SEMI-INTEGRABLE ALMOST HYPERHERMITIAN STRUCTURES

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Abstract. In this work, we introduce a family of almost hyperhermitian structures that we call semi-integrable. We subdivide them into four disjoint classes and show that each class is non-empty. Finally, we construct semi-integrable almost hyperhermitian structures on all reductive Lie algebras of compact type and dimension $4n$, with $n \geq 2$.

1. Introduction

An *almost hypercomplex structure* on a smooth manifold of real dimension $4n$ is a reduction of the structure group of the tangent bundle to $\mathrm{GL}(n, \mathbb{H})$, where $\mathbb{H}$ denote the quaternions. It can be equivalently interpreted as a 2-dimensional sphere H of almost complex structures. The structure group reduction is often described by means of a triple (I, J, K) of almost complex structures satisfying the usual identities of quaternionic units: $IJK = -\mathrm{Id}$. Such a point of view may, however, be restrictive, as it suggests that the triple (I, J, K) has a preferential role, whereas this is in general not the case.

When all the almost complex structures of the sphere are integrable one simply drops the term *almost* and refers to H as a *hypercomplex structure*. Note that this does not entail the full integrability of H as a $\mathrm{GL}(n, \mathbb{H})$-structure, which would mean that the manifold is locally isomorphic to $\mathbb{H}^n$. Indeed, as Obata showed in [23], integrability is encoded by the vanishing of the curvature of the unique torsion-free connection that preserves H. Integrable $\mathrm{GL}(n, \mathbb{H})$-structures were studied by Sommese [27].

Boyer [7] proved that in real dimension 4 there are only three classes of manifolds carrying hyperhermitian structures, up to conformal equivalence: tori, K3 surfaces and certain Hopf surfaces. Hyperhermitian 4-manifolds were also considered by Gauduchon–Tod [14] in their study of non-trivial triholomorphic Killing vector fields.

Hypercomplex geometry has been investigated at length especially in relation to compatible hyperhermitian metrics. Let (H, g) be an almost hyperhermitian structure. Each pair (L, g), where $L \in \mathsf{H}$, lies in one of the classes in the Gray–Hervella classification of almost Hermitian structures [15]. A refinement for almost hyperhermitian structures was later provided by Martìn-Cabrera–Swann [19], but they focused on the choice of a generating triple (I, J, K) for H, showing how the Gray–Hervella classes of $(I, g), (J, g)$ condition that of (K, g). On the one hand we aim to study this problem by removing the dependence on a chosen generating triple and instead investigate how the Gray–Hervella classes of two arbitrary almost complex structures in H influences those of the remaining ones. On the other hand, for the purposes of the present paper, we shall only take into

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account two Gray-Hervella classes: $\mathcal{W}_3 \oplus \mathcal{W}_4$, which corresponds to integrability of the almost complex structure, and $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$, corresponding to semi-Kähler structures, i.e. those whose associated fundamental form is coclosed.

For what concerns integrability, we recall the result of Obata [23] that if two orthogonal almost complex structures I, J in an almost hypercomplex structure H are integrable, then their composition is integrable as well. Yano and Ako [31] generalised Obata's result by showing that, in fact, every element of H is integrable. In Theorem 3.1 we drop the assumption of orthogonality of I and J and prove that whenever two non-proportional almost complex structures in H are integrable, every element of H is integrable as well.

We next focus on the case when H contains an integrable complex structure and a semi-Kähler one (possibly coinciding). The main object of investigation of our paper is thus the following class of structures:

Definition 1.1. An almost hyperhermitian structure (H, g) is called *semi-integrable* if there exist $I, L \in \mathsf{H}$, such that I is integrable and (L, g) is semi-Kähler, i.e. $\delta\omega_L = 0$, where ω_L is the associated fundamental 2-form.

First of all, in Theorem 4.3 we show that semi-integrable almost hyperhermitian structures can be divided into four disjoint classes:

- Type 1: Every almost complex structure in H is integrable and semi-Kähler.
- Type 2: Up to a sign there is a unique integrable complex structure in H and every almost complex structure in H is semi-Kähler.
- Type 3: Up to a sign there is a unique integrable complex structure and the semi-Kähler almost complex structures in H are exactly the ones orthogonal to it.
- Type 4: Up to a sign there are a unique integrable complex structure and a unique semi-Kähler almost complex structure in H (not necessarily distinct) and they are not orthogonal to each other.

We will provide examples of semi-integrable structures for each type, showing that these classes are not empty.

A class of hyperhermitian structures (H, g) of independent interest consists of those equipped with a connection with skew-symmetric torsion that preserves H and g, often called the Bismut connection [5]. We recall that the condition of having skew-symmetric torsion can be interpreted in the context of the Gray–Hervella classification as the vanishing of the component along $\mathcal{W}_2$. These structures give rise to the so-called *hyperkähler manifolds with torsion* (HKT), introduced by Howe–Papadopoulos [16]. Because they represent a relaxation of hyperkähler structures and because of their significance within the context of supersymmetric sigma models, they have attracted considerable attention in recent years (see, e.g., [1, 3, 4, 9, 10, 13, 17, 24, 20, 30] and references therein). Whenever the torsion 3-form of the Bismut connection is closed, these structures are called *strong* HKT and were studied in several papers (see, e.g., [4, 9, 10, 20, 30]). In particular, with the further assumption of having parallel Bismut torsion, building on the works [2, 8], Brienza–Fino–Grantcharov–Verbitsky [10] proved that any simply connected compact strong HKT manifold with parallel Bismut torsion splits as the product of a compact hyperkähler manifold and a compact Lie group with a bi-invariant metric. Such a result can be regarded as a particular case of the more general description provided by Moroianu–Schwahn [22] of geometries of a Riemannian connection with parallel, skew-symmetric and closed torsion.

More precisely from the analysis of Moroianu-Schwahn it follows that an almost hyperhermitian structure preserved by a metric connection with parallel, skew-symmetric and closed torsion on a complete simply connected almost hyperhermitian manifold (M, H, g) is the product of a complete hyperkähler manifold and a reductive Lie group (G, H_G, g_G) of compact type endowed with a bi-invariant metric. Furthermore, the torsion form is non-trivial only on the group part, where it coincides with a multiple of the canonical 3-form. Since this splitting is preserved by each almost complex structure belonging to H, it follows that the hyperkähler factor plays no role in the semi-integrability of (H, g), namely, (H, g) is semi-integrable if and only if (H_G, g_G) is semi-integrable. In addition, H_G is completely determined by its restriction on the Lie algebra $\mathfrak{g}$ of G, by left-translation.

Motivated by this discussion, we will carry out our investigation on reductive Lie algebras of compact type equipped with an $\mathrm{ad}_\mathfrak{g}$-invariant metric. First, we show that any semi-integrable almost hyperhermitian structure on such Lie algebras is of type 3, unless the Lie algebra is abelian (Proposition 5.3). Conversely, we prove in Theorem 5.7 that a reductive Lie algebra $\mathfrak{g}$ of compact type and dimension $4n$, equipped with an $\mathrm{ad}_\mathfrak{g}$-invariant metric g admits a semi-integrable almost hyperhermitian structure (H, g) if and only if $\mathfrak{g} \neq \mathfrak{su}(2) \times \mathbb{R}$.

Let us also mention that left-invariant hypercomplex structures on compact Lie groups were also constructed by Spindel–Sevrin–Troost–Van Proeyen [28], and their construction was later formalised and extended to certain homogeneous spaces by Joyce [18]. However, these hypercomplex structures are not semi-integrable with respect to any metric, because otherwise such metric would be balanced, which is excluded by the recent work of Fino–Grantcharov–Mainenti [12, Theorem 4.6].

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2. Preliminaries

2.1. **Riemannian geometry.** Let (M, g) be a Riemannian manifold. Throughout the manuscript we will tacitly raise and lower indices of tensors by means of the metric g. In particular we shall often identify a 2-form $\eta \in \Lambda^2 M$ with a skew-symmetric endomorphism, still denoted η, as follows:

$$g(\eta(X), Y) = \eta(X, Y)\,, \qquad X, Y \in TM\,.$$

As a special case, when J is an almost Hermitian structure on (M, g), we are identifying the fundamental 2-form

$$\omega_J(X, Y) := g(JX, Y)\,, \qquad X, Y \in TM\,,$$

with the almost complex structure J itself. In the same way, a 3-form $\tau \in \Lambda^3 M$ may be identified with a skew-symmetric tensor of type $(2,1)$ via

$$\tau(X,Y,Z) = g(\tau_X Y, Z)\,, \qquad X,Y,Z \in TM\,.$$

We will denote with δ the codifferential, i.e. the formal adjoint of the exterior differential d with respect to g. Given a local orthonormal frame (e_i) on M, we recall the well-known formula

$$\delta = -\sum_i e_i \lrcorner \nabla_{e_i}\,,$$

where ∇ is the Levi-Civita connection of g.

