On the holonomy of Weyl connections

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Aims of the talk :

- Weyl connections carrying parallel forms
- On the irreducibility of closed Weyl connections

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Part I

Weyl connections on conformal manifolds



Hermann Weyl (1885 - 1955)

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Weight bundles

Definition

Let M^n be a manifold and let P denote the principal frame bundle. The weight bundle of M is the real line bundle L associated to Pvia the representation $|\det|^{1/n}$ of $\operatorname{GL}(n,\mathbb{R})$.

More generally : $\forall k \in \mathbb{R} \quad \rightsquigarrow \quad L^k := P \times_{|\det|^{k/n}} \mathbb{R}$. Properties :

• L^k is orientable \rightsquigarrow trivial; • $L^k \otimes L^{k'} \cong L^{k+k'}$: • $L^{-n} \cong |\Lambda^n(T^*M)| = \delta M$, the bundle of densities on M. Positive k-weights \rightsquigarrow sections of $P \times_{|\det|^{k/n}} \mathbb{R}^+ \subset L^k$. Weighted tensors \rightsquigarrow sections of $TM^{\otimes a} \otimes T^*M^{\otimes b} \otimes L^k$ for some

 $a, b \in \mathbb{N}$ and $k \in \mathbb{R}$. Conformal weight = a - b + k.

Weyl connections on conformal manifolds Weyl connections with parallel forms On the irreducibility of closed Weyl connections

Conformal structures

Definition

A conformal structure (usually denoted by c) on M:

- An equivalence class of Riemannian metrics for the equivalence relation $g \simeq \tilde{g} \iff \exists f \in C^{\infty}(M), \ \tilde{g} = e^{2f}g$;
- a symmetric positive definite bilinear form on $TM \otimes L^{-1}$:
- a symmetric positive definite L^2 -valued bilinear form on TM;
- a reduction $P(CO_n)$ of P to the conformal group $\mathrm{CO}_n \cong \mathbb{R}^+ \times \mathrm{O}_n \subset \mathrm{GL}(n, \mathbb{R}).$

Once a conformal structure c is fixed, one-to-one correspondence between positive sections I of L and Riemannian metrics on M:

$$c(X, Y) = g(X, Y)l^2, \quad \forall X, Y \in TM.$$

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Weyl connections

Definition

A Weyl connection on a conformal manifold is a torsion-free connection on $P(CO_n)$.

Fundamental theorem of Weyl geometry

1-1 correspondence between Weyl connections and covariant derivatives on L.

Proof. Every connection on P induces a covariant derivative D on TM and on L. It is a Weyl structure if and only if D satisfies $D_X Y - D_Y X = [X, Y]$ and $D_X c = 0$ for all vector fields X, Y on M. Like in the Riemannian situation, these two relations are equivalent to the Koszul formula

$$2c(D_XY,Z) = D_X(c(Y,Z)) + D_Y(c(X,Z)) - D_Z(c(X,Y)) + c([X,Y],Z) + c([Z,X],Y) + c([Z,Y],X).$$

The Lee form

If D is a Weyl connection on (M, c), $g \in c$ and I is the section of L satisfying $c = g \otimes l^2$, we define the Lee form θ of D with respect to g by

$$D_X I = \theta(X)I, \quad \forall X \in TM,$$

or equivalently

$$D_Xg = -2\theta(X)g, \quad \forall X \in TM.$$

If $\tilde{g} = e^{2f}g$, $\tilde{\theta} = \theta - df$.

Weyl connections on conformal manifolds Weyl connections with parallel forms On the irreducibility of closed Weyl connections

Closed and exact Weyl connections

Definition

A Weyl structure D is called closed (exact) if one of the equivalent conditions hold :

- The Lee form wrt to an arbitrary metric in c is closed (exact).
- L carries a local (global) D-parallel section.
- D is locally (globally) the Levi-Civita connection of a Riemannian metric $g \in c$.

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The metric cone

Fundamental example

Let (N, g^N) be a Riemannian manifold and let

$$(M_0,g_0):=(\mathbb{R}^*_+\times N,dt^2+t^2g^N)$$

be the metric cone over N (note that g_0 and the product metric g on $M_0 \simeq \mathbb{R} \times N$ are conformally related by setting $t = e^s, t \in \mathbb{R}^*_+, s \in \mathbb{R}$). The multiplication by some $\lambda > 1$ on the \mathbb{R} -factor is a strict homothety of g_0 and an isometry of g. It generates a group Γ acting freely and properly discontinuously on M_0 . The metric g projects to the product metric, also denoted by g, on the quotient manifold $M := M_0/\Gamma \simeq S^1 \times N$. The Levi-Civita connection D_0 of g_0 is Γ -invariant, inducing therefore a closed, non-exact Weyl structure D on (M, [g]). Since $g_0 = e^{2s}g$, the Lee form of D wrt g is ds (on M_0 as well as on M).

