CONFORMAL KILLING 2-FORMS ON COMPACT SYMMETRIC SPACES

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Abstract. We show that a simply connected compact symmetric space of dimension $n \geq 3$ admits a conformal Killing 2-form which is not Killing if and only if it is homothetic to one of $\mathbb{S}^n$, $\mathbb{CP}^m$ or $\mathbb{HP}^q$.

1. Introduction

Let (M^n, g) be a Riemannian manifold. A differential p-form ψ is called a *conformal Killing form*, or *twistor form*, if

$$\nabla_X \psi = \frac{1}{p+1} X \lrcorner d\psi - \frac{1}{n-p+1} X^\flat \wedge \delta\psi \tag{1}$$

for every vector field X, where ∇ denotes the Levi-Civita connection of g. This notion is relevant only for $1 \leq p \leq n-1$, since (1) is tautologically satisfied by any 0- or n-form.

A conformal Killing form is called a *Killing form* if $\delta\psi = 0$ and a $*$-*Killing form* if $d\psi = 0$. The Hodge star preserves the space of conformal Killing forms and interchanges Killing forms with $*$-Killing forms. Note that parallel forms are trivial solutions of (1).

Killing forms on compact simply connected symmetric spaces were classified in [5]. The main result there states that a compact simply connected symmetric space carries a non-parallel Killing p-form, $p \geq 2$, if and only if it has a round sphere factor $\mathbb{S}^k$ with $k > p$. In particular, on an irreducible simply connected compact symmetric space which is not a sphere, every Killing form of degree at least 2 is parallel.

It is therefore natural to ask whether every conformal Killing form on a compact symmetric space is a sum of a Killing form and a $*$-Killing form, equivalently a sum of a Killing form and the Hodge dual of a Killing form. This holds on the round spheres but is false already on $\mathbb{CP}^m$, where nonconstant Killing potentials give rise to non-parallel conformal Killing 2-forms [9]. Since Killing forms of degree at least 2 on compact Kähler manifolds are parallel, these examples cannot be sums of a Killing and a $*$-Killing form. Likewise, $\mathbb{HP}^q$ carries strict conformal Killing 2-forms, and in fact the existence of such a form characterizes $\mathbb{HP}^q$ among compact quaternionic-Kähler manifolds of dimension at least eight [7].

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The purpose of this note is to classify compact simply connected symmetric spaces carrying non-parallel conformal Killing 2-forms, by isolating the algebraic mechanism behind the above exceptions. Our main result is the following:

Theorem 1.1. *Let (M, g) be a compact simply connected Riemannian symmetric space of dimension $n \geq 3$. If M admits a conformal Killing 2-form which is not Killing, then, up to homothety,*

$$M = \mathbb{S}^n, \qquad M = \mathbb{CP}^m, \qquad or \qquad M = \mathbb{HP}^q.$$

The starting point is the prolongation theory of conformal Killing forms [14]. On a locally symmetric space all curvature terms are parallel, so the prolongation becomes a finite-dimensional algebraic problem determined by the isotropy representation. For $p = 2$ one can extract from this system a particularly simple invariant: a parallel 4-form whose action on $\Lambda^2 TM$ has a prescribed eigenvalue on the orthogonal complement of the holonomy algebra.

More precisely, on an irreducible simply connected compact symmetric space $M = G/K$ with non-constant sectional curvature with simple transvection algebra $\mathfrak{g}$, one can show using the results in [5] that there is a one-to-one correspondence between conformal Killing 2-forms modulo parallel forms, and Killing vector fields. Evaluating the conformal Killing equation at a base point produces a K-invariant 4-form $\Omega \in \Lambda^4 \mathfrak{m}^*$ on the isotropy space $\mathfrak{m}$ satisfying the spectral condition

$$C_\Omega = \frac{3}{n-1}\,\mathrm{id} \qquad \text{on } \rho(\mathfrak{k})^\perp,$$

where

$$\rho : \mathfrak{k} \longrightarrow \Lambda^2 \mathfrak{m}$$

denotes the isotropy representation, and $C_\Omega : \Lambda^2 \mathfrak{m} \to \Lambda^2 \mathfrak{m}$ is the natural symmetric endomorphism defined by Ω. The Kähler and quaternionic-Kähler rigidity arguments then single out $\mathbb{CP}^m$ and $\mathbb{HP}^q$ among the non-spherical cases. The only irreducible compact symmetric spaces not covered by the simplicity assumption are the spaces of group type. They are treated separately in Section 7.1.

The main new reduction obtained below is Proposition 4.3. It should be regarded as the conformal-Killing analogue of the algebraic step in [5] which reduces the Killing-form equation to a statement about invariant subspaces of exterior powers.

In principle one could try to extend the approach in the present paper to the study of conformal Killing p-forms on symmetric spaces for $p \geq 3$. However, in this case $\delta\psi$ is not necessarily a Killing $(p-1)$-form, as is the case for $p = 2$, so new arguments have to be found in order to deal with the general case.

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2. Preliminaries

We denote by

$$\mathcal{T}^p(M), \qquad \mathcal{K}^p(M), \qquad \mathcal{P}^p(M)$$

the spaces of conformal Killing, Killing, and parallel p-forms, respectively.

Although we will not use this here, we mention that a p-form ψ is conformal Killing, i.e. satisfies (1), if and only if the projection of $\nabla\psi$ onto the Cartan summand of $T^*M \otimes \Lambda^p T^*M$ vanishes. This projection is called the *twistor operator*.

Taking the covariant derivative of (1), skew-symmetrizing, and contracting shows that the symmetric part of $\nabla(\delta\psi)$ vanishes if the metric g is Einstein. Hence, we have the following important fact (see [16], Thm. 4).

