\bar\partial(\omega\otimes\xi) = \nabla^*\nabla(\omega\otimes\xi) + i(\rho^{(p)}\omega)\otimes\xi + \frac{i}{2}\omega\otimes R^E(Je_j,e_j)\xi, \tag{63}$$

where R^E is the curvature of E.

Proof. The curvature $\tilde R$ of $\Lambda^{p,0}M\otimes E$ decomposes in a sum

$$\tilde R(X,Y)(\omega\otimes\xi) = (R(X,Y)\omega)\otimes\xi + \omega\otimes R^E(X,Y)(\xi), \tag{64}$$

where R is the Riemannian curvature. It is an easy exercise to check that the Riemannian curvature operator acts on forms by $R(X,Y)(\omega) = R(X,Y)e_k\wedge e_k \lrcorner \omega$. From Proposition (6.2) (i) we have $2\rho = R(Je_j,e_j)$ as endomorphisms of the tangent space of M. Therefore Theorem 14.2 and (64) yield the desired result. □

We are now ready to obtain the vanishing results mentioned above.

THEOREM 14.4. *Let M be a compact Kähler manifold. If the Ricci curvature of M is negative definite (i.e.* $\mathrm{Ric}(X,X) < 0$ *for all non–zero $X \in TM$) then M has no holomorphic vector field.*

PROOF. Let us take $p = 0$ and $E = T^{1,0}M$ in Proposition 14.3. If ξ is a holomorphic vector field, we have

$$0 = 2\bar\partial^*\bar\partial\xi = \nabla^*\nabla\xi + \frac{i}{2}R(Je_j, e_j)\xi = \nabla^*\nabla\xi + i\rho(\xi). \tag{65}$$

Taking the (Hermitian) scalar product with ξ in this formula and integrating over M, using the fact that $\rho\xi = \mathrm{Ric}(J\xi) = i\mathrm{Ric}(\xi)$ yields

$$0 = \int_M H(\nabla^*\nabla\xi - \mathrm{Ric}(\xi), \xi)dv = \int_M |\nabla\xi|^2 - H(\mathrm{Ric}\xi, \xi)dv.$$

Thus, if Ric is negative definite, ξ has to vanish identically. □

THEOREM 14.5. *Let M be a compact Kähler manifold. If the Ricci curvature of M vanishes, then every holomorphic form is parallel. If the Ricci curvature of M is positive definite, then there exist no holomorphic $(p,0)$–forms on M for $p > 0$.*

PROOF. We take E to be trivial and apply (63) to some holomorphic $(p,0)$–form ω. Since $\rho = 0$ we get $0 = \nabla^*\nabla\omega$. Taking the Hermitian product with ω and integrating over M yields the result.

Suppose now that Ric is positive definite. From (63) applied to some holomorphic $(p,0)$–form ω we get

$$0 = \nabla^*\nabla\omega + i\rho^{(p)}(\omega). \tag{66}$$

The interior product of a $(0,1)$–vector and ω vanishes, showing that $JX \lrcorner\, \omega = iX \lrcorner\, \omega$. We thus get

$$i\rho^{(p)}(\omega) = i\rho(e_j) \wedge e_j \lrcorner\, \omega = i\rho(Je_j) \wedge Je_j \lrcorner\, \omega = -\rho(Je_j) \wedge e_j \lrcorner\, \omega = \mathrm{Ric}(\omega).$$

Since Ric is positive, its extension to $(p,0)$–forms is positive, too, hence taking the Hermitian product with ω in (66) and integrating over M yields

$$\int_M |\nabla\omega|^2 + H(\mathrm{Ric}(\omega), \omega)dv = 0,$$

showing that ω has to vanish □

14.3. Exercises.

(1) Show that the extension to $\Lambda^p M \otimes \mathbb{C}$ of a positive definite symmetric endomorphism of TM is positive definite.

(2) Prove the following real version of the Weitzenböck formula:

$$\Delta\omega = \nabla^*\nabla\omega + \mathcal{R}\omega, \qquad \forall\ \omega \in \Omega^p M,$$

where $\mathcal{R}$ is the endomorphism of $\Omega^p M$ defined by

$$\mathcal{R}(\omega) := -e_j \wedge e_k \lrcorner (R(e_j, e_k)(\omega)).$$

(3) Applying the above identity to 1–forms, prove the Bochner formula

$$\Delta\omega = \nabla^*\nabla\omega + \mathrm{Ric}(\omega), \qquad \forall\ \omega \in \Omega^1 M.$$

(4) Prove that there are no global holomorphic forms on the complex projective space.

15. The Hirzebruch–Riemann–Roch formula

15.1. Positive line bundles. In order to state another application of the Weitzenböck formula we have to make the following

DEFINITION 15.1. *A real* $(1,1)$*–form* φ *on a complex manifold* (M,g,J) *is called* positive *(resp. negative) if the symmetric tensor* A *satisfying* $A(JX,Y) := \varphi(X,Y)$ *is positive (resp. negative) definite. A cohomology class in* $H^{1,1}M \cap H^2(M,\mathbb{R})$ *is called positive (resp. negative) if it can be represented by a positive (resp. negative)* $(1,1)$*–form. A holomorphic line bundle* L *over a compact complex manifold is called* positive *(resp. negative) if there exists a Hermitian structure on* L *with Chern connection* ∇ *and curvature form* R^∇ *such that* iR^∇ *is a positive (resp. negative)* $(1,1)$*–form.*

The positivity of a holomorphic line bundle is a topological property on Kähler manifolds:

LEMMA 15.2. *A holomorphic line bundle* L *over a compact Kähler manifold* M *is positive if and only if its first Chern class is positive.*

PROOF. One direction is clear from the definition. Suppose, conversely, that $c_1(L)$ is positive. That means that there exists a positive $(1,1)$–form ω and a Hermitian structure h on L whose Chern connection ∇ has curvature R^∇ such that $[iR^\nabla] = [\omega]$ (the factor 2π can obviously be skipped). From the global $i\partial\bar\partial$–Lemma, there exists a real function u such that $iR^\nabla = \omega + i\partial\bar\partial u$. we now use the formula (40) which gives the curvature of the Chern connection in terms of the square norm of an arbitrary local holomorphic section σ:

$$R^\nabla = -\partial\bar\partial \log h(\sigma,\sigma).$$

It is then clear that the curvature of the Chern connection $\tilde\nabla$ associated to $\tilde h := he^u$ satisfies for every local holomorphic section σ:

$$iR^{\tilde\nabla} = -i\partial\bar\partial \log \tilde h(\sigma,\sigma) = -i\partial\bar\partial \log h(\sigma,\sigma) - i\partial\bar\partial u = iR^\nabla - i\partial\bar\partial u = \omega,$$

thus showing that L is positive. □

In order to get a feeling for this notion, notice that the fundamental form of a Kähler manifold is positive, as well as the Ricci form of a Kähler manifold with positive Ricci tensor. From Lemma 11.4 we know that the canonical bundle K of a Kähler manifold has curvature $i\rho$. Thus K is negative if and only if the Ricci tensor is positive definite.

