The Heterotic G_2 System on 2-step Nilmanifolds endowed with Principal Torus Bundles

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Universidad San Antonio de Abad del Cuzco, October 21, 2025

Outline

Motivation and setting

- The heterotic system couples a G_2-structure φ on a manifold M^7, a K-principal bundle $P \to M$ with connection θ, and a bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ on the Lie algebra $\mathfrak{k}$ of K.
- We consider the case where $M = \Gamma \backslash N$ is a 2-step nilmanifold, $K = \mathbb{T}^k$ is a torus and solutions are left-invariant.
- Goal: study the existence of invariant solutions for each 2-step nilpotent Lie algebra of dimension 7, with focus on:
  - the possible dimensions of K,
  - the possible signatures of the bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$,
  - the (non)-vanishing of the cosmological constant λ.

Basic properties of G_2

- The group G_2 is the automorphism group of the octonions $\mathbb{O}$. Recall that $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$, $(a,b)(c,d) := (ac - \bar{d}b, da + b\bar{c})$.
- G_2 preserves the center $\mathbb{R}$ of $\mathbb{O}$ and $\mathrm{Im}(\mathbb{O}) = \mathrm{Im}(\mathbb{H}) \oplus \mathbb{H}$.
- It can be realized as the subgroup of $\mathrm{GL}(7,\mathbb{R})$ preserving

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{245} - e^{146} - e^{236}.$$

- Indeed, G_2 preserves the map $\mathbb{R}^7 \times \mathbb{R}^7 \to \Lambda^7(\mathbb{R}^7)^*$:

$$(X,Y) \mapsto (X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi$$

- $\mathbb{R}^7 \to (\mathbb{R}^7)^* \otimes \Lambda^7(\mathbb{R}^7)^* \quad \rightsquigarrow \quad \Lambda^7\mathbb{R}^7 \to (\Lambda^7(\mathbb{R}^7)^*)^{\otimes 8}$
- G_2 is automatically contained in $\mathrm{SO}(7)$.
- G_2 acts transitively on orthonormal pairs of vectors in $\mathbb{R}^7$.

Basic properties of G_2-structures

Equivalent expressions of φ:

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{245} - e^{146} - e^{236}.$$

$$\varphi = e^7 \wedge (e^{12} + e^{34} + e^{56}) + (e^{135} - e^{245} - e^{146} - e^{236}) = \xi \wedge \omega + \Psi,$$

where (ω, Ψ) is an $\mathrm{SU}(3)$-structure on $\xi^\perp$.

$$\begin{aligned}\varphi &= e^{567} + e^5 \wedge (e^{13} - e^{24}) + e^6 \wedge (-e^{14} - e^{23}) + e^7 \wedge (e^{12} + e^{34}) \\ &= \xi^1 \wedge \xi^2 \wedge \xi^3 + \sum_{i=1}^{3} \xi^i \wedge \omega_i,\end{aligned}$$

where $\{\omega_1, \omega_2, \omega_3\}$ is a basis of $\Lambda^+\mathbb{R}^4$ whose corresponding complex structures J_1, J_2, J_3 satisfy $J_1 \circ J_2 = -J_3$.

G_2-structures on manifolds

A G_2*-structure* on a 7-dim (compact) manifold M is a 3-form φ, s.t. in the neigbourhood of each point there exists a local frame $\{e_1, \ldots, e_7\}$ with

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{245} - e^{146} - e^{236}.$$

$\varphi \quad \rightsquigarrow \quad g^\varphi$, $*_\varphi$. The G_2-structure is *torsion-free* if

$$d\varphi = d(*_\varphi \varphi) = 0 \iff \nabla^\varphi \varphi = 0 \iff \mathrm{Hol}(\nabla^\varphi) \subseteq G_2$$

This is a very strong condition, for instance implies $\mathrm{Ric}_{g^\varphi} = 0$.