Lemma 2.1. *Let (M^n, g) be Einstein and let $\psi \in \mathcal{T}^2(M)$. Then $\delta\psi \in \mathcal{K}^1(M)$, that is, the 1-form $\delta\psi$ is dual to a Killing vector field.*

Examples of non-parallel conformal Killing 2-forms were previously constructed on $\mathbb{S}^n, \mathbb{CP}^m$ and $\mathbb{HP}^q$. We give here a short account.

On $\mathbb{S}^n, n \geq 3$, the space of conformal Killing 2-forms has the maximal possible dimension, i.e. $\binom{n+2}{3}$. Any such form is a sum of a Killing 2-form and a $*$-Killing 2-form, which itself are eigenforms for the minimal eigenvalue of the Laplace operator restricted to closed respectively coclosed 2-forms (see [14, Prop. 3.2]).

Conformal Killing 2-forms on $\mathbb{CP}^m$ were described in [9, Prop. 6.1]. Let $\mathbb{CP}^m$ be equipped with the Fubini-Study metric g_{FS}, i.e. $\mathrm{Ric} = 2(m+1)g_{FS}$. Then every non-parallel conformal Killing 2-form can be written as $\psi_f := dd^c f + 6f\omega$, where ω is the Kähler form and f is a Laplace eigenfunction with $\Delta f = 4(m+1)f$. Recall that $4(m+1)$ is the minimal eigenvalue of Δ on non-constant function and that any Killing vector field on $(\mathbb{CP}^m, g_{FS})$ is of the form $K_f := J\mathrm{grad}f$, for such a minimal eigenfunction f. An easy calculation shows $\delta\Psi_f = 2(m-1)K_f$.

The description of conformal Killing 2-forms on $\mathbb{HP}^q$ $(q \geq 2)$ is given in [7, Thm. 1]. The authors show that there is an isomorphism between the space of Killing vector fields and the space of conformal Killing 2-forms, with inverse given by the codifferential δ. The isomorphism is defined by an explicit linear combination of the two projections of the covariant derivative of a Killing vector field K onto the subbundles $\mathrm{Sym}^2 H$ and $\mathrm{Sym}^2 E$ of $\Lambda^2 TM$.

3. The case of irreducible symmetric spaces

Let $(M = G/K, g)$ be a compact simply connected irreducible Riemannian symmetric space, with symmetric decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}.$$

In view of the above discussion, we will assume that (M, g) is not a round sphere. We identify $\mathfrak{m}$ with T_oM at the base point $o = eK$ and equip $\mathfrak{g}$ with an $\mathrm{ad}(G)$-invariant inner product inducing the Riemannian metric g. Then $\mathfrak{g}$ is isomorphic to the space of Killing vector fields of g.

Since (M, g) is not a round sphere, the classification of Killing forms on symmetric spaces [5, Thm. 1.1] gives

$$\mathcal{K}^2(M) = \mathcal{P}^2(M). \tag{2}$$

Thus Lemma 2.1 yields an injective map into the space of Killing vector fields

$$\bar{\delta} : \mathcal{T}^2(M)/\mathcal{P}^2(M) \longrightarrow \mathfrak{g}, \qquad [\psi] \longmapsto (\delta\psi)^\sharp. \tag{3}$$

For clarity we consider first the case where $\mathfrak{g}$ is simple. The only irreducible compact symmetric spaces G/K of compact type with non-simple transvection algebra $\mathfrak{g}$ are the spaces of group type, that is, with $G = K \times K$. These are treated separately in Section 7.1.

Since the map defined in (3) is G-equivariant and the adjoint representation of a simple Lie algebra is irreducible as a G-module, we obtain:

Proposition 3.1. *Assume that the symmetric space $M = G/K$ is not a sphere and that $\mathfrak{g}$ is simple. If $\mathcal{T}^2(M) \neq \mathcal{P}^2(M)$, then the restriction of δ to the L^2-orthogonal complement of $\mathcal{P}^2(M)$,*

$$\delta : \mathcal{P}^2(M)^\perp \cap \mathcal{T}^2(M) \longrightarrow \mathfrak{g}$$

is a G-equivariant isomorphism.

Proof. The kernel is zero by (2). The image is a non-zero G-invariant subspace of $\mathfrak{g}$, hence an ideal. Simplicity of $\mathfrak{g}$ implies that the image is all of $\mathfrak{g}$. □

For every $B \in \mathfrak{g}$ we therefore write ψ_B for the unique conformal Killing 2-form in the chosen complement satisfying

$$(\delta\psi_B)^\sharp = B^*, \tag{4}$$

where B^* is the Killing field of (M, g) determined by B.

4. The invariant 4-form

This section contains the key construction of an invariant 4-form associated to the space $\mathcal{T}^2(M)/\mathcal{P}^2(M)$ of conformal Killing 2-forms modulo parallel forms.

Let σ be the involution of the symmetric pair. Thus

$$\sigma|_{\mathfrak{k}} = +\,\mathrm{id}, \qquad \sigma|_{\mathfrak{m}} = -\,\mathrm{id}\,.$$

and σ acts as $(-1)^r$ on $\Lambda^r\mathfrak{m}^*$. The construction of ψ_B is natural under isometries. Hence, the geodesic symmetry s_o, with $(ds_o)_o = \sigma$, satisfies $s_o^*\psi_B = \psi_{\sigma B}$. This shows that there are K-equivariant maps

$$P : \mathfrak{k} \longrightarrow \Lambda^2\mathfrak{m}^*\,, \qquad Q : \mathfrak{m} \longrightarrow \Lambda^3\mathfrak{m}^* \tag{5}$$

such that

$$\psi_A(o) = P(A)\,, \qquad d\psi_A(o) = 0 \qquad \forall A \in \mathfrak{k}\,,$$

and

$$\psi_Y(o) = 0\,, \qquad d\psi_Y(o) = Q(Y) \qquad \forall Y \in \mathfrak{m}\,.$$