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Part II

Weyl connections with parallel forms

Conformal submersions

Definition

 (M^m, c) and (N^n, c') conformal manifolds, $f: M \to N$ is called a conformal submersion if

$$df|_{(\ker df)^{\perp}}: (\ker df)^{\perp} \to T_{f(x)}N$$

is a conformal isomorphism for every $x \in M$.

Lemma

Let (M, c) be a conformal manifold and let $p: M \to N$ be a submersion onto a manifold N.

- The pull-back of the weight bundle of N is canonically isomorphic to the weight bundle of M.
- If the horizontal distribution $H := (\ker df)^{\perp}$ is parallel with respect to some Weyl structure D, \exists a unique conformal structure on N turning p into a conformal submersion.

Conformal products

Remark

If (M_1, c_1) and (M_2, c_2) are two conformal manifolds \rightsquigarrow no natural conformal structure on $M = M_1 \times M_2$. Algebraic reason : the group $CO(n_1) \times CO(n_2) \subset GL(n_1 + n_2, \mathbb{R})$ is not included in $CO(n_1 + n_2).$

Definition

A conformal structure on M is called a conformal product of (M_1, c_1) and (M_2, c_2) iff the canonical submersions $p_1: M \to M_1$ and $p_2: M \to M_2$ are orthogonal conformal submersions.

Weyl connections on conformal manifolds Weyl connections with parallel forms On the irreducibility of closed Weyl connections

Conformal products

Proposition

One-to-one correspondence between the set of conformal product structures on $M := M_1 \times M_2$ and the set of pairs of bundle homomorphisms $P_1: L \to L_1$ and $P_2: L \to L_2$, whose restrictions to each fiber are isomorphisms, such that the following diagram is commutative :



The adapted Weyl connection

Theorem

A conformal manifold has a (local) conformal product structure iff it carries a Weyl structure with reducible holonomy.

Proof. Let $TM = H_1 \oplus H_2$ be a *D*-parallel splitting of the tangent bundle. D is torsion-free \Rightarrow H₁ and H₂ integrable \Rightarrow two orthogonal foliations \Rightarrow locally, $M = M_1 \times M_2 \Rightarrow$ (previous lemma) there exist conformal structures c_1 , c_2 on M_1 , M_2 , such that the canonical projections are orthogonal conformal submersions \Rightarrow local conformal product structure.

The adapted Weyl connection

Conversely, assume that $M = M_1 \times M_2$ has a conformal product structure. It is enough to define the connection D on the weight bundle L. Using the previous diagram



define the horizontal space H_l of D at $l \in L$ as the direct sum of ker dP_1 and ker dP_2 . In terms of covariant derivatives, this definition amounts to say that the pull-back of a section of L_1 (resp. L_2) is D-parallel in the direction of H_2 (resp. H_1), where $H_1 := \ker dp_2$ and $H_2 := \ker dp_1$. D is called the adapted Weyl structure of the conformal product (M, c).

Closed conformal products

Let c be a conformal product structure on $M = M_1 \times M_2$ with adapted Weyl structure D. Then each slice $M_1 \times \{y\} \simeq M_1$ carries a Weyl structure D^{y} such that $p_{1}^{*}(D_{X}^{y}T) = D_{X}(p_{1}^{*}T)$ for all vectors X and tensor fields T on M_1 .

Definition

A conformal product (M, c) is called a closed conformal product if the adapted Weyl structure is closed.

Equivalently :

- all connections D^{y} coincide.
- the previous diagram can be completed by bundle homomorphisms $Q_i: L_i \to L_0$, isomorphic on each fiber (where L_0 is the weight bundle of the point manifold •), such that the resulting diagram is commutative :

Weyl connections on conformal manifolds Weyl connections with parallel forms On the irreducibility of closed Weyl connections

Closed conformal products



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Weyl connections on conformal manifolds Weyl connections with parallel forms On the irreducibility of closed Weyl connections

Parallel forms

Problem

Characterize Weyl manifolds (M, c, D) carrying (weighted) parallel forms.

