

CONFORMAL VECTOR FIELDS ON LCK MANIFOLDS

ANDREI MOROIANU, MIHAELA PILCA

ABSTRACT. We show that any conformal vector field on a compact lcK manifold is Killing with respect to the Gauduchon metric. Furthermore, we prove that any conformal vector field on a compact lcK manifold whose Kähler cover is neither flat, nor hyperkähler, is holomorphic.

1. INTRODUCTION

It is well known that on a compact Kähler manifold every conformal vector field is Killing [7, §90], and every Killing vector field is holomorphic. The aim of this paper is to extend these two results to compact locally conformally Kähler (lcK) manifolds.

Recall that a (compact) lcK manifold [14] is a compact complex manifold (M, J) together with a conformal class c of Riemannian metrics such that in the neighbourhood of each point of M there exists a Kähler metric in c compatible with J . Equivalently, (M, J, c) is lcK if the universal cover \widetilde{M} of M carries a Kähler metric g_K in the induced conformal class \tilde{c} compatible with the induced complex structure \tilde{J} . The simply connected Kähler manifold $(\widetilde{M}, \tilde{J}, g_K)$ will henceforth be referred to as the Kähler cover of (M, J, c) .

The interest of this notion is that many complex manifolds which for topological reasons do not carry Kähler metrics (like most complex surfaces with odd first Betti number [1], Hopf manifolds $S^1 \times S^{2n-1}$, some OT manifolds [11], etc.) have lcK structures instead.

Every compact lcK manifold (M, J, c) carries a distinguished metric $g_0 \in c$, uniquely defined up to a positive constant, called the Gauduchon metric [4]. Given a conformal vector field ξ on (M, c) , one cannot reasonably hope that it preserves any metric in the conformal class, simply because if $g \in c$ is preserved by ξ , then for any smooth function f non-constant along the flow of ξ , the conformally equivalent metric $\tilde{g} := e^{2f}g$ is no longer preserved by ξ . What one can hope, however, is to show that ξ preserves the Gauduchon metric g_0 . Note that if ξ were also holomorphic, this would be almost tautological. Indeed, since g_0 is defined up to a constant by c and J , the flow of ξ would be homothetic with respect to g_0 , and on a compact Riemannian manifold every homothetic vector field is Killing.

Our first result (Theorem 5.1 below) says that this is indeed the case: *every conformal vector field on a compact lcK manifold preserves the Gauduchon metric*. This result was conjectured and proved under some more restrictive assumptions in [9].

We then move to the next natural question: is every conformal vector field on a compact lcK manifold holomorphic? It turns out that in this generality the answer is negative. Indeed, one can easily construct lcK metrics with non-holomorphic Killing vector fields on Hopf manifolds $S^1 \times S^{2n-1}$ and on products of S^1 with 3-Sasakian manifolds (see [9, Remark 2.4 (ii)]).

However, these are basically the only possible counterexamples: our second main result (Theorem 6.2 below) states that *if (M, J, c) is not a Hopf manifold or locally conformally hyperkähler* (that is, if the Kähler metric on the universal cover is not hyperkähler or flat), then *every conformal vector field is holomorphic*.

Unlike the analogous result on Kähler manifolds, which is a simple consequence of Cartan's formula (see *e.g.* [8, Prop. 15.5]), this extension to lcK geometry is highly non-trivial, and is based on a recent result by M. Kourganoff [6, Theorem 1.5.] which describes compact lcK manifolds whose Kähler cover is reducible and non-flat.

Let us now explain in more detail the strategy of the proofs. We start by showing (in Prop. 3.1) that on a Kähler manifold (not necessarily compact), the divergence of any conformal vector field is harmonic. Note that in the compact case, this already implies Lichnerowicz' result mentioned above. We then consider a conformal vector field ξ on a compact lcK manifold (M, J, c) and apply this result to the lift $\tilde{\xi}$ of ξ to the Kähler cover $(\tilde{M}, \tilde{J}, g_K)$ of (M, J, c) . Using the theory of Weyl structures and the existence of Gauduchon metrics, we show in Prop. 4.1 that $\tilde{\xi}$ has constant divergence on \tilde{M} with respect to g_K . We then interpret this condition in terms of the Gauduchon metric on M and conclude by an integration argument, using the compactness of M .

The proof of Theorem 6.2 goes roughly as follows. If ξ is a conformal vector field on (M, J, c) , then its lift $\tilde{\xi}$ is not only conformal, but even homothetic on the Kähler cover $(\tilde{M}, \tilde{J}, g_K)$, thanks to Theorem 5.1. In particular $\tilde{\xi}$ is affine, *i.e.* preserves the Levi-Civita connection of g_K . An easy argument [9, Lemma 2.1] shows that if g_K is irreducible and not hyperkähler, then $\tilde{\xi}$ is holomorphic.

In the case where g_K is non-flat but has reducible holonomy, we make use of a deep result by M. Kourganoff, stating that (\tilde{M}, g_K) is a Riemannian product with a non-trivial flat factor \mathbb{R}^q . Using the fact that $\pi_1(M)$ acts on \tilde{M} cocompactly and properly discontinuously by similarities of the metric g_K , preserving the homothetic vector field $\tilde{\xi}$, we then show in Proposition 6.1 that the component of $\tilde{\xi}$ on \mathbb{R}^q vanishes. This is the core of the argument and uses the explicit form of homothetic vector fields on flat spaces.

The end of the proof uses a result by K.P. Tod [13, Prop. 2.2] involving Einstein-Weyl structures, and the irreducibility of non-flat cone metrics over complete manifolds proved by S. Gallot [3, Prop. 3.1].

Acknowledgments. This work was supported by the Procope Project No. 57445459 (Germany) / 42513ZJ (France).

2. PRELIMINARIES

In this preliminary section we briefly recall the main definitions and collect a few known basic results that will be needed throughout the paper.

Let M be a smooth n -dimensional manifold. For every real number r , the weight bundle L^r is the real line bundle associated to the frame bundle of M with respect to the representation $|\det|^{\frac{r}{n}}$. Two Riemannian metrics g, \tilde{g} on M are said to be conformally equivalent if there

exists a function f such that $\tilde{g} = e^{2f}g$. A conformal structure on M is an equivalence class of Riemannian metrics with respect to this equivalence relation.