The G-equivariance of the map $B \mapsto \psi_B$ means that, for the left action L_g of G on M,

$$L_g^* \psi_B = \psi_{\mathrm{Ad}_{g^{-1}} B}.$$

Taking $g = \exp(tX)$ for some $X \in \mathfrak{m}$ and $B = Y \in \mathfrak{m}$ and differentiating in $t = 0$ gives

$$\mathcal{L}_{X^*} \psi_Y = -\psi_{[X,Y]},$$

where X^* denotes as before the Killing vector field of M determined by X. Since $X^*(o) = X$ and $\psi_Y(o) = 0$ for $Y \in \mathfrak{m}$, the Lie derivative and the covariant derivative agree at o:

$$(\mathcal{L}_{X^*}\psi_Y)(o) = (\nabla_{X^*}\psi_Y)(o).$$

Indeed, the difference between these two expressions is equal to the action of the endomorphism $(\nabla X^*)(o)$ of T_oM on $\psi_Y(o)$. Finally, $[X,Y] \in \mathfrak{k}$ and hence $\psi_{[X,Y]}(o) = P([X,Y])$, which, together with the fact that $X^*(o) = X$ proves

$$\nabla_X \psi_Y(o) = -P([X,Y]), \qquad X, Y \in \mathfrak{m}. \tag{6}$$

For $p = 2$, the conformal Killing equation reads

$$\nabla_X \psi = \frac{1}{3} X \lrcorner\, d\psi - \frac{1}{n-1} X^\flat \wedge \delta\psi. \tag{7}$$

Using (4) and evaluating (6) at o for $\psi = \psi_Y$, with $Y \in \mathfrak{m}$, we obtain

$$-P([X,Y]) = \frac{1}{3} X \lrcorner\, Q(Y) - \frac{1}{n-1} X^\flat \wedge Y^\flat \tag{8}$$

for every $X, Y \in \mathfrak{m}$.

Lemma 4.1. *There exists a K-invariant 4-form*

$$\Omega \in (\Lambda^4 \mathfrak{m}^*)^K$$

such that

$$Q(Y) = Y \lrcorner\, \Omega \qquad \textit{for all } Y \in \mathfrak{m}.$$

Proof. Interchanging X and Y in (8) and adding the two equations gives

$$X \lrcorner\, Q(Y) + Y \lrcorner\, Q(X) = 0.$$

Define

$$\Omega(Y, X_1, X_2, X_3) := Q(Y)(X_1, X_2, X_3).$$

Since $Q(Y)$ is alternating in X_1, X_2, X_3 and the displayed identity says that Ω is also alternating in Y and X_1, we conclude that Ω is a 4-form. Its K-invariance follows from the K-equivariance of the map Q. ☐

Remark 4.2. Recall that on a symmetric space $M = G/K$, there is a one to one correspondence between K-invariant forms on $\mathfrak{m}$ and globally defined parallel forms on M.

For a 4-form Ω define the associated symmetric endomorphism

$$C_\Omega : \Lambda^2\mathfrak{m}^* \longrightarrow \Lambda^2\mathfrak{m}^*$$

by

$$C_\Omega(X^\flat \wedge Y^\flat) := X \lrcorner Y \lrcorner \Omega, \tag{9}$$

extended linearly. Also define the bracket map

$$b : \Lambda^2\mathfrak{m} \longrightarrow \mathfrak{k}, \qquad b(X \wedge Y) = [X, Y].$$

Equation (8) now becomes

$$P \circ b = \frac{1}{n-1}\,\mathrm{id} - \frac{1}{3} C_\Omega. \tag{10}$$

Let $\rho : \mathfrak{k} \longrightarrow \Lambda^2\mathfrak{m}$ denote the isotropy representation, where we use the metric to identify $\mathfrak{so}(\mathfrak{m})$ with $\Lambda^2\mathfrak{m}$. Since the scalar product on $\mathfrak{g}$ is $\mathrm{ad}(G)$-invariant, the adjoint b^* of the bracket map b agrees with ρ. In particular,

$$\ker b = \rho(\mathfrak{k})^\perp. \tag{11}$$

This observation leads to the following crucial result:

Proposition 4.3. *Let $M^n = G/K$ be an irreducible simply connected compact symmetric space with $\mathfrak{g}$ simple. Assume that M is not a sphere and carries a non-parallel conformal Killing 2-form. Then there exists a non-zero K-invariant 4-form $\Omega \in \Lambda^4\mathfrak{m}^*$ such that*

$$C_\Omega = \frac{3}{n-1}\,\mathrm{id} \quad \text{on } \rho(\mathfrak{k})^\perp \subset \Lambda^2\mathfrak{m}. \tag{12}$$

Proof. Equation (10) vanishes on $\ker b$. Using (11) gives

$$\frac{1}{n-1}u - \frac{1}{3}C_\Omega u = 0, \qquad u \in \rho(\mathfrak{k})^\perp,$$

which is (12).

If $\Omega \neq 0$ we are done. If $\Omega = 0$, then (12) implies $\rho(\mathfrak{k})^\perp = 0$, hence

$$\rho(\mathfrak{k}) = \Lambda^2\mathfrak{m} = \mathfrak{so}(\mathfrak{m}).$$

The isotropy is therefore the full orthogonal algebra, and the irreducible symmetric space has constant sectional curvature. In the compact simply connected case this is the round sphere, contrary to the assumption. Thus $\Omega \neq 0$. $\square$

5. The algebraic reduction statement

We now give the representation-theoretic reduction mentioned above. Namely we will show that the existence of a K-invariant 4-form $\Omega \in \Lambda^4\mathfrak{m}^*$ satisfying the condition (12) implies that the symmetric space is either Kähler or quaternionic Kähler.

The first step is particularly useful because it avoids most of Cartan's list.

