THE HETEROTIC G_2-SYSTEM ON 2-STEP NILMANIFOLDS ENDOWED WITH PRINCIPAL TORUS BUNDLES

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Abstract. We study the heterotic G_2-system on 7-dimensional 2-step nilmanifolds $M = \Gamma\backslash N$ endowed with principal torus bundles. We first prove that every invariant G_2-structure solving the system must be coclosed (under an additional calibration assumption when the dimension of the derived Lie algebra of N is 3). Then, we discuss the existence of solutions for all possible isomorphism classes of 7-dimensional 2-step nilpotent Lie algebras, and we provide examples with constant dilaton function both when the cosmological constant of the spacetime is zero and when it is nonzero.

1. Introduction

In this paper, we study the heterotic G_2-system [4, 5, 6, 7, 10, 11, 13, 14, 15, 16, 17, 19, 21, 25] on 2-step nilmanifolds endowed with principal torus bundles.

Consider a 7-dimensional manifold M endowed with a G_2-structure $\varphi \in \Omega^3(M)$ and a K-principal bundle $P \to M$, where K is a k-dimensional real Lie group. Let $\theta \in \Omega^1(P;\mathfrak{k})$ be a connection on P with curvature F_θ, and let $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ be an $\mathrm{Ad}(K)$-invariant non-degenerate symmetric bilinear form with signature (k_+, k_-) on the Lie algebra $\mathfrak{k}$ of K. Following the formalism of [5], the *heterotic* G_2*-system* is given by the following set of equations for the data $(\varphi, P, \theta, \langle\cdot,\cdot\rangle_{\mathfrak{k}})$:

$$\begin{aligned} d*\varphi &= 4\tau_1 \wedge *\varphi, \\ F_\theta \wedge *\varphi &= 0, \\ dH_\varphi &= \langle F_\theta \wedge F_\theta\rangle_{\mathfrak{k}}\,, \end{aligned} \tag{1.1}$$

where $*$ is the Hodge operator determined by the G_2-structure φ, $\tau_1 \coloneqq \frac{1}{12} *(\varphi \wedge *d\varphi) \in \Omega^1(M)$ is the intrinsic torsion 1-form of φ, and the 3-form H_φ is given by

$$H_\varphi \coloneqq \frac{1}{6}*(\varphi \wedge d\varphi)\varphi - *d\varphi + *(4\tau_1 \wedge \varphi).$$

By [5, Thm. 4.9], the intrinsic torsion form $\tau_0 \coloneqq \frac{1}{7} *(\varphi \wedge d\varphi) \in C^\infty(M)$ of the G_2-structure φ is constant for every solution of (1.1). As in [5], we introduce the notation $\lambda \coloneqq \frac{7}{12}\tau_0$.

The heterotic G_2-system arises in the study of $\mathcal{N} = 1$ supersymmetric compactifications of the heterotic string to three dimensions in the supergravity limit. In this context, the dilaton field ϕ is a potential for the intrinsic torsion 1-form of the G_2-structure, $\tau_1 = \frac{1}{2}d\phi$, and $-\lambda^2$ is a non-zero multiple of the *cosmological constant* of the 3-dimensional spacetime, which is constrained to be Minkowski (if $\lambda = 0$) or Anti de Sitter (if $\lambda \neq 0$) by supersymmetry and maximal symmetry [7]. The system admits an alternative description in terms of Killing spinor equations (see e.g. [4, 5, 7]), and thus it can be regarded as the analogue of the Hull–Strominger system [20, 26] in dimension 7. The complete bosonic equations of heterotic supergravity were presented in [23] and [24], where the first compactification backgrounds not locally isomorphic to a supersymmetric compactification background were obtained.

The first two equations of the heterotic G_2-system (1.1) are equivalent to the *gravitino equation* [5, Lemma 2.12]. The first one states that one of the components of the intrinsic torsion of φ, namely the intrinsic torsion 2-form τ_2, is zero. This condition characterizes the existence of a connection ∇ on TM that preserves the G_2-structure and has totally skew-symmetric torsion [13, Thm. 4.7]; the

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latter corresponds to the 3-form H_φ under the musical isomorphism. G_2-structures satisfying the condition $\tau_2 = 0$ are known as *integrable* G_2*-structures* [13, 14] or G_2*-structures with torsion* (shortly G_2T*-structures*) [3] in the literature. The second equation in (1.1) states that the connection θ on P is a G_2-instanton [9]. The last equation is known as the *heterotic Bianchi identity*.

Special solutions to the heterotic G_2-system are provided by *torsion-free* G_2-structures, which are characterized by the conditions $d\varphi = 0$ and $d * \varphi = 0$. In this case, H_φ vanishes and any flat bundle gives a solution to the system. More generally, G_2T-structures satisfying the condition $dH_\varphi = 0$ provide solutions to the system when the connection θ is flat. These G_2-structures are known as *strong* G_2-structures with torsion [3, 12], in analogy to the terminology used in the Hermitian case. In this context, they play the role that Calabi-Yau structures play in the Hull-Strominger system.

In the present paper, we study the heterotic G_2-system on 2-step nilmanifolds $M = \Gamma\backslash N$, where N is a simply connected 2-step nilpotent 7-dimensional Lie group and Γ and is a cocompact lattice. We will consider the case where P is a principal torus bundle over M. In this geometric setting, we refer to an *invariant* solution to the system as a solution where φ and the curvature form F_θ (which can be thought of as a 2-form on M with values in $\mathfrak{k}$) are induced by left-invariant forms on N. In this way, the system can be regarded as a system for exterior forms on the Lie algebra $\mathfrak{n}$ of N, and the analysis can be performed in an algebraic fashion.

Note that if a solution of (1.1) exists for some signature (k_+, k_-) of $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$, then one can obtain solutions for any signature (k'_+, k'_-) with $k'_+ \geq k_+$ and $k'_- \geq k_-$ simply by multiplying P with a trivial bundle with flat connection. We will thus be interested in constructing solutions with smallest possible k_+ and k_-. Note also that any solution must have $k_+ \geq 1$. This follows from [5, Thm. 3.9] and will be discussed in Section 2.2.

Nilpotent Lie algebras $\mathfrak{n}$ of dimension 7 are classified [18]. In the 2-step case, the possible dimensions of the commutator $\mathfrak{n}' := [\mathfrak{n}, \mathfrak{n}]$ of $\mathfrak{n}$ are 1, 2 or 3. In the case $\dim(\mathfrak{n}') = 3$ we focus on G_2-structures φ which calibrate $\mathfrak{n}'$, i.e., φ restricts to a volume form on $\mathfrak{n}'$. This is a technical assumption, already considered in [8], which allows us to have a global description of the 3-form φ.

As a preliminary result of independent interest, we prove the following:

Proposition 1.1. *Let φ be an invariant G_2T-structure on a 2-step nilpotent Lie algebra $\mathfrak{n}$. Then φ is coclosed, i.e., $d * \varphi = 0$, if either $\dim(\mathfrak{n}') \leq 2$, or $\dim(\mathfrak{n}') = 3$ and φ calibrates $\mathfrak{n}'$.*

Note that the result does not hold without the calibration assumption when $\dim(\mathfrak{n}') = 3$: on 2-step nilpotent Lie algebras $\mathfrak{n}$ with $\dim(\mathfrak{n}') = 3$, it is possible to construct invariant G_2T-structures φ (not calibrating $\mathfrak{n}'$) which are not coclosed, see Example 5.2.

This result allows us to restrict our analysis to 2-step nilpotent Lie algebras admitting coclosed G_2-structures; such Lie algebras are classified [1, 8]. In detail: if $\dim(\mathfrak{n}') = 1$, the only three isomorphism classes of 7-dimensional 2-step nilpotent Lie algebras are $\mathfrak{h}_3 \oplus \mathbb{R}^4$, $\mathfrak{h}_5 \oplus \mathbb{R}^2$ and $\mathfrak{h}_7$ (here $\mathfrak{h}_n$ denotes the n-dimensional Heinsenberg Lie algebra) and each class admits coclosed G_2-structures; if $\dim(\mathfrak{n}') = 2$ there are four isomorphism classes of 2-step nilpotent Lie algebras admitting a coclosed G_2-structure ($\mathfrak{n}_{5,2} \oplus \mathbb{R}^2$, $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}$, $\mathfrak{h}_3^{\mathbb{C}} \oplus \mathbb{R}$, $\mathfrak{n}_{6,2} \oplus \mathbb{R}$); if $\dim(\mathfrak{n}') = 3$ all seven isomorphism classes of 2-step nilpotent Lie algebras admit coclosed G_2-structures (see Appendix B for a detailed description of these Lie algebras).

When $\dim(\mathfrak{n}') = 3$ and the G_2-structure is coclosed, our technical assumption is not restrictive from the metric point of view: any metric induced by a coclosed G_2-structure on $\mathfrak{n}$ is also induced by a coclosed G_2-structure calibrating $\mathfrak{n}'$ [8, Lemma 4.9].

From the physical viewpoint, Proposition 1.1 implies that, in the setting we are considering, any solution to the heterotic G_2-system has constant dilaton function ϕ.

Our first main result focuses on invariant solutions in the case where $\dim(\mathfrak{n}') = 1$.

Theorem 1.2. *Let $M = \Gamma\backslash N$ be a 7-dimensional 2-step nilmanifold such that $\dim(\mathfrak{n}') = 1$.*

(*i*) *If $\mathfrak{n} \cong \mathfrak{h}_3 \oplus \mathbb{R}^4$, the heterotic G_2-system (1.1) on M has no invariant solutions with $\lambda = 0$;*

(*ii*) *If* $\mathfrak{n} \cong \mathfrak{h}_5 \oplus \mathbb{R}^2$, *there are solutions with* $\lambda = 0$, *non-trivial torus bundle* P, *and with signature* $(k_+, k_-) = (1, 0)$;

(*iii*) *If* $\mathfrak{n} \cong \mathfrak{h}_7$ *there are solutions with* $\lambda = 0$, *signature* $(k_+, k_-) = (1, 0)$ *and non-zero curvature. Each solution with* $\lambda = 0$ *and either* $k_+ = 1$ *or* $k_- = 0$ *has trivial torus bundle, and there are solutions with* $\lambda = 0$, *non-trivial torus bundle, and* $(k_+, k_-) = (2, 1)$;

(*iv*) *The heterotic* G_2-*system on* M *has no invariant solutions with* $\lambda \neq 0$ *and* $k_+ = 1$, *while there are solutions to the system with* $\lambda \neq 0$ *and* $(k_+, k_-) = (2, 0)$ *for each of the three isomorphism classes of* $\mathfrak{n}$.

Our second main result focuses on the existence of invariant solutions in the case where $\dim(\mathfrak{n}') \geq 2$.

Theorem 1.3. *Let* $M = \Gamma\backslash N$ *be a* 7-*dimensional* 2-*step nilmanifold such that* $\dim(\mathfrak{n}') \geq 2$.

(*i*) *The heterotic* G_2-*system* (1.1) *has solutions with* $\lambda = 0$ *and* $k_+ = 1$ *for every isomorphism class of* $\mathfrak{n}$ *admitting a coclosed* G_2-*structure.*

(*ii*) *If* $\dim(\mathfrak{n}') = 2$, *there are no invariant solutions to the heterotic* G_2-*system on* M *with* $\lambda \neq 0$ *and* $k = k_+ + k_- \leq 2$. *Moreover, if* $\mathfrak{n} \cong \mathfrak{n}_{5,2} \oplus \mathbb{R}^2$ *then the heterotic* G_2-*system on* M *admits invariant solutions with* $\lambda \neq 0$ *and* $k \geq 3$.

(*iii*) *If* $\dim(\mathfrak{n}') = 3$, *there are no invariant solutions to the heterotic* G_2-*system on* M *with* $\lambda \neq 0$, $k = 1$, *and* φ *calibrating* $\mathfrak{n}'$. *Moreover, the heterotic* G_2-*system on* M *admits invariant solutions with* $\lambda \neq 0$, $k = 2$ *and* φ *calibrating* $\mathfrak{n}'$ *if and only if* $\mathfrak{n} \cong \mathfrak{n}_{6,3} \oplus \mathbb{R}$.

The proof of the results is obtained as follows.

If $\dim(\mathfrak{n}') = 1$, then every G_2-structure φ on $\mathfrak{n}$ induces an SU(3)-structure (ω, Ω_+) on the orthogonal complement $\mathfrak{v}$ of $\mathfrak{n}'$. The isomorphism class of $\mathfrak{n}$ is determined by the rank of $\alpha := dz^\flat$, where z is a unit-norm generator of $\mathfrak{n}'$. The heterotic G_2-system can then regarded as a system of forms on $\mathfrak{v}$ and the decomposition of 2- and 3-forms in SU(3)-modules can be used to simplify the equations. In particular, the system can be reduced to the equation

$$\alpha_0 \wedge \alpha_0 - 4\lambda^2 \omega \wedge \omega = \sum_{r=1}^{k} \varepsilon_r \sigma^r \wedge \sigma^r \,,$$

on $\mathbb{R}^6$ with the standard SU(3)-structure, where $\alpha_0 \in \Lambda^{(1,1)}(\mathbb{R}^6)^*$, $\lambda \in \mathbb{R}$, $\varepsilon_1 \ldots, \varepsilon_k \in \mathbb{R} \smallsetminus \{0\}$, $\sigma_1, \ldots, \sigma_k \in \Lambda_0^{(1,1)}(\mathbb{R}^6)^*$.

If $\dim(\mathfrak{n}') = 2$ or if $\dim(\mathfrak{n}') = 3$ and φ calibrates $\mathfrak{n}'$, one can split $\mathfrak{n}$ as $\mathfrak{v} \oplus \langle z_1, z_2, z_3 \rangle$ with $\mathfrak{n}' \subseteq \langle z_1, z_2, z_3 \rangle$, so that the 4-dimensional space $\mathfrak{v}$ inherits an SU(2)-structure $\omega_1, \omega_2, \omega_3$. In this case the heterotic system can be written in terms of forms on $\mathfrak{v}$ and the decompositions of forms in SU(2)-modules allows us to the simplify the equations.

The paper is organized as follows. Section 2 contains preliminary material about G_2-structures and the heterotic G_2-system. Section 3 shows how to reformulate the heterotic G_2-system in the invariant case in terms of invariant forms (see system 3.1). Section 4 focuses on the case $\dim(\mathfrak{n}') = 1$ and provides a proof of Theorem 1.2, while Section 5 studies the cases $\dim(\mathfrak{n}') = 2$ and $\dim(\mathfrak{n}') = 3$ and contains the proof of Theorem 1.3. Section 6 collects some problems arising from the results of the present paper which remain open. The paper also contains two appendices: in the first one we recall how to construct principal torus bundles on 2-step nilmanifolds from integral forms, and in the second one we recall the classification of 7-dimensional 2-step nilpotent Lie algebras.

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2. Preliminaries

2.1. **Basic properties of G_2-structures.** Let M be a 7-dimensional manifold with a G_2-structure $\varphi \in \Omega^3(M)$. Recall that φ determines a Riemannian metric g_φ and an orientation on M. We denote by $*$ the corresponding Hodge operator and let $\psi \coloneqq *\varphi$.

At each point x of M there exists an oriented g_φ-orthonormal basis $\{e_1, \ldots, e_7\}$ of T_xM with dual basis $\{e^1, \ldots, e^7\}$ such that

$$
\begin{aligned}
\varphi_x &= e^{127} + e^{347} + e^{567} + e^{135} - e^{245} - e^{146} - e^{236}, \\
\psi_x &= e^{1234} + e^{1256} + e^{3456} + e^{1367} + e^{1457} + e^{2357} - e^{2467}.
\end{aligned} \tag{2.1}
$$

These bases are said to be *adapted* to the G_2-structure φ.

The spaces of 4- and 5-forms on M decompose as follows into G_2-irreducible summands

$$
\Omega^4(M) = \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}\,, \quad \Omega^5(M) = \Omega^5_7 \oplus \Omega^5_{14}\,,
$$

where

$$
\begin{aligned}
\Omega^4_1 &= \{f\,\psi \ : \ f \in C^\infty(M)\}\,, \qquad \Omega^4_7 = \{\alpha \wedge \varphi \ : \ \alpha \in \Omega^1(M)\}\,, \\
\Omega^4_{27} &= \{*\gamma \in \Omega^4(M) \ : \ \gamma \wedge \varphi = 0,\ \gamma \wedge \psi = 0\}, \\
\Omega^5_7 &= \{\alpha \wedge \psi \ : \ \alpha \in \Omega^1(M)\}\,, \qquad \Omega^5_{14} = \{*\beta \in \Omega^5(M) \ : \ \beta \wedge \varphi = - * \beta\}\,.
\end{aligned}
$$

The exterior differentials of φ and ψ decompose accordingly as

$$
\begin{aligned}
d\varphi &= \tau_0\,\psi + 3\tau_1 \wedge \varphi + *\tau_3, \\
d\psi &= 4\tau_1 \wedge \psi - *\tau_2\,,
\end{aligned} \tag{2.2}
$$

where $\tau_0 \in C^\infty(M)$, $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2_{14} = *\Omega^5_{14}$, and $\tau_3 \in \Omega^3_{27} = *\Omega^4_{27}$ are called the *intrinsic torsion forms* of φ, and the presence of τ_1 in both expressions is shown in [2, Prop. 1].

A G_2-structure φ is said to be

- *torsion-free* if all intrinsic torsion forms $\tau_0, \tau_1, \tau_2, \tau_3$ vanish, namely if φ is both closed and coclosed;
- *coclosed* if $d * \varphi = 0$, namely if both τ_1 and τ_2 are zero;
- G_2T (G_2 *with torsion*) if $\tau_2 = 0$.

From [13, Thm. 4.7], a 7-manifold M endowed with a G_2-structure φ admits a G_2-connection ∇ with totally skew-symmetric torsion if and only if $\tau_2 = 0$. In such a case, ∇ is unique and its torsion 3-form H_φ is given by

$$
H_\varphi = \frac{1}{6}\tau_0\,\varphi - \tau_1 \lrcorner\, \psi - \tau_3\,. \tag{2.3}
$$

For later use, we make the following observation:

Remark 2.1. If $\tau_1 = 0$, then the first equation in (2.2) implies $H_\varphi = \frac{7}{6}\tau_0\,\varphi - *d\varphi$.

2.2. **The heterotic G_2-system.** Let M be a connected 7-dimensional manifold. Consider a G_2-structure $\varphi \in \Omega^3(M)$ with dual 4-form $\psi \coloneqq *\varphi$ and intrinsic torsion forms $\tau_0, \tau_1, \tau_2, \tau_3$ defined in (2.2), a K-principal bundle $P \to M$, where K is a k-dimensional real Lie group with Lie algebra $\mathfrak{k}$, a connection 1-form θ on P with curvature F_θ, and an $\mathrm{Ad}(K)$-invariant non-degenerate symmetric bilinear form

$$
\langle \cdot, \cdot \rangle_{\mathfrak{k}} \colon \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}.
$$

Following [5], the quadruple $(\varphi, P, \theta, \langle\cdot,\cdot\rangle_{\mathfrak{k}})$ is a solution to the *heterotic* G_2*-system* if the following equations are satisfied:

$$\begin{aligned} \tau_2 &= 0, \\ F_\theta \wedge \psi &= 0, \\ dH_\varphi &= \langle F_\theta \wedge F_\theta \rangle_{\mathfrak{k}}\,, \end{aligned} \tag{2.4}$$

where the 3-form H_φ is defined by (2.3), and the extension of the bracket $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ to $\mathfrak{k}$-valued differential forms on M is defined as follows. Let $\{t_1,\dots,t_k\}$ be a basis of $\mathfrak{k}$. For every $\mathfrak{k}$-valued forms $\alpha = \sum \alpha^i \otimes t_i$ and $\beta = \sum \beta^j \otimes t_j$, where $\alpha^i \in \Omega^r(M)$ and $\beta^j \in \Omega^s(M)$, we set

$$\langle \alpha \wedge \beta \rangle_{\mathfrak{k}} \coloneqq \sum_{1\le i,j\le k} \alpha^i \wedge \beta^j \langle t_i, t_j\rangle_{\mathfrak{k}}.$$

In particular, if we consider an orthogonal basis $\{t_1,\dots,t_k\}$ of $\mathfrak{k}$ and let $\varepsilon_i \coloneqq \langle t_i, t_i\rangle_{\mathfrak{k}}$, then

$$\langle \alpha \wedge \beta \rangle_{\mathfrak{k}} = \sum_{i=1}^k \varepsilon_i\, \alpha^i \wedge \beta^i.$$

As recalled in the introduction, the intrinsic torsion form τ_0 of the G_2-structure φ is constant for every solution of (2.4), and we henceforth introduce the notation $\lambda \coloneqq \frac{7}{12}\tau_0$ as in [5].

The next result shows how λ is related to other quantities, such as the scalar curvature of g_φ.

Theorem 2.2. [5, Thm. 3.9] *Let $(\varphi, P, \theta, \langle\cdot,\cdot\rangle_{\mathfrak{k}})$ be a solution to the heterotic* G_2*-system on a 7-manifold M. Then*

$$\mathrm{Scal}_{g_\varphi} - \frac{1}{2}|H_\varphi|^2 + |F_\theta|^2_{\mathfrak{k}} - 8d^*\tau_1 - 16|\tau_1|^2 = 4\lambda^2,$$

where $|F_\theta|^2_{\mathfrak{k}} \coloneqq *\langle F_\theta \wedge *F_\theta\rangle_{\mathfrak{k}}$.