2.2. **Almost hypercomplex structures.**

Let M be a smooth manifold of dimension $4n$. An almost hypercomplex structure on M is a reduction of the structure group of M to $\mathrm{GL}(n,\mathbb{H}) \subset \mathrm{GL}(4n,\mathbb{R})$. Such a structure is defined by a triple (I,J,K) of almost complex structures satisfying

$$IJ = -JI = K\,. \tag{1}$$

This triple is not unique, but the 2-dimensional sphere of almost complex structures on M,

$$\mathsf{H} := \{aI + bJ + cK \mid (a,b,c) \in S^2\}$$

is uniquely determined by the almost hypercomplex structure. A set $\{I,J,K\} \subset \mathsf{H}$ satisfying (1) will be called a *basis* of H. Note that SO(3) acts simply transitively on the set of bases of H.

We will henceforth use the notation (M,H) to denote an almost hypercomplex manifold.

We say that two elements $I_1, I_2 \in \mathsf{H}$ are orthogonal if $\mathrm{tr}(I_1 I_2) = 0$. For every identification of H with S^2 by means of a basis, the orthogonal of the almost complex structure $aI + bJ + cK$ corresponding to a point $(a,b,c) \in S^2$ is the great circle in the sphere S^2 orthogonal to the vector $(a,b,c) \in \mathbb{R}^3$.

In Section 3 we will study almost hypercomplex structures H containing at least two (integrable) complex structures $P, Q \in \mathsf{H}$ with $P \neq \pm Q$. We will show that in this case every element in H is integrable, thus extending results by Obata [23] and by Yano and Ako [31].

2.3. **Almost hyperhermitian structures.**

Recall the following:

Definition 2.1. An almost *hyperhermitian* structure on a manifold M is an almost hypercomplex structure H together with a Riemannian metric g which is almost Hermitian with respect to every almost complex structure $L \in \mathsf{H}$.

If (H, g) is an almost hyperhermitian structure, each almost complex structure $L \in \mathsf{H}$ has some Gray-Hervella type with respect to g. In particular, L is integrable if (L,g) is of type $\mathcal{W}_3 \oplus \mathcal{W}_4$ (this condition actually does not depend on g), and (L,g) is semi-Kähler if it is of type $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$, or, equivalently, if $\delta\omega_L = 0$.

We now make the following:

Definition 2.2. An almost hyperhermitian structure (H, g) is called *semi-integrable* if there exist $I, L \in \mathsf{H}$, such that I is integrable and (L, g) is semi-Kähler, i.e. $\delta\omega_L = 0$.

In the last sections of the paper we will focus on semi-integrable almost hyperhermitian structures. We will show that they can be divided into 4 disjoint classes, and construct examples of such structures in each class.

3. The integrability of almost hypercomplex structures

Let $\mathsf{H} := \{aI + bJ + cK \mid (a, b, c) \in S^2\}$ be an almost hypercomplex structure on M. It was proved by Obata [23] that whenever two mutually orthogonal elements, e.g. $I, J \in \mathsf{H}$ are integrable, then so is their composition $K := IJ$. More generally, Yano and Ako [31] showed that the integrability of I and J forces any almost complex structure in H to be integrable.

However, from the result of Yano and Ako, it does not follow that the integrability of two (not necessarily orthogonal) almost complex structures in H implies the integrability of all the others. Since we are not aware of a reference proving such a fact, we present a proof below.

Theorem 3.1. *Let H be an almost hypercomplex structure. If two almost complex structures $P \neq \pm Q$ in H are integrable, then all elements of H are integrable.*

Proof. Let g be a Riemannian metric compatible with H, with Levi-Civita connection denoted ∇. Recall (see e.g. [21, Lemma 11.4]) that an almost complex structure L compatible with g is integrable if and only if $E_L(X) = 0$ for every $X \in TM$, where

$$E_L(X) := L\nabla_X L - \nabla_{LX} L\,. \tag{2}$$

Let (I, J, K) be a basis of H and $X \in TM$. We aim to show that $E_L(X)$ anti-commutes with I, J, and K for any $L \in \mathsf{H}$. It will then immediately follow that $E_L(X) = 0$. Indeed, for any endomorphism A that anti-commutes with I, J, and K we have

$$AK = AIJ = -IAJ = IJA = KA = -AK\,,$$

so $A = 0$.

We denote by $\mathrm{End}^-(TM)$ the set of g-skew-symmetric endomorphisms of TM. If A, B are endomorphisms of TM, we say that $A \equiv B \mod \mathrm{End}^-(TM)$ if $A - B$ is skew-symmetric with respect to g.

Note that I, J, K, and $E_L(X)$ all belong to $\mathrm{End}^-(TM)$, and two endomorphisms $A, B \in \mathrm{End}^-(TM)$ anti-commute if and only if $AB \equiv 0 \mod \mathrm{End}^-(TM)$.

Without loss of generality we may assume $P = I$ and $Q = aI + bJ$ for some $(a, b) \in S^1 \setminus \{(\pm 1, 0)\}$.

We shall first prove that J is integrable. Let X be any tangent vector. Since evidently we have $E_J(X)J + JE_J(X) = 0$, we need to prove $IE_J(X) \equiv 0 \mod \mathrm{End}^-(TM)$ and $KE_J(X) \equiv 0 \mod \mathrm{End}^-(TM)$.

Since they will be used multiple times, we prove here the following equivalences mod $\mathrm{End}^-(TM)$. First, using that $E_I(X)=0$ we obtain

$$I\nabla_{IX}J = I\nabla_{IX}(KI) = I\nabla_{IX}KI + IK\nabla_{IX}I = I\nabla_{IX}KI + IKI\nabla_X I = I\nabla_{IX}KI + K\nabla_X I\,,$$

and since $I\nabla_{IX}KI$ is skew-symmetric, we obtain

$$I\nabla_{IX}J \equiv K\nabla_X I \mod \mathrm{End}^-(TM)\,. \tag{3}$$

If we replace X with JX in (3) we also deduce

$$I\nabla_{KX}J \equiv K\nabla_{JX} I \mod \mathrm{End}^-(TM)\,. \tag{4}$$

From the integrability of I and Q we deduce the identity

$$\begin{aligned}
0 &= E_Q(X) \\
&= (aI+bJ)\nabla_X(aI+bJ) - \nabla_{(aI+bJ)X}(aI+bJ) \\
&= a^2E_I(X) + ab(I\nabla_X J + J\nabla_X I - \nabla_{IX}J - \nabla_{JX}I) + b^2E_J(X) \\
&= ab(I\nabla_X J + J\nabla_X I - \nabla_{IX}J - \nabla_{JX}I) + b^2E_J(X)\,.
\end{aligned}$$

Thanks to (3) we have

$$\begin{aligned}
IE_J(X) &= -\frac{a}{b}(-\nabla_X J + K\nabla_X I - I\nabla_{IX}J - I\nabla_{JX}I) \\
&\equiv \frac{a}{b}(I\nabla_{IX}J - K\nabla_X I) \mod \mathrm{End}^-(TM) \\
&\equiv 0 \mod \mathrm{End}^-(TM)\,,
\end{aligned}$$

and similarly, using (4) we obtain

$$\begin{aligned}
KE_J(X) &= -\frac{a}{b}(J\nabla_X J - I\nabla_X I - K\nabla_{IX}J - K\nabla_{JX}I) \\
&\equiv \frac{a}{b}(K\nabla_{IX}J + K\nabla_{JX}I) \mod \mathrm{End}^-(TM) \\
&= \frac{a}{b}(K\nabla_{IX}J + I\nabla_{KX}J) \mod \mathrm{End}^-(TM) \\
&\equiv \frac{a}{b}IE_J(IX) \mod \mathrm{End}^-(TM) \\
&\equiv 0 \mod \mathrm{End}^-(TM)\,,
\end{aligned}$$

where the last equality follows from the previous computation. This shows that $E_J(X)=0$ and thus J is integrable.