Lemma 5.1. *Under the hypotheses of Proposition 4.3, the isotropy algebra $\mathfrak{k}$ is not simple.*

Proof. The isotropy representation is an s-representation. The vanishing result of [11] applies here: Proposition 2.10 there shows that, if the isotropy algebra $\mathfrak{k}$ is simple, then

$$(\Lambda^4\mathfrak{m}^*)^K = 0.$$

On the other hand Proposition 4.3 produces a non-zero element $\Omega \in (\Lambda^4\mathfrak{m}^*)^K$. Hence $\mathfrak{k}$ cannot be simple. □

By Cartan's classification (see, for example, [8]), irreducible compact symmetric pairs with non-simple isotropy fall into four classes: Hermitian symmetric spaces, quaternionic-Kähler symmetric spaces (Wolf spaces), oriented real Grassmannians

$$\mathrm{Gr}_k^+(\mathbb{R}^{k+l}) := \mathrm{SO}(k+l)/(\mathrm{SO}(k)\times\mathrm{SO}(l)),$$

and quaternionic Grassmannians

$$\mathrm{Gr}_k(\mathbb{H}^{k+l}) := \mathrm{Sp}(k+l)/(\mathrm{Sp}(k)\times\mathrm{Sp}(l)).$$

The cases $k=2$ in the family of real Grassmannians are Hermitian symmetric, while the cases $k=4$ in the family of real Grassmannians and $k=1$ in the family of quaternionic Grassmannians are quaternionic-Kähler. We will refer to the remaining two families of Grassmannians as genuine real and genuine quaternionic Grasmannians.

Next we will show that for genuine Grasmannians, the condition (12) of Proposition 4.3 cannot be satisfied. The Hermitian symmetric and quaternionic-Kähler symmetric spaces will be dealt with in Section 6.

We first record the dimension of the space of invariant 4-forms in the two genuine Grassmannian families.

Lemma 5.2. *Let $M = G/K$ be either*

$$\mathrm{Gr}_k^+(\mathbb{R}^{k+l}), \qquad k,l \geq 3,$$

with $k,l \neq 4$, or

$$\mathrm{Gr}_k(\mathbb{H}^{k+l}), \qquad k,l \geq 2.$$

Then

$$\dim(\Lambda^4\mathfrak{m}^*)^K = 1.$$

Proof. Since M is a compact symmetric space, a differential form is G-invariant if and only if it is harmonic. Thus

$$\dim(\Lambda^4\mathfrak{m}^*)^K = b_4(M).$$

The required values of b_4 are given by Theorem A.1 in Appendix A. Under the assumptions of the lemma one has $b_4(M) = 1$ in both the genuine real and the genuine quaternionic Grassmannian cases. □

Remark 5.3. The restrictions in the preceding lemma are also transparent from Theorem A.1. A complex Grassmannian with $k,l \geq 2$ has $b_4 = 2$, whereas in the oriented real family a rank-4 factor contributes one additional degree-4 class. Thus the cases $k=4$ or $l=4$ are precisely the exceptional real Grassmannians relevant here. They belong to the quaternionic-Kähler part of Cartan's list and are treated separately below.

Remark 5.4. The point of Lemma 5.2 is that, in the two genuine Grassmannian families, it is enough to compute C_Ω for one non-zero invariant 4-form. Any other invariant 4-form is a scalar multiple of it, so distinct eigenvalues on two irreducible summands of $\rho(\mathfrak{k})^\perp$ rule out the scalar condition in Proposition 4.3 for every non-zero invariant 4-form.

We record the elementary representation-theoretic calculation which is needed here.

Lemma 5.5. *Let $M = G/K$ be either*

$$\mathrm{Gr}_k^+(\mathbb{R}^{k+l}), \qquad k, l \geq 3,$$

with $k, l \neq 4$, or

$$\mathrm{Gr}_k(\mathbb{H}^{k+l}), \qquad k, l \geq 2.$$

Then no non-zero invariant 4-form $\Omega \in (\Lambda^4\mathfrak{m}^*)^K$ can satisfy

$$C_\Omega = c\,\mathrm{id} \quad \text{on } \rho(\mathfrak{k})^\perp$$

with $c \neq 0$.

Proof. By Lemma 5.2, in either family every invariant 4-form is a scalar multiple of a fixed non-zero generator. We treat the real case first.

Put $V = \mathbb{R}^k$, $W = \mathbb{R}^l$ and $\mathfrak{m} = V \otimes W$. Then

$$\mathfrak{k} = \mathfrak{so}(V) \oplus \mathfrak{so}(W)$$

and

$$\Lambda^2(V \otimes W) = (\Lambda^2 V \otimes \mathrm{Sym}^2 W) \oplus (\mathrm{Sym}^2 V \otimes \Lambda^2 W).$$

Using

$$\mathrm{Sym}^2 V = \mathbb{R} g_V \oplus \mathrm{Sym}^2_0 V\,, \qquad \mathrm{Sym}^2 W = \mathbb{R} g_W \oplus \mathrm{Sym}^2_0 W\,,$$

we obtain

$$\rho(\mathfrak{k})^\perp = E_1 \oplus E_2\,,$$

where

$$E_1 = \Lambda^2 V \otimes \mathrm{Sym}^2_0 W\,, \qquad E_2 = \mathrm{Sym}^2_0 V \otimes \Lambda^2 W\,.$$

Both summands are irreducible and non-isomorphic. Let $\Omega_0 \in \Lambda^4(V \otimes W)$ be the invariant 4-form constructed in [11, Lemma 2.7]. It is obtained from

$$R(u, v, w, z) := \mathrm{tr}\left((uv^* - vu^*)(wz^* - zw^*)\right)$$

by the Bianchi map:

$$\Omega_0(u, v, w, z) := R(u, v, w, z) + R(v, w, u, z) + R(w, u, v, z).$$

Here $u, v, w, z \in V \otimes W$ are viewed via the metric as elements in the space of linear maps $\mathcal{L}(V, W)$, and their adjoints as elements in $\mathcal{L}(W, V)$.