THEOREM 15.3. *A negative holomorphic line bundle* L *over a compact Kähler manifold has no non–vanishing holomorphic section.*

PROOF. Taking $p = 0$ and $E = L$ in (63) shows that every holomorphic section ξ of E satisfies

$$0 = 2\bar\partial^*\bar\partial\xi = \nabla^*\nabla\xi + \frac{i}{2}R^\nabla(Je_j,e_j)\xi. \tag{67}$$

By hypothesis we have $iR^\nabla(X,Y) = A(JX,Y)$, with A negative definite. Thus

$$\frac{i}{2}R^\nabla(Je_j,e_j) = -\frac{1}{2}A(e_j,e_j) = -\frac{1}{2}\mathrm{Tr}(A)$$

is a strictly positive function on M. Consequently, taking the Hermitian product with ξ in (67) and integrating over M shows that ξ has to vanish. $\square$

This result is consistent with our previous calculations on $\mathbb{C}\mathrm{P}^m$. We have seen that the canonical bundle K is negative, and that K is isomorphic to the $m+1^{\mathrm{st}}$ tensor power of the tautological bundle L, which is thus negative, too. On the other hand, we have shown with a direct computation that this last bundle has no holomorphic section.

15.2. The Hirzebruch–Riemann–Roch formula. Let $E \to M$ be a holomorphic vector bundle over some compact Hermitian manifold M^{2m}. We denote by $\Omega^k(E) := \mathcal{C}^\infty(\Lambda^{0,k}M \otimes E)$ the space of E–valued $(0,k)$–forms on M. Consider the following elliptic complex

$$\Omega^0(E) \xrightarrow{\bar\partial} \Omega^1(E) \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \Omega^m(E). \tag{68}$$

We define the cohomology groups

$$H^q(M,E) := \frac{\mathrm{Ker}(\bar\partial : \Omega^q(E) \to \Omega^{q+1}E)}{\bar\partial\Omega^{q-1}(E)}.$$

By analogy with the usual (untwisted) case, we denote

$$H^{p,q}(M,E) := H^q(M, \Lambda^{p,0}M \otimes E).$$

For every Hermitian structure on E one can consider the formal adjoint $\bar\partial^*$ of $\bar\partial$, and define the space of harmonic E–valued $(0,q)$–forms on M by

$$\mathcal{H}^q(E) := \{\omega \in \Omega^q(E) \mid \bar\partial\omega = 0,\ \bar\partial^*\omega = 0\}.$$

The analog of the Dolbeault decomposition theorem holds true in this case and as a corollary we have

Theorem 15.4. *The cohomology groups $H^q(M,E)$ are isomorphic with the spaces of harmonic E–valued $(0,q)$–forms:*

$$H^q(M,E) \simeq \mathcal{H}^q(E).$$

We can view the elliptic complex (68) as an elliptic first order differential operator simply by considering

$$\bar\partial + \bar\partial^* : \Omega^{\mathrm{even}}(E) \to \Omega^{\mathrm{odd}}(E).$$

The *index* of the elliptic complex (68) is defined to be the index of this elliptic operator:

$$\mathrm{Ind}(\bar\partial + \bar\partial^*) := \dim(\mathrm{Ker}(\bar\partial + \bar\partial^*)) - \dim(\mathrm{Coker}(\bar\partial + \bar\partial^*)).$$

The holomorphic Euler characteristic $\Xi(M,E)$ is defined by

$$\Xi(M,E) := \sum_{k=0}^{m}(-1)^k \dim H^k(M,E)$$

and is nothing else but the index of the elliptic complex (68).

THEOREM 15.5. (Hirzebruch–Riemann–Roch formula) *The holomorphic Euler characteristic of E can be computed as follows*

$$\Xi(M,E) = \int_M \mathrm{Td}(M)\mathrm{ch}(E),$$

where $\mathrm{Td}(M)$ *is the Todd class of the tangent bundle of* M *and* $\mathrm{ch}(E)$ *is the Chern character of* E.

The Todd class and the Chern character are characteristic classes of the corresponding vector bundles that we will not define explicitly. The only thing that we will use in the sequel is that they satisfy the naturality axiom with respect to pull–backs. If E is the trivial line bundle, the holomorphic Euler characteristic $\Xi(M,E)$ is simply denoted by $\Xi(M) := \sum_{k=0}^{m}(-1)^k h^{0,k}(M)$.

For a proof of the Hirzebruch–Riemann–Roch formula see [**3**].

We will give two applications of the Riemann–Roch formula, both concerning the fundamental group of Kähler manifolds under suitable positivity assumptions of the Ricci tensor. The first one is a theorem due to Kobayashi:

THEOREM 15.6. *A compact Kähler manifold with positive definite Ricci tensor is simply connected.*

PROOF. Theorem 14.5 shows that there is no holomorphic $(p,0)$–form on M, so $h^{p,0}(M) = 0$ for $p > 0$. Of course, the holomorphic functions are just the constants, so $h^{0,0}(M) = 1$. Since M is Kähler we have $h^{p,0}(M) = h^{0,p}(M)$, thus $\Xi(M) = 1$.

By Myers' Theorem, the fundamental group of M is finite. Let $\tilde{M}$ be the universal cover of M, which is therefore compact, too. Applying the previous argument to $\tilde{M}$ we get $\Xi(\tilde{M}) = 1$. Now, if $\pi : \tilde{M} \to M$ denotes the covering projection, we have, by naturality, $\mathrm{Td}(\tilde{M}) = \pi^*\mathrm{Td}(M)$, and an easy exercise shows that for every top degree form ω on M one has

$$\int_{\tilde{M}} \pi^*\omega = k\int_M \omega,$$

where k denotes the number of sheets of the covering. This shows that $k = 1$, so M is simply connected. □

Our second application concerns Ricci–flat Kähler manifolds. By Theorem 11.5, a compact Kähler manifold M^{2m} is Ricci–flat if and only if the restricted holonomy group $\mathrm{Hol}_0(M)$ is a subgroup of SU_m. A compact Kähler manifold M with $\mathrm{Hol}_0(M) = \mathrm{SU}_m$ is called *Calabi–Yau* manifold.

THEOREM 15.7. *Let* M^{2m} *be a Calabi–Yau manifold. If* m *is odd, then* $\mathrm{Hol}(M) = \mathrm{SU}_m$, *so there exists a global holomorphic* $(m,0)$*–form even if* M *is not simply connected. If* m *is even, then either* M *is simply connected, or* $\pi_1(M) = \mathbb{Z}_2$ *and* M *carries no global holomorphic* $(m,0)$*–form.*

PROOF. Let $\tilde{M}$ be the universal covering of M. The Cheeger–Gromoll theorem (cf. [**1**], p. 168) shows that $\tilde{M}$ is compact (having irreducible holonomy). By Theorem 14.5, every holomorphic form on M is parallel, and thus corresponds to a fixed point of the holonomy representation. It is easy to check that SU_m has only two invariant one–dimensional complex subspaces on $(p,0)$–forms, one for $p=0$ and one for $p=m$. Thus

$$\Xi(\tilde{M}) = \begin{cases} 0 & \text{for } m \text{ odd} \\ 2 & \text{for } m \text{ even} \end{cases}$$

Moreover, $\Xi(\tilde{M}) = k\Xi(M)$, where k is the order of the fundamental group of M. This shows that $\Xi(M) = 0$ for m odd, hence $h^{m,0}M = 1$, so M has a global holomorphic $(m,0)$–form.