With the same technique we now show that the complex structure $R = cI + dJ$ is integrable for all $(c,d)\in S^1$. Clearly

$$E_R(X) = cd(I\nabla_X J + J\nabla_X I - \nabla_{IX}J - \nabla_{JX}I)$$

and thus, thanks to (3) and (4), we compute

$$\begin{aligned}
IE_R(X) &= cd(-\nabla_X J + K\nabla_X I - I\nabla_{IX} J - I\nabla_{JX} I) \\
&\equiv cd(K\nabla_X I - I\nabla_{IX} J) \mod \mathrm{End}^-(TM) \\
&\equiv 0 \mod \mathrm{End}^-(TM) \\
JE_R(X) &= cd(-K\nabla_X J - \nabla_X I - J\nabla_{IX} J - J\nabla_{JX} I) \\
&\equiv cd(-K\nabla_X J - J\nabla_{JX} I) \mod \mathrm{End}^-(TM) \\
&\equiv cd(KJ\nabla_{JX} J - \nabla_{JX}(JI) + (\nabla_{JX} J)I) \mod \mathrm{End}^-(TM) \\
&\equiv 0 \mod \mathrm{End}^-(TM) \\
KE_R(X) &= cd(J\nabla_X J - I\nabla_X I - K\nabla_{IX} J - K\nabla_{JX} I) \\
&\equiv cd(-K\nabla_{IX} J - K\nabla_{JX} I) \mod \mathrm{End}^-(TM) \\
&\equiv cd(KJ\nabla_{JIX} J - I\nabla_{KX} J) \mod \mathrm{End}^-(TM) \\
&= 0 \mod \mathrm{End}^-(TM)\,.
\end{aligned}$$

By symmetry, to conclude the proof, it is enough to show that K is integrable.

$$\begin{aligned}
IE_K(X) &= -J\nabla_X K - I\nabla_{KX} K \\
&\equiv (\nabla_X J)K + I\nabla_{KX}(JI) \mod \mathrm{End}^-(TM) \\
&\equiv -J(\nabla_{JX} J)K + K\nabla_{KX} I \mod \mathrm{End}^-(TM) \\
&\equiv (\nabla_{JX} J)JK + J\nabla_{JX} I \mod \mathrm{End}^-(TM) \\
&\equiv (\nabla_{JX} J)I + \nabla_{JX}(JI) - (\nabla_{JX} J)I \mod \mathrm{End}^-(TM) \\
&\equiv 0 \mod \mathrm{End}^-(TM) \\
JE_K(X) &= I\nabla_X K - J\nabla_{KX} K \\
&\equiv -(\nabla_X I)K - J\nabla_{KX}(IJ) \mod \mathrm{End}^-(TM) \\
&\equiv I(\nabla_{IX} I)K + K\nabla_{KX} J \mod \mathrm{End}^-(TM) \\
&= -(\nabla_{IX} I)IK - KJ\nabla_{JKX} J \mod \mathrm{End}^-(TM) \\
&\equiv -I\nabla_{IX} J + I\nabla_{IX} J \mod \mathrm{End}^-(TM) \\
&= 0 \mod \mathrm{End}^-(TM)
\end{aligned}$$

and since, obviously, $KE_K(X) \equiv 0 \mod \mathrm{End}^-(TM)$, we are done. □

4. Semi-integrable almost hyperhermitian structures

Recall that the Lee form of an almost Hermitian structure (I, g) is defined by $\theta_I := I\delta I$, where as already mentioned, we identify the skew-symmetric endomorphism I with the corresponding 2-form ω_I. We begin this section with the following observation regarding the Lee forms of an almost hyperhermitian structure. It can also be deduced from [11], but we prefer to provide a direct proof here for convenience of the reader.

Lemma 4.1. *Let (H, g) be an almost hyperhermitian structure such that $I \in \mathsf{H}$ is integrable. Then the Lee forms of (L, g) coincide for every $L \in \mathsf{H}$ orthogonal to I.*

Proof. Let e_i be a local orthonormal frame. Fix a $J \in \mathsf{H}$ orthogonal to I and set $K := IJ$. We have

$$\delta K = -\sum_i e_i \lrcorner \nabla_{e_i} K = -\sum_i e_i \lrcorner (\nabla_{e_i} I) J - \sum_i e_i \lrcorner I (\nabla_{e_i} J) = -\sum_i e_i \lrcorner (\nabla_{e_i} I) J + I \delta J \,.$$

We claim that the first summand in the right hand side term vanishes. Recall that I is integrable if and only if the tensor E_I defined in (2) vanishes. Furthermore $0 = \nabla(I^2) = (\nabla I) I + I \nabla I$ and thus

$$\sum_i e_i \lrcorner (\nabla_{e_i} I) J = \sum_i I e_i \lrcorner (\nabla_{I e_i} I) J = -\sum_i I e_i \lrcorner (\nabla_{e_i} I) I J = -\sum_i e_i \lrcorner (\nabla_{e_i} I) I J I$$
$$= -\sum_i e_i \lrcorner (\nabla_{e_i} I) J \,,$$

where in the first equality we simply replaced the basis (e_i) with $(I e_i)$. Hence the claim is proved. We thus have $\delta K = I \delta J$, whence

$$K \delta K = K I \delta J = J \delta J \,. \tag{5}$$

Now, if $L \in \mathsf{H}$ is orthogonal to I, it is of the form $L = aJ + bK$ with $a^2 + b^2 = 1$. Then, using (5) we deduce

$$\theta_L = (aJ + bK)\delta(aJ + bK) = a^2 J \delta J + ab(J \delta K + K \delta J) + b^2 K \delta K = (a^2 + b^2)\theta_J = \theta_J \,,$$

concluding the proof. □

From Lemma 4.1 we obtain immediately the following.

Corollary 4.2. *Let (H, g) be a semi-integrable structure and consider $I, L \in \mathsf{H}$ such that I is integrable and (L, g) is semi-kähler. If L is orthogonal to I, then (L', g) is semi-Kähler for all $L' \in \mathsf{H}$ orthogonal to I.*

We will now focus on the set of semi-integrable almost hyperhermitian structures (H, g), and show that it can be divided into 4 distinct classes, depending on how many almost complex structures in H are integrable, and how many are semi-Kähler with respect to g.

Theorem 4.3. *Every semi-integrable almost hyperhermitian structure (H, g) belongs to only one of the following (disjoint) classes:*

- *Type 1: Every almost complex structure in H is integrable and semi-Kähler.*
- *Type 2: Up to a sign there is a unique integrable complex structure in H and every almost complex structure in H is semi-Kähler.*
- *Type 3: Up to a sign there is a unique integrable complex structure and the semi-Kähler almost complex structures in H are exactly the ones orthogonal to it.*
- *Type 4: Up to a sign there are a unique integrable complex structure and a unique semi-Kähler almost complex structure in H (not necessarily distinct) and they are not orthogonal to each other.*

Proof. Let $I, L \in \mathsf{H}$ be almost complex structures such that I is integrable and (L, g) is semi-Kähler. Let us also fix $J \in \mathsf{H}$ orthogonal to I and denote by $K := IJ$.

Suppose first that there is another integrable complex structure $I' \neq \pm I$ in H. Then Theorem 3.1 tells us that every complex structure in H is integrable. We need to show that in this case a semi-integrable structure is of type 1. Using the SO(3) action on H, we can now assume without loss of generality that $L = J$. By Corollary 4.2, we deduce that for every a, b with $a^2 + b^2 = 1$, the structure $(aJ + bK, g)$ is semi-Kähler. Another application of Corollary 4.2 with the roles of I and J replaced by $aJ + bK$ and $bJ - aK$ implies that for every x, y with $x^2 + y^2 = 1$, the almost Hermitian structure $(xI + y(bJ - aK), g)$ is semi-Kähler. Since every element of H can be written in this form, we obtain that (H, g) is of type 1.

If (H, g) is not of type 1, then $\pm I$ are the only integrable complex structures in H.

Suppose first that L is not orthogonal to I. Using the action of SO(3) on H we may suppose that $L = aI + bJ$ with $a^2 + b^2 = 1$, and $a \neq 0$. Let $L' \in \mathsf{H}$ be any almost complex structure such that (L', g) is semi-Kähler. We write $L' = a'I + b'J + c'K$ with $(a')^2 + (b')^2 + (c')^2 = 1$. Then we compute

$$0 = \delta L = a\delta I + b\delta J\,, \tag{6}$$

which implies $\delta I = -\frac{b}{a}\delta J$.

If $\delta J = 0$, we obtain $\delta I = 0$ and $\delta K = 0$, thanks to Lemma 4.1. By linearity we get $\delta L_0 = 0$ for every $L_0 \in \mathsf{H}$, so (H, g) is of type 2.

Assume from now on that $\delta J \neq 0$. From (5) and (6) we deduce

$$0 = \delta L' = a'\delta I + b'\delta J + c'\delta K = -a'\frac{b}{a}\delta J + b'\delta J - c'I\delta J\,.$$

As δJ and $I\delta J$ are linearly independent and non-zero, we conclude that $c' = 0$ and $b' = a'\frac{b}{a}$ which forces L' to be proportional to L. Therefore in this case the semi-integrable structure is of type 4.