Compact examples with $\mathrm{Hol}(\nabla^\varphi) = G_2$ are difficult to construct [Joyce '96, Kovalev 03, Kovalev-Lee '11, Corti-Haskins-Nordström-Pacini '15, Joyce-Karigiannis '18]

Torsion of a G_2-structure

Given a G_2-structure φ, $\Lambda^5 \simeq \Lambda^2$ and $\Lambda^4 \simeq \Lambda^3$ decompose as

$$\Lambda^4 = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27}\,, \qquad \Lambda^5 = \Lambda^5_7 \oplus \Lambda^5_{14},$$

where (denoting $\psi := *\varphi \in \Lambda^4$):

$$\Lambda^4_1 = \{f\,\psi\}\,, \quad \Lambda^4_7 = \{\alpha \wedge \varphi\}\,, \quad \Lambda^4_{27} = \{*\gamma \;:\; \gamma \wedge \varphi = \gamma \wedge \psi = 0\}$$
$$\Lambda^5_7 = \{\alpha \wedge \psi\}\,, \quad \Lambda^5_{14} = \{\beta \wedge \varphi \;:\; \beta \wedge \varphi = -*\beta\}\,.$$

Correspondingly

$$\boxed{\begin{aligned} d\varphi &= \tau_0\,\psi + 3\tau_1 \wedge \varphi + *\tau_3 \\ d\psi &= 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi\,. \end{aligned}}$$

where $\tau_i \in \Omega^i(M)$ are the torsion forms.

Some Classes of G_2-structures

- torsion-free: $d\varphi = d\psi = 0$
- closed: $d\varphi = 0$
- coclosed: $d\psi = 0 \iff \tau_1 = \tau_2 = 0$. Any G_2-manifold has a coclosed G_2-structure [Crowley–Nordström '15]
- integrable/G_2T: $d\psi = 4\tau_1 \wedge \psi$ ($\iff \tau_2 = 0$). This condition is equivalent to require that there exits a G_2-connection ∇ with skew-symmetric torsion $H_\varphi = \frac{1}{6}\tau_0\varphi - \tau_1 \lrcorner \psi - \tau_3$ [Friedrich–Ivanov '02].
- strong G_2T: $\tau_2 = 0$, $dH_\varphi = 0$. The only known closed examples are $S^3 \times S^3 \times S^1$ and $S^3 \times N^4$, with N^4 hyperkähler. These are the only compact homogeneous spaces [Fino–Martın-Merchan–Raffero '24].
- nearly parallel: $d\varphi = \tau_0\psi$ with τ_0 constant($\implies d\psi = 0$).The standard G_2-structure on S^7 is nearly parallel [Gray '71].

Some induced G_2-structures

- (N^4, ω_i) hyper-symplectic:

$$\varphi := dt^1 \wedge dt^2 \wedge dt^3 + \sum_i dt^i \wedge \omega_i$$

  closed on $T^3 \times N$
- (M, ω, Ψ) Calabi-Yau: $\varphi = dt \wedge \omega + \Psi$ closed on $S^1 \times M$
- $(N^7, \xi_i, \Phi_i, \eta_i, g)$ 3-Sasakian:
  $\varphi = a\,\eta_1 \wedge \eta_2 \wedge \eta_3 + \sum_i a_i \eta_i \wedge d\eta_i$ on N [Kennon-Lotay '23],
  φ is coclosed and this family includes nearly parallel G_2-str.
  [Friedrich-Kath-Moroianu-Semmelmann '97]

The heterotic G_2-system

A solution to the *Heterotic* G_2*-system* on M^7 is a quadruple $(\varphi, P, \theta, \langle\cdot,\cdot\rangle_{\mathfrak{k}})$ where:

- φ is an integrable G_2-str. (i.e. $\tau_2 = 0$);
- $P \to M$ is a principal K-bundle with a nondeg. bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{k}} \colon \mathfrak{k} \otimes \mathfrak{k} \to \mathbb{R}$, θ is a connection on P with curvature F_θ, and

$$\begin{cases} \tau_0 = const. & \textit{cosmological constant} \\ F_\theta \wedge \psi = 0 & \mathrm{G}_2\textit{-instanton} \\ dH_\varphi = \langle F_\theta \wedge F_\theta \rangle_{\mathfrak{k}} & \textit{Anomaly cancellation} \end{cases}$$

where $H_\varphi := \frac{1}{6}\tau_0\varphi - \tau_1 \lrcorner \psi - \tau_3$ [Friedrich–Ivanov '02, de la Ossa–Larfors–Svanes '20]

Example: φ strong $\mathrm{G}_2\mathrm{T}$ ($dH_\varphi = 0$), $F_\theta = 0$. [Ivanov–Stanchev '23].