If m is even, then either M is simply connected, or $k=2$ and $\Xi(M) = 1$. In this last case, we necessarily have $h^{m,0}M = 0$, so M carries no global holomorphic $(m,0)$–form.

□

15.3. Exercises.

(1) Prove the *Kodaira–Serre duality*:

$$H^q(M,E) \simeq H^{m-q}(M, E^* \otimes K_M)$$

for every holomorphic vector bundle E over a compact Hermitian manifold M.

(2) Prove that the operator

$$\bar\partial + \bar\partial^* : \Omega^{\text{even}}(E) \to \Omega^{\text{odd}}(E)$$

is elliptic, in the sense that its principal symbol applied to any non–zero real 1–form is an isomorphism.

(3) Prove that the index of the above defined operator is equal to the holomorphic Euler characteristic $\Xi(M,E)$.

(4) Let $\pi : \tilde{M} \to M$ be a k–sheet covering projection between compact oriented manifolds. Prove that for every top degree form ω on M one has

$$\int_{\tilde{M}} \pi^*\omega = k \int_M \omega.$$

Hint: Start by showing that to any open cover $\{U_i\}$ of M one can associate a closed cover $\{C_j\}$ such that for every j there exists some i with $C_j \subset U_i$ and such that the interiors of C_j and C_k are disjoint for every $j \neq k$.

(5) Show that the representation of SU_m on $\Lambda^p\mathbb{C}^m$ has no invariant one–dimensional subspace for $1 \le p \le m-1$.

16. Further vanishing results

16.1. The Schrödinger–Lichnerowicz formula for Kähler manifolds. Let L be a holomorphic Hermitian line bundle over some Kähler manifold M^{2m} with scalar curvature S. We would like to compute the curvature term in the Weitzenböck formula on sections of $\Lambda^{0,k}M \otimes L$, and to show that this term becomes very simple in the case where L is a square root of the canonical bundle. The reader familiar with spin geometry will notice that in this case $\Lambda^{0,*}M \otimes K^{\frac{1}{2}}$ is just the *spin bundle* of M and the operator $\sqrt{2}(\partial + \bar\partial)$ is just the Dirac operator.

Let us denote by $i\alpha$ the curvature of the Chern connection of L. The first term of the curvature operator $\mathcal{R}$ applied to some section $\omega \otimes \xi \in \Omega^{0,k}M \otimes E$ can be computed as follows

$$\begin{aligned}\mathcal{R}_1(\omega\otimes\xi) &:= \frac{i}{2}\tilde R(Je_j,e_j)(\omega\otimes\xi) = \frac{i}{2}\Big(2(\rho^{(k)}\omega)\otimes\xi + i\alpha(Je_j,e_j)\omega\otimes\xi\Big)\\ &= i(\rho^{(k)}\omega)\otimes\xi - \frac{1}{2}\alpha(Je_j,e_j)\omega\otimes\xi.\end{aligned}$$

In order to compute the second curvature term we make use of the following algebraic result

LEMMA 16.1. *The Riemannian curvature operator satisfies*

$$(e_j - iJe_j)\wedge(e_k + iJe_k)\lrcorner R(e_j,e_k)\omega = 4i\rho^{(k)}\omega$$

for every $(0,k)$*–form* ω.

PROOF. Since the interior product of a $(1,0)$–vector and a $(0,k)$–form vanishes we obtain

$$X\lrcorner\omega = iJX\lrcorner\omega \qquad \forall\ \omega\in\Omega^{0,k}M. \tag{69}$$

The forms $R(e_j,e_k)\omega$ are still $(0,k)$–forms, since the connection preserves the type decomposition of forms. By changing e_j to Je_j and then e_k to Je_k we get

$$\begin{aligned}e_j\wedge(e_k+iJe_k)\lrcorner R(e_j,e_k)\omega &= Je_j\wedge(e_k+iJe_k)\lrcorner R(Je_j,e_k)\omega\\ &= -Je_j\wedge(e_k+iJe_k)\lrcorner R(e_j,Je_k)\omega\\ &= -iJe_j\wedge(e_k+iJe_k)\lrcorner R(e_j,e_k)\omega.\end{aligned}$$

Thus

$$(e_j-iJe_j)\wedge(e_k+iJe_k)\lrcorner R(e_j,e_k)\omega = 2e_j\wedge(e_k+iJe_k)\lrcorner R(e_j,e_k)\omega = 4e_j\wedge e_k\lrcorner R(e_j,e_k)\omega.$$

Now, using (69) twice we get

$$\begin{aligned}R(e_j,e_k,e_l,e_s)e_j\wedge e_k\wedge e_s\lrcorner e_l\lrcorner\omega &= -R(e_j,e_k,Je_l,Je_s)e_j\wedge e_k\wedge e_s\lrcorner e_l\lrcorner\omega\\ &= -R(e_j,e_k,e_l,e_s)e_j\wedge e_k\wedge e_s\lrcorner e_l\lrcorner\omega,\end{aligned}$$

so this expression vanishes. From the first Bianchi identity we then obtain

$$\begin{aligned}R(e_j,e_k,e_l,e_s)e_j\wedge e_s\wedge e_k\lrcorner e_l\lrcorner\omega &= R(e_j,e_l,e_k,e_s)e_j\wedge e_s\wedge e_k\lrcorner e_l\lrcorner\omega\\ &\quad +R(e_j,e_s,e_l,e_k)e_j\wedge e_s\wedge e_k\lrcorner e_l\lrcorner\omega\\ &= -R(e_j,e_l,e_k,e_s)e_j\wedge e_s\wedge e_l\lrcorner e_k\lrcorner\omega\end{aligned}$$

whence

$$R(e_j, e_l, e_k, e_s)e_j \wedge e_s \wedge e_k \lrcorner e_l \lrcorner \omega = 0.$$

Finally we get

$$\begin{aligned}(e_j - iJe_j) \wedge (e_k + iJe_k) \lrcorner R(e_j, e_k)\omega &= 4e_j \wedge e_k \lrcorner R(e_j, e_k)\omega \\ &= 4R(e_j, e_k, e_l, e_s)e_j \wedge e_k \lrcorner (e_s \wedge e_l \lrcorner \omega) \\ &= -4\mathrm{Ric}(e_j, e_l)e_j \wedge e_l \lrcorner \omega = 4i\mathrm{Ric}(e_j, Je_l)e_j \wedge e_l \lrcorner \omega \\ &= 4i\rho^{(k)}\omega.\end{aligned}$$

□

For every $(1,1)$–form α and $(0,k)$–form ω we have as before

$$\begin{aligned}(e_j - iJe_j) \wedge (e_k + iJe_k) \lrcorner \alpha(e_j, e_k)\omega &= 2e_j \wedge (e_k + iJe_k) \lrcorner \alpha(e_j, e_k)\omega = 4e_j \wedge e_k \lrcorner \alpha(e_j, e_k)\omega \\ &= -4\alpha^{(k)}(\omega).\end{aligned}$$