The only remaining case is when L is orthogonal to I, and there exists no semi-Kähler structure in H which is not orthogonal to I. Then by Corollary 4.2, for every almost complex structure $L' \in \mathsf{H}$ orthogonal to I, (L', g) is semi-Kähler, so the structure is of type 3. □

We can summarize the different classes of semi-integrable almost hyperhermitian structures with the following table:

	Integrable	Semi-Kähler
Type 1	$\mathsf{H} \simeq S^2$	$\mathsf{H} \simeq S^2$
Type 2	$\{\pm I\} \simeq S^0$	$\mathsf{H} \simeq S^2$
Type 3	$\{\pm I\} \simeq S^0$	$I^\perp \simeq S^1$
Type 4	$\{\pm I\} \simeq S^0$	$\{\pm L\} \simeq S^0,\ L \not\perp I$

In this table, each entry in the middle column represents the set of integrable elements in H, and each entry in the right column represents the set of semi-Kähler elements in H.

We will now construct examples of semi-integrable almost hyperhermitian structures. The first remark is that in dimension 4 there is no such structure of type 2. This fact follows from [14, Proposition 2]. We provide a proof for the readers' convenience:

Lemma 4.4. *If* (H, g) *is an almost hyperhermitian structure such that every* $L \in \mathsf{H}$ *is semi-Kähler and there exists* $I \in \mathsf{H}$ *which is integrable, then every* $L \in \mathsf{H}$ *is integrable.*

Proof. If M has dimension 4, any semi-Kähler Hermitian structure is Kähler. Thus I is parallel with respect to the Levi-Civita connection ∇ of g. Let I, J, K denote a basis of H. The corresponding 2-forms ω_I, ω_J and ω_K form an orthogonal basis of the space $\Lambda^2_+(M)$ of self-dual forms, which is preserved by ∇. Moreover they have constant length $\sqrt{2}$ and $\nabla I = 0$. This shows that there exists a 1-form α such that $\nabla_X \omega_J = \alpha(X)\omega_K$ for every $X \in TM$. Since ω_J is coclosed by assumption, we get $0 = \delta\omega_J = -K(\alpha)$, so $\alpha = 0$, whence J is ∇-parallel. This shows that (H, g) is hyperkähler, thus proving our claim. □

We will now construct examples of semi-integrable almost hyperhermitian structures on $\mathbb{R}^4$ of the remaining types. Since every hyperkähler structure (e.g. the flat one on $\mathbb{R}^4 \simeq \mathbb{H}$) is in particular semi-integrable of type 1, we just need to construct examples of type 3 and 4.

Example 4.5. Consider the flat hyperkähler structure (I_0, J_0, K_0, g_0) on $\mathbb{R}^4$, with corresponding fundamental 2-forms

$$\omega_{I_0} = dx_1 \wedge dx_2 + dx_3 \wedge dx_4, \qquad \omega_{J_0} = dx_1 \wedge dx_3 - dx_2 \wedge dx_4, \qquad \omega_{K_0} = dx_1 \wedge dx_4 + dx_2 \wedge dx_3\,.$$

For every non-constant function $f \in C^\infty(\mathbb{R}^4)$, we define

$$I := I_0, \qquad J := \cos f\, J_0 + \sin f\, K_0, \qquad K := -\sin f\, J_0 + \cos f\, K_0\,. \tag{7}$$

We claim that (I, J, K, g_0) is a semi-integrable almost hyperhermitian structure on $\mathbb{R}^4$ of type 4. Since I is Kähler, it suffices, by Theorem 4.3, to prove that J is non-integrable and not semi-Kähler. Indeed,

$$\delta\omega_J = \sin f\, J_0(\nabla f) - \cos f\, K_0(\nabla f) = -K(\nabla f)$$

is not identically 0 since f is non-constant. Moreover, for every $X \in TM$ we have $\nabla_X J = X(f)K$, so the tensor $E_J(X)$ defined in (2) is equal to

$$E_J(X) = J\nabla_X J - \nabla_{JX} J = X(f)I - (JX)(f)K\,.$$

Again, since f is non-constant, $E_J(X)$ cannot vanish for every X, so J is not integrable by [21, Lemma 11.4].

Example 4.6. Consider again the hypercomplex structure (I, J, K) on $\mathbb{R}^4$ defined by a function f as in (7). We claim that for $f := x_1$ and $g = e^{x_2} g_0$ (where g_0 is the standard Euclidean metric), the almost hypercomplex structure (I, J, K, g) is semi-integrable of type 3. Notice first that I is integrable, but (I, g) is not semi-Kähler since $d\omega_I = dx_2 \wedge \omega_I \neq 0$. Moreover, for every almost Hermitian structure (L, g_0), denoting by $\omega_L = g(L\cdot, \cdot)$ and $\omega^0_L = g_0(L\cdot, \cdot)$ the fundamental forms, we have that $\omega^0_{J_0}$ and $\omega^0_{K_0}$ are closed, so

$$\begin{aligned} d\omega^0_J &= -\sin x_1 dx_1 \wedge \omega^0_{J_0} + \cos x_1 dx_1 \wedge \omega^0_{K_0} \\ &= \sin x_1 dx_1 \wedge dx_2 \wedge dx_4 + \cos x_1 dx_1 \wedge dx_2 \wedge dx_3\,, \end{aligned}$$

and thus

$$\begin{aligned} d\omega_J &= d(e^{x_2}\omega_J^0) = e^{x_2}(d\omega_J^0 + dx_2\wedge\omega_J^0) = e^{x_2}(d\omega_J^0 + \cos x_1 dx_2\wedge\omega_{J_0}^0 + \sin x_1 dx_2\wedge\omega_{K_0}^0)\\ &= e^{x_2}(d\omega_J^0 + \cos x_1 dx_2\wedge dx_1\wedge dx_3 + \sin x_1 dx_2\wedge dx_1\wedge dx_4) = 0\,. \end{aligned}$$

This shows that (J,g) is non-integrable and semi-Kähler, (I,g) is integrable and not semi-Kähler, whence (I,J,K,g) is semi-integrable of type 3.

We will now consider the higher dimensional cases $n\geq 8$. Since the product of a semi-integrable almost hyperhermitian structure with a hyperkähler 4-manifold is again semi-integrable of the same type, in order to show that the 4 classes in Theorem 4.3 are non-empty for all $n\geq 8$, we just need to construct a semi-integrable almost hyperhermitian structure of type 2 in dimension 8.

Example 4.7. Let $(M,g)=(\mathbb{R}^8,g_0)$ be the flat Euclidean space. Take any function f depending only on the first 4 coordinates, and define the almost hyperhermitian structure (I,J,K) on (M,g) whose fundamental 2-forms are

$$\begin{aligned} \omega_I &= dx_1\wedge dx_2 + dx_3\wedge dx_4 + dx_5\wedge dx_6 + dx_7\wedge dx_8,\\ \omega_J &= dx_1\wedge dx_3 - dx_2\wedge dx_4 + \cos f(dx_5\wedge dx_7 - dx_6\wedge dx_8)\\ &\quad + \sin f(dx_5\wedge dx_8 + dx_6\wedge dx_7),\\ \omega_K &= dx_1\wedge dx_4 + dx_2\wedge dx_3 - \sin f(dx_5\wedge dx_7 - dx_6\wedge dx_8)\\ &\quad + \cos f(dx_5\wedge dx_8 + dx_6\wedge dx_7)\,. \end{aligned}$$

Clearly I is integrable (since (I,g) is Kähler), and by the computations in Example 4.5 it is easy to check that J is not integrable. Moreover, (I,g) is semi-Kähler (again since it is Kähler), and

$$\begin{aligned} \delta\omega_J &= -\sum_{i=1}^{8}\frac{\partial}{\partial x_i}\lrcorner\nabla_{\frac{\partial}{\partial x_i}}\omega_J\\ &= -\sum_{i=1}^{8}\frac{\partial}{\partial x_i}\lrcorner\left(\frac{\partial\cos f}{\partial x_i}(dx_5\wedge dx_7 - dx_6\wedge dx_8) + \frac{\partial\sin f}{\partial x_i}(dx_5\wedge dx_8 + dx_6\wedge dx_7)\right)\\ &= 0\,, \end{aligned}$$

by the choice of f.

5. Semi-integrable structures on reductive Lie algebras

Throughout this section, by reductive Lie group *of compact type* we mean a Lie group whose universal cover is a direct product of a compact Lie group and a Euclidean factor $\mathbb{R}^k$. Also, a reductive Lie algebra *of compact type* is simply the Lie algebra of a reductive Lie group of compact type.