The heterotic G_2-system

The moduli space of solutions is finite-dimensional [Clarke–Garcia-Fernandez–Tipler '22]

Significative properties of a solution:

- $\tau_0 = 0$ or $\tau_0 \neq 0$.
- the rank k of P.
- the signature (k_+, k_-) of $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$.
- P trivial or P nontrivial.

Some solutions to the heterotic system

- On Lie groups [Fernandez–Ivanov–Ugarte–Villacampa '11, Fernandez–Ivanov–Ugarte–Vassilev '15]
- Fu-Yau Ansatz [Clarke–Garcia-Fernandez–Tipler '22]
- On contact Calabi-Yau manifolds [Lotay–Sá Earp '23]
- Using 3-Sasakian geometry [de la Ossa–Galdeano '24]
- On 3-(α, δ)-Sasakian manifolds [Galdeano–Stecker '24]
- On 2-step nilmanifolds, when P is the ON frame bundle, and θ is the characteristic connection [Clarke–Del Barco–Moreno '23]

Invariant solutions on 2-step nilmanifolds

Let $M^7 = \Gamma\backslash N$ be a 2-step nilmanifold. A solution $(\varphi, P, \theta, \langle\cdot,\cdot\rangle_{\mathfrak{k}})$ to the heterotic G_2-system on M is *invariant* if P is a *torus bundle* and φ and F_θ are *left-invariant*.

7-dimensional 2-step nilmanifolds are classified [Gong '88]. In particular $\dim \mathfrak{n}' \in \{1, 2, 3\}$, where $\mathfrak{n}' := [\mathfrak{n}, \mathfrak{n}]$ is the derived algebra of $\mathfrak{n}$.

When $\dim \mathfrak{n}' = 3$ we assume that $\mathfrak{n}'$ calibrates φ, i.e. φ restricts to a volume on $\mathfrak{n}'$ (as in [Del Barco–Moroianu–Raffero '22]).

Prop [Moroianu–Raffero–Vezzoni '25]. *Under the assumption above, any left-invariant integrable G_2-structure on M is coclosed.*

2-step nilpotent Lie algebra decomposition

Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra, and φ a G_2-structure on $\mathfrak{n}$.

- If $\dim \mathfrak{n}' = 1$: the decomposition $\mathfrak{n} = \langle z \rangle \oplus \mathfrak{v}$ gives an induced SU(3)-structure on $\mathfrak{v}$.
- If $\dim \mathfrak{n}' = 2$: decompose $\mathfrak{n} = \langle z_1, z_2, z_3 \rangle \oplus \mathfrak{r}$, where z_1, z_2 is an ON basis of $\mathfrak{n}'$ and $z_3 := \varphi(z_1, z_2)^\sharp$;the orthogonal complement $\mathfrak{r} \cong \mathbb{R}^4$ carries an SU(2)-structure.
- If $\dim \mathfrak{n}' = 3$ and $\mathfrak{n}'$ calibrated by φ:again, the orthogonal complement $\mathfrak{r} \cong \mathbb{R}^4$ of $\mathfrak{n}'$ carries an SU(2)-structure.

Presentation of 2-step nilpotent metric Lie algebras

- If $(\mathfrak{n}, g)$ is a 2-step nilpotent metric Lie algebra with derived algebra $\mathfrak{n}'$ we denote by $\mathfrak{v}$ the orthogonal complement of $\mathfrak{n}'$.
- Since $[\mathfrak{n}, \mathfrak{n}'] = 0$, the only non-zero commutators are $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{n}'$.
- For every $\alpha \in \mathfrak{n}^*$ and $x, y \in \mathfrak{n}$ one has $d\alpha(x, y) = -\alpha([x, y])$.
- Consequently, if $\{v_1, \ldots, v_l, z_1, \ldots, z_p\}$ is a basis of $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{n}'$, then for the dual basis we have $dv_i^\flat = 0$ and $dz_j^\flat \in \Lambda^2\mathfrak{v}$.
- The Lie algebra structure of $\mathfrak{n}$ is completely described by $(0, \ldots, 0, dz_1^\flat, \ldots, dz_p^\flat)$.