The second term in (62) thus reads

$$\begin{aligned}\mathcal{R}_2(\omega \otimes \xi) &:= -\frac{1}{2}(e_j - iJe_j) \wedge (e_k + iJe_k) \lrcorner (\tilde{R}(e_j, e_k)(\omega \otimes \xi)) \\ &= -\frac{1}{2}(e_j - iJe_j) \wedge (e_k + iJe_k) \lrcorner ((R(e_j, e_k)\omega) \otimes \xi + i\alpha(e_j, e_k)\omega \otimes \xi) \\ &= -2i\rho^{(k)}(\omega) \otimes \xi + 2i\alpha^{(k)}(\omega) \otimes \xi.\end{aligned}$$

Suppose that the curvature of the line bundle L satisfies

$$R^L := i\alpha = \frac{1}{2}i\rho.$$

The formulas above show that the curvature term in the Weitzenböck formula on $\Omega^{0,k}M \otimes L$ satisfies

$$\begin{aligned}\mathcal{R}(\omega \otimes \xi) &= (\mathcal{R}_1 + \mathcal{R}_2)(\omega \otimes \xi) = i\rho^{(k)}(\omega) \otimes \xi - \frac{1}{2}\alpha(Je_j, e_j)\omega \otimes \xi \\ &\quad -2i\rho^{(k)}(\omega) \otimes \xi + 2i\alpha^{(k)}(\omega) \otimes \xi \\ &= -\frac{1}{4}\rho(Je_j, e_j)\omega \otimes \xi = \frac{S}{4}\omega \otimes \xi.\end{aligned}$$

This proves the

THEOREM 16.2. (Schrödinger–Lichnerowicz formula). *Let $L = K^{\frac{1}{2}}$ be a square root of the canonical bundle of a Kähler manifold M, in the sense that L has a Hermitian structure h such that K_M is isomorphic to $L \otimes L$ with the induced tensor product Hermitian structure. Then, if Ψ is a section of the complex vector bundle*

$$\Sigma M := (\Lambda^{0,0}M \oplus \ldots \oplus \Lambda^{0,m}M) \otimes K^{\frac{1}{2}}$$

and $D := \sqrt{2}(\bar\partial + \bar\partial^*)$ *is the Dirac operator on* ΣM*, the following formula holds*

$$D^2\Psi = \nabla^*\nabla\Psi + \frac{S}{4}\Psi.$$

The Schrödinger–Lichnerowicz formula is valid in a more general setting (on all spin manifolds, not necessarily Kähler), and it has important applications in geometry and topology (see [**4**], [**9**]).

16.2. The Kodaira vanishing theorem. Let M^{2m} be a compact Kähler manifold and let L be a positive line bundle over M. From the definition, we know that L carries a Hermitian structure whose Chern connection ∇ has curvature R^∇ with $iR^\nabla > 0$. We consider the Kähler metric on M whose fundamental form is just iR^∇. By a slight abuse of language, we denote by $\partial : \mathcal{C}^\infty(\Lambda^{p,q}M \otimes L) \to \mathcal{C}^\infty(\Lambda^{p+1,q}M \otimes L)$ the extension of $\nabla^{1,0}$ to forms. Note that, whilst $\bar\partial$ is an intrinsic operator, ∂ depends of course on the Hermitian structure on L. We apply the Weitzenböck formula to some section $\omega \otimes \xi$ of $\Lambda^{p,q}M \otimes L$:

$$2(\bar\partial^*\bar\partial + \bar\partial\bar\partial^*)(\omega \otimes \xi) = \nabla^*\nabla(\omega \otimes \xi) + \mathcal{R}(\omega \otimes \xi). \tag{70}$$

The same computation actually yields the dual formula

$$2(\partial^*\partial + \partial\partial^*)(\omega \otimes \xi) = \nabla^*\nabla(\omega \otimes \xi) + \tilde{\mathcal{R}}(\omega \otimes \xi). \tag{71}$$

Alternatively, one can apply (70) to a section $\tilde\omega \otimes \xi^*$ of $\Lambda^{q,p}M \otimes L^*$ and then take the complex conjugate. Subtracting these two equations yields

$$2(\bar\partial^*\bar\partial + \bar\partial\bar\partial^*)(\omega \otimes \xi) = 2(\partial^*\partial + \partial\partial^*)(\omega \otimes \xi) + (\mathcal{R} - \tilde{\mathcal{R}})(\omega \otimes \xi). \tag{72}$$

We now compute this curvature term.

$$\begin{aligned}
(\mathcal{R} - \tilde{\mathcal{R}})(\omega \otimes \xi) &= i\tilde R(Je_j, e_j)(\omega \otimes \xi) - \frac{1}{2}(e_j - iJe_j) \wedge (e_k + iJe_k) \lrcorner \tilde R(e_j, e_k)(\omega \otimes \xi) \\
&\quad + \frac{1}{2}(e_j + iJe_j) \wedge (e_k - iJe_k) \lrcorner \tilde R(e_j, e_k)(\omega \otimes \xi) \\
&= i\tilde R(Je_j, e_j)(\omega \otimes \xi) + iJe_j \wedge e_k \lrcorner \tilde R(e_j, e_k)(\omega \otimes \xi) \\
&\quad - ie_j \wedge Je_k \lrcorner \tilde R(e_j, e_k)(\omega \otimes \xi) \\
&= i\tilde R(Je_j, e_j)(\omega \otimes \xi) + 2iJe_j \wedge e_k \lrcorner \tilde R(e_j, e_k)(\omega \otimes \xi) \\
&= 2i\rho(\omega) \otimes \xi + i\omega \otimes R^\nabla(Je_j, e_j)\xi + 2iJe_j \wedge e_k \lrcorner R(e_j, e_k)\omega \otimes \xi \\
&\quad + 2iJe_j \wedge e_k \lrcorner \omega \otimes R^\nabla(e_j, e_k)\xi \\
&= 2i\rho(\omega) \otimes \xi - 2m\omega \otimes \xi + 2iJe_j \wedge e_k \lrcorner R(e_j, e_k)\omega \otimes \xi + 2(p+q)\omega \otimes \xi.
\end{aligned}$$