If c is a conformal structure and $g \in c$ is a Riemannian metric, its volume element vol_g is a section of L^{-n} . The volume element of a conformally equivalent metric $\tilde{g} = e^{2f}g$ is $\text{vol}_{\tilde{g}} = e^{nf}\text{vol}_g$, thus showing that $(\text{vol}_g)^{-\frac{2}{n}} \otimes g$ is a section of $L^2 \otimes \text{Sym}^2(\text{T}^*M)$ which does not depend on the choice of g . We will sometimes identify c with this section.

Let (M, c) be an n -dimensional conformal manifold. A vector field ξ on M is called conformal if its flow preserves the conformal class c , *i.e.* for any metric $g \in c$, its Lie derivative with respect to ξ is proportional to g : $\mathcal{L}_\xi g = \lambda g$, for some function $\lambda \in \mathcal{C}^\infty(M)$.

We recall that on a given Riemannian manifold (M, g) , the divergence of a vector field is the trace of the endomorphism $\nabla^g X$ of the tangent bundle: $\text{div}^g X := \text{tr}(\nabla^g X)$, where ∇^g is the Levi-Civita connection of g . The divergence of a vector field is equal to the opposite of the codifferential of its dual 1-form $X^b := g(X, \cdot)$, *i.e.* $\text{div}^g X = -\delta^g(X^b)$, where the codifferential δ^g is the formal adjoint of the exterior differential d and is expressed in terms of a local g -orthonormal basis $\{e_i\}_i$ as follows: $\delta^g \alpha = -\sum_{i=1}^n e_i \lrcorner \nabla_{e_i}^g \alpha$, for all forms α on M .

In the sequel we will drop the metric g in the notation each time the metric is clear from the given context.

Taking traces in the defining equality of a conformal vector field, $\mathcal{L}_\xi g = \lambda g$, shows that necessarily $\lambda = -\frac{2}{n}\delta^g \xi^b$, for any metric $g \in c$. Thus, if ξ is a conformal vector field on (M, c) , then

$$(1) \quad \mathcal{L}_\xi g = -\frac{2}{n}(\delta^g \xi^b)g, \quad \forall g \in c.$$

In particular, a conformal vector field ξ on (M, c) is Killing with respect to some metric $g \in c$ if and only if $\delta^g \xi^b = 0$.

The condition that a vector field ξ is conformal is also equivalent to the fact that the covariant derivative of ξ^b with respect to any metric $g \in c$ has no trace-free symmetric component, *i.e.*:

$$\nabla_X^g \xi^b = \frac{1}{2}X \lrcorner d\xi^b - \frac{1}{n}(\delta^g \xi^b)X^b, \quad \forall X \in \text{TM}.$$

Definition 2.1. A *Weyl structure* on a conformal manifold (M, c) is a torsion-free linear connection D which preserves the conformal class c . If D has reducible holonomy, then (M, c, D) is called *Weyl-reducible*.

The condition that D preserves the conformal class c means that for each metric $g \in c$, there exists a unique 1-form $\theta^g \in \Omega^1(M)$, called the *Lee form* of D with respect to g , such that

$$(2) \quad Dg = -2\theta^g \otimes g.$$

The Weyl connection D is then related to ∇^g by

$$(3) \quad D_X = \nabla_X^g + \theta^g(X)\text{Id} + \theta^g \wedge X, \quad \forall X \in \text{TM},$$

where $\theta^g \wedge X$ is the skew-symmetric endomorphism of TM defined by

$$(\theta^g \wedge X)(Y) := \theta^g(Y)X - g(X, Y)(\theta^g)^\sharp.$$

A Weyl connection D is called *closed* if it is locally the Levi-Civita connection of a (local) metric in c and is called *exact* if it is the Levi-Civita connection of a globally defined metric in c . Equivalently, D is closed (resp. exact) if its Lee form with respect to one (and hence to any) metric in c is closed (resp. exact). Note that in the particular case when the Weyl structure is exact, $D = \nabla^{\tilde{g}}$ with $\tilde{g} = e^{2f}g$, the Lee form θ^g of D with respect to g is given by $\theta^g = df$. This immediately follows from (2), since $Dg = \nabla^{\tilde{g}}(e^{-2f}\tilde{g}) = -2df \otimes (e^{-2f}\tilde{g}) = -2df \otimes g$.

If the manifold M is compact of dimension greater than 2, then for every Weyl connection D on (M, c) there exists a unique (up to homothety) metric $g_0 \in c$, called the *Gauduchon metric* of D , such that its associated Lee form θ_0 is co-closed with respect to g_0 , cf. [4].

The natural extension of (3) to the bundle of exterior k -forms reads:

$$(4) \quad D_X \alpha = \nabla_X^g \alpha - k\theta^g(X)\alpha + X \wedge (\theta^g)^\sharp \lrcorner \alpha - \theta^g \wedge (X \lrcorner \alpha), \quad \forall X \in TM, \forall \alpha \in \Omega^k(M).$$

The codifferential $\delta^D : \Omega^k(M) \rightarrow L^{-2} \otimes \Omega^{k-1}(M)$ associated to a Weyl structure D on (M, c) is defined as follows:

$$\delta^D \alpha = -\text{tr}_c(D\alpha),$$

where tr_c denotes the conformal trace with respect to c . More precisely, if $c = \ell^2 \otimes g$, then δ^D is related to δ^g by the following formula

$$(5) \quad \delta^D \alpha = \ell^{-2}(\delta^g \alpha - (n - 2k)\theta^\sharp \lrcorner \alpha),$$

which directly follows by applying (4) to any k -form α and a local g -orthonormal basis $\{e_i\}_i$:

$$\begin{aligned} -\sum_{i=1}^n e_i \lrcorner D_{e_i} \alpha &= -\sum_{i=1}^n e_i \lrcorner \nabla_{e_i}^g \alpha + k\theta^\sharp \lrcorner \alpha - n\theta^\sharp \lrcorner \alpha + (k-1)\theta^\sharp \lrcorner \alpha + \theta^\sharp \lrcorner \alpha \\ &= \delta^g \alpha - (n-2k)\theta^\sharp \lrcorner \alpha. \end{aligned}$$

An exterior form α satisfying $\delta^D \alpha = 0$ is called *D -coclosed*. According to (5), α is D -coclosed if and only if for any metric $g \in c$, the codifferential of α verifies $\delta^g \alpha - (n-2k)\theta^\sharp \lrcorner \alpha = 0$, where θ is the Lee form of D with respect to g .