With respect to orthonormal bases $\{e_i\}$ of V and $\{f_a\}$ of W, and defining $z_{ia} = e_i \otimes f_a$, an elementary calculation shows that Ω_0 satisfies

$$\Omega_0(z_{11}, z_{12}, z_{21}, z_{22}) = 2.$$

Schur's Lemma implies that C_{Ω_0} is scalar on E_1 and E_2. We now compute the two scalars explicitly. On the 4-dimensional subspace spanned by $z_{11}, z_{12}, z_{21}, z_{22}$ we have

$$\Omega_0 = 2\, z_{11}^* \wedge z_{12}^* \wedge z_{21}^* \wedge z_{22}^*.$$

Since

$$C_{\Omega_0}(X \wedge Y) = X \lrcorner Y \lrcorner \Omega_0,$$

direct contraction gives

$$C_{\Omega_0}(z_{11} \wedge z_{21}) = 2\, z_{12} \wedge z_{22},$$
$$C_{\Omega_0}(z_{12} \wedge z_{22}) = 2\, z_{11} \wedge z_{21},$$

and

$$C_{\Omega_0}(z_{11} \wedge z_{12}) = -2\, z_{21} \wedge z_{22},$$
$$C_{\Omega_0}(z_{21} \wedge z_{22}) = -2\, z_{11} \wedge z_{12}.$$

Now

$$u_1 := z_{11} \wedge z_{21} - z_{12} \wedge z_{22}$$

corresponds, via the embedding of $E_1 = \Lambda^2 V \otimes S_0^2 W$ into $\Lambda^2(V \otimes W) \simeq \mathrm{End}^-(V \otimes W)$,

$$A \otimes S \mapsto \big\{ e \otimes f \mapsto A(e) \otimes S(f) \big\}$$

to

$$(e_1 \wedge e_2) \otimes (f_1 \odot f_1 - f_2 \odot f_2),$$

and hence $u_1 \in E_1$. Since

$$C_{\Omega_0} u_1 = 2\, z_{12} \wedge z_{22} - 2\, z_{11} \wedge z_{21} = -2 u_1,$$

we obtain by the Schur Lemma

$$C_{\Omega_0}|_{E_1} = -2\ \mathrm{id}\,.$$

Similarly,

$$u_2 := z_{11} \wedge z_{12} - z_{21} \wedge z_{22}$$

corresponds to

$$(e_1 \odot e_1 - e_2 \odot e_2) \otimes (f_1 \wedge f_2),$$

so $u_2 \in E_2$. Computing

$$C_{\Omega_0} u_2 = -2\, z_{21} \wedge z_{22} + 2\, z_{11} \wedge z_{12} = 2 u_2,$$

yields

$$C_{\Omega_0}|_{E_2} = 2\ \mathrm{id}\,.$$

Thus the two eigenvalues of C_{Ω_0} on E_1 and E_2 are -2 and 2. Since every non-zero invariant 4-form is of the form $\Omega = \lambda \Omega_0$ with $\lambda \neq 0$, the corresponding eigenvalues are -2λ and 2λ. Hence C_Ω cannot act by one non-zero scalar on all of $\rho(\mathfrak{k})^\perp$.

For the quaternionic Grassmannian we complexify the isotropy representation. If E and F denote the standard complex representations of $\mathrm{Sp}(k)$ and $\mathrm{Sp}(l)$, then

$$\mathfrak{m}^{\mathbb{C}} = E \otimes F$$

and

$$\Lambda^2(E \otimes F) = (\Lambda^2 E \otimes \mathrm{Sym}^2 F) \oplus (\mathrm{Sym}^2 E \otimes \Lambda^2 F).$$

Writing

$$\Lambda^2 E = \mathbb{C}\omega_E \oplus \Lambda^2_0 E, \qquad \Lambda^2 F = \mathbb{C}\omega_F \oplus \Lambda^2_0 F,$$

shows that $\rho(\mathfrak{k})^\perp$ again contains the two non-isomorphic irreducible summands

$$E_1' = \Lambda^2_0 E \otimes \mathrm{Sym}^2 F, \qquad E_2' = \mathrm{Sym}^2 E \otimes \Lambda^2_0 F.$$

The space of invariant 4-forms is one-dimensional in the genuine quaternionic Grassmannian case. Evaluating its generator on a quaternionic 2×2 block gives opposite non-zero eigenvalues on E_1' and E_2'. Note that E_1' and E_2' are non-zero thanks to the assumption $k, l \geq 2$. Hence the restriction of C_Ω to $\rho(\mathfrak{k})^\perp$ is not scalar. This proves the claim. □

We can now formulate the key reduction statement.

Lemma 5.6. *Let $(\mathfrak{g}, \mathfrak{k})$ be an irreducible compact symmetric pair with simple transvection algebra $\mathfrak{g}$, of non-constant sectional curvature. If there exists an element $\Omega \in (\Lambda^4\mathfrak{m}^*)^K$ such that the restriction of C_Ω to $\rho(\mathfrak{k})^\perp$ is a non-zero scalar, then the symmetric space G/K is Hermitian symmetric or quaternionic-Kähler.*

Proof. By Lemma 5.1, $\mathfrak{k}$ is not simple. Cartan's list therefore reduces the possibilities to the four classes mentioned after Lemma 5.1. The two genuine Grassmannian families are excluded by Lemma 5.5. The remaining possibilities are precisely the Hermitian symmetric spaces and the quaternionic-Kähler symmetric spaces. □

6. The Kähler and quaternionic-Kähler cases

We now explain why the two expected non-spherical exceptions are precisely $\mathbb{CP}^m$ and $\mathbb{HP}^q$.

6.1. **Hermitian symmetric spaces.** Let (M^{2m}, g, J) be compact Kähler, with $m \geq 2$. The conformal Killing forms on compact Kähler manifolds were described in [9]. In degree 2, modulo parallel forms, the conformal Killing equation is equivalent to the equation for a Hamiltonian 2-form. More precisely, after adding a suitable multiple of the Kähler form, a conformal Killing 2-form gives a Hamiltonian 2-form (see [9, Prop. 3.13] and [3]. In complex dimension 2 the additional condition in the Hamiltonian 2-form description is that the codifferential be dual to a Killing vector field, which is automatic here by Lemma 2.1.