In [22, Thm. 5.1] the following result was obtained:

Theorem 5.1. *Let (M,g) be a complete simply connected Riemannian manifold of dimension $n\geq 8$, equipped with a connection ∇^τ with closed and parallel skew-symmetric torsion τ. If ∇^τ preserves a hypercomplex structure H compatible with g, then (M,g) splits* *isometrically as a product of a complete hyperkähler manifold (N, g_N) and a reductive Lie group G of compact type, equipped with a bi-invariant metric g_G. Every almost complex structure in H preserves the two factors of this decomposition. Moreover, the torsion form τ vanishes on N and its restriction to G is equal to $\pm\frac{1}{2}\sigma \in \Omega^3 G$, where σ denotes the canonical 3-form of G, defined by $\sigma(X, Y, Z) = g_G([X, Y], Z)$.*

Note that the connections ∇^τ are flat for $\tau = \pm\frac{1}{2}\sigma$. Their parallel vector fields are exactly the left-invariant vector fields on G for $\tau = -\frac{1}{2}\sigma$ and the right-invariant vector fields for $\tau = \frac{1}{2}\sigma$. In addition, we may assume without loss of generality that $\tau = -\frac{1}{2}\sigma$. Indeed, applying the automorphism $x \mapsto x^{-1}$ left-invariant and right-invariant vector fields are interchanged. Therefore, an almost hyperhermitian structure compatible with g and preserved by a connection with parallel, skew-symmetric, and closed torsion on $(M, g) = (N, g_N) \times (G, g_G)$ is determined by the choice of three almost complex structures on the Lie algebra $\mathfrak{g}$ of G, compatible with g_G and satisfying the quaternionic relations (1), and by the choice of a hyperkähler triple on N. Such a structure is semi-integrable if and only if its restriction to G is semi-integrable.

Summarising, in view of Theorem 5.1, in order to understand semi-integrable almost hyperhermitian structures preserved by a connection with parallel, skew-symmetric, and closed torsion on simply connected complete manifolds, we only need to understand left-invariant semi-integrable almost hyperhermitian structures compatible with a bi-invariant metric on simply connected reductive Lie groups of compact type. These, in turn are determined by their restriction to the Lie algebra. We thus make the following definition:

Definition 5.2. Let $(\mathfrak{g}, g)$ be a reductive Lie algebra of compact type endowed with an $\mathrm{ad}_{\mathfrak{g}}$-invariant metric g. An almost hyperhermitian structure $\mathsf{H} \subset \mathrm{End}^-(\mathfrak{g}, g)$ is called semi-integrable if there exist $I, L \in \mathsf{H}$ with I integrable and $\omega_L := g(L\cdot, \cdot)$ co-closed.

By the above discussion, semi-integrable almost hyperhermitian structures on $(\mathfrak{g}, g)$ are in one to one correspondence with left-invariant semi-integrable almost hyperhermitian structures on the simply connected Lie group G with Lie algebra $\mathfrak{g}$, endowed with the bi-invariant metric determined by g.

Let $\mathfrak{g}$ be a reductive Lie algebra of compact type. We begin by collecting here some well known facts about roots spaces, in order to fix notations and terminology. Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a Cartan subalgebra for $\mathfrak{g}$ and $R \subset \mathfrak{t}^*$ the corresponding root system. We denote with $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ the complexification of $\mathfrak{g}$ and let $\mathfrak{g}_\alpha$ denote the root space of $\mathfrak{g}^{\mathbb{C}}$ with respect to a root $\alpha \in R$. Let $R^+ \subset R$ be a system of positive roots for $\mathfrak{g}$, namely a subset of R such that $R^+ \cup (-R^+) = R$, $R^+ \cap (-R^+) = \emptyset$, $\mathrm{Span}_{\mathbb{C}}(R^+) = \mathfrak{t}^{\mathbb{C}}$ and if $\alpha, \beta \in R^+$ are such that $\alpha + \beta \in R$, then $\alpha + \beta \in R^+$.

Recall that a *Cartan-Weyl basis* for $\mathfrak{g}^{\mathbb{C}}$ with respect to R^+ is a choice of generators $E_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ and elements $H_\alpha \in \mathfrak{t}^{\mathbb{C}}$ for all $\alpha \in R^+$ such that

$$B(E_\alpha, E_{-\alpha}) = 1\,, \qquad [E_\alpha, E_{-\alpha}] = H_\alpha\,,$$
$$B(T, H_\alpha) = \alpha(T)\,, \quad \forall T \in \mathfrak{t}\,, \qquad [T, E_\alpha] = \alpha(T) E_\alpha\,, \quad \forall T \in \mathfrak{t}\,,$$

where $B(X, Y) = \mathrm{tr}(\mathrm{ad}_X \mathrm{ad}_Y)$ is the Cartan-Killing form of $\mathfrak{g}^{\mathbb{C}}$.

When the dimension of $\mathfrak{g}$ is even (and thus also its rank r), the choice of a system of positive roots and of a complex structure on $\mathfrak{t}$ determines a left-invariant integrable complex structure I on $\mathfrak{g}$ defined by decreeing the vectors E_α and $E_{-\alpha}$ to be of type $(1,0)$ and $(0,1)$ respectively. This fact was first observed by Samelson [26] and Wang [29] independently. Conversely, Pittie [25] proved that any left-invariant integrable complex structure I arises in this way, namely I singles out a Cartan subalgebra $\mathfrak{t}$ such that $I|_\mathfrak{t}$ is a complex structure on $\mathfrak{t}$ and it also induces an associated system of positive roots R^+ with respect to $\mathfrak{t}$.

Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Any bi-invariant metric on G is determined by an Ad_G-invariant scalar product g_G on $\mathfrak{g}$, and a left-invariant almost hyperhermitian structure H on (G, g_G) is determined by its restriction to $\mathfrak{g}$, also denoted by H. We choose a g_G-orthonormal basis $\{e_i\}$ of $\mathfrak{g}$, and extend it by left-invariance to M. Then $\nabla_{e_i} e_j = \frac{1}{2}[e_i, e_j]$, where ∇ is the Levi-Civita connection of g_G. For every $L \in \mathsf{H}$ we may compute the codifferential of ω_L as follows:

$$\delta\omega_L = -\sum_i e_i \lrcorner \nabla_{e_i}\omega_L = -\frac{1}{2}\sum_i e_i \lrcorner (\mathrm{ad}_{e_i})_*\omega_L = -\frac{1}{2}\sum_i [e_i, Le_i]\,, \tag{8}$$

where $(\mathrm{ad}_{e_i})_*$ denotes the extension of ad_{e_i} to the tensor algebra of $\mathfrak{g}$ as derivation. Assume now that $I \in \mathsf{H}$ is integrable. Then, once we fix a Cartan-Weyl basis $(E_\alpha)_{\alpha\in R^+}$ for $\mathfrak{g}$ and a basis (T_k) for the $(1,0)$-part of the Cartan subalgebra $\mathfrak{t}$ induced by I as before, the Lee form of (L, g) can be written as

$$\theta_L = L\delta\omega_L = -\mathrm{Re}\left(\sum_{\alpha\in R^+} L[E_\alpha, L\bar{E}_\alpha] + \sum_{k=1}^{r/2} L[T_k, L\bar{T}_k]\right)\,, \tag{9}$$

where r is the rank of $\mathfrak{g}$, which is even.

We now observe that on (non-abelian) reductive Lie algebras of compact type, among the classes of semi-integrable almost hyperhermitian structures introduced in Theorem 4.3 the only ones that can occur are those of type 3.

Proposition 5.3. *A semi-integrable almost hyperhermitian structure* (H, g_G) *on a non-abelian reductive Lie algebra* $\mathfrak{g}$ *of compact type is necessarily of type* 3.

Proof. Let $I \in \mathsf{H}$ be integrable and $L \in \mathsf{H}$ be semi-Kähler. By transitivity of the SO(3)-action on bases of H, there exists a basis (I, J, K) such that $L = aI + bJ$, for some constants a, b such that $a^2 + b^2 = 1$. As L is semi-Kähler, we have

$$0 = \delta\omega_L = a\delta\omega_I + b\delta\omega_J\,. \tag{10}$$

But now, using (9) we deduce

$$\begin{aligned}\delta\omega_I &= -\mathrm{Re}\left(\sum_{\alpha\in R^+}[E_\alpha, I\bar{E}_\alpha] + \sum_{k=1}^{r/2}[T_k, I\bar{T}_k]\right) = \mathrm{Re}\left(\sqrt{-1}\sum_{\alpha\in R^+}[E_\alpha, \bar{E}_\alpha]\right)\\ &= \mathrm{Re}\left(\sqrt{-1}\sum_{\alpha\in R^+} H_\alpha\right) \in \mathfrak{t}\,,\end{aligned}$$

because $[T_k, \bar{T}_k] = 0$ as $\mathfrak{t}$ is abelian. On the other hand

$$\delta\omega_J = -\mathrm{Re}\left(\sum_{\alpha\in R^+}[E_\alpha, J\bar{E}_\alpha] + \sum_{k=1}^{r/2}[T_k, J\bar{T}_k]\right) \in \bigoplus_{\beta\in R}\mathfrak{g}_\beta\,.$$