We observe that this theorem readily implies the following.

Corollary 2.3. *Let $(\varphi, P, \theta, \langle\cdot,\cdot\rangle_{\mathfrak{k}})$ be a solution to the heterotic* G_2*-system with negative definite $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$, $d^*\tau_1 \ge 0$ and $\mathrm{Scal}_{g_\varphi} \le 0$. Then, the* G_2*-structure is torsion-free and the connection θ is flat.*

Proof. Using Theorem 2.2 and the hypothesis, we have

$$0 = \mathrm{Scal}_{g_\varphi} - \frac{1}{2}|H_\varphi|^2 + |F|^2_{\mathfrak{k}} - 8d^*\tau_1 - 16|\tau_1|^2 - 4\lambda^2 \le 0.$$

Since all summands are non-positive, this implies that the intrinsic torsion forms $\tau_0 = \frac{12}{7}\lambda$, τ_1 and τ_3 are zero and that the connection θ has vanishing curvature F. ◻

Corollary 2.4. *Let $(\varphi, P, \theta, \langle\cdot,\cdot\rangle_{\mathfrak{k}})$ be a solution to the heterotic* G_2 *system on a 7-dimensional unimodular solvable Lie group* G*. If the* G_2*-structure φ is left-invariant and the bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ is negative definite, then the* G_2*-structure is torsion-free, the induced metric g_φ is flat, and the connection θ is flat.*

Proof. Since φ is left-invariant, it induces a left-invariant metric g_φ and the corresponding intrinsic torsion forms τ_0, τ_1 and τ_3 are left-invariant, too. Since G is unimodular, every left-invariant form of degree $6 = \dim(\mathrm{G}) - 1$ on it is closed, thus $d^*\tau_1 = 0$. Since G is solvable, then either g_φ is flat or $\mathrm{Scal}_{g_\varphi} < 0$ [22]. The thesis then follows from Corollary 2.3. ◻

In the following, we shall look for invariant solutions to the heterotic G_2-system on 7-dimensional 2-step nilmanifolds endowed with principal torus bundles. The last corollary shows that, in this setting, there are no solutions admitting a negative definite bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$. In other words, if we denote by (k_+, k_-) the signature of $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$, then $k_+ \ge 1$ whenever a solution of the heterotic G_2-system exists.

2.3. **2-step nilpotent metric Lie algebras.** A real n-dimensional Lie algebra $\mathfrak{n}$ with derived algebra $\mathfrak{n}' := [\mathfrak{n},\mathfrak{n}]$ is 2-*step nilpotent* if

$$\{0\} \neq \mathfrak{n}' \subseteq \mathfrak{z},$$

where $\mathfrak{z}$ is the center of $\mathfrak{n}$.

Given a metric g on $\mathfrak{n}$, one can consider the orthogonal splitting $\mathfrak{n} = \mathfrak{r} \oplus \mathfrak{n}'$, where $\mathfrak{r} := (\mathfrak{n}')^{\perp}$. Choosing a basis $\{z_1,\dots,z_{n'}\}$ of $\mathfrak{n}'$ and a basis $\{e_1,\dots,e_{n-n'}\}$ of $\mathfrak{r}$, it is possible to describe the structure of $\mathfrak{n}$ in terms of the differentials of the dual basis $\{e^1,\dots,e^{n-n'},z^1,\dots,z^{n'}\}$ of $\mathfrak{n}^*$:

$$\begin{cases} de^i = 0, & 1 \leq i \leq n-n', \\ dz^r =: \alpha_j, & 1 \leq j \leq n', \end{cases}$$

where $\alpha_j \in \Lambda^2\mathfrak{r}^*$ for $1 \leq j \leq n'$. Typically, the properties of the forms α_j allow one to distinguish between isomorphic and non-isomorphic Lie algebras (see, e.g., Remark 4.3 below).

3. The heterotic G_2-system on 2-step nilmanifolds

We now consider the heterotic G_2-system (2.4) on 2-step nilmanifolds endowed with principal torus bundles. In this case, $M = \Gamma\backslash N$ is the quotient of a 7-dimensional, simply connected, 2-step nilpotent Lie group N by a cocompact discrete subgroup Γ, and the Lie group $K = \mathbb{T}^k$ is a k-dimensional torus, so $\mathfrak{k} \cong \mathbb{R}^k$ and $F_\theta = d\theta$.

Given a 7-dimensional nilmanifold $M = \Gamma\backslash N$, we look for solutions $(\varphi, P, \theta, \langle\cdot,\cdot\rangle_{\mathfrak{k}})$ of (2.4) such that both the 3-form $\varphi \in \Omega^3(M)$ and the curvature F_θ, thought as a $\mathfrak{k}$-valued 2-form on M, are induced by left-invariant forms on N. This allows us to study the system (2.4) for a G_2-structure $\varphi \in \Lambda^3\mathfrak{n}^*$ and a closed $\mathfrak{k}$-valued 2-form $F_\theta \in \Lambda^2\mathfrak{n}^* \otimes \mathfrak{k}$ on the Lie algebra $\mathfrak{n}$ of N. We refer to such solutions as *invariant solutions* to the heterotic G_2-system.

Note that every bilinear form on $\mathfrak{k}$ is $\mathrm{Ad}(\mathbb{T}^k)$-invariant. It is convenient to fix an $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$-orthogonal basis $\{t_1,\dots,t_k\}$ of $\mathfrak{k}$ and set

$$\varepsilon_r := \langle t_r, t_r\rangle_{\mathfrak{k}}, \;\; 1 \leq r \leq k.$$

We then have

$$F_\theta = \sum_{r=1}^{k} F^r\, t_r,$$

where $F^r \in \Lambda^2\mathfrak{n}^*$ are closed 2-forms on $\mathfrak{n}$, for $1 \leq r \leq k$, and

$$\langle F_\theta \wedge F_\theta\rangle_{\mathfrak{k}} = \sum_{r=1}^{k} \varepsilon_r\, F^r \wedge F^r.$$

If φ is a left-invariant G_2-structure on N, then its intrinsic torsion forms are left-invariant, too. In particular, τ_0 is constant independently of φ being a solution to the heterotic G_2-system. The study of invariant solutions to the heterotic G_2-system on $M = \Gamma\backslash N$ can then be reduced to the study of the following set of equations on the Lie algebra $\mathfrak{n}$ of N for a G_2-structure $\varphi \in \Lambda^3\mathfrak{n}^*$, 2-forms $F^1,\dots,F^r \in \Lambda^2\mathfrak{n}^*$ and non-zero constants $\varepsilon_1,\dots,\varepsilon_r \in \mathbb{R}\smallsetminus\{0\}$:

$$\begin{cases} \tau_2 = 0\,, \\ dF^r = 0, & 1 \leq r \leq k\,, \\ F^r \wedge \psi = 0, & 1 \leq r \leq k\,, \\ dH_\varphi = \sum_{r=1}^{k} \varepsilon_r\, F^r \wedge F^r\,. \end{cases} \tag{3.1}$$

On the other hand, a suitable solution $(\varphi, F^1,\dots,F^k,\varepsilon_1,\dots,\varepsilon_k)$ to system (3.1) on a 7-dimensional 2-step nilpotent Lie algebra $\mathfrak{n}$ gives rise to an invariant solution to the heterotic G_2-system on a certain 2-step nilmanifold endowed with a principal torus bundle. In detail, consider the g_φ-orthogonal splitting $\mathfrak{n} = (\mathfrak{n}')^\perp \oplus \mathfrak{n}'$ and a basis $\mathcal{B} = \{e_1, \dots, e_{7-n'}, z_1, \dots, z_{n'}\}$ of $\mathfrak{n}$ for which the structure constants are integers and such that $\{z_1, \dots, z_{n'}\}$ is a basis of $\mathfrak{n}'$ $(1 \le n' \le 3)$. Then

$$\Gamma := \exp\left(\mathrm{span}_{\mathbb{Z}}(6e_1, \dots, 6e_{7-n'}, z_1, \dots, z_{n'})\right)$$

is a cocompact lattice of the simply connected nilpotent Lie group $N = \exp(\mathfrak{n})$ and thus $M = \Gamma\backslash N$ is a 2-step nilmanifold. The choice of this particular lattice is explained in Remark A.1.

The G_2-structure $\varphi \in \Lambda^3\mathfrak{n}^*$ solves the first equation of (3.1), so it gives rise to a left-invariant G_2T-structure on N with constant intrinsic torsion form τ_0. This in turn defines a G_2-structure of the same type on the quotient $M = \Gamma\backslash N$. If the 2-forms $F^1, \dots, F^k$ solving (3.1) are *integral* with respect to $\mathcal{B}$, namely $F^r(x, y) \in \mathbb{Z}$ for every $x, y \in \mathcal{B}$ and $1 \le r \le k$, then there exists a principal $\mathbb{T}^k$-bundle $P \to M$ endowed with a connection θ whose curvature is $F_\theta = \sum_{r=1}^k F^r t_r$ (see Theorem A.3 of Appendix A), and we can define

$$\langle \cdot, \cdot \rangle_{\mathfrak{k}} := \sum_{r=1}^k \varepsilon_r t^r \otimes t^r ,$$

where $\{t_1, \dots, t_k\}$ is a basis of the Lie algebra of $\mathbb{T}^k$ and $\{t^1, \dots, t^k\}$ is its dual basis. The data $(\varphi, P, \theta, \langle \cdot, \cdot \rangle_{\mathfrak{k}})$ is then an invariant solution to the heterotic G_2-system on $M = \Gamma\backslash N$.

We can then focus on the system (3.1). We study it by considering the cases $\dim(\mathfrak{n}') = 1$ and $\dim(\mathfrak{n}') = 2$ or 3 separately.

4. The case $\dim(\mathfrak{n}') = 1$

Assume that $\mathfrak{n}' := [\mathfrak{n}, \mathfrak{n}]$ is 1-dimensional. Let φ be a G_2-structure on $\mathfrak{n}$ and let g_φ be the associated metric. Denote by $\mathfrak{v}$ the g_φ-orthogonal complement to $\mathfrak{n}'$. Then, once a unit vector $z \in \mathfrak{n}'$ is fixed, φ induces an SU(3)-structure $(\omega, \Omega_+) \in \Lambda^2\mathfrak{v}^* \times \Lambda^3\mathfrak{v}^*$ on $\mathfrak{v}$ via the relation

$$\varphi = \omega \wedge z^\flat + \Omega_+ , \tag{4.1}$$

where $z^\flat$ is the dual covector of z with respect to g_φ. Using the splitting $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{n}' = \mathfrak{v} \oplus \langle z \rangle$, we can write any 2-form F on $\mathfrak{n}$ as

$$F = F_{\mathfrak{v}} + \eta \wedge z^\flat,$$

for some $\eta \in \Lambda^1\mathfrak{v}^*$, where $F_{\mathfrak{v}}$ is the projection of F onto $\Lambda^2\mathfrak{v}^*$. By using this approach we can rewrite system (3.1) as a system of forms in $\mathfrak{v}$. This is the goal of the present subsection.

We adopt the following notation:

- J is the complex structure on $\mathfrak{v}$ induced by (ω, Ω_+);
- $*_{\mathfrak{v}}$ is the Hodge star operator on $\mathfrak{v}$ determined by the metric $\omega(\cdot, J\cdot)$ and the orientation ω^3;
- $\Omega_- := *_{\mathfrak{v}}\Omega_+$.

From (4.1), we obtain

$$\psi = *\varphi = \frac{1}{2}\omega \wedge \omega + \Omega_- \wedge z^\flat. \tag{4.2}$$

Since (ω, Ω_+) is an SU(3)-structure on $\mathfrak{v}$, $\Lambda^2\mathfrak{v}^*$ splits into orthogonal SU(3)-irreducible summands as follows:

$$\Lambda^2\mathfrak{v}^* = \langle \omega \rangle \oplus \Lambda^{(2,0)+(0,2)}\mathfrak{v}^* \oplus \Lambda^{(1,1)}_0\mathfrak{v}^* , \tag{4.3}$$

where

$$\langle \omega \rangle = \mathbb{R}\omega , \quad \Lambda^{(2,0)+(0,2)}\mathfrak{v}^* := \{ *_{\mathfrak{v}}(\eta \wedge \Omega_+) \ : \ \eta \in \Lambda^1\mathfrak{v}^* \} , \quad \Lambda^{(1,1)}_0\mathfrak{v}^* := \{ \sigma \in \Lambda^2\mathfrak{v}^* \ : \ \sigma \wedge \omega = - *_{\mathfrak{v}} \sigma \} .$$

Remark 4.1. The space $\Lambda^{(1,1)}_0\mathfrak{v}^*$ can be described as the space of real 2-forms that are of type $(1,1)$ with respect to J (i.e. commuting with J when viewed as skew-symmetric endomorphisms) and primitive with respect to ω, i.e., their wedge product with $\omega\wedge\omega$ is zero. Similarly, the space $\Lambda^{(2,0)+(0,2)}\mathfrak{v}^*$ is the space of J anti-invariant real 2-forms (identified with skew-symmetric endomorphisms anti-commuting with J).

Lemma 4.2. *The following formulas hold:*

$$*_{\mathfrak{v}}(\sigma\wedge\sigma\wedge\omega) = -|\sigma|^2, \quad \textit{for all } \sigma\in\Lambda^{(1,1)}_0\mathfrak{v}^*, \tag{4.4}$$

$$(*_{\mathfrak{v}}(\eta\wedge\Omega_+))\wedge(*_{\mathfrak{v}}(\eta\wedge\Omega_+)) = 2*_{\mathfrak{v}}(\eta\wedge J\eta), \tag{4.5}$$

$$|*_{\mathfrak{v}}(\eta\wedge\Omega_+)|^2 = 2|\eta|^2, \tag{4.6}$$

for every $\eta\in\Lambda^1\mathfrak{v}^*$, *where by definition* $(J\eta)(\cdot) := -\eta(J\cdot)$.

Proof. For every SU(3)-structure (g,J,ω,Ω_+) on a 6-dimensional vector space $\mathfrak{v}$, there exists an oriented orthonormal basis $\{e_1,\dots,e_6\}$ of $\mathfrak{v}$ with dual basis $\{e^1,\dots,e^6\}$ such that

$$J(e_{2i-1}) = e_{2i}, \qquad \omega = e^{12}+e^{34}+e^{56}, \qquad \Omega_+ = e^{135}-e^{245}-e^{146}-e^{236}. \tag{4.7}$$

In order to check (4.4), we notice that the left hand side is proportional to $|\sigma|^2$ by the Schur Lemma, since $\Lambda^{(1,1)}_0\mathfrak{v}^*$ is an irreducible SU(3) representation. In order to find the proportionality factor, we pick an arbitrary $\sigma\in\Lambda^{(1,1)}_0\mathfrak{v}^*$, e.g. $\sigma = e^{12}-e^{34}$.

In order to check the other two relations, we use the fact that SU(3) acts transitively on the spheres of $\mathfrak{v}$, so for every $\eta\in\mathfrak{v}^*$ one can choose the above basis so that $\eta = ce^1$, for some $c\in\mathbb{R}$. Then both formulas can be checked directly from (4.7). □

Notice that any r-form $\gamma\in\Lambda^r\mathfrak{n}^*$ can be written as $\gamma = \gamma_1+\gamma_2\wedge z^\flat$, for some $\gamma_1\in\Lambda^r\mathfrak{v}^*$ and $\gamma_2\in\Lambda^{r-1}\mathfrak{v}^*$. Since $z\in\mathfrak{n}'$ and $\mathfrak{n}$ is 2-step nilpotent, we have that $\alpha := dz^\flat$ lies in $\Lambda^2\mathfrak{v}^*$, so it decomposes according to the splitting $\Lambda^2\mathfrak{v}^* = \langle\omega\rangle\oplus\langle\omega\rangle^\perp$ as follows

$$\alpha := dz^\flat = b\omega+\alpha_0. \tag{4.8}$$

In particular, $\alpha_0\in\Lambda^{(2,0)+(0,2)}_0\mathfrak{v}^*\oplus\Lambda^{(1,1)}_0\mathfrak{v}^*$.

Remark 4.3. The 2-form $\alpha\in\Lambda^2\mathfrak{v}^*$ is nonzero, since $\mathfrak{n}$ is not abelian, and its rank determines the isomorphism class of $\mathfrak{n}$: $\mathfrak{n}\cong\mathfrak{h}_3\oplus\mathbb{R}^4$ if $\mathrm{rank}(\alpha) = 2$, $\mathfrak{n}\cong\mathfrak{h}_5\oplus\mathbb{R}^2$ if $\mathrm{rank}(\alpha)=4$ and $\mathfrak{n}\cong\mathfrak{h}_7$ if $\mathrm{rank}(\alpha)=6$.

Lemma 4.4. *Consider the* G_2*-structure* φ *given by* (4.1)*. Then, denoting as before* $\lambda := \frac{7}{12}\tau_0$, *one has* $b = 2\lambda$ *in the expression* (4.8) *of* α*. In addition, the condition* $\tau_2 = 0$ *implies* $\tau_1 = 0$ *and is equivalent to*

$$\alpha\in\Lambda^{(1,1)}\mathfrak{v}^* = \mathbb{R}\omega\oplus\Lambda^{(1,1)}_0\mathfrak{v}^*.$$

Proof. Using the expressions (4.1) and (4.2), we obtain

$$d\varphi = \omega\wedge dz^\flat = \omega\wedge\alpha, \qquad d\psi = -\Omega_-\wedge dz^\flat = -\Omega_-\wedge\alpha.$$

Hence

$$7\tau_0*1 = d\varphi\wedge\varphi = \omega\wedge\omega\wedge dz^\flat\wedge z^\flat = 6b*1.$$

From this, the first assertion follows.

Now, the first equation in (2.2) gives $12\tau_1 = *(\varphi\wedge*d\varphi)$, which in the present situation implies

$$\tau_1 = \frac{1}{12}*\Big((\omega\wedge z^\flat+\Omega_+)\wedge*(\omega\wedge dz^\flat)\Big).$$

This shows that $\tau_1\in\Lambda^1\mathfrak{v}^*$. The condition $\tau_2 = 0$ is equivalent to $d\psi = 4\tau_1\wedge\psi$, i.e., to

$$d\psi = 4\tau_1\wedge*\varphi = 4\tau_1\wedge\left(\frac{1}{2}\omega\wedge\omega+\Omega_-\wedge z^\flat\right). \tag{4.9}$$

Since $d\psi = -\Omega_- \wedge \alpha \in \Lambda^4\mathfrak{v}^*$, (4.9) gives

$$\tau_1 \wedge \Omega_- = 0\,,$$

which in turn implies that $\tau_1 = 0$, so that φ is coclosed. Now, from

$$0 = d\psi = -\Omega_- \wedge \alpha,$$

we deduce that $\alpha \in \Lambda^{(1,1)}\mathfrak{v}^*$. □

Lemma 4.4 proves Proposition 1.1 when $\dim(\mathfrak{n}') = 1$: every G_2T-structure $\varphi \in \Lambda^3\mathfrak{n}^*$ is coclosed. Moreover, it shows that

$$\alpha = dz^\flat = 2\lambda\omega + \alpha_0,$$

with $\alpha_0 \in \Lambda_0^{(1,1)}\mathfrak{v}^*$.

We next focus on the torsion form H_φ of φ and write its differential in terms of α_0, ω and λ.

Lemma 4.5. *Assume that the* G_2*-structure* φ *given by* (4.1) *has* $\tau_2 = 0$. *Then*

$$H_\varphi = 2\lambda\varphi - (4\lambda\omega - \alpha_0) \wedge z^\flat,$$

and

$$dH_\varphi = \alpha_0 \wedge \alpha_0 - 4\lambda^2\omega \wedge \omega\,.$$

Proof. From Lemma 4.4 we have $\tau_1 = 0$, so from (2.2) we get:

$$*\tau_3 = d\varphi - \frac{12}{7}\lambda * \varphi = \omega \wedge \alpha - \frac{12}{7}\lambda * \varphi,$$

and thus

$$\tau_3 = *\,(\omega \wedge \alpha) - \frac{12}{7}\lambda\varphi = (-\alpha_0 + 4\lambda\omega) \wedge z^\flat - \frac{12}{7}\lambda\varphi.$$

We then obtain

$$H_\varphi = \frac{2}{7}\lambda\varphi - \tau_3 = \frac{2}{7}\lambda\varphi - (-\alpha_0 + 4\lambda\omega) \wedge z^\flat + \frac{12}{7}\lambda\varphi = 2\lambda\varphi - (-\alpha_0 + 4\lambda\omega) \wedge z^\flat\,,$$

and therefore

$$\begin{aligned} dH_\varphi &= 2\lambda\omega \wedge \alpha - (-\alpha_0 + 4\lambda\omega) \wedge \alpha = -2\lambda\omega \wedge (\alpha_0 + 2\lambda\omega) + \alpha_0 \wedge (\alpha_0 + 2\lambda\omega) \\ &= -4\lambda^2\omega \wedge \omega + \alpha_0 \wedge \alpha_0\,, \end{aligned}$$

as required. □

In order to study the instanton equation, we prove the following.