It is convenient to use the equivalent endomorphism notation. A Hamiltonian 2-form φ corresponds to a Hermitian endomorphism F by

$$\varphi(X, Y) = g(JFX, Y).$$

Then F satisfies the c-projective mobility equation

$$\nabla_X F = X^\flat \otimes \xi_F + \xi_F^\flat \otimes X + (JX)^\flat \otimes J\xi_F + (J\xi_F)^\flat \otimes JX, \tag{13}$$

where $J\xi_F$ is a Killing vector field (see [4, Rem. 1]).

We now prove the rigidity statement for Hermitian symmetric spaces.

Proposition 6.1. *Let (M^{2m}, g, J), $m \geq 2$, be a compact simply connected irreducible Hermitian symmetric space. If M carries a non-parallel conformal Killing 2-form, then up to homothety,*

$$M \simeq \mathbb{CP}^m.$$

Proof. A compact simply connected irreducible Hermitian symmetric space is Kähler–Einstein with positive scalar curvature. Its curvature tensor decomposes as

$$R = B + R_{\mathrm{cst}}, \tag{14}$$

where B is the Bochner tensor and R_{cst} is an algebraic Kähler curvature tensor of constant positive holomorphic sectional curvature. By Proposition 3.1, a non-parallel conformal Killing 2-form gives, modulo parallel forms, a G-module of such forms isomorphic to the transvection algebra $\mathfrak{g}$. Under the correspondence above this produces a G-module of solutions of (13). The degree of c-projective mobility (that is, the dimension of the solution space of (13)) is thus at least $1 + \dim(\mathfrak{g}) \geq 3$.

We may therefore apply the curvature-nullity theorem of Calderbank–Matveev–Rosemann [4, Thms. 1 and 2] stating that there is a constant b such that for every non-parallel solution F of (13), the corresponding vector field ξ_F lies in the b-nullity of the curvature. Equivalently, for a suitable constant b one has

$$\big(R - bR_{\mathrm{cst}}\big)(X, Y)\xi_F = 0 \qquad \text{for all } X, Y \in TM, \tag{15}$$

so by the decomposition (14),

$$\big(B - (b-1)R_{\mathrm{cst}}\big)(X, Y)\xi_F = 0 \qquad \text{for all } X, Y \in TM, \tag{16}$$

Since the Ricci contraction of B vanishes and the Ricci contraction of R_{cst} is $c\,g$, with $c \neq 0$, the Ricci contraction in (16) leads to the equation

$$\big(b - 1\big)\xi_F = 0,$$

Thus $b = 1$, and hence (16) becomes

$$B(X, Y)\xi_F = 0 \qquad \text{for all } X, Y \in TM. \tag{17}$$

On the other hand, by Proposition 3.1, the set of vector fields $J\xi_F$ satisfying (13) is equal to the set $\mathfrak{g}$ of Killing vector fields of M. Thus at every point $x \in M$, the set of $\xi_F(x)$ is equal to T_xM. Equation (17) therefore implies

$$B = 0.$$

Finally, a Kähler–Einstein metric with vanishing Bochner tensor has constant holomorphic sectional curvature. Since (M, g) is compact and simply connected it has to be biholomorphic to $\mathbb{CP}^m$ with the Fubini–Study metric, up to homothety. □

Conversely, as explained in Section 2, every nonconstant Killing potential on $\mathbb{CP}^m$ produces a non-parallel conformal Killing 2-form (see also [9]).

6.2. **Quaternionic-Kähler symmetric spaces.** The quaternionic case is completely rigid by a theorem of David and Pontecorvo.

Theorem 6.2 (David–Pontecorvo [7])**.** *Let (M^{4q}, g), $q \geq 2$, be a compact quaternionic-Kähler manifold. Then M admits a conformal Killing 2-form which is not Killing if and only if, up to homothety, M is isomorphic as a quaternionic-Kähler manifold, to $\mathbb{HP}^q$.*

As mentioned in Section 2, every Killing vector field on $\mathbb{HP}^q$ defines a non-parallel conformal Killing 2-form (see also [7]).

The prolongation connection constructed by David [6] gives a particularly transparent explanation of this rigidity. Its curvature is controlled by the quaternionic Weyl tensor and the prolongation is flat exactly when that tensor vanishes. Thus the quaternionic projective space is the maximally symmetric quaternionic model for the conformal Killing 2-form equation.

7. The classification

7.1. **The case of non-simple transvection algebra.** We first remove the simplicity assumption on the transvection algebra. Note that an irreducible compact symmetric space has non-simple transvection algebra if and only if is of group type.

Proposition 7.1. *Let M be a compact simply connected irreducible symmetric space whose transvection algebra is not simple. If M is not a round sphere, then every conformal Killing 2-form on M is parallel.*

Proof. By the structure theorem for irreducible compact symmetric pairs [8, Ch. X], there is a compact simple Lie algebra $\mathfrak{h}$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}, \qquad \mathfrak{k} = \operatorname{diag} \mathfrak{h}, \qquad \mathfrak{m} = \{(X, -X) : X \in \mathfrak{h}\},$$

and the symmetric involution exchanges the two simple ideals. Geometrically, M is a compact simple Lie group with a bi-invariant metric. The case $\mathfrak{h} \simeq \mathfrak{su}(2)$ gives the round 3-sphere and has already been excluded.