Indeed, for every $\alpha \in R^+$ and $k = 1, \dots, r/2$ we can write

$$J\bar{E}_\alpha = F_\alpha + S_\alpha\,, \qquad J\bar{T}_k = F_k + S_k\,, \qquad F_\alpha, F_k \in \bigoplus_{\beta\in R^+}\mathfrak{g}_\beta\,,\ S_\alpha, S_k \in \mathfrak{t}\,,$$

and thus

$$[E_\alpha, J\bar{E}_\alpha] = [E_\alpha, F_\alpha] + [E_\alpha, S_\alpha] \in \bigoplus_{\beta\in R^+}\mathfrak{g}_\beta\,, \qquad [T_k, J\bar{T}_k] = [T_k, F_k] \in \bigoplus_{\beta\in R^+}\mathfrak{g}_\beta\,,$$

because $[\mathfrak{t}, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_\beta$. Hence, both summands on the right-hand side of (10) vanish. Suppose by contradiction that $a \neq 0$. Then we must have $\delta\omega_I = 0$, i.e. $\sum_{\alpha\in R^+} H_\alpha = 0$, which is impossible since G is not abelian. Thus $a = 0$ and $L = \pm J$, concluding that the semi-integrable almost hyperhermitian structure can only be of type 3. □

The remaining part of the paper is devoted to show that, except for $\mathfrak{su}(2) \oplus \mathbb{R}$, we can always find semi-integrable almost hyperhermitian structures on any reductive Lie algebra of compact type of dimension $4n$. Before proceeding, we make the following definition:

Definition 5.4. A pair of positive roots $(\alpha, \beta) \in R^+ \times R^+$ is called *sum-free* if the sum $\alpha + \beta$ is not a root. In particular, if (α, β) is sum-free, then $[E_\alpha, E_\beta] = 0$.

As the Lie algebra of a reductive Lie group decomposes into a direct sum of its centre and its simple ideals, we first focus on proving the existence of a certain involution on compact simple Lie groups.

Lemma 5.5. *Let G be a compact simple Lie group of rank r, and $R^+(G)$ a system of positive roots. Then there exists a decomposition*

$$R^+(G) = R_0^+(G) \coprod R_1^+(G)$$

with

$$|R_0^+(G)| = \begin{cases} 0 & \text{if } \dim(G) - r \equiv 0 \mod 4\,,\\ 1 & \text{if } \dim(G) - r \equiv 2 \mod 4\,,\end{cases}$$

and an involution without fixed points $f\colon R_1^+(G) \to R_1^+(G)$ such that $(\alpha, f(\alpha))$ is sum-free for all $\alpha \in R_1^+(G)$.

Proof. Note that $\dim(G) - r = 2|R^+(G)|$. Therefore we will go through the list of all compact simple Lie groups discussing them according to the parity of $|R^+(G)|$. In each case, we will define the value of $f\colon R_1^+(G) \to R_1^+(G)$ using the following notation

$$\alpha \longleftrightarrow \beta$$

to mean $f(\alpha) = \beta$ and $f(\beta) = \alpha$.

We recall here the simple root systems following the conventions of Bourbaki [6]:

Dynkin type	Positive roots R^+	
A_n	$e_i - e_j$	$1 \leq i < j \leq n+1$
B_n	e_i $e_i - e_j$ $e_i + e_j$	$1 \leq i \leq n$ $1 \leq i < j \leq n$
C_n	$e_i - e_j$ $e_i + e_j$ $2e_i$	$1 \leq i < j \leq n$ $1 \leq i \leq n$
D_n	$e_i - e_j$ $e_i + e_j$	$1 \leq i < j \leq n$
E_6	$-e_i + e_j$ $e_i + e_j$ $\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5}(-1)^{\nu(i)} e_i)$	$1 \leq i < j \leq 5$ $\sum_{i=1}^{5} \nu(i)$ even
E_7	$-e_i + e_j$ $e_i + e_j$ $\frac{1}{2}(e_8 - e_7 + \sum_{i=1}^{6}(-1)^{\nu(i)} e_i)$	$1 \leq i < j \leq 6$ or $(i,j) = (7,8)$ $1 \leq i < j \leq 6$ $\sum_{i=1}^{6} \nu(i)$ odd
E_8	$-e_i + e_j$ $e_i + e_j$ $\frac{1}{2}(e_8 + \sum_{i=1}^{7}(-1)^{\nu(i)} e_i)$	$1 \leq i < j \leq 8$ $\sum_{i=1}^{7} \nu(i)$ even
F_4	e_i $e_i - e_j$ $e_i + e_j$ $\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$	$1 \leq i \leq 4$ $1 \leq i < j \leq 4$
G_2	$\alpha_1,\ \alpha_2,\ \alpha_1 + \alpha_2,\ 2\alpha_1 + \alpha_2,\ 3\alpha_1 + \alpha_2,\ 3\alpha_1 + 2\alpha_2$	

TABLE 1. Simple root systems in the convention of [6]. For G_2 we express the roots in a basis of simple roots α_1, α_2.

Case 1. $|R^+(G)| \equiv 0 \mod 2$. In this case we set $R_0^+(G) = \emptyset$ hence we only need to define an involution f of $R^+(G)$ such that for each each element $\alpha \in R^+(G)$, the pair $(\alpha, f(\alpha))$ is sum-free.

- **Case** A_n **with** $n \equiv 0, 3 \mod 4$**.** We set

$$e_{2i-1} - e_j \longleftrightarrow e_{2i} - e_j\,, \qquad 1 < 2i < j \leq n+1\,,$$
$$e_{4i-3} - e_{4i-2} \longleftrightarrow e_{4i-1} - e_{4i}\,, \qquad 1 \leq i \leq k\,,$$

where $k := [\frac{n+1}{4}]$.

- **Case** B_{2n}**.** We define

$$e_i - e_{2n} \longleftrightarrow e_i\,, \qquad 1 \le i < 2n\,,$$
$$e_i - e_j \longleftrightarrow e_i + e_j\,, \qquad 1 \le i < j < 2n\,,$$

and pair the remaining roots $e_1 + e_{2n}, \dots e_{2n-1} + e_{2n}, e_{2n}$ arbitrarily.
- **Case** C_{2n}**.** We define

$$e_i - e_{2n} \longleftrightarrow 2e_i\,, \qquad 1 \le i < 2n\,,$$
$$e_i - e_j \longleftrightarrow e_i + e_{j-1}\,, \qquad 1 \le i < j < 2n\,,$$

and pair the remaining roots $e_1 + e_2, e_2 + e_3 \dots e_{2n-1} + e_{2n}, 2e_{2n}$ arbitrarily.
- **Case** D_n**.** We set

$$e_i - e_j \longleftrightarrow e_i + e_j\,, \qquad 1 \le i < j \le n\,.$$

- **Case** E_6**.** We define the pairing

$$e_i + e_j \longleftrightarrow -e_i + e_j\,, \qquad 1 \le i < j \le 5\,,$$

and pair the remaining roots $\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5}(-1)^{\nu(i)}e_i)$ arbitrarily.
- **Case** E_8**.** We set

$$e_i + e_j \longleftrightarrow -e_i + e_j\,, \qquad 1 \le i < j \le 8\,,$$
$$\frac{1}{2}\left(e_8 + \sum_{i=1}^{7}(-1)^{\nu(i)}e_i\right) \longleftrightarrow \frac{1}{2}\left(e_8 - (-1)^{\nu(1)}e_1 - (-1)^{\nu(2)}e_2 + \sum_{i=3}^{7}(-1)^{\nu(i)}e_i\right)\,.$$

- **Case** F_4**.** We define

$$e_i - e_{2n} \longleftrightarrow e_i\,, \qquad 1 \le i < 2n\,,$$
$$e_i - e_j \longleftrightarrow e_i + e_j\,, \qquad 1 \le i < j < 2n\,,$$
$$\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \longleftrightarrow \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm (-e_4))\,,$$

and pair the remaining roots $e_1 + e_4, e_2 + e_4, e_3 + e_4, e_4$ arbitrarily.
- **Case** G_2**.** There are exactly 4 possibilities. The first is

$$\alpha_1 \longleftrightarrow 3\alpha_1 + \alpha_2\,,$$
$$\alpha_2 \longleftrightarrow 2\alpha_1 + \alpha_2\,,$$
$$\alpha_1 + \alpha_2 \longleftrightarrow 3\alpha_1 + 2\alpha_2\,.$$

The second possibility is

$$\alpha_1 \longleftrightarrow 3\alpha_1 + 2\alpha_2\,,$$
$$\alpha_2 \longleftrightarrow 2\alpha_1 + \alpha_2\,,$$
$$\alpha_1 + \alpha_2 \longleftrightarrow 3\alpha_1 + \alpha_2\,.$$

The third possibility is

$$\alpha_1 \longleftrightarrow 3\alpha_1 + \alpha_2\,,$$
$$\alpha_2 \longleftrightarrow \alpha_1 + \alpha_2\,,$$
$$2\alpha_1 + \alpha_2 \longleftrightarrow 3\alpha_1 + 2\alpha_2\,.$$

The fourth possibility is

$$\begin{aligned} \alpha_1 &\longleftrightarrow 3\alpha_1 + 2\alpha_2 \,, \\ \alpha_2 &\longleftrightarrow \alpha_1 + \alpha_2 \,, \\ 2\alpha_1 + \alpha_2 &\longleftrightarrow 3\alpha_1 + \alpha_2 \,. \end{aligned}$$

Case 2. $|R^+| \equiv 1 \mod 2$. In this case we first define the involution f on all but one positive root γ and we set $R_0^+(G) = \{\gamma\}$.