On the other hand, the expression $Je_j \wedge e_k \lrcorner R(e_j, e_k)\omega$ can be simplified as follows:

$$\begin{aligned}
Je_j \wedge e_k \lrcorner R(e_j, e_k)\omega &= Je_j \wedge e_k \lrcorner R(e_j, e_k)e_l \wedge e_l \lrcorner \omega \\
&= -\mathrm{Ric}(e_j, e_l)Je_j \wedge e_l \lrcorner \omega - Je_j \wedge R(e_j, e_k)e_l \wedge e_k \lrcorner e_l \lrcorner \omega \\
&= -\rho(\omega) - R(e_j, e_k, e_l, e_s)Je_j \wedge e_s \wedge e_k \lrcorner e_l \lrcorner \omega,
\end{aligned}$$

and from the Bianchi identity

$$\begin{aligned}
2R(e_j,e_k,e_l,e_s)Je_j\wedge e_s\wedge e_k\lrcorner e_l\lrcorner\omega &= R(e_j,e_k,e_l,e_s)Je_j\wedge e_s\wedge e_k\lrcorner e_l\lrcorner\omega\\
&\quad +R(e_j,e_l,e_k,e_s)Je_j\wedge e_s\wedge e_l\lrcorner e_k\lrcorner\omega\\
&= R(e_l,e_k,e_j,e_s)Je_j\wedge e_s\wedge e_k\lrcorner e_l\lrcorner\omega\\
&= -R(e_l,e_k,Je_j,e_s)e_j\wedge e_s\wedge e_k\lrcorner e_l\lrcorner\omega=0,
\end{aligned}$$

where the last expression vanishes because $R(\cdot,\cdot,J\cdot,\cdot)$ is symmetric in the last two arguments. This shows that $Je_j\wedge e_k\lrcorner R(e_j,e_k)\omega=-\rho(\omega)$, so from the previous calculation we get

$$2(\bar\partial^*\bar\partial+\bar\partial\bar\partial^*)(\omega\otimes\xi)=2(\partial^*\partial+\partial\partial^*)(\omega\otimes\xi)+2(p+q-m)(\omega\otimes\xi).$$

After taking the Hermitian product with $\omega\otimes\xi$ (which we denote by σ for simplicity) and integrating over M we get

$$\int_M|\bar\partial\sigma|^2+|\bar\partial^*\sigma|^2dv=\int_M|\partial\sigma|^2+|\partial^*\sigma|^2+(p+q-m)|\sigma|^2dv. \tag{73}$$

If σ is a harmonic L–valued form, the left hand side term in (73) vanishes, thus proving the

Theorem 16.3. (Kodaira vanishing theorem). *If L is a positive holomorphic line bundle on a compact Kähler manifold M, one has $H^{p,q}(M,L)=0$ whenever $p+q>m$.*

Part 6

Calabi–Yau manifolds

17. Ricci–flat Kähler metrics

17.1. Hyperkähler manifolds. In order to obtain the classification (up to finite coverings) of compact Ricci–flat Kähler manifolds, we make the following

DEFINITION 17.1. *A Riemannian manifold (M^n, g) is called* hyperkähler *if there exist three complex structures I, J, K on M satisfying $K = IJ$ such that g is a Kähler metric with respect to each of these complex structures.*

It is clear that a metric is hyperkähler if and only if it is Kähler with respect to two anti–commuting complex structures. In the irreducible case, this can be weakened as follows:

PROPOSITION 17.2. *Let (M^n, g) be a locally irreducible Riemannian manifold. If g is Kähler with respect to two complex structures J and J_1, and if J_1 is different from J and $-J$, then (M, g) is hyperkähler.*

PROOF. The endomorphism $JJ_1 + J_1J$ is symmetric and parallel on M, so by local irreducibility it has to be constant:

$$JJ_1 + J_1J = \alpha \mathrm{Id}, \qquad \alpha \in \mathbb{R}. \tag{74}$$

From the Cauchy–Schwartz inequality we get

$$\alpha^2 = |\alpha \mathrm{Id}|^2 = |JJ_1 + J_1J|^2 \leq 2(|JJ_1|^2 + |J_1J|^2) \leq 4|J|^2|J_1|^2 = 4,$$

where the norm considered here is the operator norm. The equality case can only hold if $JJ_1 = \beta J_1 J$ for some real number β. Together with (74) this shows that $JJ_1 = \gamma \mathrm{Id}$ for some real number γ, so $J_1 = \pm J$, which was excluded in the hypothesis. Therefore we have $\alpha^2 < 4$. We then compute using (74)

$$(J_1 + JJ_1J)^2 = (\alpha^2 - 4)\mathrm{Id},$$

so the parallel skew–symmetric endomorphism

$$I := \frac{1}{\sqrt{4 - \alpha^2}}(J_1 + JJ_1J)$$

defines a complex structure anti–commuting with J, with respect to which g is Kähler. □

Consider the identification of $\mathbb{C}^{2k}$ with $\mathbb{H}^k$ given by $(z_1, z_2) \mapsto z_1 + jz_2$. We denote by I, J and K the right product on $\mathbb{H}^k$ with i, j and k respectively, which correspond to the following endomorphisms of $\mathbb{C}^{2k}$:

$$I(z_1, z_2) = (iz_1, iz_2) \qquad J(z_1, z_2) = (-\bar{z}_2, \bar{z}_1) \qquad K(z_1, z_2) = (-i\bar{z}_2, i\bar{z}_1).$$

Let us denote by Sp_k the group of unitary transformations of $\mathbb{C}^{2k}$ (that is, preserving the canonical Hermitian product and commuting with I), which also commute with J (and thus also with K). Clearly we have

$$\mathrm{Sp}_k = \left\{ M = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \in \mathcal{M}_{2k}(\mathbb{C}) \,\middle|\, M\bar{M}^t = I_{2k} \right\}.$$

It is tautological that a $4k$–dimensional manifold is hyperkähler if and only if the bundle of orthonormal frames has a reduction to Sp_k.

Lemma 17.3. $\mathrm{Sp}_k \subset \mathrm{SU}_{2k}$.

Proof. By definition we have $\mathrm{Sp}_k \subset \mathrm{U}_{2k}$, so every matrix in Sp_k is diagonalizable as complex matrix and its eigenvalues are complex numbers of unit norm. If v is an eigenvector of some $M \in \mathrm{Sp}_k$ with eigenvalue $\lambda \in S^1$ then

$$MJv = JMv = J\lambda v = \bar{\lambda}Jv = \lambda^{-1}Jv,$$

showing that the determinant of M equals 1. □

This shows that every hyperkähler manifold is Ricci–flat. A hyperkähler manifold is called *strict* if it is locally irreducible.

Let now M be an arbitrary compact Ricci–flat Kähler manifold. The Cheeger–Gromoll theorem ([**1**], p.168) says that M is isomorphic to a quotient

$$M \simeq (M_0 \times \mathbb{T}^l)/\Gamma,$$

where M_0 is a compact simply connected Kähler manifold, $\mathbb{T}^l$ is a complex torus and Γ is a finite group of holomorphic transformations. Let $M_0 = M_1 \times \ldots \times M_s$ be the De Rham decomposition of M_0. Then M_j are compact Ricci–flat simply connected Kähler manifolds with irreducible holonomy for all j. A symmetric space which is Ricci–flat is automatically flat, so the M_j's are not symmetric. The Berger holonomy theorem then shows that M_j is either Calabi–Yau or strict hyperkähler for every j. We thus have the following

Theorem 17.4. *A compact Ricci–flat Kähler manifold M is isomorphic to the quotient*

$$M \simeq (M_1 \times \ldots \times M_s \times M_{s+1} \ldots \times M_r \times \mathbb{T}^l)/\Gamma,$$

where M_j are simply connected compact Calabi–Yau manifolds for $j \le s$, simply connected compact strict hyperkähler manifolds for $s+1 \le j \le r$ and Γ is a finite group of holomorphic transformations.

17.2. Projective manifolds. A compact complex manifold (M^{2m}, J) is called *projective* if it can be holomorphically embedded in some complex projective space $\mathbb{C}\mathrm{P}^N$. A well–known result of Chow states that a projective manifold is *algebraic*, that is, defined by a finite set of homogeneous polynomials in the complex projective space.