The Weyl Laplacian $\Delta^D : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(L^{-2})$ is defined by

$$\Delta^D \varphi := \delta^D d\varphi = -\text{tr}_c(Dd\varphi), \quad \forall \varphi \in \mathcal{C}^\infty(M).$$

For every metric $g \in c$ written as $g = \ell^{-2} \otimes c$, (5) applied to the 1-form $\alpha = d\varphi$ yields

$$(6) \quad \Delta^D \varphi = \ell^{-2}(\Delta^g \varphi + (2-n)g(\theta, d\varphi)), \quad \forall \varphi \in \mathcal{C}^\infty(M).$$

A function $\varphi \in \mathcal{C}^\infty(M)$ satisfying $\Delta^D \varphi = 0$ is called *D -harmonic*.

Lemma 2.2. *On a Riemannian manifold (M, g) the commutator between the Lie derivative and the codifferential acting on 1-forms satisfies the following equation:*

$$(7) \quad [\delta, \mathcal{L}_X]Y^\flat = \delta((\mathcal{L}_X g)(Y)) - g(Y^\flat, d(\delta X^\flat)), \quad \forall X, Y \in \Gamma(TM).$$

Proof. We can assume that M is oriented, up to passing to a double cover. If vol_g denotes the volume form of g , then using the well known formula $\mathcal{L}_X \text{vol}_g = -\delta X^b \text{vol}_g$ (see for instance [5, Appendix 6]) we compute for all vector fields X and Y :

$$\begin{aligned} \delta([X, Y]^b) \text{vol}_g &= -\mathcal{L}_{[X, Y]} \text{vol}_g = -[\mathcal{L}_X, \mathcal{L}_Y] \text{vol}_g = \mathcal{L}_X(\delta Y^b \text{vol}_g) - \mathcal{L}_Y(\delta X^b \text{vol}_g) \\ &= X(\delta Y^b) \text{vol}_g - Y(\delta X^b) \text{vol}_g, \end{aligned}$$

and thus

$$(8) \quad \delta([X, Y]^b) = X(\delta Y^b) - Y(\delta X^b) = \mathcal{L}_X(\delta Y^b) - g(Y^b, d(\delta X^b)).$$

We now compute the commutator as follows:

$$\begin{aligned} [\delta, \mathcal{L}_X](Y^b) &= \delta(\mathcal{L}_X Y^b) - \mathcal{L}_X(\delta Y^b) = \delta([X, Y]^b) + \delta((\mathcal{L}_X g)(Y)) - \mathcal{L}_X(\delta Y^b) \\ &\stackrel{(8)}{=} \delta((\mathcal{L}_X g)(Y)) - g(Y^b, d(\delta X^b)). \end{aligned}$$

□

Recall that a complex manifold (M, J, c) endowed with a conformal structure c is called locally conformally Kähler (lcK) if around each point of M , every metric $g \in c$ can be conformally rescaled to a Kähler metric. Equivalently, (M, J, c) is lcK if every $g \in c$ is Hermitian with respect to J and the fundamental 2-form $\Omega := g(J\cdot, \cdot)$ satisfies $d\Omega = -2\theta \wedge \Omega$ for some closed 1-form θ called the Lee form of (M, c, J) with respect to g .

If (M, J, c) is lcK, the universal cover $\pi: \widetilde{M} \rightarrow M$, endowed with the induced complex structure \widetilde{J} and conformal structure \widetilde{c} , admits a Kähler metric in \widetilde{c} with respect to which $\pi_1(M)$ acts by holomorphic homotheties.

Indeed, if $g \in c$ is any metric on (M, J) with Lee form θ , then the pull-back $\widetilde{\theta}$ is exact on \widetilde{M} , i.e. $\widetilde{\theta} = d\varphi$, for some function $\varphi \in \widetilde{M}$, and the metric $g_K := e^{2\varphi} \widetilde{g}$ is Kähler. Moreover, $\pi_1(M)$ acts on $(\widetilde{M}, \widetilde{J}, g_K)$ by holomorphic homotheties. Hence, the Levi-Civita connection of g_K projects to a closed, non-exact, Weyl connection D on M , the so-called *standard Weyl connection* of the lcK manifold (M, J, c) , whose Lee form in the sense of (2) is exactly θ .

3. CONFORMAL VECTOR FIELDS ON KÄHLER MANIFOLDS

In this section we show that the divergence of a conformal vector field on a (not necessarily compact) Kähler manifold is a harmonic function with respect to the Kähler metric.

Let us first recall some well known results in Kähler geometry, whose proofs can be found for instance in [8]. Let (M, J, g, Ω) be an n -dimensional Kähler manifold.

In the sequel $\{e_i\}_i$ denotes a local orthonormal basis with respect to the metric g . Then the Kähler 2-form can be written as $\Omega = \frac{1}{2} \sum_{i=1}^n e_i \wedge J e_i$, where here and in the sequel we identify vectors and 1-forms using the metric g . We denote by L the wedge product with Ω :

$$L: \Omega^k(M) \rightarrow \Omega^{k+2}(M), \quad L(\alpha) = \Omega \wedge \alpha.$$

The natural extension of J acting as a derivation on forms is given by

$$J: \Omega^k(M) \rightarrow \Omega^k(M), \quad J(\alpha) = \sum_{i=1}^n J e_i \wedge e_i \lrcorner \alpha.$$

The twisted differential d^c is defined as follows:

$$d^c: \Omega^k(M) \rightarrow \Omega^{k+1}(M), \quad d^c(\alpha) = \sum_{i=1}^n J e_i \wedge \nabla_{e_i} \alpha,$$

and its formal adjoint is

$$\delta^c: \Omega^{k+1}(M) \rightarrow \Omega^k(M), \quad \delta^c(\alpha) = - * d^c * = - \sum_{i=1}^n J e_i \lrcorner \nabla_{e_i} \alpha,$$

where ∇ denotes the Levi-Civita connexion of g . The twisted Laplace operator is then defined by $\Delta^c := d^c \delta^c + \delta^c d^c$.