Remarks

- If D is exact \rightsquigarrow Riemannian situation.
- One may restrict to weightless forms (i.e. in $\Lambda_0^k M := \Lambda^k M \otimes L^k).$
- There exists at most one Weyl structure with respect to which a given form is parallel. More precisely, to each nowhere vanishing section of $\Lambda_0^k M$ one can associate a minimal Weyl connection D^{ω} , which is the unique candidate for $D\omega = 0$.

Examples

Examples of Weyl-parallel forms

- Closed Weyl structures (i.e. locally Riemannian). Parallel form \iff fix point of the holonomy representation on $\Lambda^k M$.
 - powers of the Kähler form on Kähler manifolds,
 - the complex volume form on Calabi-Yau manifolds,
 - special 3- and 4-forms on G_2 and Spin₇-manifolds,
 - tensor products of the above on Riemannian products,...
- The weightless volume forms of the factors on a conformal product.
- If c is any compatible conformal structure on a complex surface (M^4, J) , the weightless 2-form defined by J is parallel wrt the minimal Weyl connection D^{J} .

The classification

Other examples?

Theorem [Belgun - M., 2009]	
No !	

Sketch of the proof :

- If D is irreducible, Schwachhöfer-Merkulov's classification of possible holonomies of irreducible torsion-free connections \rightarrow either n = 4, or D has full holonomy $CO^+(n)$, or D is closed.
- If D has reducible holonomy \Rightarrow conformal product. The point is to show that the parallel form is necessarily one of the volume forms of the factors.

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The classification

Parallel decomposition

 $\Lambda_0^k M \simeq \bigoplus_{k_1+k_2-k} p_1^*(\Lambda_0^{k_1} M_1) \otimes p_2^*(\Lambda_0^{k_2} M_2)$. Correspondingly, $\omega = \omega_0 + \cdots + \omega_k$. The lack of symmetry by pairs of the curvature R^D forces the components ω_i to vanish for $1 \le i \le k-1$ (provided that D is not closed).

• It remains to show that $k = \dim(M_1)$ or $k = \dim(M_2)$. Assume $\omega_k \neq 0$. Since ω_k is *D*-parallel, it is the pull-back of some τ on M_1 . For every $y \in M_2$, τ is D^y -parallel \Rightarrow all D^y coincide $\Rightarrow D$ is closed (unless τ has maximal degree)...

More details in : Weyl-parallel Forms and conformal Products (with F. Belgun), arXiv :0901.3647.

Part III

On the irreducibility of closed Weyl connections

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Theorem A [Gallot, 1979]

A metric cone over a compact Riemannian manifold (N, h) is either irreducible or flat.

Equivalently : The Weyl structure D_0 on $N \times S^1$ whose Lee form wrt the product metric $h + ds^2$ is ds. is either irreducible or flat.

Conjecture

Is every closed, non-exact, Weyl structure on a compact conformal manifold irreducible or flat?

Theorem B [Belgun - M., 2009]

Yes, provided it is tame.

Definition

A connection is called tame if the life-time of incomplete geodesics is bounded on compact subsets of $TM \setminus \{0\}$.

The minimal Riemannian cover

Basic idea : Closed Weyl connections on compact conformal manifolds \iff co-compact groups of homotheties on Riemannian manifolds.

More precisely : A closed Weyl connection D on a compact conformal manifold (M, c) induces an exact Weyl connection \tilde{D} on $\tilde{M} \rightsquigarrow \tilde{D}$ is the LC connection of \tilde{g} (defined up to a constant).

Let \mathcal{I} denote the subgroup of $\pi_1(M)$ preserving \tilde{g} . Then \tilde{g} projects to a metric g_0 on $M_0 := M/\mathcal{I}$. The Abelian group $\Gamma := \pi_1(M)/\mathcal{I}$ acts by proper homotheties on (M_0, g_0) and $M = M_0/\Gamma$.

Definition

 (M_0, g_0) is the minimal Riemannian cover of (M, c, D).

The minimal Riemannian cover

Theorem [Belgun - M., 2009]

If M is compact and d denotes the geodesic distance induced on M_0 by g_0 , then the metric completion of (M_0, d) has exactly one extra-point ω , called the *singularity* of M_0 .

Remark

If the homothety constant $\rho(f)$ of some $f \in \Gamma$ is < 1 then $\{f^n(x)\}$ is a non-convergent Cauchy sequence of (M_0, d) .

Key argument : For every $x \in M_0$ there exists $K_x \in \mathbb{R}$ such that $d(x, f(x)) < K_x \ \forall f \text{ with } \rho(f) \leq 1.$

Remark

A closed Weyl structure D on (M, c) is tame iff there exists a constant k such that for every $x \in M_0$, the life-time of every incomplete geodesic through x is less than $k d(x, \omega)$.