Proposition 17.5. *Every projective manifold has a positive holomorphic line bundle.*

Proof. Let φ be the fundamental form of the Fubini–Study metric on $\mathbb{C}\mathrm{P}^N$. It is easy to check (*e.g.* using (22)) that the hyperplane bundle H on $\mathbb{C}\mathrm{P}^N$ has a connection with curvature $-i\varphi$. The restriction of this line bundle to any complex submanifold of $\mathbb{C}\mathrm{P}^N$ is thus positive. □

Conversely, we have the celebrated

Theorem 17.6. (Kodaira embedding theorem). *A compact complex manifold M with a positive holomorphic line bundle L is projective.*

A proof can be found in [**2**], p. 176. The main idea is to show that a suitable positive power L^k of L has a basis of holomorphic sections $\{\sigma_0, \dots, \sigma_N\}$ such that the holomorphic mapping

$$M \to \mathbb{C}\mathrm{P}^N \qquad x \mapsto [\sigma_0(x) : \ldots : \sigma_N(x)]$$

is an embedding.

Corollary 17.7. *Every Calabi–Yau manifold of complex dimension $m \geq 3$ is projective.*

Proof. For every compact manifold M, let $\mathcal{A}$ and $\mathcal{A}^*$ be the sheaves of smooth functions on M with values in $\mathbb{C}$ and $\mathbb{C}^*$ respectively. The exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{A} \stackrel{\exp}{\to} \mathcal{A}^* \to 0$$

induces an exact sequence in Čech cohomology

$$\to H^1(M, \mathcal{A}) \to H^1(M, \mathcal{A}^*) \stackrel{c_1}{\to} H^2(M, \mathbb{Z}) \to H^2(M, \mathcal{A}) \to .$$

The sheaf $\mathcal{A}$ is *fine* (that is, it admits a partition of unity), so $H^1(M, \mathcal{A}) = 0$ and $H^2(M, \mathcal{A}) = 0$, thus proving that

$$H^1(M, \mathcal{A}^*) \simeq H^2(M, \mathbb{Z}).$$

Notice that $H^1(M, \mathcal{A}^*)$ is just the set of equivalence classes of complex line bundles over M, and the isomorphism above is given by the first Chern class.

The above argument shows that for every integer cohomology class $\gamma \in H^2(M, \mathbb{Z})$, there exists a complex line bundle L with $c_1(L) = \gamma$. Moreover, if ω is any complex 2–form representing γ in real cohomology, there exists a connection ∇ on L such that $\frac{i}{2\pi} R^{\nabla} = \omega$. To see this, take any connection $\tilde{\nabla}$ on L with curvature $R^{\tilde{\nabla}}$. Then since $[\omega] = c_1(L)$ we get $[2\pi\omega] = [iR^{\tilde{\nabla}}]$, so there exists some 1–form θ such that $2\pi\omega = i(R^{\tilde{\nabla}} + d\theta)$. Clearly the curvature of $\nabla := \tilde{\nabla} + i\theta$ satisfies the desired equation. If the form ω is real and of type $(1, 1)$, then the complex bundle L has a holomorphic structure, given by the $(0, 1)$–part of the connection whose curvature is ω.

Let now M^{2m} be a Calabi–Yau manifold, $m > 2$. Since SU_m has no fixed point on $\Lambda^{2,0}\mathbb{C}^m$, we deduce that there are no parallel $(2, 0)$–forms on M, so by Theorem 14.5 we get $h^{2,0}(M) = 0$. By the Dolbeault decomposition theorem we obtain that any harmonic 2–form on M is of type $(1, 1)$. Consider the fundamental form φ of M. Since $H^2(M, \mathbb{Q})$ is dense in $H^2(M, \mathbb{R})$, and the space of positive harmonic $(1, 1)$–forms is open in $\mathcal{H}^{1,1}(M, \mathbb{R}) = \mathcal{H}^2(M, \mathbb{R})$, we can find a positive harmonic $(1, 1)$–form ω such that $[\omega] \in H^2(M, \mathbb{Q})$. By multiplying with the common denominator, we may suppose that $[\omega] \in H^2(M, \mathbb{Z})$. Then the argument above shows that there exists a holomorphic line bundle whose first Chern class is ω, thus a positive holomorphic line bundle on M. By the Kodaira embedding theorem, M is then projective. $\square$

18. Constructions of Calabi–Yau manifolds

18.1. Divisors. Let M be a complex manifold. An *analytic hypersurface* of M is a subset $V \in M$ such that for every $x \in V$ there exists an open set $U_x \subset M$ containing x and a holomorphic function f_x defined on U_x such that $V \cap U_x$ is the zero–set of f_x. Such an f_x is called a *local defining function* for V near x. The quotient of any two local defining functions around x is a non–vanishing holomorphic function around x.

An analytic hypersurface V is called *irreducible* if it can not be written as the union of two smaller analytic hypersurfaces. Every analytic hypersurface is a finite union of its irreducible components.

If V is an irreducible analytic hypersurface, with defining function φ_x around some $x \in V$, then for every holomorphic function f around x, the order of f along V at x is defined to be the largest positive integer a such that $\frac{f}{\varphi_x^a}$ is holomorphic around x. It can be shown that the order of f is a well–defined positive integer, which does not depend on x, and is denoted by $o(f, V)$.

Definition 18.1. *A* divisor D *in a compact complex manifold M is a finite formal sum with integer coefficients of irreducible analytic hypersurfaces of M.*

$$D := \sum_i a_i V_i, \qquad a_i \in \mathbb{Z}.$$

A divisor D is called effective *if all $a_i \geq 0$ for all i.*

The set of divisors is clearly a commutative group under formal sums.

A *meromorphic function* on a complex manifold M is an equivalence class of collections $(U_\alpha, f_\alpha, g_\alpha)$ where $\{U_\alpha\}$ is an open covering of M, and f_α, g_α are holomorphic functions defined on U_α such that $f_\alpha g_\beta = f_\beta g_\alpha$ on $U_\alpha \cap U_\beta$ for all α, β. Two such collections $(U_\alpha, f_\alpha, g_\alpha)$ and $(U'_\beta, f'_\beta, g'_\beta)$ are equivalent if $f_\alpha g'_\beta = f'_\beta g_\alpha$ on $U_\alpha \cap U'_\beta$ for all α, β. A meromorphic function can be always expressed locally as $\frac{f}{g}$, where f and g are locally defined holomorphic functions.

We define similarly a *meromorphic section of a holomorphic line bundle L* as an equivalence class of collections $(U_\alpha, \sigma_\alpha, g_\alpha)$ where σ_α is a local holomorphic section of L over U_α and g_α is a holomorphic function on U_α, such that $\sigma_\alpha g_\beta = \sigma_\beta g_\alpha$ on $U_\alpha \cap U_\beta$ for all α, β.

A meromorphic function h defines a divisor (h) in a canonical way by

$$(h) := (h)_0 - (h)_\infty,$$

where $(h)_0$ and $(h)_\infty$ denote the zero–locus (resp. the pole–locus) of h taken with multiplicities.