Lemma 4.6. *Consider the* G_2*-structure* φ *given by* (4.1). *A 2-form* $F = F_\mathfrak{v} + \eta \wedge z^\flat \in \Lambda^2\mathfrak{n}^* = \Lambda^2\mathfrak{v}^* \oplus \mathfrak{v}^* \wedge z^\flat$ *satisfies*

$$\begin{cases} dF = 0\,, \\ F \wedge \psi = 0\,, \end{cases}$$

if and only if

$$\eta \wedge \alpha = 0\,, \quad \text{and} \quad F_\mathfrak{v} = \frac{1}{2} *_\mathfrak{v} (\eta \wedge \Omega_+) + \sigma\,, \quad \sigma \in \Lambda_0^{(1,1)}\mathfrak{v}^*\,.$$

In particular, if $\operatorname{rank}(\alpha) \in \{4, 6\}$, *then* $\eta = 0$ *and* $F = F_\mathfrak{v} \in \Lambda_0^{(1,1)}\mathfrak{v}^*$.

Proof. Since both $F_\mathfrak{v}$ and η are closed, the condition $dF = 0$ is equivalent to

$$\eta \wedge \alpha = 0\,,$$

which implies $\eta = 0$ if $\operatorname{rank}(\alpha) \in \{4, 6\}$. Next we impose $F \wedge \psi = 0$. Since

$$F \wedge \psi = \frac{1}{2}F_\mathfrak{v} \wedge \omega \wedge \omega + \frac{1}{2}\eta \wedge z^\flat \wedge \omega \wedge \omega + F_\mathfrak{v} \wedge \Omega_- \wedge z^\flat\,,$$

we get

$$F \wedge \psi = 0 \iff \begin{cases} F_{\mathfrak{v}} \wedge \omega \wedge \omega = 0\,, \\ F_{\mathfrak{v}} \wedge \Omega_- = -\frac{1}{2}\eta \wedge \omega \wedge \omega\,. \end{cases}$$

The first equation shows that one can write $F_{\mathfrak{v}} = *_{\mathfrak{v}}(x \wedge \Omega_+) + \sigma$ for some $\sigma \in \Lambda_0^{(1,1)}\mathfrak{v}^*$ and $x \in \mathfrak{v}^*$. Using the general formula $*_{\mathfrak{v}}(x \wedge \Omega_+) \wedge \Omega_- = -x \wedge \omega \wedge \omega$, we get $x = \frac{1}{2}\eta$. □

Lemma 4.7. *Assume that the* G_2*-structure* φ *given by* (4.1) *has* $\tau_2 = 0$*, and let* H_φ *be the corresponding torsion* 3*-form. Let* $\varepsilon_1, \ldots, \varepsilon_k \in \mathbb{R} \smallsetminus \{0\}$*. Then the* 2*-forms* $F^r = F^r_{\mathfrak{v}} + \eta^r \wedge z^\flat \in \Lambda^2\mathfrak{n}^*$*,* $r = 1, \ldots, k$*, satisfy*

$$\begin{cases} dF^r = 0 \quad \text{for every } r\,, \\ F^r \wedge \psi = 0 \quad \text{for every } r\,, \\ dH_\varphi = \sum_{r=1}^k \varepsilon_r\, F^r \wedge F^r\,, \end{cases} \tag{4.10}$$

if and only if

$$\begin{cases} \eta^r \wedge \alpha = 0, \\ \sum_{r=1}^k \varepsilon_r *_{\mathfrak{v}} (\eta^r \wedge \Omega_+) \wedge \sigma^r = 0 = \sum \varepsilon_r (*_{\mathfrak{v}}(\eta^r \wedge \Omega_+)) \wedge (*_{\mathfrak{v}}(\eta^r \wedge \Omega_+)), \\ \alpha_0 \wedge \alpha_0 - 4\lambda^2 \omega \wedge \omega = \sum_{r=1}^k \varepsilon_r \sigma^r \wedge \sigma^r, \\ \sum_{r=1}^k \varepsilon_r \left(*_{\mathfrak{v}}(\eta^r \wedge \Omega_+) \wedge \eta^r + 2\sigma^r \wedge \eta^r\right) = 0\,, \end{cases} \tag{4.11}$$

where σ^r *is the component of* F^r *in* $\Lambda_0^{(1,1)}\mathfrak{v}^*$ *according to the decomposition* (4.3)*.*

Proof. Lemma 4.6 implies that each F^r can be written as

$$F^r = \frac{1}{2} *_{\mathfrak{v}} (\eta^r \wedge \Omega_+) + \sigma^r + \eta^r \wedge z^\flat\,,$$

with $\eta^r \wedge \alpha = 0$. It follows

$$F^r \wedge F^r = *_{\mathfrak{v}}(\eta^r \wedge \Omega_+) \wedge \eta^r \wedge z^\flat + 2\sigma^r \wedge \eta^r \wedge z^\flat + *_{\mathfrak{v}}(\eta^r \wedge \Omega_+) \wedge \sigma^r + \frac{1}{4}(*_{\mathfrak{v}}(\eta^r \wedge \Omega_+)) \wedge (*_{\mathfrak{v}}(\eta^r \wedge \Omega_+)) + \sigma^r \wedge \sigma^r.$$

By Lemma 4.5, the last equation of system (3.1) is equivalent to the system

$$\begin{cases} \alpha_0 \wedge \alpha_0 - 4\lambda^2 \omega \wedge \omega = \sum_{r=1}^k \varepsilon_r \left(\frac{1}{4}(*_{\mathfrak{v}}(\eta^r \wedge \Omega_+)) \wedge (*_{\mathfrak{v}}(\eta^r \wedge \Omega_+)) + \sigma^r \wedge \sigma^r + *_{\mathfrak{v}}(\eta^r \wedge \Omega_+) \wedge \sigma^r\right), \\ 0 = \sum_{r=1}^k \varepsilon_r \left(*_{\mathfrak{v}}(\eta^r \wedge \Omega_+) \wedge \eta^r + 2\sigma^r \wedge \eta^r\right)\,. \end{cases}$$

Since the forms

$$\alpha_0 \wedge \alpha_0\,, \quad \omega \wedge \omega\,, \quad (*_{\mathfrak{v}}(\eta^r \wedge \Omega_+)) \wedge (*_{\mathfrak{v}}(\eta^r \wedge \Omega_+))\,, \quad \sigma^r \wedge \sigma^r$$

belong to $\Lambda^{(2,2)}\mathfrak{v}^*$, while

$$*_{\mathfrak{v}}(\eta^r \wedge \Omega_+) \wedge \sigma^r$$

belongs to $\Lambda^{(3,1)+(1,3)}\mathfrak{v}^*$, the above system splits into

$$\begin{cases} 0 = \sum_{r=1}^k \varepsilon_r *_{\mathfrak{v}} (\eta^r \wedge \Omega_+) \wedge \sigma^r, \\ \alpha_0 \wedge \alpha_0 - 4\lambda^2 \omega \wedge \omega = \sum_{r=1}^k \varepsilon_r \left(\frac{1}{4}(*_{\mathfrak{v}}(\eta^r \wedge \Omega_+)) \wedge (*_{\mathfrak{v}}(\eta^r \wedge \Omega_+)) + \sigma^r \wedge \sigma^r\right), \\ 0 = \sum_{r=1}^k \varepsilon_r \left(*_{\mathfrak{v}}(\eta^r \wedge \Omega_+) \wedge \eta^r + 2\sigma^r \wedge \eta^r\right)\,. \end{cases} \tag{4.12}$$

Taking the wedge product of the last equation with Ω_+ and using $\sigma^r \wedge \Omega_+ = 0$ and (4.6) we get

$$\sum_{r=1}^k \varepsilon_r |\eta^r|^2 = 0\,. \tag{4.13}$$

Next we show that $\sum_{r=1}^k \varepsilon_r (*_{\mathfrak{v}}(\eta^r \wedge \Omega_+)) \wedge (*_{\mathfrak{v}}(\eta^r \wedge \Omega_+)) = 0$. We can assume that $\operatorname{rank}(\alpha) = 2$ since otherwise $\eta^r = 0$ for all r. From the fact that $\alpha \wedge \eta^r = 0$ and $\alpha \in \Lambda^{(1,1)}\mathfrak{v}^*$, we also get $\alpha \wedge J\eta^r = 0$ for every r, thus showing that for every r there exists c_r with $c_r\alpha = \eta^r \wedge J\eta^r$. Taking the scalar product with ω yields

$$|\eta^r|^2 = 6\lambda c_r\,, \tag{4.14}$$

so from (4.13) we obtain $\lambda \sum \varepsilon_r c_r = 0$. Note that we can assume $\lambda \neq 0$ since otherwise (4.14) yields $\eta^r = 0$ for all r. Therefore, we get

$$\sum_{r=1}^{k} \varepsilon_r \eta^r \wedge J\eta^r = 0. \tag{4.15}$$

On the other hand, using (4.5), we have that (4.15) yields

$$\sum \varepsilon_r (*_{\mathfrak{v}}(\eta^r \wedge \Omega_+)) \wedge (*_{\mathfrak{v}}(\eta^r \wedge \Omega_+)) = 0, \tag{4.16}$$

showing that the middle equation in the system (4.12) decouples into (4.16) and

$$\alpha_0 \wedge \alpha_0 - 4\lambda^2 \omega \wedge \omega = \sum_{r=1}^{k} \varepsilon_r \sigma^r \wedge \sigma^r. \tag{4.17}$$

Hence the claim follows. □

Remark 4.8. We observe that Equation (4.17) has no solution if all ε_r's are negative. Indeed, wedging both sides of equation (4.17) by ω, we obtain

$$-|\alpha_0|^2 - 24\lambda^2 = -\sum_{r=1}^{k} \varepsilon_r |\sigma^r|^2\,.$$

If all ε_r's are negative, we then get $|\alpha_0| = \lambda = 0$, whence $\alpha = 0$ by (4.8) and Lemma 4.4, contradicting the assumption that $\mathfrak{n}$ is not abelian. This is consistent with Corollary 2.4.

If $\eta^k = 0$, for all $1 \leq k \leq r$, then the system (4.11) reduces to Equation (4.17). This happens, for instance, when $\operatorname{rank}(\alpha) \in \{4, 6\}$ by Lemma 4.6. In this case, in order to find solutions to the heterotic G$_2$-system, we can focus on Equation (4.17) on $\mathbb{R}^6$ equipped with the standard basis $\mathcal{B} = \{e_1, \ldots, e_6\}$ and the standard SU(3)-structure

$$\omega = e^{12} + e^{34} + e^{56}\,, \quad \Omega_+ = e^{135} - e^{146} - e^{236} - e^{245}\,, \tag{4.18}$$

and use the next result.

Proposition 4.9. *Consider the vector space $\mathbb{R}^6$ endowed with the with the standard* SU(3)*-structure* (4.18)*. Assume that $\alpha_0, \sigma^1, \ldots, \sigma^k \in \Lambda_0^{(1,1)}(\mathbb{R}^6)^*$, $\lambda \in \mathbb{R}$, $\varepsilon_1 \ldots, \varepsilon_k \in \mathbb{R} \smallsetminus \{0\}$, is a solution to* (4.17)*, and that the forms $2\lambda\omega + \alpha_0$ and $\sigma^1, \ldots, \sigma^k$ are integral with respect to the standard basis $\mathcal{B}$ of $\mathbb{R}^6$. Then, this solution gives rise to a 2-step nilmanifold endowed with a k-torus bundle and admitting an invariant solution to the heterotic* G$_2$*-system.*

Proof. Consider the 7-dimensional vector space $\mathfrak{n} := \mathbb{R}^6 \oplus \langle z \rangle$ and endow it with the product metric g for which z is a unit vector. Then, imposing

$$dz^\flat := 2\lambda\omega + \alpha_0, \quad de^i = 0,\ 1 \leq i \leq 6,$$

gives $\mathfrak{n}$ the structure of a 2-step nilpotent Lie algebra with 1-dimensional derived algebra $\mathfrak{n}' = \langle z \rangle$. Since the 2-form $2\lambda\omega + \alpha_0$ is integral, the simply connected 2-step nilpotent Lie group $N = \exp(\mathfrak{n})$ has a cocompact lattice given by $\Gamma = \operatorname{span}_{\mathbb{Z}}(6e_1, \ldots, 6e_6, z)$, and $M = \Gamma\backslash N$ is a 2-step nilmanifold. Moreover, since the σ^r's are integral, Theorem A.3 ensures the existence of a principal $\mathbb{T}^k$-bundle $P \to M = \Gamma\backslash N$ endowed with a connection θ whose curvature is $F_\theta = \sum_{r=1}^{k} \sigma^r t_r$, where $\{t_1, \ldots, t_k\}$ is a basis of the Lie algebra of $\mathbb{T}^k$. The 3-form $\varphi := \omega \wedge z^\flat + \Omega_+$ defines a G$_2$-structure on $\mathfrak{n}$ inducing the product metric g. By Lemma 4.4, φ is coclosed and has $\tau_0 = \frac{12}{7}\lambda$, so it gives rise to a G$_2$-structure of the same type on $M = \Gamma\backslash N$. If we define $\langle\cdot,\cdot\rangle_{\mathfrak{k}} := \sum_{r=1}^{k} \varepsilon_r t^r \otimes t^r$, we then obtain an invariant solution $(\varphi, P, \theta, \langle\cdot,\cdot\rangle_{\mathfrak{k}})$ to the heterotic G_2-system on $M = \Gamma\backslash N$.

□

Motivated by the last result, we now consider a 6-dimensional vector space $\mathfrak{v}$ equipped with an SU(3)-structure (ω, Ω_+), and we focus on Equation (4.17)

$$\alpha_0 \wedge \alpha_0 - 4\lambda^2 \omega \wedge \omega = \sum_{r=1}^{k} \varepsilon_r \sigma^r \wedge \sigma^r$$

for the unknowns $\alpha_0, \sigma^1, \ldots, \sigma^k \in \Lambda_0^{(1,1)}\mathfrak{v}^*$, $\lambda \in \mathbb{R}$, and $\varepsilon_1, \ldots, \varepsilon_k \in \mathbb{R} \smallsetminus \{0\}$. Note that λ and α_0 cannot be both zero, as otherwise the Lie algebra $\mathfrak{n}$ would be abelian.

In order to study Equation (4.17), which involves 4-forms on $\mathfrak{v}$, we rewrite it in terms of endomorphisms of $\mathfrak{v}$. To do this, we use the dual Lefschetz operator, which can we written using an orthonormal basis $\{e_1 \ldots, e_6\}$ of $\mathfrak{v}$ as

$$\Lambda = \frac{1}{2} \sum_{i=1}^{6} J e_i \lrcorner e_i \lrcorner \,.$$

This operator is an isomorphism from $\Lambda^4\mathfrak{v}^*$ to $\Lambda^2\mathfrak{v}^*$.

One can easily check that $\Lambda(\omega \wedge \omega) = 4\omega$ and $\Lambda(\sigma \wedge \sigma)(\cdot,\cdot) = 2\langle A^2 J \cdot, \cdot\rangle$, for every $\sigma \in \Lambda_0^{(1,1)}\mathfrak{v}^*$, where A is the skew-symmetric endomorphism commuting with J defined by $\sigma(\cdot,\cdot) = \langle A\cdot,\cdot\rangle$. Applying Λ, Equation (4.17) can be rewritten in the following, equivalent, form

$$L_0^2 - \sum_{r=1}^{k} \varepsilon_r L_r^2 = -8\lambda^2 \mathrm{Id}\,, \tag{4.19}$$

where the L_r's are the trace-free symmetric endomorphisms commuting with J defined by

$$\alpha_0(\cdot,\cdot) = \langle L_0 J\cdot,\cdot\rangle\,, \quad \sigma^r(\cdot,\cdot) = \langle L_r J\cdot,\cdot\rangle\,.$$

We now discuss some properties of the solutions of (4.19) that will be useful in the proof of Theorem 1.2. Assume that one of the ε_r's, say ε_1, is positive, and all the others are negative. In this case, by setting

$$A = \sqrt{\frac{\varepsilon_1}{8}}\, L_1\,, \quad B_1 = \sqrt{\frac{1}{8}}\, L_0\,, \quad B_r = \sqrt{\frac{-\varepsilon_r}{8}}\, L_r\,, \quad 2 \le r \le k\,, \tag{4.20}$$

we can rewrite (4.19) as

$$A^2 = \lambda^2\, \mathrm{Id} + \sum_{r=1}^{k} B_r^2\,. \tag{4.21}$$

Proposition 4.10. *Let $A, B_1, \ldots, B_k$ be trace-free symmetric endomorphisms of a 6-dimensional Hermitian vector space $(\mathfrak{v}, J)$ commuting with J and satisfying Equation (4.21). Then $\lambda = 0$ and the endomorphisms $A^2, B_1^2, \ldots, B_k^2$ are positively collinear.*

Proof. Composing A^2 with itself and using (4.21), we get

$$A^4 = \lambda^4 \mathrm{Id} + 2\lambda^2 \sum_{r=1}^{k} B_r^2 + \sum_{r=1}^{k} B_r^4 + \sum_{r \ne s} B_r^2 B_s^2\,. \tag{4.22}$$

Next, we observe that if X is a trace-free symmetric endomorphism, then

$$\mathrm{tr}\,(X^4) = \frac{(\mathrm{tr}(X^2))^2}{4}. \tag{4.23}$$

Indeed, if $x_1, x_2, -x_1-x_2 \in \mathbb{R}$ are the eigenvalues of X (all of algebraic multiplicity 2), we have

$$\mathrm{tr}\,(X^2) = 2(x_1^2 + x_2^2 + (x_1+x_2)^2) = 4(x_1^2 + x_2^2 + x_1x_2)$$

and

$$\mathrm{tr}\,(X^4) = 2(x_1^4 + x_2^4 + (x_1+x_2)^4) = 4(x_1^4 + x_2^4 + 2x_1^3x_2 + 2x_1x_2^3 + 3x_1^2x_2^2) = 4(x_1^2 + x_2^2 + x_1x_2)^2\,,$$

from which (4.23) follows. Taking the trace of both sides of (4.22) and applying (4.23), we obtain:

$$\frac{(\mathrm{tr}(A^2))^2}{4} = \mathrm{tr}(A^4) = 6\lambda^4 + 2\lambda^2\sum_{r=1}^k \mathrm{tr}\,(B_r^2) + \sum_{r=1}^k \mathrm{tr}\,(B_r^4) + \sum_{r\neq s}\mathrm{tr}(B_r^2B_s^2)\,.$$

Moreover, from (4.21) and (4.23), we have

$$\begin{aligned}\frac{(\mathrm{tr}(A^2))^2}{4} &= \frac{1}{4}\left(\mathrm{tr}\left(\lambda^2\mathrm{Id} + \sum_{r=1}^k B_r^2\right)\right)^2 = 9\lambda^4 + 3\lambda^2\sum_{r=1}^k\mathrm{tr}(B_r^2) + \frac{1}{4}\sum_{r=1}^k(\mathrm{tr}(B_r^2))^2 + \frac{1}{4}\sum_{r\neq s}\mathrm{tr}(B_r^2)\mathrm{tr}(B_s^2)\\ &= 9\lambda^4 + 3\lambda^2\sum_{r=1}^k\mathrm{tr}(B_r^2) + \sum_{r=1}^k\mathrm{tr}(B_r^4) + \frac{1}{4}\sum_{r\neq s}\mathrm{tr}(B_r^2)\mathrm{tr}(B_s^2)\,.\end{aligned}$$

Comparing these two equations yields

$$3\lambda^4 + \lambda^2\sum_{r=1}^k \mathrm{tr}\,(B_r^2) + \frac{1}{4}\sum_{r\neq s}\mathrm{tr}(B_r^2)\mathrm{tr}(B_s^2) - \sum_{r\neq s}\mathrm{tr}(B_r^2B_s^2) = 0\,. \tag{4.24}$$

Now, we observe that (4.23) and the Cauchy-Schwarz inequality imply:

$$\mathrm{tr}(B_s^2B_r^2) \leq \sqrt{\mathrm{tr}(B_r^4)}\,\sqrt{\mathrm{tr}(B_s^4)} = \frac{1}{4}\mathrm{tr}(B_r^2)\mathrm{tr}(B_s^2)\,.$$

so (4.24) yields

$$3\lambda^4 + \lambda^2\sum_{r=1}^k\mathrm{tr}(B_r^2) \leq 0$$

which forces $\lambda = 0$.

Moreover, the equality case in the Cauchy-Schwarz inequality implies that the symmetric endomorphisms B_r^2 are positively collinear for $1 \leq r \leq k$. Finally, since $A^2 = \sum_{r=1}^k B_r^2$, the same conclusion holds for A^2 as well. □

We apply the previous proposition in our case. Since $\lambda = 0$, the 2-form α_0 is equal to the 2-form $\alpha = dz^\flat$, which is non-zero. Thus the endomorphism B_1 defined in (4.20) must be non-zero. Consequently, there exist $a_2, \ldots, a_k \in \mathbb{R}$ such that

$$B_r^2 = a_r^2 B_1^2\,,$$

for all $2 \leq r \leq k$, and therefore

$$A^2 = \left(1 + \sum_{r=2}^k a_r^2\right)B_1^2.$$

Lemma 4.11. *Let A and B be trace-free symmetric endomorphisms of $\mathfrak{v}$ commuting with the complex structure J and satisfying*

$$A^2 = a^2B^2,$$

for a some $a \in \mathbb{R}$. If B is invertible, then $A = \pm aB$.