Proof of the main result

Proof of Theorem B. Step 1. We need to show that a reducible closed, tame Weyl structure D on a compact conformal manifold (M, c) is flat. Equivalently, the minimal Riemannian cover (M_0, g_0) is irreducible or flat. Assume that $TM_0 = V_1 \oplus V_2$ is a parallel decomposition. (M_0, g_0) carries a co-compact group Γ of strict homotheties. Every $f \in \Gamma$ preserves the sets of integral leaves of V_i .

If the integral leafs of V_i were all complete and mutually isometric (which is obviously not the case!), one could conclude by the following :

Lemma [Kobayashi - Nomizu]

A complete Riemannian manifold carrying a strict homothety f is flat.

$$\begin{array}{l} \mathsf{Proof}: f^*g = \rho^2 g \Rightarrow |R_{f(x)}| = \rho^{-2} |R_x|, \text{ so } \forall x, \ f^n(x) \to x_0, \\ |R_x| = \rho^{2n} |R_{f^n(x)}| \to 0. \end{array}$$

Proof of the main result

Do complete maximal leaves of V_i exist?

Step 2. If M_1 is an incomplete leaf of V_1 , then every maximal leaf of V_2 which intersects M_1 is complete.

Step 3. If M_1 is a maximal leaf of V_1 which is incomplete, then all maximal leaves of V_2 which intersect M_1 are isometric. (consequence of the local de Rham theorem, using the fact that geodesics starting on M_1 parallel to V_2 are complete).

Step 4. It remains to show that there exists $f \in \Gamma$ and a maximal leaf M_2 of V_2 which intersects M_1 , such that $f(M_2)$ also intersects M_1 .

Proof of the main result

Lemma

Let $\gamma : (0, a] \rightarrow M_0$ be an incomplete g_0 -geodesic parametrized by arc-length, such that $\lim_{t\to 0} d(\gamma(t), \omega) = 0$. There exist positive real numbers ρ , $q \in (0, 1)$ such that the open balls $B_n := B_{\gamma(q^n)}(\rho q^n), n \in \mathbb{N}$, are pairwise disjoint.

Choose a metric on M. The projection to M of B_n has volume bounded by below. There exists $f \in \Gamma$ and $y, z \in B := \bigcup_{n \geq 1} B_n$ such that y = f(z). Moreover, one can choose ρ small enough (in terms of the injectivity radius of a compact fundamental domain of Γ) so that the maximal leaf of V_2 through any point of B intersects M_{1} .

Examples of tame connections

Definition

A Weyl structure D on a conformal manifold (M, c) is called analytically tame if there exists a complete Riemannian metric $g \in c$ and a positive real number $\varepsilon > 0$ such that

$$|\theta|^2 g(X,X) + (
abla_X heta)(X) \geq 2 \varepsilon g(X,X), \qquad orall \ X \in TM,$$

where ∇ denotes the Levi-Civita covariant derivative of g and θ denotes the Lee form of D with respect to g.

Example

The standard Weyl structure D_0 on a metric cone is analytically tame. Indeed, the Lee form of D_0 with respect to the complete metric g is $\theta_0 := ds$ and it is parallel for $\nabla = \nabla^g$. Every C^1 -close Weyl structure is thus analytically tame as well.

Examples of tame connections

Theorem C [Belgun - M., 2009]

Any analytically tame Weyl connection is tame.

Sketch of proof. Consider along each geodesic $\gamma : (a, b) \rightarrow M$ the functions

- $F(t) := g(\dot{\gamma}(t), \dot{\gamma}(t))^{-\frac{1}{2}}$ ("distance to the singularity")
- $H(t) := \theta(\dot{\gamma}(t))$ ("slope wrt radial direction")

Lemma

If $b < \infty$ then $\lim_{t \to b} F(t) = 0$.

The analytically tame condition reads :

•
$$F'(t) = F(t)H(t)$$

• $H'(t) \ge 2\varepsilon F(t)^{-2} - 2H(t)^2$

Examples of tame connections

Key Lemma

F has at most one critical point. Such a critical point exists iff γ is complete.

Technical Lemma

If F has no critical point, $H(t) \leq -\frac{\sqrt{\varepsilon}}{F(t)}$, $\forall t \in I$ (assuming that H < 0).

Thus $F'(t) \leq -\sqrt{\varepsilon}$, so the life-time of γ is bounded on compact subsets of $TM \setminus \{0\}$.

More details in : Holonomy of tame Weyl Structures (with F. Belgun), arXiv :0907.3182.