Assume that a non-parallel conformal Killing 2-form exists and consider

$$\mathcal{I} := \big\{(\delta\psi)^\sharp : \psi \in \mathcal{P}^2(M)^\perp \cap \mathcal{T}^2(M)\big\} \subset \mathfrak{g}.$$

As in Proposition 3.1, the kernel of the restriction of δ to $\mathcal{P}^2(M)^\perp \cap \mathcal{T}^2(M)$ is zero by (2). The space $\mathcal{I}$ is non-zero and G-invariant, hence it is an ideal of $\mathfrak{h} \oplus \mathfrak{h}$. Thus it is one of

$$\mathfrak{h} \oplus 0, \qquad 0 \oplus \mathfrak{h}, \qquad \mathfrak{h} \oplus \mathfrak{h}.$$

Now use the geodesic symmetry s_o at the identity. Pull-back by s_o preserves conformal Killing forms, parallel forms and the L^2-orthogonal complement of the parallel forms, and δ is natural under isometries. Therefore $\mathcal{I}$ is invariant under the induced automorphism. On the other hand, this automorphism exchanges the two simple ideals of $\mathfrak{g}$, so the only non-zero possibility is

$$\mathcal{I} = \mathfrak{h} \oplus \mathfrak{h} = \mathfrak{g}.$$

Thus the map δ is surjective also in the group-type case.

The construction of Section 4 uses only this surjectivity and equivariance. It therefore produces a non-zero invariant form

$$\Omega \in (\Lambda^4\mathfrak{m}^*)^K$$

satisfying condition (12) of Proposition 4.3. But here $\mathfrak{k} = \operatorname{diag}\mathfrak{h} \simeq \mathfrak{h}$ is simple. By [11, Prop. 2.10] one has

$$(\Lambda^4\mathfrak{m}^*)^K = 0,$$

a contradiction. Hence no non-parallel conformal Killing 2-form exists. □

7.2. **Proof of Theorem 1.1.** Combining the preceding discussion yields our classification theorem. Consider first the case where M is irreducible.

The spherical case is one of the stated alternatives, so assume that M is not a round sphere. If the transvection algebra $\mathfrak{g}$ is not simple, Proposition 7.1 shows that every conformal Killing 2-form is parallel, contrary to the hypothesis. Thus $\mathfrak{g}$ is simple and the arguments of Sections 3–5 apply.

If the invariant 4-form Ω produced in Section 4 vanishes, the proof of Proposition 4.3 shows that M has constant sectional curvature, hence is a round sphere. If $\Omega \neq 0$, Lemma 5.6 reduces the problem to the Hermitian symmetric and quaternionic-Kähler cases. In the Hermitian symmetric case, Proposition 6.1 shows that the only possibility is $\mathbb{CP}^m$, while in the quaternionic-Kähler case, Theorem 6.2 shows that the only possibility is $\mathbb{HP}^q$.

Finally, the reducible case is easily shown to be impossible. Indeed, assume that $(M, g) = (M_1, g_1) \times (M_2, g_2)$ is a compact simply connected product manifold, and that ψ is a conformal Killing 2-form on (M, g) which is not Killing. Denote by $\pi_1 : M \to M_1$ and $\pi_2 : M \to M_2$ the standard projections. By [10, Thm. 2.1], conformal Killing forms on compact Riemannian products are sums of parallel forms, pull-backs of Killing forms on the factors, and their Hodge duals. We thus can write

$$\psi = \psi_0 + \pi_1^*\psi_1 + \pi_2^*\psi_2 + *(\pi_1^*\varphi_1) + *(\pi_2^*\varphi_2),$$

where ψ_0 is a parallel 2-form on M, ψ_1, ψ_2 are Killing 2-forms on M_1, M_2, and φ_1, φ_2 are Killing $(n-2)$-forms on M_1, M_2 respectively. In particular, since ψ is not Killing, at least one of φ_1 or φ_2 is non-parallel. Assume that φ_1 is non-parallel. Then $\dim M_1 \geq n-1$, so $\dim M_2 \leq 1$. Since M_2 has positive dimension, this forces $\dim M_2 = 1$, which is impossible since there exists no compact simply connected 1-dimensional manifold.

a

Appendix A. The fourth Betti number of Grassmannians

In this appendix we record the fourth rational Betti number of the oriented real, complex and quaternionic Grassmannians. As strange as it might look, we were unable to find a convincing reference in the literature for this computation.

Theorem A.1. *Let $1 \leq k \leq n-1$ and $l = n-k$. Then the following formulas hold.*

(i) For the oriented real Grassmannian,

$$b_4\left(\mathrm{Gr}_k^+(\mathbb{R}^n)\right)=\begin{cases}1+\delta_{k,4}+\delta_{l,4}, & k,l\geq 2,\\ 1, & \{k,l\}=\{1,4\},\\ 0, & \min\{k,l\}=1,\ \max\{k,l\}\neq 4.\end{cases}$$

(ii) For the complex Grassmannian,

$$b_4\left(\mathrm{Gr}_k(\mathbb{C}^n)\right)=\begin{cases}2, & k,l\geq 2,\\ 1, & \min\{k,l\}=1,\ n\geq 3,\\ 0, & n=2.\end{cases}$$

(iii) For the quaternionic Grassmannian,

$$b_4\left(\mathrm{Gr}_k(\mathbb{H}^n)\right)=1.$$

Proof. We first recall Borel's equal-rank formula [1, Thm. 26.1(c), p. 191]. Let G be a compact connected Lie group and $K\subset G$ a connected closed subgroup of the same rank. If $d_1(G),\dots,d_r(G)$ and $d_1(K),\dots,d_r(K)$ are the degrees of homogeneous generators of the corresponding Weyl-group invariant polynomial algebras, then

$$P_{G/K}(t)=\prod_{i=1}^{r}\frac{1-t^{2d_i(G)}}{1-t^{2d_i(K)}}. \tag{18}$$

Note that Borel in his formula uses the cohomological degrees s_i which are related to the polynomial degrees d_i by $s_i=2d_i$.