On a Kähler manifold, the following relations hold (for a proof see for instance [8, §14]):

$$(9) \quad [J, d] = d^c, \quad [J, \delta^c] = -\delta,$$

$$(10) \quad \delta^c d + d \delta^c = \delta^c \delta + \delta \delta^c = d^c \delta + \delta d^c = 0,$$

$$(11) \quad [L, \delta^c] = -d,$$

$$(12) \quad \Delta^c = \Delta.$$

After these preliminaries we can now prove the announced result:

Proposition 3.1. *Let (M, J, g, Ω) be a (not necessarily compact) Kähler manifold of dimension $n > 2$. If η is a conformal Killing 1-form on (M, g) , then its codifferential $\delta\eta$ is a g -harmonic function.*

Proof. Let η be a conformal Killing 1-form on (M, g) , i.e. the dual 1-form of a conformal vector field on M . The covariant derivative of η in the direction of any vector field X is given as follows (see [12]):

$$\nabla_X \eta = \frac{1}{2} X \lrcorner d\eta - \frac{1}{n} (\delta\eta) X.$$

We thus compute using the above commutator relations:

$$\begin{aligned} d^c \eta &= \sum_{i=1}^n J e_i \wedge \nabla_{e_i} \eta = \frac{1}{2} \sum_{i=1}^n J e_i \wedge e_i \lrcorner d\eta - \frac{1}{n} (\delta\eta) \sum_{i=1}^n J e_i \wedge e_i \\ &= \frac{1}{2} J d\eta + \frac{2}{n} (\delta\eta) \Omega \stackrel{(9)}{=} \frac{1}{2} d J \eta + \frac{1}{2} d^c \eta + \frac{2}{n} L(\delta\eta), \end{aligned}$$

hence $d^c \eta = d J \eta + \frac{4}{n} L(\delta\eta)$. Applying δ^c to this equality yields

$$\delta^c d^c \eta = \delta^c d J \eta + \frac{4}{n} \delta^c L(\delta\eta) \stackrel{(10),(11)}{=} -d \delta^c J \eta + \frac{4}{n} L \delta^c(\delta\eta) + \frac{4}{n} d(\delta\eta) \stackrel{(9)}{=} -d \delta\eta + \frac{4}{n} d(\delta\eta),$$

and thus $\delta^c d^c \eta + \frac{n-4}{n} d(\delta \eta) = 0$. Applying now δ to this equality yields

$$0 = \delta \delta^c d^c \eta + \frac{n-4}{n} \delta d(\delta \eta) \stackrel{(10)}{=} \delta^c d^c(\delta \eta) + \frac{n-4}{n} \delta d(\delta \eta) = \Delta^c(\delta \eta) + \frac{n-4}{n} \Delta(\delta \eta) \stackrel{(12)}{=} \frac{2n-4}{n} \Delta(\delta \eta).$$

Since $n > 2$, it follows that $\Delta(\delta \eta) = 0$, so $\delta \eta$ is g -harmonic. \square

Remark 3.2. In terms of vector fields, Proposition 3.1 can be reformulated as follows:

The divergence of a conformal vector field on a Kähler manifold of real dimension greater than 2 is a harmonic function with respect to the Kähler metric.

Remark 3.3. If the manifold M is moreover assumed to be compact, a direct consequence of Proposition 3.1 is the well-known result of A. Lichnerowicz [7, §90] stating that a conformal vector field on a compact Kähler manifold of real dimension greater than 2 is necessarily Killing with respect to the Kähler metric.

4. WEYL-HARMONIC FUNCTIONS

In this section we prove that harmonic functions with respect to a Weyl structure on a compact conformal manifold are necessarily constant.

Proposition 4.1. *Let (M, c) be a compact conformal manifold of dimension $n > 2$ endowed with a Weyl structure D . Then any D -harmonic function on M is constant.*

Proof. We consider the Gauduchon metric $g_0 \in c$, which is (up to homothety) the unique metric in c whose associated 1-form θ_0 is g_0 -coclosed. If $c = \ell_0^2 \otimes g_0$, then (6) yields:

$$\Delta^D \varphi = \ell_0^{-2} (\Delta^{g_0} \varphi + (2-n)g_0(d\varphi, \theta_0)), \quad \text{for all } \varphi \in \mathcal{C}^\infty(M).$$

Thus a function $\varphi \in \mathcal{C}^\infty(M)$ is D -harmonic if and only if

$$\Delta^{g_0} \varphi = (n-2)g_0(d\varphi, \theta_0).$$

Multiplying this equality with φ and integrating over the compact manifold M yields

$$\int_M |d\varphi|_{g_0}^2 \text{vol}_0 = \int_M \varphi \Delta^{g_0} \varphi \text{vol}_0 = \frac{n-2}{2} \int_M g_0(d\varphi^2, \theta_0) \text{vol}_0 = \frac{n-2}{2} \int_M g_0(\varphi^2, \delta_0 \theta_0) \text{vol}_0 = 0,$$

which implies that $d\varphi = 0$. Thus φ is constant, since M is compact. \square

Remark 4.2. Let (M^n, c) be a conformal manifold of dimension $n > 2$ endowed with a Weyl structure D . To each vector field ξ can be associated the following function

$$(13) \quad f_\xi := \text{div}^{\nabla^g} \xi + n\theta(\xi),$$

where $g \in c$ and θ is its associated 1-form. Then f_ξ is independent of the choice of the metric $g \in c$, as shown by a direct computation. Namely, if $\tilde{g} = e^{2f}g$, then $\tilde{\theta} = \theta - df$. Taking a local orthonormal basis $\{e_i\}_i$ with respect to g , then $\{\tilde{e}_i := e^{-f}e_i\}_i$ is a local orthonormal

basis with respect to \tilde{g} and we obtain using [2, Thm. 1.159 a)]:

$$\begin{aligned}
\operatorname{div}^{\nabla^{\tilde{g}}}\xi + n\tilde{\theta}(\xi) &= \sum_{i=1}^n \tilde{g}(\tilde{e}_i, \nabla_{\tilde{e}_i}^{\tilde{g}}\xi) + n(\theta - \operatorname{d}f)(\xi) \\
&= \sum_{i=1}^n g(e_i, \nabla_{e_i}^{\tilde{g}}\xi) + n(\theta - \operatorname{d}f)(\xi) \\
&= \sum_{i=1}^n g(e_i, \nabla_{e_i}^g\xi + \operatorname{d}f(e_i)\xi + \operatorname{d}f(\xi)e_i - g(e_i, \xi)\operatorname{grad}f) + n(\theta - \operatorname{d}f)(\xi) \\
&= \operatorname{div}^{\nabla^g}\xi + \operatorname{d}f(\xi) + n\operatorname{d}f(\xi) - \operatorname{d}f(\xi) + n(\theta - \operatorname{d}f)(\xi) \\
&= \operatorname{div}^{\nabla^g}\xi + n\theta(\xi).
\end{aligned}$$

5. CONFORMAL VECTOR FIELDS ON LCK MANIFOLDS

We are now ready to prove the counterpart in lcK geometry of the above mentioned result of A. Lichnerowicz for compact Kähler manifolds. More precisely, we show the following:

Theorem 5.1. *Let (M, J, c) be a compact lcK manifold. Then every conformal vector field on (M, c) is Killing with respect to the Gauduchon metric and the induced vector field on the universal cover is homothetic with respect to the Kähler metric.*

Proof. Let ξ be a conformal vector field on (M, c) and let $\eta_0 := g_0(\xi, \cdot)$ be its dual 1-form with respect to the Gauduchon metric g_0 . Then ξ is Killing with respect to g_0 if and only if $\delta_0\eta_0 = 0$.