Proof. If $a = 0$ the statement is obvious. We can then focus on the case $a \neq 0$, where it is not restrictive assuming $a = 1$, so that $A^2 = B^2$. One can find a basis of $\mathfrak{v}$ such that the matrices representing B and J with respect to it are the following: $B = \begin{pmatrix} D & 0_3 \\ 0_3 & D \end{pmatrix}$, where $D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$ with $d_1 + d_2 + d_3 = 0$ and $d_1 d_2 d_3 \neq 0$, and $J = \begin{pmatrix} 0_3 & -I_3 \\ I_3 & 0_3 \end{pmatrix}$. Since A is symmetric, trace-free, and commutes with J, it is represented by a block-diagonal matrix of the form $A = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$, where X is a symmetric and trace-free 3×3 matrix, and Y is a skew-symmetric 3×3 matrix. The condition $A^2 = B^2$ reads then

$$\begin{cases} X^2 - Y^2 = D^2, \\ XY + YX = 0_3. \end{cases} \tag{4.25}$$

If $X = 0$, the first equation of (4.25) shows that Y is invertible, which is impossible for a skew-symmetric 3×3 matrix. Since X is trace-free, its kernel can be at most 1-dimensional. Assume that $\ker(X)$ is a line. The second equation of the system (4.25) shows that Y preserves the kernel of X, and since Y is skew-symmetric, it must vanish on $\ker(X)$. This contradicts the first equation, since D would then also vanish on $\ker(X)$. Therefore, X is invertible.

Since $\mathrm{tr}(X) = 0$, if λ is an eigenvalue of X, then $-\lambda$ is not an eigenvalue of X. The second equation of (4.25) thus shows that Y vanishes on the eigenspaces of X, so $Y = 0_3$ and $X^2 = D^2$. The spectrum of X is therefore $\{\varepsilon_1 d_1, \varepsilon_2 d_2, \varepsilon_3 d_3\}$, where $\varepsilon_i \in \{\pm 1\}$. Since X and D are trace-free, all signs ε_i are equal. If D^2 has three different eigenvalues, clearly X preserves the three eigenspaces, so it has diagonal form in the given basis, whence $X = \pm D$. If D^2 has a double eigenvalue, then one can assume $d_1 = d_2$ and $d_3 = -2d_1$, so X has block-diagonal form $X = \begin{pmatrix} U & 0 \\ 0 & \varepsilon d_3, \end{pmatrix}$, where $\varepsilon \in \{\pm 1\}$ and U is a 2×2 symmetric matrix satisfying $\mathrm{tr}(U) = -\varepsilon d_3 = 2\varepsilon d_1$ and $U^2 = d_1^2 I_2$. Then clearly $U = \varepsilon d_1 I_2$, so $X = \varepsilon D$, and thus $A = \varepsilon B$. ∎

We are now ready to give the proof of Theorem 1.2.

4.1. **Proof of Theorem 1.2-(i).** If $M = \Gamma \backslash N$ admits an invariant solution $(\varphi, P, \theta, \langle \cdot, \cdot \rangle_{\mathfrak{k}})$ to the heterotic G_2-system and $\dim(\mathfrak{n}') = 1$, then the above discussion shows that there exists a g_φ-orthogonal decomposition $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{n}'$, with $\mathfrak{n}' = \langle z \rangle$ and $dz^\flat = \alpha = 2\lambda\omega + \alpha_0$, where $\alpha_0 \in \Lambda_0^{(1,1)}\mathfrak{v}^*$ is a primitive form of type $(1,1)$. If $\lambda = 0$, then $\alpha = \alpha_0$, so $\alpha \wedge \alpha \neq 0$ by (4.4). Consequently, $\mathfrak{n}$ cannot be isomorphic to $\mathfrak{h}_3 \oplus \mathbb{R}^4$, and thus $\mathfrak{n} \cong \mathfrak{h}_5 \oplus \mathbb{R}^2$ if $\mathrm{rank}(\alpha) = 4$ or $\mathfrak{n} \cong \mathfrak{h}_7$ if $\mathrm{rank}(\alpha) = 6$. Note that, in both cases, the heterotic G_2-system reduces to Equation (4.17) by Lemma 4.6, as all of the η^r's appearing in (4.11) must be zero.

4.2. **Proof of Theorem 1.2-(ii).** By Proposition 4.9, if we consider $\mathbb{R}^6$ endowed with the standard $\mathrm{SU}(3)$-structure, then a solution to Equation (4.17) with $\lambda = 0$, $k = k_+ = 1$, $\alpha_0, \sigma^1 \in \Lambda_0^{(1,1)}(\mathbb{R}^6)^*$ integral 2-forms with respect to the standard basis of $\mathbb{R}^6$, and $\mathrm{rank}(\alpha_0) = 4$ gives rise to a solution to the heterotic G_2-system on a 2-step nilmanifold endowed with a principal S^1-bundle and with corresponding Lie algebra $\mathfrak{n} \cong \mathfrak{h}_5 \oplus \mathbb{R}^2$. The S^1-bundle is non-trivial if the curvature form $F = \sigma^1$ is not exact, namely if α_0 and σ^1 are not proportional.

Equation (4.17) is equivalent to (4.19), which now reads

$$L_0^2 = \varepsilon_1 L_1^2, \tag{4.26}$$

where L_0 and L_1 are trace-free symmetric endomorphisms of $\mathbb{R}^6$ commuting with J such that $\alpha_0 = \langle L_0 J \cdot, \cdot \rangle$ and $\sigma^1 = \langle L_1 J \cdot, \cdot \rangle$. We can take $\varepsilon_1 = 1$ and search for solutions for which L_0 has rank 4 and L_1 is not proportional to L_0.

A solution is given, for instance, by the endomorphisms represented by the following integer matrices with respect to the standard basis of $\mathbb{R}^6$:

$$L_0 = \operatorname{diag}(5,5,-5,-5,0,0)\,, \qquad L_1 = \begin{pmatrix} 0 & 0 & 3 & 4 & 0 & 0 \\ 0 & 0 & -4 & 3 & 0 & 0 \\ 3 & -4 & 0 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding 2-forms are

$$\alpha_0 = 5\,e^{12} - 5\,e^{34}, \qquad F = \sigma^1 = -4\,e^{13} + 3\,e^{14} - 3\,e^{23} - 4\,e^{24}.$$

We can now conclude as in the proof of Proposition 4.9. For the reader's convenience, we sketch the argument here. The 7-dimensional 2-step nilpotent Lie algebra is obtained considering the vector space $\mathfrak{n} := \mathbb{R}^6 \oplus \langle z \rangle$ endowed with the product metric g for which z has unit length, and setting

$$de^i = 0,\ 1 \le i \le 6, \qquad dz^\flat = \alpha_0 = 5\,e^{12} - 5\,e^{34}.$$

The coclosed G$_2$-structure on $\mathfrak{n}$ is defined by the 3-form $\varphi = \omega \wedge z^\flat + \Omega_+$, which induces the metric g. Since both α_0 and the closed 2-form F are integral with respect to the basis $\{e_1, \dots, e_6, z\}$ of $\mathfrak{n}$, we can apply Theorem A.3 and obtain the desired solution on the 2-step nilmanifold $M = \Gamma\backslash N$, where $N = \exp(\mathfrak{n})$ and $\Gamma = \exp(\operatorname{span}_{\mathbb{Z}}(6e_1, 6e_2, 6e_3, 6e_4, 6e_5, 6e_6, z))$.

4.3. **Proof of Theorem 1.2-(iii).** We now study the existence of solutions to equation (4.17) satisfying the requirements of Proposition 4.9 and such that $\operatorname{rank}(\alpha_0) = 6$, so that $\mathfrak{n} \cong \mathfrak{h}_7$. By Proposition 4.10, if we take either $k = k_+ = 1$ or $k > 1$ and $k_+ = 1$, then every solution must have $\lambda = 0$. Without loss of generality, we may assume $\varepsilon_1 > 0$ and, if $k > 1$, $\varepsilon_2, \dots, \varepsilon_k < 0$. We can then rewrite equation (4.17) as (4.21) with $\lambda = 0$, i.e.,

$$A^2 = \sum_{r=1}^{k} B_r^2\,,$$

where A and B_r's are the trace-free symmetric endomorphisms of $\mathbb{R}^6$ commuting with J defined in (4.20). If $\operatorname{rank}(\alpha_0) = 6$, the endomorphism B_1 must be invertible. In particular, A must be non-zero, so the component $F^1 = \sigma^1$ of the curvature is non-zero. Combining Proposition 4.10 and Lemma 4.11 (or applying just the latter if $k = k_+ = 1$), we obtain that A, and B_r, for $2 \le r \le k$, in (4.21), are proportional to B_1, showing that all curvature forms F^r are proportional to the exact 2-form $\alpha = dz^\flat$. It follows that every solution to the heterotic G$_2$-system has trivial torus bundle and non-zero curvature if $k_+ = 1$.

Assume now that $\lambda = 0$ and $k_- = 0$. Then, Equation (4.19) becomes

$$L_0^2 = \sum_{r=1}^{k} \varepsilon_r L_r^2,$$

with $\varepsilon_r > 0$, for $1 \le r \le k$. Since L_0 is invertible, at least one of the endomorphisms L_r is non-zero, so at least one component of the curvature is non-zero. Moreover, Proposition 4.10 implies that the endomorphisms $L_0^2, \varepsilon_1 L_1^2, \dots, \varepsilon_k L_k^2$ are positively collinear. We then have

$$L_r^2 = \frac{1}{\varepsilon_r}\, a_r^2 L_0^2,$$

for certain $a_r \in \mathbb{R}$, $1 \le r \le k$, with at least one of them different from zero. By Lemma 4.11, the endomorphisms L_r are either zero or non-zero multiples of L_0. This shows that all non-zero curvature forms F^r are proportional to the exact 2-form $\alpha = dz^\flat$, whence it follows that every solution to the heterotic G$_2$-system has trivial torus bundle and non-zero curvature.

To conclude the proof, it is sufficient to construct an example with $k_+ = 2$, $k_- = 1$ and non-trivial torus bundle. This is equivalent to finding a suitable solution to Equation (4.19) with $k = 3$ and with two positive and one negative ε_r, $1 \leq r \leq 3$. Without loss of generality, we assume $\varepsilon_1, \varepsilon_2 > 0$, $\varepsilon_3 < 0$, and look for solutions of the equation

$$L_0^2 - \varepsilon_3 L_3^2 = \varepsilon_1 L_1^2 + \varepsilon_2 L_2^2,$$

such that $\mathrm{rank}(L_0) = 6$ and at least one of the endomorphisms L_1, L_2, L_3 is not proportional to L_0. We can take, for instance, $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = 1$ and the endomorphisms represented by the following matrices with respect to the standard basis of $\mathbb{R}^6$

$$L_0 = L_2 = \mathrm{diag}\,(1, 1, -2, -2, 1, 1)\,, \qquad L_1 = L_3 = \mathrm{diag}\,(1, 1, -1, -1, 0, 0)\,.$$

They correspond to the following 2-forms

$$\alpha_0 = \sigma^2 = e^{12} - 2\,e^{34} + e^{56}, \qquad \sigma^1 = \sigma^3 = e^{12} - e^{34}.$$

We can now conclude as in the proof of Proposition 4.9.

4.4. **Proof of Theorem 1.2-(iv).** Proposition 4.10 implies that there are no invariant solutions to the heterotic G_2-system with $k_+ = 1$ and $\lambda \neq 0$ on a 2-step nilmanifold with $\dim(\mathfrak{n}') = 1$.

To conclude the proof of Theorem 1.2-(iv) it is then sufficient to show that when $k = k_+ = 2$, the heterotic G_2-system can be solved for each of the three isomorphism classes of 2-step nilpotent Lie algebras $\mathfrak{n}$ with $\dim(\mathfrak{n}') = 1$. In view of Proposition 4.9, the problem reduces to finding a solution $(\lambda, \alpha_0, \sigma^1, \sigma^2, \varepsilon_1, \varepsilon_2)$ to equation (4.17) on $\mathbb{R}^6$ equipped the standard $\mathrm{SU}(3)$-structure (4.18), where $\lambda \in \mathbb{R} \smallsetminus \{0\}$ and the forms $\alpha = 2\lambda\omega + \alpha_0, \sigma^1, \sigma^2$ are integral. The rank of the form $\alpha = 2\lambda\omega + \alpha_0$ then determines the isomorphism class of the corresponding Lie algebra.

A straightforward computation shows that we have the following solutions:

- $\lambda = \frac{1}{2}\,, \alpha_0 = -e^{12} - e^{34} + 2\,e^{56}, \sigma_1 = e^{12} - e^{56}$, $\sigma_2 = e^{34} - e^{56}$, $\varepsilon_1 = \varepsilon_2 = 3$.
  In this case $\alpha = 3e^{56}$, so the corresponding Lie algebra is isomorphic to $\mathfrak{h}_3 \oplus \mathbb{R}^4$.
- $\lambda = \frac{1}{2}\,, \alpha_0 = -e^{12} + e^{56}$, $\sigma_1 = -e^{12} + 2\,e^{34} - e^{56}$, $\sigma_2 = e^{12} - e^{56}$, $\varepsilon_1 = \frac{1}{2}\,, \varepsilon_2 = \frac{5}{2}$.
  In this case $\alpha = e^{34} + 2\,e^{56}$, so the corresponding Lie algebra is isomorphic to $\mathfrak{h}_5 \oplus \mathbb{R}^2$.
- $\lambda = \frac{1}{2}\,, \alpha_0 = 0\,, \sigma_1 = -2\,e^{12} + e^{34} + e^{56}\,, \sigma_2 = e^{34} - e^{56}$, $\varepsilon_1 = \frac{1}{2}\,, \varepsilon_2 = \frac{3}{2}$.
  In this case $\alpha = e^{12} + e^{34} + e^{56}$, so the corresponding Lie algebra is isomorphic to $\mathfrak{h}_7$.

5. The case $\dim(\mathfrak{n}') = 2$ or 3.

We now consider the case $\dim(\mathfrak{n}') \geq 2$. We use an approach that is similar to one of the previous section, but in this case we will decompose the Lie algebra $\mathfrak{n}$ as the orthogonal direct sum of a 3-dimensional subspace calibrated by φ and its orthogonal complement $\mathfrak{r}$. The latter naturally inherits an $\mathrm{SU}(2)$-structure induced by φ, which can be used to rewrite (3.1) as a system of exterior forms on $\mathfrak{r}$.

Let φ be a G_2-structure on $\mathfrak{n}$. If $\dim(\mathfrak{n}') = 3$, we assume that φ calibrates $\mathfrak{n}'$, that is, there exists an orthonormal basis $\{z_1, z_2, z_3\}$ of $\mathfrak{n}'$ such that $|\varphi(z_1, z_2, z_3)| = 1$. Recall that $\varphi(z_1, z_2, z_3) = 1$ if and only if $z_1 \times_\varphi z_2 = z_3$, where $\times_\varphi$ denotes the vector cross product induced by φ.

Let us consider two orthonormal vectors $z_1, z_2 \in \mathfrak{n}'$ and let

$$z_3 \coloneqq \varphi(z_1, z_2, \cdot)^\sharp = z_1 \times_\varphi z_2.$$

The vector z_3 has unit length and is orthogonal to z_1 and z_2. The subspace $\langle z_1, z_2, z_3\rangle$ is calibrated by φ and it is contained in the center of $\mathfrak{n}$, but does not necessarily coincide with it. When $\dim(\mathfrak{n}') = 2$, we have $\mathfrak{n}' = \langle z_1, z_2\rangle$ and we will show that $dz_3^\flat = 0$. If $\dim(\mathfrak{n}') = 3$ and φ calibrates $\mathfrak{n}'$, we have $\mathfrak{n}' = \langle z_1, z_2, z_3\rangle$ and $dz_3^\flat \neq 0$.

Let $\mathfrak{r} \coloneqq \langle z_1, z_2, z_3\rangle^\perp$, so that $\mathfrak{n} = \mathfrak{r} \oplus \langle z_1, z_2, z_3\rangle$. Since G_2 acts transitively on orthonormal pairs of vectors, there exists an adapted basis $\{e_1, \dots, e_7\}$ of $\mathfrak{n}$ (i.e. a basis where φ is expressed by (2.1)), such that $e_5 = z_1$ and $e_6 = z_2$. Then by (2.1) we get $z_3 = \varphi(z_1, z_2, \cdot)^\sharp = e_7$, and denoting by $\{e^1, e^2, e^3, e^4, z^1, z^2, z^3\}$ the dual basis, we have

$$\varphi = \sum_{i=1}^{3} \omega_i \wedge z^i + z^1 \wedge z^2 \wedge z^3, \tag{5.1}$$

where $\omega_1 = e^{13} - e^{24}$, $\omega_2 = -e^{14} - e^{23}$ and $\omega_3 = e^{12} + e^{34}$ define an SU(2)-structure on $\mathfrak{r}$. The metric corresponding to the SU(2)-structure is the restriction of g_φ to $\mathfrak{r}$. Moreover, since $\langle z_1, z_2, z_3\rangle$ is calibrated by φ, its orthogonal complement $\mathfrak{r}$ is calibrated by $\psi = *\varphi$, so it is oriented by the volume form $\psi|_\mathfrak{r}$. We denote by $*_\mathfrak{r}$ the Hodge operator corresponding to this metric and orientation on $\mathfrak{r}$. The closed 2-forms $\omega_1, \omega_2, \omega_3$ are self-dual and pairwise orthogonal:

$$*_\mathfrak{r}\omega_i = \omega_i, \qquad \omega_i \wedge \omega_j = 2\delta_{ij} *_\mathfrak{r} 1\,, \qquad \forall i, j \in \{1, 2, 3\}\,.$$

From this observation and the expression of φ, we deduce that

$$\psi = *\varphi = \omega_1 \wedge z^2 \wedge z^3 + \omega_2 \wedge z^3 \wedge z^1 + \omega_3 \wedge z^1 \wedge z^2 + *_\mathfrak{r} 1\,. \tag{5.2}$$

We introduce the following notation

$$\alpha_i \coloneqq dz^i \in \Lambda^2\mathfrak{r}^*, \quad 1 \le i \le 3,$$

and

$$a_{ij} \coloneqq \langle \omega_i, \alpha_j\rangle = *_\mathfrak{r}(\omega_i \wedge \alpha_j)\,, \quad 1 \le i, j \le 3\,.$$

If $\dim(\mathfrak{n}') = 2$, then α_1 and α_2 are linearly independent and $\alpha_3 = 0$, since z_3 is orthogonal to $\mathfrak{n}'$. Consequently, $a_{i3} = 0$ for all $1 \le i \le 3$. If $\dim(\mathfrak{n}') = 3$, then α_1, α_2 and α_3 are linearly independent.

Lemma 5.1. *Consider the G_2-structure φ given by (5.1). Then*

$$\lambda = \frac{1}{6}\,(a_{11} + a_{22} + a_{33})\,. \tag{5.3}$$

Moreover, the condition $\tau_2 = 0$ is equivalent to

$$a_{ij} = a_{ji}\,, \qquad \forall i, j \in \{1, 2, 3\}\,. \tag{5.4}$$

and implies $\tau_1 = 0$.