For $\mathrm{U}(m)$ the degrees are $1,2,\dots,m$. Hence

$$P_{\mathrm{Gr}_k(\mathbb{C}^n)}(t)=\frac{\prod_{j=1}^{n}(1-t^{2j})}{\prod_{j=1}^{k}(1-t^{2j})\prod_{j=1}^{l}(1-t^{2j})}.$$

Up to degree 4,

$$P_{\mathrm{Gr}_k(\mathbb{C}^n)}(t)=\begin{cases}1+t^2+2t^4+O(t^6), & k,l\geq 2,\\ 1+t^2+t^4+O(t^6), & \min\{k,l\}=1,\ n\geq 3.\end{cases}$$

For $n=2$ one has $\mathrm{Gr}_1(\mathbb{C}^2)=\mathbb{CP}^1$. This proves (ii).

For $\mathrm{Sp}(m)$ the degrees are $2,4,\dots,2m$. Consequently

$$P_{\mathrm{Gr}_k(\mathbb{H}^n)}(t)=\frac{\prod_{j=1}^{n}(1-t^{4j})}{\prod_{j=1}^{k}(1-t^{4j})\prod_{j=1}^{l}(1-t^{4j})}=1+t^4+O(t^8),$$

which proves (iii).

We now turn to the oriented real Grassmannian and consider first the case where k and l are not both odd. Since

$$\mathrm{rk}\,\mathrm{SO}(m)=\left\lfloor\frac{m}{2}\right\rfloor,$$

the groups $\mathrm{SO}(n)$ and $\mathrm{SO}(k)\times\mathrm{SO}(l)$ have the same rank in these cases, hence (18) applies.

Only the degrees $d_i = 1$ and $d_i = 2$ can contribute to the Poincaré polynomial up to degree 4, since a basic invariant of degree d_i contributes a factor $1 - t^{2d_i}$ in (18). For $\mathrm{SO}(m)$, the degree 1 occurs only for $m = 2$. The multiplicity of the degree 2 among the basic Weyl-group invariant degrees is

$$c(m) := \#\{i \mid d_i(\mathrm{SO}(m)) = 2\} = \begin{cases} 0, & m \le 2, \\ 2, & m = 4, \\ 1, & m \ge 3,\ m \ne 4. \end{cases}$$

If $k, l \ge 3$ and are not both odd, the groups $\mathrm{SO}(k+l)$ and $\mathrm{SO}(k) \times \mathrm{SO}(l)$ have the same rank, so Borel's Poincaré-polynomial formula gives

$$P_{\mathrm{Gr}_k^+(\mathbb{R}^n)}(t) = (1 - t^4)^{\,c(n)-c(k)-c(l)} + O(t^6).$$

Since $n \ge 6$, one has $c(n) = 1$, and therefore

$$b_4 = c(k) + c(l) - 1 = 1 + \delta_{k,4} + \delta_{l,4}.$$

Suppose next that $k = 2$ and $l \ge 3$. Then

$$P_{\mathrm{Gr}_2^+(\mathbb{R}^{l+2})}(t) = (1 - t^2)^{-1}(1 - t^4)^{\,c(l+2)-c(l)} + O(t^6),$$

so

$$b_4 = 1 + c(l) - c(l+2) = 1 + \delta_{l,4}.$$

For $k = l = 2$,

$$P_{\mathrm{Gr}_2^+(\mathbb{R}^4)}(t) = (1 - t^2)^{-2}(1 - t^4)^2 = 1 + 2t^2 + t^4 + O(t^6),$$

and hence $b_4 = 1$. Thus every equal-rank real case with $k, l \ge 2$ satisfies

$$b_4 = 1 + \delta_{k,4} + \delta_{l,4}.$$

If one of k, l equals 1, then

$$\mathrm{Gr}_1^+(\mathbb{R}^n) \cong S^{n-1},$$

which gives the remaining cases in (i).

It remains to consider the real Grassmannians for which k and l are both odd and at least 3. In this case

$$\mathrm{rk}\big(\mathrm{SO}(k) \times \mathrm{SO}(l)\big) = \mathrm{rk}\,\mathrm{SO}(k+l) - 1.$$

so Borel's Poincaré-polynomial formula does not hold. We replace it by a homotopy argument.

For odd $m \ge 3$,

$$\pi_3(\mathrm{SO}(m)) \otimes \mathbb{Q} \cong \mathbb{Q}, \qquad \pi_4(\mathrm{SO}(m)) \otimes \mathbb{Q} = 0\,,$$

see [15, §§ 22–23, pp. 116ff.].

Let $M := \mathrm{Gr}_k^+(\mathbb{R}^n)$. We will use long exact homotopy sequence associated to the fibration

$$\mathrm{SO}(k) \times \mathrm{SO}(l) \longrightarrow \mathrm{SO}(n) \longrightarrow M\,,$$

The two block inclusions of $\mathrm{SO}(k)$ and $\mathrm{SO}(l)$ into $\mathrm{SO}(k+l)$ induce non-zero maps at the level of $\pi_3 \otimes \mathbb{Q}$, so after rescaling the two rational generators, the relevant part of the long exact sequence reads

$$0 \longrightarrow \pi_4(M) \otimes \mathbb{Q} \longrightarrow \mathbb{Q} \oplus \mathbb{Q} \xrightarrow{(a,b) \mapsto a+b} \mathbb{Q} \longrightarrow \pi_3(M) \otimes \mathbb{Q} \longrightarrow 0.$$

It follows that

$$\pi_4(M) \otimes \mathbb{Q} \cong \mathbb{Q}, \qquad \pi_3(M) \otimes \mathbb{Q} = 0.$$

Moreover, M is simply connected and $\pi_2(M)$ is finite. Hence the rational Hurewicz Theorem, in the form of Serre [2, Chapter III, Theorem 1], yields

$$H_4(M;\mathbb{Q}) \cong \pi_4(M) \otimes \mathbb{Q} \cong \mathbb{Q}.$$

Thus $b_4(M) = 1$, completing the proof of (i). □

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