We consider the universal cover $\pi: \tilde{M} \rightarrow M$ endowed with the pull-back $(\tilde{J}, \tilde{g}_0, \tilde{\theta}_0)$ of the lcK structure (J, g_0, θ_0) , where θ_0 is the Lee form defined by $\operatorname{d}\Omega_0 = -2\theta_0 \wedge \Omega_0$. If $\varphi \in C^\infty(\tilde{M})$ is a primitive of $\tilde{\theta}_0$, i.e. $\tilde{\theta}_0 = \operatorname{d}\varphi$, then the metric $g_K := e^{2\varphi}\tilde{g}_0$ is Kähler.

We denote by $\tilde{\xi}$ the vector field induced by ξ on \tilde{M} , i.e. $\pi_*\tilde{\xi} = \xi$. Then $\tilde{\xi}$ is a conformal vector field with respect to the conformal class $[\tilde{g}_0] = [g_K]$, and thus its dual 1-form $\eta_K := g_K(\tilde{\xi}, \cdot)$ is a conformal Killing 1-form on the Kähler manifold $(\tilde{M}, \tilde{J}, g_K)$. The pull-back $\tilde{\eta}_0$ of η_K is related to η_K by $\tilde{\eta}_0 = e^{-2\varphi}\eta_K$.

We claim that $\delta_{g_K}\eta_K = -\pi^*f_\xi$, where f_ξ is the function associated to the vector field ξ , as defined by (13). Indeed, we compute using the formula for the conformal change of the codifferential [2, Thm. 1.159 i)]:

$$\begin{aligned}
\delta_{g_K}\eta_K &= e^{-2\varphi}(\delta_{\tilde{g}_0}\eta_K - (n-2)\tilde{g}_0(\eta_K, \operatorname{d}\varphi)) = e^{-2\varphi}\delta_{\tilde{g}_0}(e^{2\varphi}\tilde{\eta}_0) - (n-2)\tilde{g}_0(\tilde{\eta}_0, \operatorname{d}\varphi) \\
&= \delta_{\tilde{g}_0}\tilde{\eta}_0 - 2\tilde{g}_0(\tilde{\eta}_0, \operatorname{d}\varphi) - (n-2)\tilde{g}_0(\tilde{\eta}_0, \operatorname{d}\varphi) = \delta_{\tilde{g}_0}\tilde{\eta}_0 - n\tilde{g}_0(\tilde{\eta}_0, \tilde{\theta}_0) \\
&= \pi^*(\delta_{g_0}\eta_0 - n g_0(\eta_0, \theta_0)) = \pi^*(-\operatorname{div}^{\nabla^{g_0}}\xi - n\theta_0(\xi)) = -\pi^*(f_\xi).
\end{aligned}$$

Since by Proposition 3.1, the function $\pi^*(f_\xi) = -\delta_{g_K}\eta_K$ is g_K -harmonic, it follows that f_ξ is D -harmonic, where D is the standard Weyl structure of the lcK structure (M, J, c) . Applying Proposition 4.1 we obtain that f_ξ is constant, so $f_\xi = C \in \mathbb{R}$. On the other hand, using the

Gauduchon metric g_0 with its associated 1-form θ_0 , we express f_ξ as follows:

$$C = f_\xi = \operatorname{div}^{\nabla^{g_0}} \xi + n\theta_0(\xi) = -\delta_0\eta_0 + n\theta_0(\xi),$$

hence

$$(14) \quad \theta_0(\xi) = \frac{C}{n} + \frac{1}{n}\delta_0\eta_0.$$

By Cartan's formula we further compute:

$$\mathcal{L}_\xi\theta_0 = d(\xi \lrcorner \theta_0) + \xi \lrcorner d\theta_0 = d(\theta_0(\xi)) = \frac{1}{n}d(\delta_0\eta_0).$$

Applying now the codifferential δ_0 to this equality and using Lemma 2.2, we obtain:

$$\begin{aligned} \frac{1}{n}\Delta_0(\delta_0\eta_0) &= \frac{1}{n}\delta_0 d(\delta_0\eta_0) = \delta_0(\mathcal{L}_\xi\theta_0) \\ &\stackrel{(7)}{=} \mathcal{L}_\xi(\delta_0\theta_0) + \delta_0((\mathcal{L}_\xi g)(\theta_0^\sharp)) - g_0(\theta_0, d(\delta_0\eta_0)) \\ &\stackrel{(1)}{=} -\frac{2}{n}\delta_0((\delta_0\eta_0)\theta_0) - g_0(\theta_0, d(\delta_0\eta_0)) \\ &= -\frac{2}{n}(\delta_0\eta_0)\delta_0\theta_0 + \frac{2}{n}g_0(\theta_0, d(\delta_0\eta_0)) - g_0(\theta_0, d(\delta_0\eta_0)) \\ &= \frac{2-n}{n}g_0(\theta_0, d(\delta_0\eta_0)), \end{aligned}$$

since $\delta_0\theta_0 = 0$ by the definition of the Gauduchon metric. Thus, we obtain:

$$(15) \quad \Delta_0(\delta_0\eta_0) = (2-n)g_0(\theta_0, d(\delta_0\eta_0)).$$

Multiplying (15) with the function $\delta_0\eta_0$ and integrating over the compact manifold M yields:

$$\int_M (\delta_0\eta_0)\Delta_0(\delta_0\eta_0)\operatorname{vol}_{g_0} = \frac{2-n}{2} \int_M g_0(\theta_0, d((\delta_0\eta_0)^2))\operatorname{vol}_{g_0},$$

so we obtain:

$$\int_M |d(\delta_0\eta_0)|_0^2 \operatorname{vol}_{g_0} = \frac{2-n}{2} \int_M g_0(\delta_0\theta_0, (\delta_0\eta_0)^2) \operatorname{vol}_{g_0} = 0,$$

showing that $d(\delta_0\eta_0) = 0$. As M is compact, it follows that the function $\delta_0\eta_0$ is constant and hence vanishes, since $\int_M \delta_0\eta_0 \operatorname{vol}_{g_0} = 0$. Thus $\delta_0\eta_0 = 0$, so ξ is a Killing vector field with respect to the Gauduchon metric g_0 .