Proof. We have

$$d\varphi = \sum_{i=1}^{3} \omega_i \wedge \alpha_i + \alpha_1 \wedge z^2 \wedge z^3 + \alpha_2 \wedge z^3 \wedge z^1 + \alpha_3 \wedge z^1 \wedge z^2\,, \tag{5.5}$$

which together with (5.1) and the fact that $\omega_i \wedge \alpha_i = a_{ii} *_\mathfrak{r} 1$ gives

$$\varphi \wedge d\varphi = 2(a_{11} + a_{22} + a_{33}) * 1\,. \tag{5.6}$$

On the other hand, from the first equation in (2.2) we obtain $\varphi\wedge d\varphi = 7\tau_0 *1 = 12\lambda *1$, which together with (5.6) implies (5.3). Moreover,

$$\begin{aligned} d\psi &= \omega_1 \wedge \alpha_2 \wedge z^3 - \omega_1 \wedge \alpha_3 \wedge z^2 + \omega_2 \wedge \alpha_3 \wedge z^1 - \omega_2 \wedge \alpha_1 \wedge z^3 + \omega_3 \wedge \alpha_1 \wedge z^2 - \omega_3 \wedge \alpha_2 \wedge z^1 \\ &= (a_{12} - a_{21}) *_\mathfrak{r} 1 \wedge z^3 + (a_{31} - a_{13}) *_\mathfrak{r} 1 \wedge z^2 + (a_{23} - a_{32}) *_\mathfrak{r} 1 \wedge z^1, \end{aligned}$$

and

$$12\tau_1 = \psi \lrcorner\, d\psi = (a_{12} - a_{21})z^3 + (a_{31} - a_{13})z^2 + (a_{23} - a_{32})z^1,$$

which implies

$$12\tau_1 \wedge \psi = ((a_{23} - a_{32})\omega_1 + (a_{31} - a_{13})\omega_2 + (a_{12} - a_{21})\omega_3) \wedge z^1 \wedge z^2 \wedge z^3\,.$$

Hence, the condition $\tau_2 = 0$, which by (2.2) is equivalent to $4\tau_1 \wedge \psi = d\psi$, is satisfied if and only if

$$a_{ij} = a_{ji}\,, \qquad \forall i, j \in \{1, 2, 3\}\,,$$

and, consequently, implies $\tau_1 = 0$. □

The previous result proves Proposition 1.1: if $\dim(\mathfrak{n}') = 2$ or $\dim(\mathfrak{n}') = 3$ and $\mathfrak{n}'$ is calibrated by φ, every G_2T-structure on $\mathfrak{n}$ is coclosed. Note however that, when $\dim(\mathfrak{n}') = 3$, there exist G_2T-structures with $\mathfrak{n}'$ not calibrated by φ that are not coclosed:

Example 5.2. For any real numbers x, y consider the 2-step nilpotent Lie algebra structure on $\mathbb{R}^7$ with $\mathfrak{n}' = \langle e_4, e_6, e_7\rangle$ and structure 2-forms

$$de^4 = e^{13} + xe^{12} + ye^{23}, \qquad de^6 = e^{15} - ye^{25}, \qquad de^7 = -e^{35} + xe^{25}.$$

Then the G_2 structure defined by (2.1) satisfies $d\psi = e^2 \wedge \psi$, so it has $\tau_2 = 0$ and $\tau_1 = \frac{1}{4}e^2$. Note that the Lie algebra is isomorphic to $\mathfrak{n}_{6,3} \oplus \mathbb{R}$ if $x = y = 0$, and to $\mathfrak{n}_{7,3,B}$ otherwise.

Combining Lemma 5.1 with [8, Prop. 4.4], we obtain the following.

Corollary 5.3. *If $\dim(\mathfrak{n}') = 2$ and φ is a G_2T-structure on $\mathfrak{n}$, then $\mathfrak{n}$ is decomposable. In particular, $\mathfrak{n}$ is isomorphic to one of the following Lie algebras: $\mathfrak{n}_{5,2} \oplus \mathbb{R}^2$, $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}$, $\mathfrak{h}_3^{\mathbb{C}} \oplus \mathbb{R}$, $\mathfrak{n}_{6,2} \oplus \mathbb{R}$.*

Lemma 5.4. *Assume that the G_2-structure φ given by (5.1) has $\tau_2 = 0$ (and thus $\tau_1 = 0$ by Lemma 5.1). Then*

$$dH_\varphi = -4\lambda\,\left(\alpha_1 \wedge z^2 \wedge z^3 + \alpha_2 \wedge z^3 \wedge z^1 + \alpha_3 \wedge z^1 \wedge z^2\right) + \left(12\lambda^2 - \sum_{i=1}^3 |\alpha_i|^2\right) *_{\mathfrak{r}} 1\,.$$

Proof. Using (5.5) and (5.3), we compute

$$\begin{aligned} *d\varphi &= *\left(\sum_{i=1}^3 \omega_i \wedge \alpha_i + \alpha_1 \wedge z^2 \wedge z^3 + \alpha_2 \wedge z^3 \wedge z^1 + \alpha_3 \wedge z^1 \wedge z^2\right) \\ &= \sum_{i=1}^3 a_{ii}\, z^1 \wedge z^2 \wedge z^1 + \sum_{i=1}^3 (*_{\mathfrak{r}}\alpha_i) \wedge z^i \\ &= 6\lambda\, z^1 \wedge z^2 \wedge z^3 + \sum_{i=1}^3 (*_{\mathfrak{r}}\alpha_i) \wedge z^i\,. \end{aligned}$$

Since $H_\varphi = 2\lambda\varphi - *d\varphi$ (cf. Remark 2.1), it follows that

$$H_\varphi = -6\lambda\, z^1 \wedge z^2 \wedge z^3 - \sum_{i=1}^3 (*_{\mathfrak{r}}\alpha_i) \wedge z^i + 2\lambda \sum_{i=1}^3 \omega_i \wedge z^i + 2\lambda\, z^1 \wedge z^2 \wedge z^3\,.$$

We then compute using again (5.3):

$$\begin{aligned} dH_\varphi =& -6\lambda\,\left(\alpha_1 \wedge z^2 \wedge z^3 - \alpha_2 \wedge z^1 \wedge z^3 + \alpha_3 \wedge z^1 \wedge z^2\right) - \sum_{i=1}^3 |\alpha_i|^2 *_{\mathfrak{r}} 1 \\ &+ 2\lambda \sum_{i=1}^3 \omega_i \wedge \alpha_i + 2\lambda\alpha_1 \wedge z^2 \wedge z^3 - 2\lambda\alpha_2 \wedge z^1 \wedge z^3 + 2\lambda\alpha_3 \wedge z^1 \wedge z^2 \\ =& \; -4\lambda\left(\alpha_1 \wedge z^2 \wedge z^3 + \alpha_2 \wedge z^3 \wedge z^1 + \alpha_3 \wedge z^1 \wedge z^2\right) + \left(12\lambda^2 - \sum_{i=1}^3 |\alpha_i|^2\right) *_{\mathfrak{r}} 1\,, \end{aligned}$$

□

We now focus on the instanton equation. We first note that any $F \in \Lambda^2\mathfrak{n}^*$ can be written as

$$F = F_0 + v_1 \wedge z^1 + v_2 \wedge z^2 + v_3 \wedge z^3 + a_1\, z^2 \wedge z^3 + a_2\, z^3 \wedge z^1 + a_3\, z^1 \wedge z^2\,, \tag{5.7}$$

for some $F_0 \in \Lambda^2\mathfrak{r}^*$, $v_1, v_2, v_3 \in \mathfrak{r}^*$, and $a_1, a_2, a_3 \in \mathbb{R}$.

Lemma 5.5. *Let $F \in \Lambda^2\mathfrak{n}^*$ be any 2-form, expressed as in (5.7). Then*

$$\begin{cases} dF = 0\,, \\ F \wedge \psi = 0, \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} F_0 \wedge \omega_i = 0\,, & \forall i \in \{1,2,3\}, \\ \sum_{i=1}^3 v_i \wedge \alpha_i = 0, \\ \sum_{i=1}^3 v_i \wedge \omega_i = 0, \\ a_1 = a_2 = a_3 = 0. \end{cases}$$

Proof. Since F_0, v_1, v_2, v_3 are closed, we have

$$dF = -v_1 \wedge \alpha_1 - v_2 \wedge \alpha_2 - v_3 \wedge \alpha_3 + (a_2\,\alpha_3 - a_3\alpha_2) \wedge z^1 + (a_3\,\alpha_1 - a_1\,\alpha_3) \wedge z^2 + (a_1\,\alpha_2 - a_2\,\alpha_1) \wedge z^3,$$

and then

$$dF = 0 \iff \begin{cases} v_1 \wedge \alpha_1 + v_2 \wedge \alpha_2 + v_3 \wedge \alpha_3 = 0, \\ a_2\,\alpha_3 - a_3\,\alpha_2 = 0, \\ a_3\,\alpha_1 - a_1\,\alpha_3 = 0, \\ a_1\,\alpha_2 - a_2\,\alpha_1 = 0. \end{cases}$$

If $\dim(\mathfrak{n}') = 2$, then $\alpha_3 = 0$ and α_1 and α_2 are linearly independent. If $\dim(\mathfrak{n}') = 3$, then α_1, α_2 and α_3 are linearly independent. In both cases, the previous system shows that $a_1 = a_2 = a_3 = 0$, hence

$$dF = 0 \iff \begin{cases} v_1 \wedge \alpha_1 + v_2 \wedge \alpha_2 + v_3 \wedge \alpha_3 = 0, \\ a_1 = a_2 = a_3 = 0. \end{cases}$$

Moreover,

$$\begin{aligned} F \wedge \psi &= (F_0 + v_1 \wedge z^1 + v_2 \wedge z^2 + v_3 \wedge z^3) \wedge (\omega_1 \wedge z^2 \wedge z^3 + \omega_2 \wedge z^3 \wedge z^1 + \omega_3 \wedge z^1 \wedge z^2 + *_{\mathfrak{r}}1) \\ &= F_0 \wedge \omega_1 \wedge z^2 \wedge z^3 + F_0 \wedge \omega_2 \wedge z^3 \wedge z^1 + F_0 \wedge \omega_3 \wedge z^1 \wedge z^2 + \left(\sum_{i=1}^3 \omega_i \wedge v_i\right) \wedge z^1 \wedge z^2 \wedge z^3\,, \end{aligned}$$

whence

$$F \wedge \psi = 0 \iff \begin{cases} F_0 \wedge \omega_i = 0\,, & \forall i \in \{1,2,3\} \\ \sum_{i=1}^3 v_i \wedge \omega_i = 0 \end{cases}$$

which implies the claim. □

Proposition 5.6. *Assume that the G$_2$-structure φ given by (5.1) has $\tau_2 = 0$ (and thus $\tau_1 = 0$ by Lemma 5.1). Consider $\varepsilon_1, \ldots, \varepsilon_k \in \mathbb{R} \setminus \{0\}$. Then the 2-forms $F^r = F_0^r + \sum_{i=1}^3 v_i^r \wedge z^i \in \Lambda^2\mathfrak{n}^*$, $1 \le r \le k$, satisfy*

$$\begin{cases} dF^r = 0\,, \\ F^r \wedge \psi = 0\,, \\ dH_\varphi = \sum_{r=1}^k \varepsilon_r\, F^r \wedge F^r\,, \end{cases} \tag{5.8}$$

if and only if the following equations are satisfied

$$F_0^r \wedge \omega_i = 0\,, \qquad \forall i \in \{1,2,3\},\ \forall r \in \{1,\ldots,k\}\,, \tag{5.9}$$

$$\textstyle\sum_{r=1}^k \varepsilon_r\, F_0^r \wedge v_i^r = 0\,, \qquad \forall i \in \{1,2,3\}\,, \tag{5.10}$$

$$\textstyle\sum_{i=1}^3 v_i^r \wedge \alpha_i = 0\,, \qquad \forall r \in \{1,\ldots,k\}, \tag{5.11}$$

$$\textstyle\sum_{i=1}^3 v_i^r \wedge \omega_i = 0\,, \qquad \forall r \in \{1,\ldots,k\}, \tag{5.12}$$

$$\textstyle\sum_{r=1}^k \varepsilon_r\, |F_0^r|^2 = \sum_{i=1}^3 |\alpha_i|^2 - 12\lambda^2\,, \tag{5.13}$$

$$\textstyle\sum_{r=1}^k \varepsilon_r\, v_1^r \wedge v_2^r = 2\lambda\,\alpha_3\,, \tag{5.14}$$

$$\textstyle\sum_{r=1}^k \varepsilon_r\, v_3^r \wedge v_1^r = 2\lambda\,\alpha_2\,, \tag{5.15}$$

$$\textstyle\sum_{r=1}^k \varepsilon_r\, v_2^r \wedge v_3^r = 2\lambda\,\alpha_1\,. \tag{5.16}$$

Proof. Equations (5.9), (5.11), (5.12) come from Lemma 5.5. The first one shows that $F_0^r \in \Lambda^2\mathfrak{r}^*$ is anti-self-dual, so $F_0^r \wedge F_0^r = -|F_0^r|^2 *_\mathfrak{r} 1$, for all $1 \le r \le k$. We then have

$$\begin{aligned} F^r \wedge F^r = &- |F_0^r|^2 *_\mathfrak{r} 1 - 2v_1^r \wedge v_2^r \wedge z^1 \wedge z^2 - 2v_1^r \wedge v_3^r \wedge z^1 \wedge z^3 \\ &- 2v_2^r \wedge v_3^r \wedge z^2 \wedge z^3 + \sum_{i=1}^3 F_0^r \wedge v_i^r \wedge z^i. \end{aligned} \tag{5.17}$$

Taking into account Lemma 5.4, we obtain that the equation $dH_\varphi = \sum_{r=1}^k \varepsilon_r F^r \wedge F^r$ is equivalent to (5.10) and (5.13)–(5.16).

□

Remark 5.7.

- Equation (5.12) implies that, for every r, v_3^r is determined by v_1^r and v_2^r. Indeed, if we denote by $\{J_1, J_2, J_3\}$ the hypercomplex structure on $\mathfrak{r}$ induced by $\{\omega_1, \omega_2, \omega_3\}$, then (5.12) is equivalent to
$$v_3^r = -J_2 v_1^r + J_1 v_2^r\,.$$
- When $\lambda \neq 0$, Equations (5.14)–(5.16) together with (5.12) imply (5.4). Indeed, we have
$$\begin{aligned} \alpha_1 \wedge \omega_2 &= \frac{1}{2\lambda}\left(\sum_{r=1}^k \varepsilon_r\, v_2^r \wedge v_3^r\right) \wedge \omega_2 = -\frac{1}{2\lambda}\sum_{r=1}^k \varepsilon_r\, v_3^r \wedge v_2^r \wedge \omega_2 \\ &= -\frac{1}{2\lambda}\sum_{r=1}^k \varepsilon_r\, v_3^r \wedge (-v_1^r \wedge \omega_1 - v_3^r \wedge \omega_3) = \frac{1}{2\lambda}\sum_{r=1}^k \varepsilon_r\, v_3^r \wedge v_1^r \wedge \omega_1 = \alpha_2 \wedge \omega_1, \end{aligned}$$
and, similarly, $\alpha_1 \wedge \omega_3 = \alpha_3 \wedge \omega_1$ and $\alpha_2 \wedge \omega_3 = \alpha_3 \wedge \omega_2$.

In summary, if a 2-step nilmanifold $M = \Gamma\backslash N$ admits an invariant solution $(\varphi, P, \theta, \langle\cdot,\cdot\rangle_\mathfrak{k})$ to the heterotic G_2-system and either $\dim(\mathfrak{n}') = 2$ or $\dim(\mathfrak{n}') = 3$ and φ calibrates $\mathfrak{n}'$, then equations (5.3), (5.4) and (5.9)-(5.16) must hold. This will allow us to determine certain constraints on the Lie algebra $\mathfrak{n}$ or on the signature of the bilinear form $\langle\cdot,\cdot\rangle_\mathfrak{k}$ imposed by the existence of solutions. Conversely, we can state the following existence result, which is analogous to Proposition 4.9.

Proposition 5.8. *Consider the vector space $\mathbb{R}^4$ endowed with the standard basis $\{e_1, e_2, e_3, e_4\}$ and the standard $\mathrm{SU}(2)$-structure $(\omega_1 = e^{13} - e^{24},\ \omega_2 = -e^{14} - e^{23},\ \omega_3 = e^{12} + e^{34})$. Then, any solution of the equations (5.3), (5.4) and (5.9)-(5.16) such that the 1-forms $v_1, v_2, v_3 \in (\mathbb{R}^4)^*$, the 2-forms $\alpha_1, \alpha_2, \alpha_3 \in \Lambda^2(\mathbb{R}^4)^*$ and $F_0^r \in \Lambda^2(\mathbb{R}^4)^*$, $1 \le r \le k$, are integral with respect to the standard basis of $\mathbb{R}^4$, gives rise to a 2-step nilmanifold endowed with a k-torus bundle and admitting an invariant solution to the heterotic G_2-system.*

Proof. Consider the 7-dimensional vector space $\mathfrak{n} \coloneqq \mathbb{R}^4 \oplus \langle z_1, z_2, z_3\rangle$ and endow it with the product metric g for which the vectors z_1, z_2, z_3 are orthonormal. Then, if $\{e^1, e^2, e^3, e^4, z^1, z^2, z^3\}$ denotes the dual basis, $\mathfrak{n}$ becomes a 2-step nilpotent Lie algebra by imposing

$$(de^1, de^2, de^3, de^4, dz^1, dz^2, dz^3) = (0, 0, 0, 0, \alpha_1, \alpha_2, \alpha_3),$$

where the 2-forms $\alpha_1, \alpha_2, \alpha_3$ are extended to $\mathfrak{n}$ in the obvious way. The derived algebra of $\mathfrak{n}$ is $\mathfrak{n}' = \langle z_1, z_2\rangle$ if $\alpha_3 = 0$ and $\mathfrak{n}' = \langle z_1, z_2, z_3\rangle$ otherwise. Since the 2-forms $\alpha_1, \alpha_2, \alpha_3$ are integral with respect to the basis $\mathcal{B} = \{e_1, e_2, e_3, e_4, z_1, z_2, z_3\}$ of $\mathfrak{n}$, the simply connected 2-step nilpotent Lie group $N = \exp(\mathfrak{n})$ has the cocompact lattice $\Gamma \coloneqq \exp(\mathrm{span}_\mathbb{Z}(6e_1, 6e_2, 6e_3, 6e_4, z_1, z_2, z_3))$, and we can consider the 2-step nilmanifold $M = \Gamma\backslash N$. Moreover, the hypothesis on v_1, v_2, v_3 and on the F_0^r's ensures that the 2-forms $F^r = F_0^r + \sum_{i=1}^3 v_i^r \wedge z^i$, $1 \le r \le k$, are integral with respect to the basis $\mathcal{B}$. Therefore, Theorem A.3 ensures then the existence of a principal $\mathbb{T}^k$-bundle $P \to M = \Gamma\backslash N$ endowed with a connection θ whose curvature is $F_\theta = \sum_{r=1}^k F^r t_r$, where $\{t_1, \dots, t_k\}$ is a basis of the Lie algebra of $\mathbb{T}^k$. The 3-form φ given by (5.1),

$$\varphi = \omega_1 \wedge z^1 + \omega_2 \wedge z^2 + \omega_3 \wedge z^3 + z^1 \wedge z^2 \wedge z^3\,,$$

defines a G_2-structure on $\mathfrak{n}$ inducing the product metric g. Since equations (5.3) and (5.4) hold, φ is coclosed and has $\tau_0 = \frac{12}{7}\lambda$, so it gives rise to a G_2-structure of the same type on $M = \Gamma\backslash N$. If we define $\langle\cdot,\cdot\rangle_{\mathfrak{k}} := \sum_{r=1}^{k} \varepsilon_r t^r \otimes t^r$, we then obtain an invariant solution $(\varphi, P, \theta, \langle\cdot,\cdot\rangle_{\mathfrak{k}})$ to the heterotic G_2-system on $M = \Gamma\backslash N$. □

5.1. **Proof of Theorem 1.3-(i).** As we observed in the introduction, it is sufficient to construct a solution to the heterotic G_2-system when $k_+ = 1$, $k_- = 0$, and with $\mathfrak{n}'$ calibrated by φ if $\dim(\mathfrak{n}') = 3$. We consider $\mathbb{R}^4$ endowed with the standard SU(2)-structure ($\omega_1 = e^{13} - e^{24}$, $\omega_2 = -e^{14} - e^{23}$, $\omega_3 = e^{12} + e^{34}$), and we apply Proposition 5.8. We first express α_1, α_2 (and α_3) by solving (5.3) with $\lambda = 0$ and (5.4). Then, we choose $v_1 = v_2 = v_3 = 0$, so that most of the equations given in Proposition 5.6 are satisfied. We are left with equations (5.9) and (5.13), which are easy to solve: it is sufficient to consider any self-dual 2-form $F_0^1 \in \Lambda^2(\mathbb{R}^4)^*$ and scale it appropriately. If $\dim(\mathfrak{n}') = 2$, we obtain the following solutions to (5.3) and (5.4):

- $\alpha_1 = e^{13}$, $\alpha_2 = e^{23}$, which gives a solution for $\mathfrak{n} \cong \mathfrak{n}_{5,2} \oplus \mathbb{R}^2$;
- $\alpha_1 = e^{13} + e^{14}$, $\alpha_2 = -e^{23} + e^{24}$, which gives a solution for $\mathfrak{n} \cong \mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}$;
- $\alpha_1 = e^{13} - e^{24}$, $\alpha_2 = e^{14} + e^{23}$, which gives a solution for $\mathfrak{n} \cong \mathfrak{h}_3^{\mathbb{C}} \oplus \mathbb{R}$;
- $\alpha_1 = 2\,e^{13}$, $\alpha_2 = e^{14} + e^{23}$, which gives a solution for $\mathfrak{n} \cong \mathfrak{n}_{6,2} \oplus \mathbb{R}$.