Classification of 2-step nilpotent Lie algebras in dim. 7

$$
\begin{aligned}
\mathfrak{h}_3 \oplus \mathbb{R}^4 &= (0,0,0,0,0,0,e^{12}) \\
\mathfrak{h}_5 \oplus \mathbb{R}^2 &= (0,0,0,0,0,0,e^{12}+e^{34}) \\
\mathfrak{h}_7 &= (0,0,0,0,0,0,e^{12}+e^{34}+e^{56})
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{n}_{5,2} \oplus \mathbb{R}^2 &= (0,0,0,0,0,e^{12},e^{13}) \\
\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R} &= (0,0,0,0,0,e^{12},e^{34}) \\
\mathfrak{h}_3^{\mathbb{C}} \oplus \mathbb{R} &= (0,0,0,0,0,e^{13}-e^{24},e^{14}+e^{23}) \\
\mathfrak{n}_{6,2} \oplus \mathbb{R} &= (0,0,0,0,0,e^{12},e^{14}+e^{23}) \\
\mathfrak{n}_{7,2,A} &= (0,0,0,0,0,e^{12},e^{14}+e^{35}) \\
\mathfrak{n}_{7,2,B} &= (0,0,0,0,0,e^{12}+e^{34},e^{15}+e^{23})
\end{aligned}
$$

Classification of 2-step nilpotent Lie algebras in dim. 7

$$
\begin{aligned}
\mathfrak{n}_{6,3} \oplus \mathbb{R} &= (0,0,0,0,e^{12},e^{13},e^{23}) \\
\mathfrak{n}_{7,3,A} &= (0,0,0,0,e^{12},e^{23},e^{24}) \\
\mathfrak{n}_{7,3,B} &= (0,0,0,0,e^{12},e^{23},e^{34}) \\
\mathfrak{n}_{7,3,B_1} &= (0,0,0,0,e^{12}-e^{34},e^{13}+e^{24},e^{14}) \\
\mathfrak{n}_{7,3,C} &= (0,0,0,0,e^{12}+e^{34},e^{23},e^{24}) \\
\mathfrak{n}_{7,3,D} &= (0,0,0,0,e^{12}+e^{34},e^{13},e^{24}) \\
\mathfrak{n}_{7,3,D_1} &= (0,0,0,0,e^{12}-e^{34},e^{13}+e^{24},e^{14}-e^{23})
\end{aligned}
$$

The Case $\dim \mathfrak{n}' = 1$

Thm [Moroianu–Raffero–Vezzoni '25]. *Assume* $\dim \mathfrak{n}' = 1$ (*i.e.* $\mathfrak{n} = \mathfrak{h}_3 \oplus \mathbb{R}^4$, $\mathfrak{h}_5 \oplus \mathbb{R}^2$, *or* $\mathfrak{h}_7$). *Then:*

$k_+ = 0$ *there are no invariant solutions;*

$k_+ = 1$ *there are invariant solutions only when* $\mathfrak{n} = \mathfrak{h}_5 \oplus \mathbb{R}^2$ *or* $\mathfrak{n} = \mathfrak{h}_7$. *For such solutions* $\tau_0 = 0$. *Moreover,*

- *if* $\mathfrak{n} = \mathfrak{h}_7$: P *is necessarily trivial (but* $F_\theta \neq 0$*),*
- *if* $\mathfrak{n} = \mathfrak{h}_5 \oplus \mathbb{R}^2$: *there are solutions with* P *nontrivial;*

$k_+ \geq 2$ *for every* $l \geq 2$ *there are solutions with* $\tau_0 \neq 0$ *and* $k_+ = l$ *for all* $\mathfrak{n}$.