In order to prove the last statement, we remark that

$$\mathcal{L}_{\tilde{\xi}}g_K = \mathcal{L}_{\tilde{\xi}}(e^{2\varphi}\tilde{g}_0) = 2\tilde{\theta}_0(\tilde{\xi})g_K = 2\widetilde{\theta_0}(\tilde{\xi})g_K,$$

and $\theta_0(\xi)$ is constant by (14) and the fact that $\delta_0\eta_0 = 0$. □

6. HOLOMORPHICITY OF CONFORMAL VECTOR FIELDS

In this section we prove that on a compact lcK manifold, whose Kähler cover is neither flat nor hyperkähler, every conformal vector field is holomorphic. We first show a more general result about homothetic invariant vector fields on Riemannian products endowed with a cocompact and properly discontinuous action of a group of similarities.

Let us introduce the notation needed in the sequel. We denote the group of similarities of a Riemannian manifold (M, g) by

$$\text{Sim}(M, g) := \{\phi: M \rightarrow M \mid \phi \text{ is a diffeomorphism and } \phi^*g = \lambda g, \text{ for some } \lambda \in \mathbb{R}_{>0}\}.$$

A vector field ξ on (M, g) is called homothetic if its flow consists of similarities. Equivalently, ξ is homothetic if $\mathcal{L}_\xi g = kg$ for some real number k .

Proposition 6.1. *Let $(N, g_N) \times (\mathbb{R}^q, g_{\text{flat}})$ be the Riemannian product of a non-flat incomplete Riemannian manifold (N, g_N) with irreducible holonomy and the Euclidean space $(\mathbb{R}^q, g_{\text{flat}})$. We assume that there exists a subgroup $\Gamma \subset \text{Sim}(N \times \mathbb{R}^q, g_N + g_{\text{flat}})$ which acts cocompactly and properly discontinuously on $N \times \mathbb{R}^q$. Then every Γ -invariant homothetic vector field on $(N \times \mathbb{R}^q, g_N + g_{\text{flat}})$ is tangent to N and constant in the direction of \mathbb{R}^q .*

Proof. Let us denote in this proof by $\pi: N \times \mathbb{R}^q \rightarrow (N \times \mathbb{R}^q)/\Gamma$ the projection given by the action of Γ . Let X be a Γ -invariant homothetic vector field on $(N \times \mathbb{R}^q, g_N + g_{\text{flat}})$. We write $X = X_1 + X_2$, with $X_1 \in \text{TN}$ and $X_2 \in \text{T}\mathbb{R}^q$. The flow $(\psi_t)_t$ of X preserves the decomposition $\text{TN} \oplus \mathbb{R}^q$, because any similarity preserves the flat factor of the de Rham decomposition. Thus, the following inclusions hold: $\psi_{t*}(\text{TN}) \subset \text{TN}$ and $\psi_{t*}(\mathbb{R}^q) \subset \mathbb{R}^q$, so $[X, \text{TN}] \subset \text{TN}$ and $[X, \mathbb{R}^q] \subset \mathbb{R}^q$, which further imply that $\nabla_{\text{TN}}X \subset \text{TN}$ and $\nabla_{\mathbb{R}^q}X \subset \mathbb{R}^q$, where ∇ denotes the Levi-Civita connection of $g_N + g_{\text{flat}}$. Hence,

$$(16) \quad \nabla_{\text{TN}}X_2 = 0 \quad \text{and} \quad \nabla_{\mathbb{R}^q}X_1 = 0,$$

showing that X_1 and X_2 are homothetic vector fields on the factors and are constant in the direction of the other factor. Clearly X_1 and X_2 are Γ -invariant. We need to show that $X_2 = 0$.

Since X_2 is a homothetic vector field on the Euclidean space \mathbb{R}^q , there exists a skew-symmetric matrix $A \in M(q, \mathbb{R})$, a real number $\lambda \in \mathbb{R}$ and a vector $v \in \mathbb{R}^q$ such that $(X_2)_p = Cp + v$ at each point $p \in \mathbb{R}^q$, where $C := \lambda I_q + A$ and $I_q \in M(q, \mathbb{R})$ denotes the identity matrix.

We claim that by applying a translation in \mathbb{R}^q one can assume that $v \in \text{Ker}(C)$. In order to prove this we distinguish the following two cases:

Case 1. If $\lambda \neq 0$, then C is invertible and $(X_2)_p = C(p + C^{-1}v)$, so choosing the origin of the flat factor \mathbb{R}^q at $-C^{-1}v$, we may assume that $v = 0$.

Case 2. If $\lambda = 0$, then $C = A$ is a skew-symmetric matrix. Considering the orthogonal splitting $\mathbb{R}^q = \text{Im}(C) \oplus \text{Ker}(C)$, we decompose correspondingly $v = Cv_1 + v_2$, with $v_2 \in \text{Ker}(C)$. Thus $(X_2)_p = C(p + v_1) + v_2$, so choosing the origin of the flat factor \mathbb{R}^q at $-v_1$, we may assume that $v \in \text{Ker}(C)$.

The flow of X_2 is given as follows: $\varphi_t(p) = e^{tC} \left(p + \int_0^t e^{-sC} v ds \right)$. Since $v \in \text{Ker}(C)$, this further simplifies to $\varphi_t(p) = e^{tC} p + tv$.

Now, for every element $\gamma \in \Gamma$, we write $\gamma(x, y) = (\gamma'(x), \gamma''(y))$, for all $(x, y) \in N \times \mathbb{R}^q$, with $\gamma''(y) = B_\gamma y + w_\gamma$, $B_\gamma \in \text{CO}(q)$ and $w_\gamma \in \mathbb{R}^q$. Since X_2 is Γ -invariant, it follows that its flow $(\varphi_t)_t$ commutes with γ'' , that is:

$$B_\gamma(e^{tC} p + tv) + w_\gamma = e^{tC}(B_\gamma p + w_\gamma) + tv, \quad \forall p \in \mathbb{R}^q, \forall t \in \mathbb{R},$$

or, equivalently:

$$\begin{cases} [B_\gamma, e^{tC}] = 0 \\ t(B_\gamma - I_q)v = (I_q - e^{-tC})w_\gamma. \end{cases}$$

Differentiating at $t = 0$ yields for all $\gamma \in \Gamma$:

$$(17) \quad \begin{cases} [B_\gamma, C] = 0 \\ (B_\gamma - I_q)v = Cw_\gamma. \end{cases}$$

We claim that $Cw_\gamma = 0$, for all $\gamma \in \Gamma$. We show this separately for the two cases introduced above. In the first case, if $\lambda \neq 0$, then we have already seen that we may assume $v = 0$. Hence, (17) directly implies that $Cw_\gamma = 0$ for all γ . In the second case, if $\lambda = 0$, then C is a skew-symmetric matrix and we have shown that one may assume $v \in \text{Ker}(C)$. Therefore, since B_γ and C commute, it follows that the left-hand side of the second equality in (17) also belongs to the kernel of C , $(B_\gamma - I_q)v \in \text{Ker}(C)$. The right-hand side belongs to $\text{Im}(C)$ and since $\text{Im}(C) \perp \text{Ker}(C)$, because C is skew-symmetric, we conclude that both sides of this equality vanish, so, in particular, $Cw_\gamma = 0$.