- **Case** A_n **with** $n \equiv 1, 2 \mod 4$**.** We define the pairing

$$\begin{aligned} e_{2i-1} - e_j &\longleftrightarrow e_{2i} - e_j \,, && 1 < 2i < j \leq n+1 \,, \\ e_{4i-3} - e_{4i-2} &\longleftrightarrow e_{4i-1} - e_{4i} \,, && 1 \leq i \leq k \,, \end{aligned}$$

  where $k := [\frac{n-1}{4}]$, and the remaining root is $\gamma = e_{4k+1} - e_{4k+2}$.
- **Case** B_{2n+1}**.** We define the pairing

$$\begin{aligned} e_i - e_{2n+1} &\longleftrightarrow e_i \,, && 1 \leq i < 2n+1 \,, \\ e_i - e_j &\longleftrightarrow e_i + e_j \,, && 1 \leq i < j < 2n+1 \,, \end{aligned}$$

  choose γ to be any of the roots $e_1 + e_{2n+1}, \ldots e_{2n-1} + e_{2n+1}, e_{2n+1}$ and pair the remaining ones arbitrarily.
- **Case** C_{2n+1}**.** We set

$$\begin{aligned} e_i - e_{2n+1} &\longleftrightarrow 2e_i \,, && 1 \leq i < 2n+1 \,, \\ e_i - e_j &\longleftrightarrow e_i + e_{j-1} \,, && 1 \leq i < j < 2n+1 \,, \end{aligned}$$

  choose γ to be any of the roots $e_1 + e_2, e_2 + e_3 \ldots e_{2n} + e_{2n+1}, 2e_{2n+1}$ and pair the remaining ones arbitrarily.
- **Case** E_7**.** We define the pairing

$$e_i + e_j \longleftrightarrow -e_i + e_j \,, \qquad 1 \leq i < j \leq 6 \,,$$

$$\frac{1}{2}\left(e_8 - e_7 + \sum_{i=1}^{6}(-1)^{\nu(i)} e_i\right) \longleftrightarrow \frac{1}{2}\left(e_8 - e_7 - (-1)^{\nu(1)} e_1 - (-1)^{\nu(2)} e_2 + \sum_{i=3}^{6}(-1)^{\nu(i)} e_i\right) \,,$$

  and the remaining root is $\gamma = -e_7 + e_8$. □

We can now extend Lemma 5.5 to the general case of reductive Lie groups of compact type.

Lemma 5.6. *Let G be a reductive Lie group of compact type, dimension $4n$, and rank r, and $R^+(G)$ a system of positive roots. Then there exists a decomposition*

$$R^+(G) = R_0^+(G) \coprod R_1^+(G)$$

with

$$|R_0^+(G)| = \begin{cases} 0 & \textit{if } r \equiv 0 \mod 4 \,, \\ 1 & \textit{if } r \equiv 2 \mod 4 \,, \end{cases}$$

and an involution without fixed points $f \colon R_1^+(G) \to R_1^+(G)$ such that the pair $(\alpha, f(\alpha))$ is sum-free for all $\alpha \in R_1^+(G)$.

Proof. The Lie algebra $\mathfrak{g}$ of G decomposes as

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s\,,$$

where $\mathfrak{z}(\mathfrak{g})$ is the centre of $\mathfrak{g}$ and $\mathfrak{g}_i$ is the Lie algebra of a compact simple Lie group G_i for all $i = 1, \ldots, s$. Therefore

$$R^+(G) = \coprod_{i=1}^{s} R^+(G_i)\,.$$

For all $i = 1, \ldots, s$ there exists a decomposition $R^+(G_i) = R_0^+(G_i) \coprod R_1^+(G_i)$ and an involution $f_i\colon R^+(G_i) \to R^+(G_i)$ with the properties stated in Lemma 5.5. Let $\mathcal{J}$ be the set of indices in $\{1, \ldots, s\}$ such that $|R_0^+(G_i)| = 1$. Pick any $j_0 \in \mathcal{J}$ and define

$$R_0^+(G) := \begin{cases} \emptyset, & \text{if } |\mathcal{J}| \equiv 0 \mod 2\,, \\ R_0^+(G_{j_0}), & \text{if } |\mathcal{J}| \equiv 1 \mod 2\,, \end{cases}$$

and of course

$$R_1^+(G) := R^+(G) \setminus R_0^+(G)\,.$$

We claim that $R_0^+(G)$ satisfies the requirements of the lemma. Indeed, since $\dim(G) = 4n$ we have that r is a multiple of 4 if and only if the number of roots of G is a multiple of 4. This, in turn, is equivalent to $|R^+(G)|$ being even. Finally, $|R^+(G)|$ has the same parity of $|\mathcal{J}|$ concluding the proof of the claim.

Now, let

$$\tilde{\mathcal{J}} := \begin{cases} \mathcal{J}, & \text{if } |\mathcal{J}| \equiv 0 \mod 2\,, \\ \mathcal{J} \setminus \{j_0\}, & \text{if } |\mathcal{J}| \equiv 1 \mod 2\,. \end{cases}$$

and choose any involution $\tilde{f}$ of $\tilde{\mathcal{J}}$ such that $\tilde{f}(l) \neq l$ for all $l \in \tilde{\mathcal{J}}$. Denote γ_i the root in $R_0^+(G_i)$. Finally, we define the involution $f\colon R_1^+(G) \to R_1^+(G)$ by setting $f|_{R_1^+(G_i)} = f_i$, for all $i = 1, \ldots, s$, and $f(\gamma_l) = \gamma_{\tilde{f}(l)}$, for any $l \in \tilde{\mathcal{J}}$. The fact that for every $\alpha \in R_1^+(G)$ the pair $(\alpha, f(\alpha))$ is sum-free follows from Lemma 5.5 and the fact that for $i \neq j$ the pair (γ_i, γ_j) is sum-free, since γ_i and γ_j belong to different simple summands. □

We are ready to prove the existence of semi-integrable almost hyperhermitian structures.

Theorem 5.7. *Let $\mathfrak{g}$ be a reductive Lie algebra of compact type and dimension $4n$, equipped with an $\mathrm{ad}_{\mathfrak{g}}$-invariant metric g_G. Then there exists a semi-integrable almost hyperhermitian structure (H, g_G) on $\mathfrak{g}$ if and only if $\mathfrak{g} \neq \mathfrak{su}(2) \times \mathbb{R}$.*

Proof. It is known that $\mathfrak{g} = \mathfrak{su}(2) \times \mathbb{R}$ cannot admit any semi-Kähler structure, see [22, Theorem 4.4]. Therefore we assume $\mathfrak{g} \neq \mathfrak{su}(2) \times \mathbb{R}$.

The choice of a Cartan subalgebra $\mathfrak{t}$ and a system of positive roots $R^+(G)$ on $\mathfrak{g}$ determines a (not necessarily unique) complex structure I on $\mathfrak{g}$. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ be the splitting of the complexified Lie algebra with respect to I. The elements in $\mathfrak{g}^{1,0}$ are generated by $\mathfrak{t}^{1,0}$ and the positive root spaces. Any almost hyperhermitian structure H containing I is completely determined by an almost Hermitian structure J anti-commuting with I, or, equivalently, satisfying $J\mathfrak{g}^{1,0} = \mathfrak{g}^{0,1}$.

Suppose for the moment that $\mathfrak{g} \neq \mathfrak{su}(3)$; we will treat the case of $\mathfrak{su}(3)$ separately. Let $R^+(G) = R_0^+(G) \coprod R_1^+(G)$ be a decomposition of the positive roots as in Lemma 5.6 and $f\colon R_1^+(G) \to R_1^+(G)$ an associated involution without fixed points. We define J by setting

$$JE_\alpha := E_{-f(\alpha)}\,, \qquad JE_{-\alpha} := -E_{f(\alpha)}\,, \qquad \text{for all } \alpha \in R_1^+(G)\,,$$

and

$$JE_\gamma := \bar{T}\,, \quad JT := -\bar{E}_\gamma\,, \qquad \gamma \in R_0^+(G)\,,$$

where $T \in \mathfrak{t}^{1,0}$ is such that $\gamma(T) = 0$ and $g_G(T, \bar{T}) = g_G(E_\gamma, \bar{E}_\gamma)$. Note that, since we assumed $\mathfrak{g} \neq \mathfrak{su}(3)$, the Cartan subalgebra $\mathfrak{t}$ has dimension strictly bigger than 2, and thus there always exists such a T. Finally, J is extended to the remainder of the Cartan subalgebra arbitrarily in such a way that it is compatible with the metric g_G and $J\mathfrak{t}^{1,0} = \mathfrak{t}^{0,1}$.