More precisely, for every x in M, one can write the function h as $h = \frac{f_x}{g_x}$ near x. If V is an irreducible analytic hypersurface containing x, we define the order of h along V at x to be $o(f_x, V) - o(g_x, V)$, and this is a well–defined integer independent on x, denoted by $o(h, V)$. Then

$$(h) = \sum_V o(h, V) V,$$

where the above sum is finite since for every open set U_x where $h = \frac{f_x}{g_x}$, there are only finitely many irreducible analytic hypersurfaces along which f_x of g_x have non–vanishing order.

Similarly, if σ is a global meromorphic section of a line bundle L, one can define the order $o(\sigma, V)$ of σ along any irreducible analytic hypersurface V using local trivializations of L. This clearly does not depend on the chosen trivialization, since the transition maps do not vanish, so they do not contribute to the order. As before, one defines a divisor (σ) on M by

$$(\sigma) = \sum_V o(\sigma, V)V.$$

If $D = \sum a_i V_i$ and f_i are local defining functions for V_i near some $x \in M$ (of course we can take $f_i = 1$ if V_i does not contain x), then the meromorphic function

$$\prod f_i^{a_i}$$

is called a local defining function for D around x.

Definition 18.2. *Two divisors D and D' are called* linearly equivalent *if there exists some meromorphic function h such that*

$$D = D' + (h).$$

In this case we write $D \equiv D'$.

Clearly two meromorphic sections σ and σ' of L define linearly equivalent divisors $(\sigma) = (\sigma') + (h)$, where h is the meromorphic function defined by $\sigma = \sigma' h$.

18.2. Line bundles and divisors. To any divisor D we will associate a holomorphic line bundle $[D]$ on M in the following way. Take an open covering U_α of M and local defining meromorphic functions h_α for D defined on U_α. We define $[D]$ to be the holomorphic line bundle on M with transition functions $g_{\alpha\beta} := \frac{h_\alpha}{h_\beta}$. It is easy to check that $g_{\alpha\beta}$ are non–vanishing holomorphic functions on $U_\alpha \cap U_\beta$ satisfying the cocycle conditions, and that the equivalence class of $[D]$ does not depend on the local defining functions h_α.

Example. Let H denote the hyperplane $\{z_0 = 0\}$ in $\mathbb{C}\mathrm{P}^m$ and consider the usual open covering $U_\alpha = \{z_\alpha \neq 0\}$ of $\mathbb{C}\mathrm{P}^m$. Then 1 is a local defining function for H on U_0 and $\frac{z_0}{z_\alpha}$ are local defining functions on U_α. The line bundle $[H]$ has thus transition functions $g_{\alpha\beta} = \frac{z_\beta}{z_\alpha}$, which are exactly the transition function of the *hyperplane line bundle* introduced in Section 3, which justifies its denomination.

If D and D' are divisors, then clearly $[-D] = [D]^{-1}$ and $[D + D'] = [D] \otimes [D']$. We call $\mathrm{Div}(M)$ the group of divisors on M, and $\mathrm{Pic}(M) := H^1(M, \mathcal{O})$ the *Picard group* of equivalence classes of holomorphic line bundles (where $\mathcal{O}$ denotes the sheaf of holomorphic functions). Then the arguments above show that there exists a group homomorphism

$$[\ ] : \mathrm{Div}(M) \to \mathrm{Pic}(M) \qquad D \mapsto [D].$$

Notice that the line bundle associated to a divisor (h) is trivial for every meromorphic function h. This follows directly from the definition: for any open cover U_α on M, $h|_{U_\alpha}$ is a local defining function for the divisor (h) on U_α, so the transition functions for the line bundle $[(h)]$ are equal to 1 on any intersection $U_\alpha \cap U_\beta$. Thus $[\,]$ descends to a group homomorphism

$$[\,] : \mathrm{Div}(M)/_{\equiv} \longrightarrow \mathrm{Pic}(M).$$

Suppose now that $[D] = 0$ for some divisor D on M. That means that the line bundle $[D]$ is trivial, so there exists an open cover $\{U_\alpha\}$ of M and holomorphic non–vanishing functions $f_\alpha : U_\alpha \to \mathbb{C}^*$ such that

$$\frac{f_\alpha}{f_b} = g_{\alpha\beta} = \frac{h_\alpha}{h_b} \qquad \text{on } U_\alpha \cap U_\beta,$$

where h_α is a local defining meromorphic function for D on U_α. This shows the existence of a global meromorphic function H on M such that $H|_{U_\alpha} = \frac{h_\alpha}{f_\alpha}$. Moreover, as f_α does not vanish on U_α, the divisor associated to H is just D. This proves the injectivity of $[\,]$ on isomorphism classes of divisors.

Every holomorphic line bundle of a projective manifold has a global meromorphic section (see [**2**] p.161). If $L \in \mathrm{Pic}(M)$ is a holomorphic line bundle, we have seen that a global meromorphic section σ of L defines a divisor (σ) on M. We claim that $[(\sigma)] = L$. If $g_{\alpha\beta}$ denote the transition functions of L with respect to some trivialization (U_α, ψ_α), the meromorphic section σ defines meromorphic functions σ_α on U_α such that $g_{\alpha\beta} = \frac{\sigma_\alpha}{\sigma_\beta}$. From the definition, σ_α is a defining meromorphic section for (σ) on U_α, thus L is just the line bundle associated to (σ). We have proved the

Theorem 18.3. *If the manifold M is projective, the homomorphism $[\,]$ descends to an isomorphism*

$$\mathrm{Div}(M)/_{\equiv} \xrightarrow{\cong} \mathrm{Pic}(M).$$

18.3. Adjunction formulas. Let $V \subset M$ be a smooth complex hypersurface of a compact complex manifold M. We will show that the normal and co–normal bundles of V in M can be computed in terms of the divisor V.

Proposition 18.4. (First adjunction formula) *The restriction to V of the line bundle $[V]$ associated to the divisor V is isomorphic to the holomorphic normal bundle of V in M:*

$$N_V = [V]|_V.$$

Proof. Let $i : V \to M$ be the inclusion of V into M. By definition, the normal bundle N_V is the co–kernel of the inclusion $i_* : T^{1,0}V \to T^{1,0}M|_V$ and its dual, the co–normal bundle N_V^*, is defined as the kernel of the projection $i^* : \Lambda^{1,0}M|_V \to \Lambda^{1,0}V$. Thus N_V^* is spanned by holomorphic $(1,0)$–forms on M vanishing on V.