Since $Cw_\gamma = 0$ and there exists at least a strict homothety B_γ , it follows from the second equality in (17) that $v = 0$. Thus, $(X_2)_p = Cp$, for all $p \in \mathbb{R}^q$.

Assume, for a contradiction, that $X_2 \neq 0$, *i.e.* $C \neq 0$. Let us now fix some $p \in \text{Im}(C) \setminus \{0\}$ and $x \in N$ and consider the sequence $\tilde{z}_n := (x, np)$ in $N \times \mathbb{R}^q$, as well as its image $z_n := \pi(\tilde{z}_n)$ in $(N \times \mathbb{R}^q)/\Gamma$. Since $M := (N \times \mathbb{R}^q)/\Gamma$ is compact, there exists a convergent subsequence of $(z_n)_n$, *i.e.* there exists a strictly increasing sequence $(k_n)_n \subset \mathbb{N}$ and $z_0 \in M$ such that $z_{k_n} \rightarrow z_0$. Let V be a neighbourhood of z_0 , such that there exists $\tilde{V} \subset N \times \mathbb{R}^q$ with $\pi|_{\tilde{V}}: \tilde{V} \rightarrow V$ diffeomorphism.

Let $\tilde{z}_0 := (\pi|_{\tilde{V}})^{-1}(z_0)$ and, for all $n \in \mathbb{N}$ sufficiently large in order to ensure that $z_{k_n} \in V$, define $\tilde{y}_n := (\pi|_{\tilde{V}})^{-1}(z_{k_n})$. Then the sequence $(\tilde{y}_n)_n$ converges to \tilde{z}_0 . Since $z_{k_n} = \pi(\tilde{z}_{k_n}) = \pi(\tilde{y}_n)$, there exists a sequence $(\gamma_n)_n \in \Gamma$, such that $\tilde{y}_n = \gamma_n(\tilde{z}_{k_n})$. We may now write according to the results obtained above that $\gamma_n(x, y) = (\gamma'_n(x), \gamma''_n(y))$, for all $(x, y) \in N \times \mathbb{R}^q$, where $\gamma''_n(y) = B_n y + w_n$, with $B_n \in \text{CO}(q)$ and $w_n \in \mathbb{R}^q$, such that $[B_n, C] = 0$ and $Cw_n = 0$. The equality $\tilde{y}_n = \gamma_n(\tilde{z}_{k_n})$ thus yields

$$\tilde{y}_n = \gamma_n(\tilde{z}_{k_n}) = (\gamma'_n(x), \gamma''_n(k_n p)) = (\gamma'_n(x), k_n B_n p + w_n) \xrightarrow{n \rightarrow \infty} \tilde{z}_0.$$

Since $k_n B_n p \in \text{Im}(C)$ and $w_n \in \text{Ker}(C)$, one can write $\tilde{z}_0 = (x_0, Cy_0 + w_0)$ for some $y_0 \in \mathbb{R}^q$ and $w_0 \in \text{Ker}(C)$.

Using that $\text{Im}(C) \oplus \text{Ker}(C) = \mathbb{R}^q$, we deduce that $\gamma'_n(x) \rightarrow x_0$, $k_n B_n p \rightarrow C y_0$ and $w_n \rightarrow w_0$. In particular, $\gamma_n(x, 0) = (\gamma'_n(x), w_n) \rightarrow (x_0, w_0)$. From this convergence and the fact that Γ acts properly discontinuously on $N \times \mathbb{R}^q$, it follows that the sequence $(\gamma_n)_n$ is stationary, *i.e.* there exists n_0 such that $\gamma_n = \gamma_{n_0}$, for all $n \geq n_0$. In particular, $B_n = B_{n_0}$, for $n \geq n_0$, so from $k_n B_n p = k_n B_{n_0} p \rightarrow C y_0$, with $(k_n)_n \subset \mathbb{N}$ strictly increasing, we conclude that $B_{n_0} p = 0$, so $p = 0$, which contradicts the fact that $p \in \text{Im}(C) \setminus \{0\}$. Thus we conclude that $X_2 = 0$. \square

We are now ready for the second main result of this paper.

Theorem 6.2. *Let (M, J, c) be a compact lcK manifold. If the Kähler cover $(\widetilde{M}, \widetilde{J}, g_K)$ is neither flat nor hyperkähler, then every conformal vector field on (M, c) is holomorphic.*

Proof. Let ξ be a conformal vector field on (M, c) . Then, according to Theorem 5.1, ξ is a Killing vector field with respect to the Gauduchon metric $g_0 \in c$.

If the Lee form θ_0 of g_0 vanishes identically, then (M, g_0, J) is Kähler, and a standard argument shows that ξ is holomorphic. Indeed, the Kähler form Ω_0 of (g_0, J) is harmonic and so is its Lie derivative with respect to the Killing vector field ξ . On the other hand, since $d\Omega_0 = 0$, Cartan's formula shows that $\mathcal{L}_\xi \Omega_0 = d(\xi \lrcorner \Omega_0)$ is also exact. A harmonic form which is exact vanishes identically, so $0 = \mathcal{L}_\xi \Omega_0 = g_0(\mathcal{L}_\xi J \cdot, \cdot)$, whence ξ is holomorphic.

We thus assume for the rest of the proof that θ_0 is not identically zero.