Evidently $J^2 = -\mathrm{Id}$ and $IJ = -JI$, so I and J define an almost hyperhermitian structure H on $(\mathfrak{g}, g)$ which extends to a left-invariant almost hyperhermitian structure on G. Being defined in such a way, it is straightforward to check that each term in the sum (9) applied to $L = J$ vanishes, thereby showing that (H, g) forms a semi-integrable almost hyperhermitian structure.

To conclude the proof we only need to show the result for $\mathfrak{g} = \mathfrak{su}(3)$. We consider the Cartan subalgebra $\mathfrak{t}$ of diagonal matrices. Since the bi-invariant metric on $\mathfrak{g}$ is unique, up to rescaling, we may assume that it is $\langle\cdot,\cdot\rangle = -B$. Set $e_i \in \mathfrak{t}^*$ to be the linear form

$$e_i \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} := t_i\,,$$

for $i = 1, 2, 3$. A system of positive roots for $\mathfrak{g}$ with respect to $\mathfrak{t}$ is then given by

$$\alpha_1 = e_1 - e_2, \qquad \alpha_2 = e_1 - e_3, \qquad \alpha_3 = e_2 - e_3\,.$$

Let $E_{ij} \in \mathfrak{g} \otimes \mathbb{C} = \mathfrak{sl}(3, \mathbb{C})$ be the matrix with 1 in the (i, j) entry and 0 elsewhere. It is straightforward to check that $E_{ij} \in \mathfrak{g}_{e_i - e_j}$. It is well known that the Cartan-Killing form of $\mathfrak{g}$ and $\mathfrak{g} \otimes \mathbb{C}$ is $B(X, Y) = 6\mathrm{tr}(XY)$. To see this, simply observe that $\mathrm{tr}(XY)$ is bi-invariant and so it must be a constant multiple of B, by Schur's Lemma. The right constant can now be found by evaluation on two arbitrary elements X, Y. We therefore compute $B(E_{ij}, E_{ji}) = 6$. Now, the normalized matrices

$$E_{\alpha_1} = \frac{1}{\sqrt{6}} E_{12}, \qquad E_{\alpha_2} = \frac{1}{\sqrt{6}} E_{13}, \qquad E_{\alpha_3} = \frac{1}{\sqrt{6}} E_{23}$$

form a Cartan-Weyl basis of $\mathfrak{g} \otimes \mathbb{C}$ and satisfy the commutation relations

$$[E_{\alpha_1}, E_{\alpha_2}] = 0, \qquad [E_{\alpha_1}, E_{\alpha_3}] = \frac{1}{\sqrt{6}} E_{\alpha_2}, \qquad [E_{\alpha_2}, E_{\alpha_3}] = 0\,.$$

Let $T \in \mathfrak{t}^{1,0}$ be an element such that $\langle T, \bar{T}\rangle = 1$, where we extended $\langle\cdot,\cdot\rangle$ to $\mathfrak{g} \otimes \mathbb{C}$ bilinearly. Set

$$JT = a\bar{T} + \sum_i b_i \bar{E}_{\alpha_i}, \qquad JE_{\alpha_j} = c_j \bar{T} + \sum_i d_{ji} \bar{E}_{\alpha_i}\,, \qquad a, b_i, c_j, d_{ji} \in \mathbb{C}\,,$$

for the generic endomorphism J that anticommutes with I. We observe that

$$E_{\alpha_1} = \frac{1}{2\sqrt{6}}\left(E_{12} - E_{21}\right) - \frac{\sqrt{-1}}{2\sqrt{6}}\left(\sqrt{-1}E_{12} + \sqrt{-1}E_{21}\right),$$

where each of the summands between braces on the right-hand side lies in $\mathfrak{su}(3)$. Thus

$$\begin{aligned}\langle E_{\alpha_1}, \bar{E}_{\alpha_1}\rangle &= \frac{1}{24}\langle E_{12} - E_{21}, E_{12} - E_{21}\rangle + \frac{1}{24}\langle \sqrt{-1}E_{12} + \sqrt{-1}E_{21}, \sqrt{-1}E_{12} + \sqrt{-1}E_{21}\rangle \\ &= -\frac{1}{4}\operatorname{tr}\left((E_{12} - E_{21})^2\right) - \frac{1}{4}\operatorname{tr}\left((\sqrt{-1}E_{12} + \sqrt{-1}E_{21})^2\right) = \frac{1}{2} + \frac{1}{2} = 1\,.\end{aligned}$$

Similarly, one sees that $\langle E_\alpha, \bar{E}_\alpha\rangle = 1$. Therefore, in order for the metric to be compatible with J we must have

$$\begin{aligned} b_i &= \langle E_{\alpha_i}, JT\rangle = -\langle JE_{\alpha_i}, T\rangle = c_i, \\ d_{ji} &= \langle E_{\alpha_i}, JE_{\alpha_j}\rangle = -\langle E_{\alpha_j}, JE_{\alpha_i}\rangle = -d_{ij}, \\ 1 &= \langle T, \bar{T}\rangle = \langle JT, J\bar{T}\rangle = |a|^2 + \sum_i |b_i|^2, \\ 1 &= \langle E_{\alpha_j}, \bar{E}_{\alpha_j}\rangle = \langle JE_{\alpha_j}, J\bar{E}_{\alpha_j}\rangle = |c_j|^2 + \sum_i |d_{ij}|^2\,. \end{aligned}$$

Using these identities, the condition $J^2 = -\mathrm{Id}$ is then easily seen to be equivalent to

$$a = 0, \qquad \sum_l b_l \bar{d}_{kl} = 0, \qquad -b_j\bar{b}_i + \sum_l d_{lj}\bar{d}_{il} = -\delta_{ij}, \qquad i, j, k = 1, 2, 3\,.$$

Finally to impose that J is semi-Kähler, by (9), we must have

$$0 = \sum_i [E_{\alpha_i}, J\bar{E}_{\alpha_i}] + [T, J\bar{T}] = 2\sum_i \alpha_i(T)\bar{b}_i E_{\alpha_i} - \frac{2}{\sqrt{6}}\bar{d}_{13}E_{\alpha_2}\,,$$

which holds if and only if

$$b_1 = 0, \qquad b_3 = 0, \qquad \alpha_2(T)\bar{b}_2 - \frac{1}{\sqrt{6}}\bar{d}_{13} = 0\,.$$

Summing up, the almost complex structures I, J, IJ form a basis of an almost hypercomplex structure H that, together with $\langle\cdot,\cdot\rangle$, is semi-integrable if and only if

$$JT = b_2\bar{E}_{\alpha_2}, \qquad JE_{\alpha_1} = -d_{13}\bar{E}_{\alpha_3}, \qquad JE_{\alpha_2} = -b_2\bar{T}, \qquad JE_{\alpha_3} = d_{13}\bar{E}_{\alpha_1},$$

$$|b_2|^2 = 1, \qquad |d_{13}|^2 = 1, \qquad \alpha_2(T)\bar{b}_2 - \frac{1}{\sqrt{6}}\bar{d}_{13} = 0\,.$$

Note that these equations can be solved if and only if

$$|\alpha_2(T)| = \frac{1}{\sqrt{6}}\,. \tag{11}$$

To show that (11) is satisfied we write

$$T = \begin{pmatrix} s & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & -s-t \end{pmatrix}$$

with $s, t \in \mathbb{C}$. Since $T \in \mathfrak{t}^{1,0}$, we have

$$0 = \langle T, T\rangle = 6(s^2 + t^2 + (s+t)^2) = 12(s^2 + t^2 + st)$$

so s/t must be a primitive third root of unity ε. On the other hand,

$$1 = \langle T, \bar{T} \rangle = 6(|s|^2 + |t|^2 + |s+t|^2) = 6(|\varepsilon t|^2 + |t|^2 + |\varepsilon t + t|^2)) = 18|t|^2 \,,$$

where we also used that

$$\varepsilon + \bar{\varepsilon} = \varepsilon + \frac{1}{\varepsilon} = \frac{\varepsilon^2 + 1}{\varepsilon} = -1 \,.$$

From this, we conclude

$$|\alpha_2(T)|^2 = |2s + t|^2 = |2\varepsilon t + t|^2 = 3|t|^2 = \frac{1}{6} \,,$$

which shows that (11) is satisfied and thus $\mathfrak{g}$ admits a semi-integrable hyperhermitian structure, concluding the proof. □

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