If $\dim(\mathfrak{n}') = 3$, we have

- $\alpha_1 = e^{13}$, $\alpha_2 = 2e^{23}$, $\alpha_3 = e^{12}$, which gives a solution for $\mathfrak{n} \cong \mathfrak{n}_{6,3} \oplus \mathbb{R}$;
- $\alpha_1 = e^{24}$, $\alpha_2 = e^{23}$, $\alpha_3 = 2e^{12}$, which gives a solution for $\mathfrak{n} \cong \mathfrak{n}_{7,3,A}$;
- $\alpha_1 = e^{12} + e^{13}$, $\alpha_2 = 2e^{14}$, $\alpha_3 = -e^{24} + e^{34}$, which gives a solution for $\mathfrak{n} \cong \mathfrak{n}_{7,3,B}$;
- $\alpha_1 = -e^{14}$, $\alpha_2 = 2e^{13} + e^{24}$, $\alpha_3 = e^{12} - e^{34}$, which gives a solution for $\mathfrak{n} \cong \mathfrak{n}_{7,3,B_1}$;
- $\alpha_1 = e^{24}$, $\alpha_2 = e^{23}$, $\alpha_3 = e^{12} + e^{34}$, which gives a solution for $\mathfrak{n} \cong \mathfrak{n}_{7,3,C}$;
- $\alpha_1 = e^{12} + e^{14} + e^{34}$, $\alpha_2 = -e^{13}$, $\alpha_3 = 2e^{24}$, which gives a solution for $\mathfrak{n} \cong \mathfrak{n}_{7,3,D}$;
- $\alpha_1 = e^{12} - e^{34}$, $\alpha_2 = e^{13} + e^{24}$, $\alpha_3 = e^{14} - e^{23}$, which gives a solution for $\mathfrak{n} \cong \mathfrak{n}_{7,3,D_1}$.

By Proposition 5.8, the above algebraic data provide a solution to the heterotic G_2-system on nilmanifolds associated to each of the Lie algebras above and endowed with principal S^1-bundles.

5.2. **Proof of Theorem 1.3-(ii).** Assume $\dim(\mathfrak{n}') = 2$.

5.2.1. *The case $k = 1$ and $\lambda \neq 0$.* Since $k = 1$, we can drop the superscript in $F_0^r, v_1^r, v_2^r, v_3^r$, in equations (5.9)–(5.16), and consider them as a system in the unknowns F_0, v_1, v_2, v_3.

It is easy to check that there are no solutions in this case. Indeed, since $F_0 \in \Lambda^2\mathfrak{r}^*$ is anti-self-dual, Equation (5.10) implies that either $F_0 = 0$ or all v_i's are 0. The condition $v_i = 0$ for every i implies that the α_i's vanish, which gives a contradiction, since $\mathfrak{n}$ is not abelian. Hence, we must have $F_0 = 0$. Combining (5.14)–(5.16) with (5.11) we get

$$v_1 \wedge v_2 \wedge v_3 = 0\,.$$

This means that v_1, v_2, v_3 are linearly dependent. Hence, by (5.14)–(5.16), α_1, α_2 and α_3 are collinear, which contradicts the assumption $\dim(\mathfrak{n}') = 2$.

Remark 5.9. The same argument shows that there are no invariant solutions with $k = 1$ and $\lambda \neq 0$ when $\dim(\mathfrak{n}') = 3$ and $\mathfrak{n}'$ is calibrated by φ.

5.2.2. *The case $k = 2$ and $\lambda \neq 0$.* We show that there are no solutions in this case either. Here we consider equation (5.3) (see Remark 5.7) together with the system (5.9)–(5.16) with $\alpha_3 = 0$ in the variables $v_i := v_i^1$ and $w_i := v_i^2$:

$$F_0^r \wedge \omega_i = 0\,, \qquad \forall i \in \{1,2,3\},\ \forall r \in \{1,2\}\,, \tag{5.18}$$

$$\varepsilon_1\, F_0^1 \wedge v_i + \varepsilon_2\, F_0^2 \wedge w_i = 0\,, \quad \forall i \in \{1,2,3\}, \tag{5.19}$$

$$v_1 \wedge \alpha_1 + v_2 \wedge \alpha_2 = w_1 \wedge \alpha_1 + w_2 \wedge \alpha_2 = 0\,, \tag{5.20}$$

$$v_1 \wedge \omega_1 + v_2 \wedge \omega_2 + v_3 \wedge \omega_3 = w_1 \wedge \omega_1 + w_2 \wedge \omega_2 + w_3 \wedge \omega_3 = 0\,, \tag{5.21}$$

$$\varepsilon_1\, |F_0^1|^2 + \varepsilon_2\, |F_0^2|^2 = -12\lambda^2 + |\alpha_1|^2 + |\alpha_2|^2\,, \tag{5.22}$$

$$\varepsilon_1\, v_1 \wedge v_2 + \varepsilon_2\, w_1 \wedge w_2 = 0\,, \tag{5.23}$$

$$\varepsilon_1\, v_3 \wedge v_1 + \varepsilon_2\, w_3 \wedge w_1 = 2\lambda\, \alpha_2\,, \tag{5.24}$$

$$\varepsilon_1\, v_2 \wedge v_3 + \varepsilon_2\, w_2 \wedge w_3 = 2\lambda\, \alpha_1\,. \tag{5.25}$$

Equation (5.23) implies that v_1, v_2, w_1, w_2 belong to the same plane, which we denote by P. Moreover, since we are assuming $\lambda \neq 0$, (5.24) and (5.25) give

$$\alpha_2 = -\frac{1}{2\lambda}(\varepsilon_1\, v_1 \wedge v_3 + \varepsilon_2\, w_1 \wedge w_3)\,, \qquad \alpha_1 = \frac{1}{2\lambda}(\varepsilon_1\, v_2 \wedge v_3 + \varepsilon_2\, w_2 \wedge w_3)\,. \tag{5.26}$$

Let $\{f_1, f_2, f_3, f_4\}$ be an orthonormal basis of $\mathfrak{r}^*$ such that $\{f_1, f_2\}$ is a basis of P and write

$$v_1 \wedge v_2 = a\, f_1 \wedge f_2\,, \quad v_1 \wedge w_2 + w_1 \wedge v_2 = 2b\, f_1 \wedge f_2\,. \tag{5.27}$$

Lemma 5.10. *The following relation holds*

$$\varepsilon_1\, a^2 + \varepsilon_2\, b^2 = 0\,.$$

Proof. Equations (5.20) and (5.26) imply the two relations

$$2\varepsilon_1\, v_1 \wedge v_2 \wedge v_3 + \varepsilon_2 w_3 \wedge (v_1 \wedge w_2 + w_1 \wedge v_2) = 0\,, \tag{5.28}$$

$$2\varepsilon_2\, w_1 \wedge w_2 \wedge w_3 + \varepsilon_1 v_3 \wedge (v_1 \wedge w_2 + w_1 \wedge v_2) = 0\,. \tag{5.29}$$

Using (5.23), equation (5.29) becomes

$$2\, v_1 \wedge v_2 \wedge w_3 - v_3 \wedge (v_1 \wedge w_2 + w_1 \wedge v_2) = 0\,. \tag{5.30}$$

We can rewrite (5.28) and (5.30) as

$$(\varepsilon_1\, a\, v_3 + \varepsilon_2\, b\, w_3) \wedge f_1 \wedge f_2 = 0, \tag{5.31}$$

$$(a\, w_3 - b\, v_3) \wedge f_1 \wedge f_2 = 0\,. \tag{5.32}$$

Moreover by setting

$$\begin{aligned} v_3 &= v_3' + x f_3 + x' f_4\,, \\ w_3 &= w_3' + y f_3 + y' f_4\,, \end{aligned} \tag{5.33}$$

with $v_3', w_3' \in P$ and $x, x', y, y' \in \mathbb{R}$, system (5.31)–(5.32) becomes

$$\begin{cases} \varepsilon_1\, a\, x + \varepsilon_2\, b\, y = 0 \\ b\, x - a\, y = 0 \\ \varepsilon_1\, a\, x' + \varepsilon_2\, b\, y' = 0 \\ b\, x' - a\, y' = 0 \end{cases} \tag{5.34}$$

which implies

$$x = y = x' = y' = 0 \quad \text{ or } \quad \varepsilon_1 a^2 + \varepsilon_2 b^2 = 0\,.$$

If $x = y = x' = y' = 0$, then both v_3 and w_3 belong to P, so by (5.26) the structure 2-forms α_1 and α_2 are proportional, which contradicts $\dim(\mathfrak{n}') = 2$. Hence, $\varepsilon_1 a^2 + \varepsilon_2 b^2 = 0$, as claimed. ☐

Lemma 5.11. *The forms $v_1 \wedge v_2$ and $v_1 \wedge w_2 + w_1 \wedge v_2$ are nonzero, namely $ab \neq 0$ in Equation (5.27).*

Proof. Assume by contradiction that $ab = 0$. Then, Lemma 5.10 implies $a = b = 0$. Thus

$$v_1 \wedge v_2 = w_1 \wedge w_2 = v_1 \wedge w_2 + w_1 \wedge v_2 = 0\,. \tag{5.35}$$

Therefore, v_1 is collinear with v_2 and w_1 is collinear with w_2.

We claim that in this case the four vectors v_1, v_2, w_1, w_2 are all collinear. Assume first that v_1 and w_1 are nonzero. Then, we can write

$$v_2 = s v_1\,, \quad w_2 = t w_1,$$

for some $s, t \in \mathbb{R}$. Hence

$$0 = v_1 \wedge w_2 + w_1 \wedge v_2 = (t - s) v_1 \wedge w_1.$$

Since α_1 and α_2 are not proportional, $t \neq s$ and then v_1 and w_1 have to be collinear. Hence, the four vectors v_1, v_2, w_1, w_2 are collinear. The case $v_1 = w_1 = 0$ is impossible since it would imply $\alpha_2 = 0$. If $v_1 = 0$, $w_1 \neq 0$, we get $w_1 \wedge w_2 = w_1 \wedge v_2 = 0$ from (5.35), which implies that $w_2, v_2 \in \langle w_1 \rangle$. The case $v_1 \neq 0$, $w_1 = 0$ is similar. This proves the claim.

Using the SU(2)-freedom on $\mathfrak{r}^*$, we can choose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{r}^*$ in order to have $\omega_1 = e^{13} - e^{24}$, $\omega_2 = -e^{14} - e^{23}$, $\omega_3 = e^{12} + e^{34}$, and

$$\begin{aligned} v_1 &= a_1 e_1\,, \quad v_2 = a_2 e_1\,, \quad v_3 = a_1 e_4 + a_2 e_3\,, \\ w_1 &= b_1 e_1\,, \quad w_2 = b_2 e_1\,, \quad w_3 = b_1 e_4 + b_2 e_3\,, \end{aligned}$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $a_1 b_1 \neq 0 \neq a_2 b_2$. The expressions of v_3 and w_3 follow from Remark 5.7.

Now, the second equation of (5.26) gives

$$\begin{aligned} \alpha_1 =& \frac{1}{2\lambda}(\varepsilon_1\, v_2 \wedge v_3 + \varepsilon_2\, w_2 \wedge w_3) = \frac{1}{2\lambda}(\varepsilon_1 a_2\, e_1 \wedge (a_1 e_4 + a_2 e_3) + \varepsilon_2 b_2\, e_1 \wedge (b_1 e_4 + b_2 e_3)) \\ =& \frac{1}{2\lambda}\left((a_2^2 \varepsilon_1 + b_2^2 \varepsilon_2)\, e_1 \wedge e_3 + (a_1 a_2 \varepsilon_1 + b_1 b_2 \varepsilon_2)\, e_1 \wedge e_4\right)\,, \end{aligned}$$

and, analogously, the first equation in (5.26) implies

$$\alpha_2 = -\frac{1}{2\lambda}\left((a_1^2 \varepsilon_1 + b_1^2 \varepsilon_2)\, e_1 \wedge e_4 + (a_1 a_2 \varepsilon_1 + b_1 b_2 \varepsilon_2)\, e_1 \wedge e_3\right)\,.$$

Hence, from (5.3) we obtain

$$6\lambda = \langle \omega_1, \alpha_1 \rangle + \langle \omega_2, \alpha_2 \rangle = \frac{1}{2\lambda}(\varepsilon_1 a_2^2 + \varepsilon_2 b_2^2) + \frac{1}{2\lambda}(\varepsilon_1 a_1^2 + \varepsilon_2 b_1^2)\,,$$

whence it follows that

$$12\lambda^2 = \varepsilon_1 (a_1^2 + a_2^2) + \varepsilon_2 (b_1^2 + b_2^2)\,. \tag{5.36}$$

We now consider Equation (5.19), which gives

$$e_1 \wedge (a_1 F_0^1 + b_1 F_0^2) = 0, \qquad e_1 \wedge (a_2 F_0^1 + b_2 F_0^2) = 0\,.$$

Since the wedge product with a non-zero vector maps the space $\Lambda^- \mathfrak{r}^*$ of anti-self-dual forms on $\mathfrak{r}$ injectively into $\Lambda^3 \mathfrak{r}^*$, we get

$$a_1 F_0^1 + b_1 F_0^2 = 0 = a_2 F_0^1 + b_2 F_0^2\,.$$

The determinant of the matrix $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ is non-zero, as otherwise α_1 and α_2 would be proportional. Therefore, we obtain $F_0^1 = F_0^2 = 0$, so (5.22) becomes

$$|\alpha_1|^2 + |\alpha_2|^2 = 12\lambda^2\,. \tag{5.37}$$

From (5.36), (5.37), and the explicit formulas of α_1 and α_2 above, it follows that

$$\begin{aligned}
12\lambda^2 = |\alpha_1|^2 + |\alpha_2|^2 &= \frac{1}{4\lambda^2}\left((a_1^2\varepsilon_1 + b_1^2\varepsilon_2)^2 + (a_2^2\varepsilon_1 + b_2^2\varepsilon_2)^2 + 2(a_1a_2\varepsilon_1 + b_1b_2\varepsilon_2)^2\right)\\
&= \frac{1}{4\lambda^2}\left(\left((a_1^2\varepsilon_1 + b_1^2\varepsilon_2) + (a_2^2\varepsilon_1 + b_2^2\varepsilon_2)\right)^2 - 2\varepsilon_1\varepsilon_2(a_1b_2 - a_2b_1)^2\right)\\
&= \frac{1}{4\lambda^2}\left(\left(\varepsilon_1(a_1^2 + a_2^2) + \varepsilon_2(b_1^2 + b_2^2)\right)^2 - 2\varepsilon_1\varepsilon_2(a_1b_2 - a_2b_1)^2\right)\\
&= \frac{1}{4\lambda^2}\left(144\lambda^4 - 2\varepsilon_1\varepsilon_2(a_1b_2 - a_2b_1)^2\right)\\
&= 36\lambda^2 - \frac{1}{2\lambda^2}\varepsilon_1\varepsilon_2(a_1b_2 - a_2b_1)^2\,,
\end{aligned}$$

thus

$$\varepsilon_1\varepsilon_2(a_1b_2 - a_2b_1)^2 = 48\lambda^4\,. \tag{5.38}$$

In particular, (5.37) and (5.38) show that $\varepsilon_1\varepsilon_2 > 0$. Using this and (5.36), we get

$$|\varepsilon_1\varepsilon_2(a_1b_2 - a_2b_1)^2| \leq |\varepsilon_1(a_1^2 + a_2^2)| \cdot |\varepsilon_2(b_1^2 + b_2^2)| \leq \frac{1}{4}(\varepsilon_1(a_1^2 + a_2^2) + \varepsilon_2(b_1^2 + b_2^2))^2 = 36\lambda^4,$$

which contradicts (5.38). This finishes the proof. □

Corollary 5.12. *If* $\dim(\mathfrak{n}') = 2$*, there are no invariant solutions to the heterotic* G_2*-system with* $k = 2$*,* $\lambda \neq 0$ *and definite* $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$*.*

Proof. If $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ is definite, Lemma 5.10 implies $a = b = 0$ which contradicts Lemma 5.11. □

We now claim that having a solution to the system (5.18)–(5.25) implies

$$\alpha_1 \wedge \alpha_1 = \alpha_2 \wedge \alpha_2 = \alpha_1 \wedge \alpha_2 = 0,$$

so that $\mathfrak{n}$ is isomorphic to $\mathfrak{n}_{5,2} \oplus \mathbb{R}^2$ (cf. Appendix B). Since $\lambda \neq 0$, from the expressions of α_1 and α_2 given in (5.26) we obtain

$$\alpha_1 \wedge \alpha_1 = \frac{\varepsilon_1\varepsilon_2}{2\lambda^2}\, v_2 \wedge v_3 \wedge w_2 \wedge w_3\,, \tag{5.39}$$

$$\alpha_2 \wedge \alpha_2 = \frac{\varepsilon_1\varepsilon_2}{2\lambda^2}\, v_1 \wedge v_3 \wedge w_1 \wedge w_3\,, \tag{5.40}$$

$$\alpha_1 \wedge \alpha_2 = \frac{\varepsilon_1\varepsilon_2}{4\lambda^2}\, v_3 \wedge w_3 \wedge (v_2 \wedge w_1 + v_1 \wedge w_2)\,. \tag{5.41}$$

Equation (5.20) implies then

$$\begin{aligned}
0 &= w_3 \wedge (v_1 \wedge \alpha_1 + v_2 \wedge \alpha_2) = \frac{\varepsilon_1}{2\lambda} w_3 \wedge v_3 \wedge (v_1 \wedge v_2 - v_2 \wedge v_1) = \frac{\varepsilon_1}{\lambda} w_3 \wedge v_3 \wedge v_1 \wedge v_2\,,\\
0 &= w_3 \wedge (w_1 \wedge \alpha_1 + w_2 \wedge \alpha_2) = \frac{\varepsilon_1}{2\lambda} w_3 \wedge v_3 \wedge (v_1 \wedge w_2 + w_1 \wedge v_2).
\end{aligned}$$

By Lemma 5.11 we know that $v_1 \wedge v_2$ and $v_1 \wedge w_2 + w_1 \wedge v_2$ are non-zero multiples of $f_1 \wedge f_2$, so we have $w_3 \wedge v_3 \wedge f_1 \wedge f_2 = 0$. Since v_1, v_2, w_1, w_2 belong to $P = \langle f_1, f_2\rangle$, we deduce that $\alpha_1 \wedge \alpha_1 = \alpha_2 \wedge \alpha_2 = \alpha_1 \wedge \alpha_2 = 0$, whence $\mathfrak{n} \cong \mathfrak{n}_{5,2} \oplus \mathbb{R}^2$.

To conclude the proof, we show that there are no solutions in the only remaining case: $\mathfrak{n} \cong \mathfrak{n}_{5,2} \oplus \mathbb{R}^2$ and $\varepsilon_1\varepsilon_2 < 0$. We consider the decomposition of α_1 and α_2 into self-dual and anti-self-dual parts:

$$\alpha_1 = \alpha_1^+ + \alpha_1^-\,, \qquad \alpha_2 = \alpha_2^+ + \alpha_2^-\,.$$

Then

$$0 = \alpha_1 \wedge \alpha_1 = (|\alpha_1^+|^2 - |\alpha_1^-|^2) *_{\mathfrak{r}} 1,$$

implies $|\alpha_1^+|^2 = |\alpha_1^-|^2$. Similarly, $|\alpha_2^+|^2 = |\alpha_2^-|^2$. From (5.3) it follows that

$$|\alpha_1|^2 + |\alpha_2|^2 = 2(|\alpha_1^+|^2 + |\alpha_2^+|^2) \geq \langle \alpha_1^+, \omega_1\rangle^2 + \langle \alpha_2^+, \omega_2\rangle^2 = \langle \alpha_1, \omega_1\rangle^2 + \langle \alpha_2, \omega_2\rangle^2$$
$$\geq \frac{1}{2}(\langle \alpha_1, \omega_1\rangle + \langle \alpha_2, \omega_2\rangle)^2 = 18\lambda^2.$$

Using (5.22) we then have

$$\varepsilon_1\, |F_0^1|^2 + \varepsilon_2\, |F_0^2|^2 = -12\lambda^2 + |\alpha_1|^2 + |\alpha_2|^2 \geq 6\lambda^2,$$

whence we deduce that at least one of the anti-self-dual forms F_0^1 and F_0^2 is non-vanishing. We assume $F_0^2 \neq 0$. Taking the Hodge dual of (5.19) yields

$$\varepsilon_1\, F_0^1(v_i) + \varepsilon_2\, F_0^2(w_i) = 0\,, \qquad \forall i \in \{1,2,3\}\,,$$

showing that there exists an endomorphism $A := -\frac{\varepsilon_1}{\varepsilon_2}(F_0^2)^{-1} \circ F_0^1 \in \mathrm{End}(\mathfrak{r}^*)$ such that $w_i = A(v_i)$ for all $i \in \{1,2,3\}$. Moreover, since F_0^1 and F_0^2 belong to $\Lambda^-\mathfrak{r}^*$, the endomorphism A is of the form $\mu I_4 + B$, with $\mu \in \mathbb{R}$ and $B \in \Lambda^-\mathfrak{r}^*$.

In view of Lemma 5.11 $v_1 \wedge v_2 \neq 0$, so from (5.23) it follows that A preserves the plane P and its orthogonal complement $P^\perp$. Therefore, using the fact that $w_3 = A(v_3)$ together with (5.33), we obtain $A(xf_3 + x'f_4) = yf_3 + y'f_4$. On the other hand, the second and fourth equations in the system (5.34) together with Lemma 5.11 show that the couples (x, x') and (y, y') are proportional: $(x, x') = \frac{a}{b}(y, y')$. Moreover, we must have $(y, y') \neq (0, 0)$, since otherwise v_3, w_3 would belong to P and α_1, α_2 would be proportional. Therefore, $xf_3 + x'f_4 \in P^\perp$ is an eigenvector for A, and thus also for $B = A - \mu I_4$. Since B is skew-symmetric, its only possible real eigenvalue is 0, so B has non-trivial kernel. However, the only element in $\Lambda^-\mathfrak{r}^*$ with non-trivial kernel is 0. This shows that $B = 0$ and thus $w_i = \mu v_i$ for all $i \in \{1,2,3\}$. By (5.23) we thus get $\varepsilon_1 + \mu^2\varepsilon_2 = 0$, so (5.24) gives $\alpha_2 = 0$, which is a contradiction. This concludes the proof of Theorem 1.3-(ii): there are no invariant solutions to the heterotic G$_2$-system with $k = 2$ and $\lambda \neq 0$ when $\dim(\mathfrak{n}') = 2$.