The Case $\dim \mathfrak{n}' = 2$ or 3 and $\tau_0 \neq 0$

Thm [Moroianu–Raffero–Vezzoni '25]. *Assume* $\dim \mathfrak{n}' = 2$ *or* 3 *and* $\tau_0 \neq 0$. *Then:*

$k = 1$ *there are no invariant solutions.*

$k = 2$ *the only invariant solutions occur on* $\mathfrak{n} = \mathfrak{n}_{6,3} \oplus \mathbb{R}$, *where there are solutions with positive definite bilinear form* $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$.

$k = 3$ *there are invariant solutions on* $\mathfrak{n} = \mathfrak{n}_{5,2} \oplus \mathbb{R}^2$ *with signature* $(3,0)$ *or* $(2,1)$.

The Case $\dim \mathfrak{n}' = 2$ or 3 and $\tau_0 = 0$

If $\tau_0 = 0$ there is more flexibility:

Thm [Moroianu–Raffero–Vezzoni '25]. *Assume* $\dim \mathfrak{n}' = 2$ *or* 3 *and* $\tau_0 = 0$. *Then:*

$k_+ = 0$ *there are no invariant solutions.*

$k_+ \geq 1$ *there are invariant solutions for every* $\mathfrak{n}$ *admitting an invariant coclosed* G_2*-structure.The corresponding algebras were classified in* [Del Barco–Moroianu–Raffero '22]*: all Lie algebras above, except* $\mathfrak{n}_{7,2,A}$ *and* $\mathfrak{n}_{7,2,B}$.

The system as a system of invariant forms

The heterotic system for *invariant solutions* reduces to the following system

$$\begin{cases} \tau_2(\varphi) = 0\,, \\ dF^r = 0 \quad \text{for every } r\,, \\ F^r \wedge \psi = 0 \quad \text{for every } r\,, \\ dH_\varphi = \sum_{r=1}^k \varepsilon_r\, F^r \wedge F^r\,, \end{cases}$$

where

- φ is a left-invariant G_2-structure;
- F^r are *integral* left-invariant forms;
- ε_r are non-zero real numbers.

The case $\dim \mathfrak{n}' = 1$

Idea: Using $SU(3)$-Geometry

$$\mathfrak{n} = \langle z \rangle \oplus \mathfrak{v}\,, \quad \varphi = \omega \wedge z^\flat + \Psi\,, \quad (\omega, \Psi) \quad \text{SU(3)-structure on } \mathfrak{v}$$

Denote $dz^b =: \alpha$, $\lambda := \frac{1}{3} g(\alpha, \omega)$. For $r = 1, \ldots, k$ there exist $\mu^r, \eta^r \in \mathfrak{v}^*$, $\sigma^r \in \Lambda^{1,1}\mathfrak{v}^*$ s.t.

$$F^r = *_{\mathfrak{v}}(\mu^r \wedge \Psi) + \sigma^r + \eta^r \wedge z^\flat\,.$$

$$\begin{cases} \tau_2(\varphi) = 0 \\ dF^r = 0 \\ F^r \wedge \psi = 0 \\ dH_\varphi = \sum_{r=1}^k \varepsilon_r\, F^r \wedge F^r \end{cases} \iff \begin{cases} \alpha \in \Lambda^{1,1}\mathfrak{v}^* (\Longrightarrow d\psi = 0) \\ \eta^r = 2\mu^r \text{ and } \eta^r \wedge \alpha = 0 \\ \sum_{r=1}^k \varepsilon_r *_{\mathfrak{v}} (\eta^r \wedge \Psi) \wedge \sigma^r = 0 \\ \sum_{r=1}^k \varepsilon_r \left(*_{\mathfrak{v}}(\eta^r \wedge \Psi) \wedge \eta^r + 2\sigma^r \wedge \eta^r\right) = 0 \\ \alpha_0^2 - 4\lambda^2\omega^2 = \sum_{r=1}^k \varepsilon_r (\sigma^r)^2. \end{cases}$$