Let $\tilde{\xi}$ denote the vector field induced by ξ on \widetilde{M} , *i.e.* $\pi_*(\tilde{\xi}) = \xi$, where $\pi: \widetilde{M} \rightarrow M$ is the projection of the universal cover. By the last part of Theorem 5.1, $\tilde{\xi}$ is homothetic with respect to the Kähler metric g_K . In particular, $\tilde{\xi}$ is affine with respect to the Levi-Civita connection ∇^{g_K} . We distinguish the following two cases:

Case 1. If $\text{Hol}(g_K)$ is irreducible, then any transformation in the connected component of the identity of the group of affine transformations of ∇^{g_K} is holomorphic, *cf.* [9, Lemma 2.1.], because $(\widetilde{M}, \widetilde{J}, g_K)$ is assumed to be neither flat nor hyperkähler. Applying this result to the flow of $\tilde{\xi}$, yields that $\tilde{\xi}$ is a holomorphic vector field on $(\widetilde{M}, \widetilde{J})$, which finishes the proof in the first case.

Case 2. If $\text{Hol}(g_K)$ is reducible, then a result of M. Kourganoff, [6, Theorem 1.5.], implies that the Kähler cover splits as a Riemannian product $(\widetilde{M}, g_K) \simeq (N, g_N) \times (\mathbb{R}^q, g_{\text{flat}})$, where q is even and the metric g_N is non-flat, incomplete and has irreducible holonomy. Applying Proposition 6.1 to the action of $\Gamma := \pi_1(M)$ on (\widetilde{M}, g_K) , we conclude that $\tilde{\xi}$ is tangent to N and constant in the direction of \mathbb{R}^q , *i.e.* there exists a homothetic vector field ζ on (N, g_N) $\tilde{\xi}_{(x,y)} = \zeta_x$, for all $(x, y) \in N \times \mathbb{R}^q$.

We argue by contradiction and assume that ζ is not holomorphic. Since g_N has irreducible holonomy, we conclude, applying again Lemma 2.1. from [9], that (N, g_N) is hyperkähler, so, in particular, g_N is Ricci-flat. Thus also (\widetilde{M}, g_K) is Ricci-flat and, consequently, the standard Weyl connection D on (M, c) is Weyl-Einstein. By a result of K.P. Tod [13, Prop. 2.2] it follows that the dual vector field $T := \theta_0^\sharp$ is Killing with respect to g_0 , which implies that (M, J, g_0) is Vaisman, *i.e.* $\nabla^{g_0} T = 0$. Writing $g_K = e^{2\varphi} \tilde{g}_0$, with $d\varphi = \tilde{\theta}_0$, yields $\mathcal{L}_{\tilde{T}} g_K = 2\tilde{g}_0(\tilde{T}, \tilde{T})g_K$.

Since $\tilde{g}_0(\tilde{T}, \tilde{T})$ is constant, the induced vector field \tilde{T} is homothetic on (\tilde{M}, g_K) . By Proposition 6.1 again, \tilde{T} is tangent to N and is constant in the direction of \mathbb{R}^q . Such a vector field can only be homothetic if it is Killing. Thus $\tilde{g}_0(\tilde{T}, \tilde{T}) = 0$, so $\theta_0 = 0$, which was excluded.

Our assumption is thus false, showing that ζ is holomorphic on N , so $\tilde{\xi}$ is holomorphic on $N \times \mathbb{R}^q$, and therefore ξ is a holomorphic vector field on (M, J) . \square

Remark 6.3. An alternative argument for the second case in the proof of Theorem 6.2 is the following. Assuming that ξ is not holomorphic, and that (\tilde{M}, g_K) has reducible holonomy, we obtain as before that (M, J, g_0) is Vaisman, so the universal cover (\tilde{M}, \tilde{g}_0) carries a non-trivial parallel 1-form $\tilde{\theta}_0$. Consequently, it splits as a Riemannian product $(\tilde{M}, \tilde{g}_0) = (\mathbb{R}, d\varphi^2) \times (S, g_S)$, with $\tilde{\theta}_0 = d\varphi$ and (S, g_S) complete. Then after the change of variable $r := e^\varphi$, the Kähler metric on \tilde{M} reads $g_K = e^{2\varphi}\tilde{g}_0 = dr^2 + r^2g_S$. Thus (\tilde{M}, g_K) is isometric to the Riemannian cone of (S, g_S) , so it is irreducible by S. Gallot's result [3, Prop. 3.1]. This contradiction shows that ξ is holomorphic.

REFERENCES

- [1] F. Belgun, *On the structure of non-Kähler complex surfaces*, Math. Ann. **317** (2000), 1–40.
- [2] A. L. Besse, *Einstein manifolds*, Classics in Mathematics, Springer-Verlag, Berlin, 2008.
- [3] S. Gallot, *Équations différentielles caractéristiques de la sphère*, Ann. Sci. École Norm. Sup. (4), **12**(2) (1979), 235–267.
- [4] P. Gauduchon, *La 1-forme de torsion d'une variété hermitienne compacte*, Math. Ann. **267** (1984), 495–518.
- [5] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry I*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1996.
- [6] M. Kourganoff, *Similarity structures and de Rham decomposition*, Math. Ann. **373** (2019), 1075–1101.
- [7] A. Lichnerowicz, *Géométrie des groupes de transformations*, Dunod, Paris, 1958.
- [8] A. Moroianu, *Lectures on Kähler geometry*, London Mathematical Society Student Texts **69**, Cambridge University Press, Cambridge, 2007.
- [9] A. Moroianu, L. Ornea, *Transformations of locally conformally Kähler manifolds*, Manuscripta Math. **130** (2009), 93–100.
- [10] A. Moroianu, U. Semmelmann, *Twistor forms on Kähler manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **II** (4) (2003), 823–845.
- [11] K. Oeljeklaus, M. Toma, *Non-Kähler compact complex manifolds associated to number fields*, Ann. Inst. Fourier (Grenoble), **55** (1) (2005), 161–171.
- [12] U. Semmelmann, *Conformal Killing forms on Riemannian manifolds*, Math. Z. **245** (2003), 503–527.
- [13] K. P. Tod, *Compact 3-dimensional Einstein-Weyl structures*, J. London Math. Soc. (2) **45** (1992), 341–351.
- [14] I. Vaisman, *Locally conformal Kähler manifolds with parallel Lee form*, Rend. Mat. **12** (2) (1979), 263–284.

ANDREI MOROIANU, UNIVERSITÉ PARIS-SACLAY, CNRS, LABORATOIRE DE MATHÉMATIQUES D'ORSAY, 91405, ORSAY, FRANCE

Email address: andrei.moroianu@math.cnrs.fr

MIHAELA PILCA, FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT REGENSBURG, UNIVERSITÄTSSTR. 31 D-93040 REGENSBURG, GERMANY

Email address: mihaela.pilca@mathematik.uni-regensburg.de