5.2.3. *The case $k = 3$ and $\lambda \neq 0$.* In this case, we show that there is a solution when $\mathfrak{n} \cong \mathfrak{n}_{5,2} \oplus \mathbb{R}^2$. We may assume $\alpha_3 = 0$ and rewrite system (5.9)–(5.16) in the variables $v_i := v_i^1$, $w_i := v_i^2$, and $u_i := v_i^3$, $1 \leq i \leq 3$ as follows:

$$F_0^r \wedge \omega_i = 0\,, \qquad \forall i, r \in \{1,2,3\}\,, \tag{5.42}$$
$$\varepsilon_1\, F_0^1 \wedge v_i + \varepsilon_2\, F_0^2 \wedge w_i + \varepsilon_3\, F_0^3 \wedge u_i = 0\,, \qquad \forall i \in \{1,2,3\}\,, \tag{5.43}$$
$$v_1 \wedge \alpha_1 + v_2 \wedge \alpha_2 = w_1 \wedge \alpha_1 + w_2 \wedge \alpha_2 = u_1 \wedge \alpha_1 + u_2 \wedge \alpha_2 = 0\,, \tag{5.44}$$
$$\textstyle\sum_{i=1}^3 v_i \wedge \omega_i = \sum_{i=1}^3 w_i \wedge \omega_i = \sum_{i=1}^3 u_i \wedge \omega_i = 0\,, \tag{5.45}$$
$$\varepsilon_1\, |F_0^1|^2 + \varepsilon_2\, |F_0^2|^2 + \varepsilon_3\, |F_0^3|^2 = -12\lambda^2 + |\alpha_1|^2 + |\alpha_2|^2\,, \tag{5.46}$$
$$\varepsilon_1\, v_1 \wedge v_2 + \varepsilon_2\, w_1 \wedge w_2 + \varepsilon_3\, u_1 \wedge u_2 = 0\,, \tag{5.47}$$
$$\varepsilon_1\, v_3 \wedge v_1 + \varepsilon_2\, w_3 \wedge w_1 + \varepsilon_3\, u_3 \wedge u_1 = 2\lambda\alpha_2\,, \tag{5.48}$$
$$\varepsilon_1\, v_2 \wedge v_3 + \varepsilon_2\, w_2 \wedge w_3 + \varepsilon_3\, u_2 \wedge u_3 = 2\lambda\alpha_1\,. \tag{5.49}$$

According to Proposition 5.8 and Remark 5.7, it is sufficient to provide a suitable solution to equations (5.3) and (5.42)-(5.49) on the vector space $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4\rangle$ endowed with the standard SU(2)-structure ($\omega_1 = e^{13} - e^{24}$, $\omega_2 = -e^{14} - e^{23}$, $\omega_3 = e^{12} + e^{34}$). One such solution is the following:

$$v_1 = e^1,\ v_2 = 0,\ v_3 = e^4, \qquad w_1 = 0,\ w_2 = e^1,\ w_3 = e^3, \qquad u_1 = u_2 = u_3 = 0,$$
$$\varepsilon_1 = 12\lambda^2 - t, \quad \varepsilon_2 = t, \quad \varepsilon_3 = 12\lambda^2 - 3t + \frac{1}{4\lambda^2}\, t^2, \quad (t \in \mathbb{R}),$$
$$F_0^1 = F_0^2 = 0, \qquad F_0^3 = e^{13} + e^{24}.$$

On the 7-dimensional vector space $\mathfrak{n} = \langle e_1, e_2, e_3, e_4\rangle \oplus \langle z_1, z_2, z_3\rangle$ we then have

$$\alpha_1 = \frac{t}{2\lambda}\, e^{13}, \quad \alpha_2 = \frac{t - 12\lambda^2}{2\lambda}\, e^{14},$$

and

$$F^1 = e^1 \wedge z^1 + e^4 \wedge z^3, \quad F^2 = e^1 \wedge z^2 + e^3 \wedge z^3, \quad F^3 = e^{13} + e^{24}.$$

We note that the 2-forms F^1, F^2, F^3 are integral with respect to the considered basis of $\mathfrak{n}$. Since we also want the 2-forms α_1 and α_2 to be integral with respect to the same basis, we let $t = 2\lambda$, with $\lambda \in \mathbb{Z} \smallsetminus \{0\}$. The 2-step nilpotent Lie algebra $\mathfrak{n}$ has then structure equations

$$(de^1, de^2, de^3, de^4, dz^1, dz^2, dz^3) = (0, 0, 0, 0, e^{13}, (1 - 6\lambda)\, e^{14}, 0),$$

2-dimensional derived algebra $\mathfrak{n}'$, and it is easily seen to be isomorphic to $\mathfrak{n}_{5,2} \oplus \mathbb{R}^2$. We also note that the choice $t = 2\lambda$ implies

$$\varepsilon_1 = 12\lambda^2 - 2\lambda, \quad \varepsilon_2 = 2\lambda, \quad \varepsilon_3 = 12\lambda^2 - 6\lambda + 1,$$

so that the possible bilinear forms $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ on $\mathfrak{k} = \mathrm{Lie}(\mathbb{T}^3)$ corresponding to this solution have signature $(k_+, k_-) = (3, 0)$ when $\lambda \geq 1$ and $(k_+, k_-) = (2, 1)$ when $\lambda \leq -1$.

5.3. **Proof of Theorem 1.3-(iii).** Finally we assume $\dim(\mathfrak{n}') = 3$ and that φ calibrates $\mathfrak{n}'$.

By Remark 5.9, there are no invariant solutions with $\lambda \neq 0$ and $k = 1$, so we directly focus on the case $k = 2$ and $\lambda \neq 0$. In this case, the system (5.9)–(5.16) can be written in the variables $v_i := v_i^1$ and $w_i := v_i^2$ as follows

$$F_0^r \wedge \omega_i = 0\,, \qquad \forall i \in \{1,2,3\},\ \forall r \in \{1,2\}\,, \tag{5.50}$$

$$\varepsilon_1\, F_0^1 \wedge v_i + \varepsilon_2\, F_0^2 \wedge w_i = 0\,, \qquad \forall i \in \{1,2,3\} \tag{5.51}$$

$$v_1 \wedge \alpha_1 + v_2 \wedge \alpha_2 + v_3 \wedge \alpha_3 = w_1 \wedge \alpha_1 + w_2 \wedge \alpha_2 + w_3 \wedge \alpha_3 = 0\,, \tag{5.52}$$

$$v_1 \wedge \omega_1 + v_2 \wedge \omega_2 + v_3 \wedge \omega_3 = w_1 \wedge \omega_1 + w_2 \wedge \omega_2 + w_3 \wedge \omega_3 = 0\,, \tag{5.53}$$

$$\varepsilon_1\, |F_0^1|^2 + \varepsilon_2\, |F_0^2|^2 = -12\lambda^2 + |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2\,, \tag{5.54}$$

$$\varepsilon_1\, v_1 \wedge v_2 + \varepsilon_2\, w_1 \wedge w_2 = 2\lambda\alpha_3\,, \tag{5.55}$$

$$\varepsilon_1\, v_3 \wedge v_1 + \varepsilon_2\, w_3 \wedge w_1 = 2\lambda\alpha_2\,, \tag{5.56}$$

$$\varepsilon_1\, v_2 \wedge v_3 + \varepsilon_2\, w_2 \wedge w_3 = 2\lambda\alpha_1\,. \tag{5.57}$$

We first prove that the existence of a solution of the system with $\lambda \neq 0$ forces the 7-dimensional Lie algebra $\mathfrak{n}$ to be isomorphic either to $\mathfrak{n}_{6,3} \oplus \mathbb{R}$ or to $\mathfrak{n}_{7,3,A}$. Then, we provide an example for the first possibility and show that the second one cannot occur.

Since we assume $\lambda \neq 0$, we can write from (5.55)-(5.57):

$$\alpha_1 = \frac{1}{2\lambda}(\varepsilon_1\, v_2 \wedge v_3 + \varepsilon_2\, w_2 \wedge w_3),\ \ \alpha_2 = \frac{1}{2\lambda}(\varepsilon_1\, v_3 \wedge v_1 + \varepsilon_2\, w_3 \wedge w_1),\ \ \alpha_3 = \frac{1}{2\lambda}(\varepsilon_1\, v_1 \wedge v_2 + \varepsilon_2\, w_1 \wedge w_2).$$

Using these expressions in (5.52), we obtain

$$3\varepsilon_1\, v_1 \wedge v_2 \wedge v_3 + \varepsilon_2\, v_1 \wedge w_2 \wedge w_3 + \varepsilon_2\, v_2 \wedge w_3 \wedge w_1 + \varepsilon_2\, v_3 \wedge w_1 \wedge w_2 \;=\; 0, \tag{5.58}$$

$$3\varepsilon_2\, w_1 \wedge w_2 \wedge w_3 + \varepsilon_1\, w_1 \wedge v_2 \wedge v_3 + \varepsilon_1\, w_2 \wedge v_3 \wedge v_1 + \varepsilon_1\, w_3 \wedge v_1 \wedge v_2 \;=\; 0. \tag{5.59}$$

Next we show that $\alpha_i \wedge \alpha_i = 0$ for $1 \leq i \leq 3$. Assume by contradiction that

$$\alpha_3 \wedge \alpha_3 = \frac{\varepsilon_1 \varepsilon_2}{2\lambda^2} v_1 \wedge w_1 \wedge w_2 \wedge v_2 \neq 0.$$

Then $\{v_1, v_2, w_1, w_2\}$ is a basis of $\mathfrak{r}^*$, and we can write

$$v_3 = a_1 v_1 + a_2 v_2 + a_3 w_1 + a_4 w_2, \quad w_3 = b_1 v_1 + b_2 v_2 + b_3 w_1 + b_4 w_2,$$

for some real numbers a_k, b_k, $1 \leq k \leq 4$. Substituting these expressions into (5.58) and (5.59), we obtain

$$\begin{aligned} 0 &= (3a_3\varepsilon_1 - b_1\varepsilon_2)\, v_1 \wedge v_2 \wedge w_1 + (3a_4\varepsilon_1 - b_2\varepsilon_2)\, v_1 \wedge v_2 \wedge w_2 \\ &\quad + (b_3 - a_1)\varepsilon_2\, v_1 \wedge w_2 \wedge w_1 + (b_4 - a_2)\varepsilon_2\, v_2 \wedge w_2 \wedge w_1, \\ 0 &= (3b_1\varepsilon_2 - a_3\varepsilon_1)\, w_1 \wedge w_2 \wedge v_1 + (3b_2\varepsilon_2 - a_4\varepsilon_1)\, w_1 \wedge w_2 \wedge v_2 \\ &\quad + (a_1 - b_3)\varepsilon_1\, w_1 \wedge v_2 \wedge v_1 + (a_2 - b_4)\varepsilon_1\, w_2 \wedge v_2 \wedge v_1. \end{aligned}$$

These equations give

$$b_4 = a_2, \quad b_3 = a_1, \quad b_1 = b_2 = a_3 = a_4 = 0,$$

and thus

$$v_3 = a_1 v_1 + a_2 v_2, \qquad w_3 = a_1 w_1 + a_2 w_2.$$

From (5.56) and (5.57) we then get

$$\begin{aligned} 2\lambda\alpha_2 &= \varepsilon_1\, v_3 \wedge v_1 + \varepsilon_2\, w_3 \wedge w_1 = a_2(\varepsilon_1\, v_2 \wedge v_1 + \varepsilon_2\, w_2 \wedge w_1)\,, \\ 2\lambda\alpha_1 &= \varepsilon_1\, v_2 \wedge v_3 + \varepsilon_2\, w_2 \wedge w_3 = a_1(\varepsilon_1\, v_2 \wedge v_1 + \varepsilon_2\, w_2 \wedge w_1)\,. \end{aligned}$$

Since $\lambda \neq 0$, this contradicts the fact that α_1 and α_2 are linearly independent. Consequently, $\alpha_3 \wedge \alpha_3 = 0$. By similar arguments, we also obtain $\alpha_2 \wedge \alpha_2 = \alpha_1 \wedge \alpha_1 = 0$.

Replacing the orthonormal basis $\{z_1, z_2, z_3\}$ of $\mathfrak{n}'$ with $\left\{\frac{1}{\sqrt{2}}(z_1 + z_2), \frac{1}{\sqrt{2}}(z_1 - z_2), z_3\right\}$, the structure form α_1 is replaced by $\frac{1}{\sqrt{2}}(\alpha_1 + \alpha_2)$, so the same argument as before shows that $0 = (\alpha_1 + \alpha_2) \wedge (\alpha_1 + \alpha_2) = 2\alpha_1 \wedge \alpha_2$, and similarly $0 = \alpha_1 \wedge \alpha_3 = \alpha_2 \wedge \alpha_3$. This shows that either $\mathfrak{n} \cong \mathfrak{n}_{6,3} \oplus \mathbb{R}$ or $\mathfrak{n} \cong \mathfrak{n}_{7,3,A}$.

To conclude the discussion, we provide an invariant solution to the heterotic G_2-system with $\lambda \neq 0$ and $k = 2$ for $\mathfrak{n} \cong \mathfrak{n}_{6,3} \oplus \mathbb{R}$ and show that there are no such solutions when $\mathfrak{n} \cong \mathfrak{n}_{7,3,A}$.

5.3.1. *The case* $\mathfrak{n} \cong \mathfrak{n}_{6,3} \oplus \mathbb{R}$. We describe here an invariant solution to the heterotic G_2-system with $\lambda \neq 0$ on a 2-step nilmanifold $M = \Gamma\backslash N$ corresponding to a Lie algebra $\mathfrak{n} \cong \mathfrak{n}_{6,3} \oplus \mathbb{R}$ and endowed with a $\mathbb{T}^2$-bundle. By Proposition 5.8 and Remark 5.7, it is sufficient to provide a suitable solution to equations (5.3) and (5.50)-(5.57) on the vector space $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4 \rangle$ endowed with the standard SU(2)-structure $(\omega_1, \omega_2, \omega_3)$. We consider the following solution:

$$v_1 = 2e^1, \quad v_2 = e^2, \quad v_3 = e^4, \qquad w_1 = 0, \quad w_2 = 2e^4, \quad w_3 = 2e^2,$$

$$\varepsilon_1 = 2\lambda^2, \quad \varepsilon_2 = \frac{3}{2}\lambda^2, \qquad F_0^1 = F_0^2 = 0,$$

so that on the 7-dimensional vector space $\mathfrak{n} = \langle e_1, e_2, e_3, e_4 \rangle \oplus \langle z_1, z_2, z_3 \rangle$ we have

$$\alpha_1 = -2\lambda\, e^{24}, \quad \alpha_2 = -2\lambda\, e^{14}, \quad \alpha_3 = 2\lambda\, e^{12},$$

$$F^1 = 2e^1 \wedge z^1 + e^2 \wedge z^2 + e^4 \wedge z^3, \qquad F^2 = 2e^4 \wedge z^2 + 2e^2 \wedge z^3.$$

Now, $\mathfrak{n}$ is a 7-dimensional 2-step nilpotent Lie algebra with structure equations

$$(de^1, de^2, de^3, de^4, dz^1, dz^2, dz^3) = (0, 0, 0, 0, -2\lambda\, e^{24}, -2\lambda\, e^{14}, 2\lambda\, e^{12}),$$

and 3-dimensional derived algebra $\mathfrak{n}' = \langle e_5, e_6, e_7 \rangle$; it is isomorphic to the decomposable Lie algebra $\mathfrak{n}_{6,2} \oplus \mathbb{R}$. The choice $\lambda \in \mathbb{Z} \smallsetminus \{0\}$ ensures that the 2-forms α_1, α_2 and α_3 are integral with respect to the basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^4$. Since also the 1-forms v_i, w_i, $1 \leq i \leq 3$, and the 2-forms F_0^1 and F_0^2 are integral with respect to the same basis, the existence of the solution follows from Proposition 5.8. Note that the bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ on $\mathfrak{k} = \mathrm{Lie}(\mathbb{T}^2)$ has signature $(2, 0)$.

5.3.2. *The case* $\mathfrak{n}\cong\mathfrak{n}_{7,3,A}$. We finally show that there are no solutions when $\mathfrak{n}\cong\mathfrak{n}_{7,3,A}$, $\lambda\neq 0$ and $k=2$.

Proposition 5.13. *Every invariant solution* $(\varphi,P,\theta,\langle\cdot,\cdot\rangle_{\mathfrak{k}})$ *to the heterotic* G_2*-system on a* 2*-step nilmanifold* $M=\Gamma\backslash N$*, where* P *is a principal* $\mathbb{T}^2$*-bundle,* $\mathfrak{n}\cong\mathfrak{n}_{7,3,A}$ *and* φ *calibrates* $\mathfrak{n}'$*, has* $\lambda=0$.

Proof. Assume that there exists a solution to the system (5.50)–(5.57) with $\lambda\neq0$. Since $\mathfrak{n}\cong\mathfrak{n}_{7,3,A}$, there exists a basis $\{f_0,f_1,f_2,f_3\}$ of $\mathfrak{r}^*$ such that the structure forms α_i are given by $\alpha_i=f_0\wedge f_i$ for all $i\in\{1,2,3\}$. We decompose

$$v_i=a_if_0+v_i',\qquad w_i=b_if_0+w_i',$$

with $a_i,b_i\in\mathbb{R}$ and $v_i',w_i'\in\langle f_1,f_2,f_3\rangle$. Then the system (5.55)–(5.57) decouples into

$$\varepsilon_1\,(a_1v_2'-a_2v_1')+\varepsilon_2\,(b_1w_2'-b_2w_1')=2\lambda f_3\,,\tag{5.60}$$
$$\varepsilon_1\,(a_3v_1'-a_1v_3')+\varepsilon_2\,(b_3w_1'-b_1w_3')=2\lambda f_2\,,\tag{5.61}$$
$$\varepsilon_1\,(a_2v_3'-a_3v_2')+\varepsilon_2\,(b_2w_3'-b_3w_2')=2\lambda f_1\,,\tag{5.62}$$

and

$$\varepsilon_1\,v_1'\wedge v_2'+\varepsilon_2\,w_1'\wedge w_2'=0\,,\tag{5.63}$$
$$\varepsilon_1\,v_3'\wedge v_1'+\varepsilon_2\,w_3'\wedge w_1'=0\,,\tag{5.64}$$
$$\varepsilon_1\,v_2'\wedge v_3'+\varepsilon_2\,w_2'\wedge w_3'=0\,.\tag{5.65}$$

Assume that v_1' and w_1' are linearly independent, spanning a plane $P\subset\mathfrak{r}^*$. Then by (5.63), v_2' and w_2' belong to P, and by (5.64) v_3' and w_3' belong to P as well. Then by (5.60)–(5.62), f_1,f_2,f_3 belong to P, a contradiction. Consequently, there exists a nonzero form $x_1\in\mathfrak{r}^*$ and real numbers $t_1,s_1\in\mathbb{R}$ such that $v_1'=t_1x_1$ and $w_1'=s_1x_1$. A similar discussion shows the existence of nonzero forms $x_2,x_3\in\mathfrak{r}^*$ and real numbers $t_2,t_3,s_2,s_3\in\mathbb{R}$ such that $v_i'=t_ix_i$ and $w_i'=s_ix_i$ for $i\in\{2,3\}$.

Note that x_1,x_2,x_3 are linearly independent, as otherwise (5.60)–(5.62) would again contradict the linear independence of f_1,f_2,f_3. Substituting the new expressions of the v_i''s and w_i''s in (5.63)–(5.65), we get

$$\varepsilon_1t_it_j+\varepsilon_2s_is_j=0\,,\qquad\forall i\neq j\in\{1,2,3\}\,.\tag{5.66}$$

We also remark that $(t_i,s_i)\neq(0,0)$ for every $i\in\{1,2,3\}$. Indeed, otherwise we would get from (5.60)–(5.62) that f_1,f_2,f_3 belong to the plane generated by x_j,x_k, where $\{j,k\}\coloneqq\{1,2,3\}\smallsetminus\{i\}$.

We claim that $s_it_i\neq0$ for every $i\in\{1,2,3\}$. Assume, for instance, that $s_1=0$. Then we must have $t_1\neq0$, so from (5.66) we get $t_2=t_3=0$, and thus $s_2s_3=0$, which is impossible since $(t_2,s_2)\neq(0,0)\neq(t_3,s_3)$. This proves our claim for $i=1$; the proof in the remaining cases is analogous.