The case $\dim \mathfrak{n}' = 2$ or 3

Idea: Using $\mathrm{SU}(2)$-Geometry

$$\mathfrak{n} = \langle z_1, z_2, \varphi(z_1, z_2)^\sharp \rangle \oplus \mathfrak{r}\,, \quad \varphi = \sum_{i=1}^{3} \omega_i \wedge z_i^\flat + z_1^\flat \wedge z_2^\flat \wedge z_3^\flat\,,$$

$(\omega_1, \omega_2, \omega_3)$ is an SU(2)-structure on $\mathfrak{r}$.Let $\alpha_i := dz_i^\flat$ and $\lambda := \frac{1}{3}\sum g(\alpha_i, \omega_i)$.
Then $F^r = F_0^r + \sum v_i^r \wedge z_i^\flat$ with $v_i^r \in \mathfrak{r}^*$, $F_0^r \in \Lambda^2\mathfrak{r}^*$.

$$\begin{cases} \tau_2(\varphi) = 0 \\ dF^r = 0 \\ F^r \wedge \psi = 0 \\ dH_\varphi = \sum_{r=1}^k \varepsilon_r\, F^r \wedge F^r \end{cases} \iff \begin{cases} F_0^r \wedge \omega_i = 0\,, \quad i = 1,2,3 \\ \sum_{i=1}^3 v_i^r \wedge \alpha_i = 0\,, \quad i = 1,2,3 \\ \sum_{i=1}^3 v_i^r \wedge \omega_i = 0\,, \quad i = 1,2,3 \\ \sum_{r=1}^k \varepsilon_r\, |F_0^r|^2 = -12\lambda^2 + \sum_{i=1}^3 |\alpha_i|^2 \\ \sum_{r=1}^k \varepsilon_r\, F_0^r \wedge v_i^r = 0\,, \quad i = 1,2,3 \\ \sum_{r=1}^k \varepsilon_r\, v_1^r \wedge v_2^r = +2\lambda\alpha_3 \\ \sum_{r=1}^k \varepsilon_r\, v_1^r \wedge v_3^r = -2\lambda\alpha_2 \\ \sum_{r=1}^k \varepsilon_r\, v_2^r \wedge v_3^r = +2\lambda\alpha_1 \end{cases}$$

Explicit construction of solutions

- Once a solution to the algebraic system is constructed on the Lie algebra $\mathfrak{n}$, one still nedds to check whether it induces a solution on the compact quotients of the corresponding simply connected Lie group N.
- One first needs to construct co-compact lattices of N.This is done using the Baker-Campbell-Hausdorff formula.
- For 2-step nilpotent Lie groups, the BCH formula is very simple:

$$\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]}$$

  for every $A, B \in \mathfrak{n}$.
- Therefore, if $\Lambda \subset \mathfrak{n}$ is a lattice determined by a basis $\mathcal{B}$ with integer structure constants, then $\Gamma := e^{2\Lambda}$ is a lattice of N.

Explicit construction of solutions

Question: if $F \in \Omega^2 N$ is a closed left-invariant 2-form, when is the induced 2-form on $M := \Gamma\backslash N$ integral in cohomology?

Thm: Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra having integer structure constants with respect to a basis $\mathcal{B}$. Let N be the simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$, and let $\Gamma := e^{2\Lambda}$ be the lattice of N determined by $\mathcal{B}$. Then, for every $\mathcal{B}$-integral closed 2-form $F \in \Lambda^2\mathfrak{n}^*$ there exists a principal S^1-bundle over $M = \Gamma\backslash N$ admitting a connection 1-form θ with curvature $F_\theta = F$.

Open questions

- T^2 bundles: are there examples with $\mathfrak{n} \cong \mathfrak{n}_{6,3} \oplus \mathbb{R}$ and signature $(k_+, k_-) = (1, 1)$?
- T^3 bundles, $\dim\,\mathfrak{n}' = 2$:
  - is the Lie algebra necessarily isomorphic to $\mathfrak{n}_{5,2} \oplus \mathbb{R}^2$?
  - are there examples with signature $(k_+, k_-) = (1, 2)$ on this Lie algebra?
- T^3 bundles, $\dim\,\mathfrak{n}' = 3$: does the system impose that $\mathfrak{n}'$ must be calibrated by φ?

Thank you

Thank you!
Muchas gracias!