Let f_α be local defining functions for V on some open covering U_α. By definition, the quotients $g_{\alpha\beta} := \frac{f_\alpha}{f_\beta}$ are the transition functions of $[V]$ on $U_\alpha \cap U_\beta$. Moreover, since f_α vanishes along V which is smooth, we see that $df_\alpha|_V$ is a non–vanishing local section of N_V^*. Now, since $f_\alpha = g_{\alpha\beta} f_\beta$, we get

$$df_\alpha|_V = (f_\beta dg_{\alpha\beta} + g_{\alpha\beta} df_\beta)|_V = g_{\alpha\beta}|_V df_\beta|_V.$$

Thus the collection $(U_\alpha, df_\alpha|_V)$ defines a global holomorphic section of $N_V^*\otimes[V]|_V$, showing that this tensor product bundle is trivial. This proves that $N_V^* = [-V]|_V$ and consequently $N_V = [V]|_V$. $\square$

Consider now the exact fibre bundle sequence

$$0 \to N_V^* \to \Lambda^{1,0}M|_V \to \Lambda^{1,0}V \to 0.$$

Taking the maximal exterior power in this exact sequence yields

$$K_M|_V \simeq K_V \otimes N_V^* = K_V - [V]|_V,$$

so

$$K_V \simeq (K_M \otimes [V])|_V.$$

This is the *second adjunction formula*.

We will use the following theorem whose proof, based on the Kodaira vanishing theorem, can be found in [**2**], p. 156.

Theorem 18.5. (Lefschetz Hyperplane Theorem). *Let V be a smooth analytic hypersurface in a compact complex manifold M^{2m} such that $[V]$ is positive. Then the linear maps $H^i(M,\mathbb{C}) \to H^i(V,\mathbb{C})$ induced by the inclusion $V \to M$ are isomorphisms for $i \le m-2$ and injective for $i = m-1$. If $m \ge 3$ then $\pi_1(M) = \pi_1(V)$.*

Our main application will be the following result on complete intersections in the complex projective space.

Theorem 18.6. *Let $P_1, \cdots P_k$ be homogeneous irreducible relatively prime polynomials in $m+1$ variables of degrees $d_1, \dots d_k$. Let N denote the subset in $\mathbb{C}\mathrm{P}^m$ defined by these polynomials:*

$$N := \{[z_0 : \dots : z_m] \in \mathbb{C}\mathrm{P}^m \mid P_i(z_0, \dots, z_m) = 0,\ \forall 1 \le i \le k\}.$$

Then, if N is smooth, we have $K_N \simeq [qH]|_N$, where $q = (d_1 + \dots + d_k) - (m+1)$ and H is the hyperplane divisor in $\mathbb{C}\mathrm{P}^m$.

Proof. Notice first that N is smooth for a generic choice of the polynomials P_i. We denote by V_i the analytic hypersurface in $\mathbb{C}\mathrm{P}^m$ defined by P_i and claim that

$$V_i \cong d_i H. \tag{75}$$

This can be seen as follows. While the homogeneous polynomial P_i is not a well–defined function on $\mathbb{C}\mathrm{P}^m$, the quotient $h_i := \frac{P_i}{z_0^{d_i}}$ is a meromorphic function. More precisely, h_i is defined by the collection $(U_\alpha, \frac{P_i}{z_\alpha^{d_i}}, \frac{z_0^{d_i}}{z_\alpha^{d_i}})$. Clearly the zero–locus of h_i is $(h_i)_0 = V_i$ and the pole–locus is $(h_i)_\infty = d_i H_0$, where H_0 is just the hyperplane $\{z_0 = 0\}$. This shows that $(h_i) = V_i - d_i H_0$, thus proving our claim.

Let now, for $i = 1, \dots, k$, N_i denote the intersection of $V_1, \dots, V_i$. Since $N_{i+1} = N_i \cap V_{i+1}$, we have

$$[N_{i+1}]|_{N_i} = [d_{i+1}H]|_{N_i}. \tag{76}$$

This follows from the fact that if V is an irreducible hypersurface in a projective manifold M and N is any analytic submanifold in M then

$$[V]|_N \simeq [V \cap N].$$

We claim that

$$K_{N_i} \simeq [n_i H]|_{N_i}, \tag{77}$$

where $n_i := (d_1 + \ldots + d_i) - (m+1)$. For $i = 1$ this follows directly from the second adjunction formula together with (75), using the fact that $K_{\mathbb{C}\mathrm{P}^m} = [-(m+1)H]$. Suppose that the formula holds for some $i \geq 1$. The second adjunction formula applied to the hypersurface N_{i+1} of N_i, together with (76) yields

$$K_{N_{i+1}} = ([N_{i+1}] \otimes K_{N_i})_{N_{i+1}} = ([d_{i+1}H] \otimes [n_i H])|_{N_{i+1}} = [n_{i+1}H]|_{N_{i+1}}.$$

Thus (77) is true for every i, and in particular for $i = k$. This finishes the proof. □

Corollary 18.7. *Let $d_1, \ldots, d_k$ be positive integers and denote their sum by $m+1 := d_1 + \ldots + d_k$. Suppose that $m \geq k+3$. If $P_1, \cdots P_k$ are generic homogeneous irreducible polynomials in $m+1$ variables of degrees $d_1, \ldots d_k$, then the manifold*

$$N := \{[z_0 : \ldots : z_m] \in \mathbb{C}\mathrm{P}^m \mid P_i(z_0, \ldots, z_m) = 0,\ \forall 1 \leq i \leq k\}$$

carries a unique (up to rescaling) Ricci–flat Kähler metric compatible with the complex structure induced from $\mathbb{C}\mathrm{P}^m$. Endowed with this metric, N is Calabi–Yau.

Proof. Theorem 18.6 shows that the first Chern class of N vanishes. The condition $m \geq k+3$ together with Lefschetz Hyperplane Theorem applied inductively to the analytic hypersurfaces $N_i \subset N_{i+1}$ show that N is simply connected and $b_2(N) = b_2(\mathbb{C}\mathrm{P}^m) = 1$, and moreover the restriction of the Kähler form of $\mathbb{C}\mathrm{P}^m$ to N is a generator of the second cohomology group of N. The Calabi conjecture shows that there exists a unique Ricci–flat metric on N up to rescaling. If this metric were reducible, we would have at least two independent elements in the second cohomology of N, defined by the Kähler forms of the two factors. Since $b_1(N) = 1$ this is impossible. Thus N is either Calabi–Yau or hyperkähler. The latter case is however impossible, since every compact hyperkähler manifold has a parallel $(2,0)$–form, thus its second Betti number cannot be equal to 1. □

Bibliography

[1] A. Besse, *Einstein Manifolds*, Ergeb. Math. Grenzgeb., **10**, Springer, Berlin, 1981.

[2] P. Griffith, J. Harris, *Principles of Algebraic Geometry,* Wiley, New York, 1978.

[3] F. Hirzebruch, *Topological methods in algebraic geometry,* Springer, New York, 1966.

[4] M. Gromov, B. Lawson, Jr., *The classification of simply connected manifolds of positive scalar curvature,* Ann. of Math. **111** (1980), 423–434.

[5] L. Hörmander, *An introduction to complex analysis in several variables,* third edition, North-Holland Publishing Co., Amsterdam, 1990.

[6] D. Joyce, *Compact Manifolds with Special Holonomy,* Oxford University Press, 2000.

[7] E. Kähler, *Über eine bemerkenswerte Hermitesche Metrik,* Abh. Math. Sem. Hamburg Univ. **9** (1933), 173–186.

[8] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry I, II*, Interscience Publishers, 1963, 1969.

[9] A. Lichnerowicz, *Spineurs harmoniques,* C. R. Acad. Sci. Paris **257** (1963), 7–9.