Dividing each equation of (5.66) by t_it_j immediately shows that $\frac{s_i}{t_i}=\frac{s_j}{t_j}$ for every $i,j\in\{1,2,3\}$, so there exists a real number c such that $s_i=ct_i$, and consequently $w_i'=cv_i'$ for every $i\in\{1,2,3\}$.

Using this, the system (5.60)–(5.62) becomes

$$c_1v_2'-c_2v_1'=2\lambda f_3\,,\tag{5.67}$$
$$c_3v_1'-c_1v_3'=2\lambda f_2\,,\tag{5.68}$$
$$c_2v_3'-c_3v_2'=2\lambda f_1\,,\tag{5.69}$$

where $c_i\coloneqq\varepsilon_1a_i+c\varepsilon_2b_i$ for $i\in\{1,2,3\}$. Multiplying (5.67) by c_3, (5.68) by c_2, (5.69) by c_1 and summing the results, gives $0=2\lambda(c_1f_1+c_2f_2+c_3f_3)$. Since we assumed $\lambda\neq0$, by the linear independence of f_1,f_2,f_3 we must have $c_1=c_2=c_3=0$, which by (5.67)–(5.69) again gives $f_1=f_2=f_3=0$, a contradiction. Thus, $\lambda=0$. □

Remark 5.14. We notice that in fact we only used the equations (5.55)–(5.57) in order to prove Proposition 5.13.

6. Open questions

In this final section, we collect some open problems related to our results:

- **Theorem 1.3-ii**. When $\mathfrak{n} \cong \mathfrak{n}_{5,2} \oplus \mathbb{R}^2$, we prove the existence of invariant solutions to the heterotic G$_2$-system with $\lambda \neq 0$ and $k = 3$ (see Subsection 5.2.3). We suspect that the existence of invariant solutions to the system when $\dim(\mathfrak{n}') = 2$, $\lambda \neq 0$ and $k = 3$ might force the Lie algebra $\mathfrak{n}$ to be isomorphic to $\mathfrak{n}_{5,2} \oplus \mathbb{R}^2$. Moreover, the solution described in Subsection 5.2.3 depends on 1 integer parameter which determines the signature of $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$: it can be $(2, 1)$ or $(3, 0)$. The existence of invariant solutions for which $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ has signature $(1, 2)$ is an open problem.
- **Theorem 1.3-iii**. The heterotic G$_2$-system admits invariant solutions on a 7-dimensional 2-step nilmanifold $M = \Gamma \backslash N$ with $\lambda \neq 0$, $k = 2$ and φ calibrating $\mathfrak{n}'$ if and only if $\mathfrak{n} \cong \mathfrak{n}_{6,3} \oplus \mathbb{R}$. In Subsection 5.3.1 we described an invariant solution such that $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ has signature $(2, 0)$. The existence of invariant solutions for which $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ has signature $(1, 1)$ is an open problem.
- $\dim(\mathfrak{n}') = \mathbf{3}$. In this case we only considered invariant solutions with φ calibrating $\mathfrak{n}'$. It might be interesting investigating whether solutions occur only in this case or whether there exist invariant solutions such that $\mathfrak{n}'$ is not calibrated by φ. Moreover, the existence of invariant solutions with a principal $\mathbb{T}^k$-bundle is an open problem for $k \geq 3$.

Appendix A. A construction of principal torus bundles over 2-step nilmanifolds

The aim of this section is to show how to construct a principal torus bundle over a 2-step nilmanifold starting with a suitable 2-form. We first introduce the setting, and then we state and prove the result.

Let N be a simply connected n-dimensional 2-step nilpotent Lie group with Lie algebra $\mathfrak{n}$. Consider a complementary subspace $\mathfrak{v}$ of the derived algebra $\mathfrak{n}' := [\mathfrak{n}, \mathfrak{n}]$, so that $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{n}'$. Assume that $\mathfrak{n}$ has integer structure constants with respect to a basis $\mathcal{B} = \{e_1, \dots, e_{n-n'}, z_1, \dots, z_{n'}\}$ adapted to the splitting $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{n}'$, where $n' = \dim(\mathfrak{n}')$. Then

$$\Gamma := \exp\left(\operatorname{span}_{\mathbb{Z}}(6e_1, \dots, 6e_{n-n'}, z_1, \dots, z_{n'})\right) \tag{A.1}$$

is a cocompact lattice in N, and $M = \Gamma \backslash N$ is a 2-step nilmanifold. We shall refer to Γ as the cocompact lattice *determined* by the basis $\mathcal{B}$.

Remark A.1. Clearly, the cocompact lattice Γ defined in (A.1) is not the only one contained in N that one can construct starting from the basis $\mathcal{B}$. It is however the simplest one for which the construction discussed in this appendix works, as we will see in the proof of Lemma A.10.

Definition A.2. An $\mathbb{R}^k$-valued p-form $F \in \Lambda^p \mathfrak{n}^* \otimes \mathbb{R}^k$ is said to be *integral* with respect to the basis $\mathcal{B}$ of $\mathfrak{n}$ if there exists a basis $\{t_1, \dots, t_k\}$ of $\mathbb{R}^k$ such that $F = \sum_{r=1}^k F^r \otimes t_r$, where $F^r \in \Lambda^p \mathfrak{n}^*$, and $F^r(x_1, \dots, x_p) \in \mathbb{Z}$ for every $x_1, \dots, x_p \in \mathcal{B}$ and $1 \leq r \leq k$.

The main result of this appendix is the following.

Theorem A.3. *Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra having integer structure constants with respect to a basis $\mathcal{B}$. Let N be the simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$, and let $\Gamma \subseteq N$ be the cocompact lattice determined by $\mathcal{B}$. Then, for every integral closed 2-form $F \in \Lambda^2 \mathfrak{n}^* \otimes \mathbb{R}^k$ there exists a principal $\mathbb{T}^k$-bundle over $M = \Gamma \backslash N$ admitting a connection 1-form θ with curvature $F_\theta = F$.*

It is sufficient to prove Theorem A.3 for principal S^1-bundles, namely for $F \in \Lambda^2 \mathfrak{n}^*$. In the following, we shall denote a k-form or a vector on the Lie algebra $\mathfrak{n}$ and the corresponding left-invariant form or left-invariant vector field on N using the same symbol.

Since N is contractible, we have $F = d\sigma$ for some (not necessarily left-invariant) 1-form σ on N. Let $\gamma \in \Gamma$. Since

$$d\sigma = \gamma^*(d\sigma) = d(\gamma^*(\sigma)),$$

we have $d(\gamma^*(\sigma)-\sigma)=0$. Hence there exists a function $f_\gamma\in C^\infty(N)$ such that

$$df_\gamma=\gamma^*(\sigma)-\sigma. \tag{A.2}$$

Moreover, we can take $f_{1_N}\equiv 0$.

For $\gamma_1,\gamma_2\in\Gamma$, we have both

$$(\gamma_1\gamma_2)^*(\sigma)=\sigma+df_{\gamma_1\gamma_2}$$

and

$$(\gamma_1\gamma_2)^*(\sigma)=(\gamma_2)^*(\sigma+df_{\gamma_1})=\sigma+df_{\gamma_2}+d\gamma_2^*f_{\gamma_1}\,,$$

whence it follows that

$$df_{\gamma_2}+d\gamma_2^*f_{\gamma_1}-df_{\gamma_1\gamma_2}=0.$$

Consequently,

$$f_{\gamma_2}+\gamma_2^*f_{\gamma_1}-f_{\gamma_1\gamma_2}=c_{\gamma_1\gamma_2}\in\mathbb{R} \tag{A.3}$$

is constant.

In summary: for every $\gamma\in\Gamma$, the potential $\sigma\in\Omega^1(N)$ gives rise to a function f_γ through the identity (A.2). Moreover, the functions corresponding to different elements of Γ are related by the identity (A.3).

Next consider $N\times S^1$. For $\gamma\in\Gamma$ let $\tilde\gamma\colon N\times S^1\to N\times S^1$ be the map

$$\tilde\gamma(x,\mathrm{e}^{2\pi it}):=\left(\gamma x,\mathrm{e}^{2\pi i(t-f_\gamma(x))}\right)\,. \tag{A.4}$$

Lemma A.4. *If $c_{\gamma_1\gamma_2}$ is an integer for all $\gamma_1,\gamma_2\in\Gamma$, then (A.4) defines an action of Γ on $N\times S^1$.*

Proof. Let $\gamma_1,\gamma_2\in\Gamma$. We have

$$\widetilde{\gamma_1\gamma_2}(x,\mathrm{e}^{2\pi it})=\left(\gamma_1\gamma_2x,\mathrm{e}^{2\pi i(t-f_{\gamma_1\gamma_2}(x))}\right)=\left(\gamma_1\gamma_2x,\mathrm{e}^{2\pi i(t-f_{\gamma_2}(x)-f_{\gamma_1}(\gamma_2x)+c_{\gamma_1\gamma_2})}\right)$$

and

$$\tilde\gamma_1\left(\tilde\gamma_2(x,\mathrm{e}^{2\pi it})\right)=\tilde\gamma_1\left(\gamma_2x,\mathrm{e}^{2\pi i(t-f_{\gamma_2}(x))}\right)=\left(\gamma_1\gamma_2x,\mathrm{e}^{2\pi i(t-f_{\gamma_2}(x)-f_{\gamma_1}(\gamma_2x))}\right),$$

whence it follows that

$$\widetilde{\gamma_1\gamma_2}(x,\mathrm{e}^{2\pi it})=\tilde\gamma_1\left(\tilde\gamma_2(x,\mathrm{e}^{2\pi it})\right)\iff c_{\gamma_1\gamma_2}\in\mathbb{Z}\,.$$

□

Proposition A.5. *If $c_{\gamma_1\gamma_2}$ is an integer for all $\gamma_1,\gamma_2\in\Gamma$, then there exists a principal S^1-bundle $\Gamma\backslash(N\times S^1)\to\Gamma\backslash N$ endowed with a connection form θ with curvature $d\theta=F$.*

Proof. We know that Γ acts on $N\times S^1$ as in (A.4) and it is clear that this action commutes with the action of S^1 on itself by left multiplication. Thus, $\Gamma\backslash(N\times S^1)$ is the total space of a principal S^1 bundle over $\Gamma\backslash N$. The 1-form $\theta=\sigma+dt$ on $N\times S^1$, with $\sigma\in\Omega^1(N)$ such that $F=d\sigma$, is Γ-invariant. Indeed, for all $\gamma\in\Gamma$

$$\gamma^*(\theta)=\gamma^*(\sigma)+d\gamma^*(t)=\sigma+df_\gamma+d(t-f_\gamma)=dt+\sigma=\theta\,.$$

This shows that θ defines a connection 1-form on $\Gamma\backslash(N\times S^1)$ with curvature $d\theta=d\sigma=F$. □

We now show that the assumption of the previous result holds when the closed 2-form F is integral with respect to the basis $\mathcal{B}=\{e_1,\dots,e_{n-n'},z_1,\dots,z_{n'}\}$ of $\mathfrak{n}=\mathfrak{v}\oplus\mathfrak{n}'$. This will prove Theorem A.3 for S^1 bundles.

We denote by $\{e^1,\dots,e^{n-n'},z^1,\dots,z^{n'}\}$ the dual basis of $\mathcal{B}$, so that $\{z^1,\dots,z^{n'}\}$ is a basis of $(\mathfrak{n}')^*$ and $\{e^1,\dots,e^{n-n'}\}$ is a basis of $\mathfrak{v}^*$. Since $\mathfrak{n}$ is 2-step nilpotent, we have $dz^r\in\Lambda^2\mathfrak{v}^*$ and $de^i=0$.

Since $\Lambda^2\mathfrak{n}^*\cong\Lambda^2(\mathfrak{n}')^*\oplus\Lambda^2\mathfrak{v}^*\oplus\mathfrak{v}^*\otimes(\mathfrak{n}')^*$, a 2-form $F\in\Lambda^2\mathfrak{n}^*$ is closed if and only if $F\in\Lambda^2\mathfrak{v}^*\oplus\mathfrak{v}^*\otimes(\mathfrak{n}')^*$. We can then write an integral closed 2-form F as follows

$$F=F_1+F_2,$$

where

$$F_1 = \sum_{1\leq r<s\leq n-n'} F_{rs} e^r \wedge e^s \in \Lambda^2 \mathfrak{v}^*, \qquad F_2 = \sum_{1\leq i\leq n-n'} \sum_{1\leq r\leq n'} \tilde{F}_{ir} e^i \wedge z^r \in \mathfrak{v}^* \otimes (\mathfrak{n}')^*,$$

and $F_{rs}, \tilde{F}_{ir} \in \mathbb{Z}$.

We claim that the left-invariant 2-form F has a potential $\sigma \in \Omega^1(N)$ giving rise to functions f_γ for which the constants $c_{\gamma_1\gamma_2}$ given by (A.3) are integers for all $\gamma_1, \gamma_2 \in \Gamma$. By linearity, it is sufficient to prove this for F_1 and F_2 separately. This will be done in Lemma A.7 and Lemma A.10, respectively, and will complete the proof of Theorem A.3 for S^1 bundles.

We begin with a preliminary result.

Lemma A.6. *Let $\beta \in \mathfrak{n}^*$ and define the scalar function $u_\beta \in C^\infty(N)$ via the identity*

$$u_\beta(\mathrm{e}^X) = \beta(X),$$

for all $X \in \mathfrak{n}$. Then $d\beta = \frac{1}{2}\sum_{1\leq i,j\leq n-n'} b_{ij} e^i \wedge e^j = \sum_{1\leq i,j\leq n-n'} b_{ij} e^i \otimes e^j$, for certain $b_{ij} = -b_{ji} \in \mathbb{R}$, and

$$\beta = du_\beta + \frac{1}{2} \sum_{1\leq i,j\leq n-n'} b_{ij} u_{e^i} e^j.$$

In particular, if $\beta \in \mathfrak{v}^*$, so that $d\beta = 0$, then

$$\beta = du_\beta.$$

Proof. Since $\mathfrak{n}$ is 2-step nilpotent, for every $A, B \in \mathfrak{n}$ the following identity holds

$$\mathrm{e}^A \mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]},$$

by the Baker-Campbell-Hausdorff formula.

Let Y be a left-invariant vector field on N and $p = \mathrm{e}^Z \in N$. Then, we have

$$\begin{aligned} Y_p(u_\beta) &= \frac{d}{dt}\Big|_{t=0} u_\beta\left(p\,\mathrm{e}^{tY}\right) = \frac{d}{dt}\Big|_{t=0} u_\beta\left(\mathrm{e}^{Z+tY+\frac{1}{2}t[Z,Y]}\right) = \frac{d}{dt}\Big|_{t=0} \beta\left(Z + tY + \frac{1}{2}t[Z,Y]\right) \\ &= \beta(Y) + \frac{1}{2}\beta([Z,Y]) = \beta(Y) - \frac{1}{2}d\beta(Z,Y) = \beta(Y) - \frac{1}{2}\sum_{1\leq i,j\leq n-n'} b_{ij} e^i(Z) e^j(Y) \\ &= \beta(Y) - \frac{1}{2}\sum_{1\leq i,j\leq n-n'} b_{ij} u_{e^i}(p) e^j(Y) = \left(\beta - \frac{1}{2}\sum_{1\leq i,j\leq n-n'} b_{ij} u_{e^i} e^j\right)_p (Y). \end{aligned}$$

The thesis then follows. ☐

Lemma A.7. *The left-invariant integral closed 2-form F_1 has the following potential*

$$\sigma_1 = \sum_{1\leq r<s\leq n-n'} F_{rs} u_{e^r}\, e^s \in \Omega^1(N).$$

It gives rise to functions f_γ for which $c_{\gamma_1\gamma_2} \in \mathbb{Z}$, for all $\gamma_1, \gamma_2 \in \Gamma$.

Proof. By Lemma A.6 we have $e^r = du_{e^r}$, so we can write

$$F_1 = \sum_{1\leq r<s\leq n-n'} F_{rs} e^r \wedge e^s = \sum_{1\leq r<s\leq n-n'} F_{rs} d(u_{e^r}\, e^s) = d\sigma_1,$$

where

$$\sigma_1 \coloneqq \sum_{1\leq r<s\leq n-n'} F_{rs} u_{e^r}\, e^s.$$

Let us focus on the 1-form $\sigma_{rs} \coloneqq u_{e^r}\, e^s$ and let us consider $\gamma = \mathrm{e}^C \in \Gamma$. We claim that

$$\gamma^* u_{e^r} = u_{e^r} + e^r(C). \tag{A.5}$$

Indeed, for all $p = \mathrm{e}^Y \in N$, we have

$$\gamma^* u_{e^r}(p) = u_{e^r}(\gamma p) = u_{e^r}(\mathrm{e}^C \mathrm{e}^Y) = u_{e^r}\left(\mathrm{e}^{C+Y+\frac{1}{2}[C,Y]}\right) = e^r(C) + e^r(Y) = u_{e^r}(p) + e^r(C),$$

since $e^r|_{\mathfrak{n}'} = 0$. We now compute

$$\gamma^*\sigma_{rs} = \gamma^*(u_{e^r}\, e^s) = (\gamma^* u_{e^r})\, e^s = (u_{e^r} + e^r(C))\, e^s = \sigma_{rs} + e^r(C)e^s = \sigma_{rs} + e^r(C)du_{e^s}\,.$$

Using this, we obtain

$$\gamma^*\sigma_1 - \sigma_1 = \sum_{1\leq r<s\leq n-n'} F_{rs}e^r(C)du_{e^s} = df_\gamma,$$

where

$$f_\gamma \coloneqq \sum_{1\leq r<s\leq n-n'} F_{rs}e^r(C)u_{e^s}\,.$$

For every $\gamma_1 = \mathrm{e}^{C_1}, \gamma_2 = \mathrm{e}^{C_2} \in \Gamma$, we have $\gamma_1\gamma_2 = \mathrm{e}^{C_1+C_2+\frac{1}{2}[C_1,C_2]}$, and we then obtain

$$\begin{aligned} c_{\gamma_1\gamma_2} = f_{\gamma_2} + \gamma_2^* f_{\gamma_1} - f_{\gamma_1\gamma_2} &= \sum_{1\leq r<s\leq n-n'} F_{rs}\left(e^r(C_2)u_{e^s} + e^r(C_1)\gamma_2^* u_{e^s} - e^r(C_1+C_2)u_{e^s}\right) \\ &= \sum_{1\leq r<s\leq n-n'} F_{rs}e^r(C_1)e^s(C_2). \end{aligned}$$

Since $F_{rs} \in \mathbb{Z}$ and $e^r(C) \in \mathbb{Z}$ for all $C \in \mathrm{span}_{\mathbb{Z}}(6e_1, \ldots, 6e_{n-n'}, z_1, \ldots, z_{n'})$, the thesis follows. □

We now focus on the summand $F_2 \in \mathfrak{v}^* \otimes (\mathfrak{n}')^*$. First, we rewrite it as follows

$$F_2 = \sum_{1\leq i\leq n-n'}\sum_{1\leq r\leq n'} \tilde{F}_{ir}e^i \wedge z^r = \sum_{1\leq i\leq n-n'} e^i \wedge \eta^i,$$

where

$$\eta^i \coloneqq \sum_{1\leq r\leq n'} \tilde{F}_{ir}z^r \in (\mathfrak{n}')^*.$$

For $1 \leq i \leq n-n'$, we let

$$\alpha^i \coloneqq d\eta^i = \frac{1}{2}\sum_{1\leq j,k\leq n-n'} c_{ijk}e^j \wedge e^k = \sum_{1\leq j,k\leq n-n'} c_{ijk}e^j \otimes e^k, \tag{A.6}$$

where the components $c_{ijk} \in \mathbb{Z}$ are skew symmetric in the last two indices: $c_{ijk} = -c_{ikj}$.

Remark A.8. Since $dz^r = -\frac{1}{2}\sum_{1\leq j,k\leq n-n'} c^r_{jk}e^j \wedge e^k$, where $c^r_{jk} \in \mathbb{Z}$ are the structure constants of $\mathfrak{n}$ with respect to the basis $\mathcal{B}$, we have

$$c_{ijk} = -\sum_{r=1}^{n'} \tilde{F}_{ir}c^r_{jk}.$$

The closure of F_2 gives

$$0 = dF_2 = -\sum_{1\leq i\leq n-n'} e^i \wedge \alpha^i = -\frac{1}{2}\sum_{1\leq i,j,k\leq n-n'} c_{ijk}e^i \wedge e^j \wedge e^k,$$

hence it is equivalent to the cocycle condition

$$0 = c_{[ijk]} = c_{ijk} + c_{jki} + c_{kij}, \tag{A.7}$$

for all $1 \leq i, j, k \leq n-n'$.

The next lemma will be useful in the proof of the last result.

Lemma A.9. *Let $\gamma = e^C \in \Gamma$, then for all $1 \leq i \leq n-n'$ the following identity holds*

$$\gamma^* u_{\eta^i} = u_{\eta^i} + \eta^i(C) - \frac{1}{2}\sum_{j,k} c_{ijk}e^j(C)u_{e^k}.$$

Proof. Let $p = e^Y \in N$, then

$$\begin{aligned}\gamma^* u_{\eta^i}(p) &= u_{\eta^i}(\gamma p) = u_{\eta^i}(e^C e^Y) = u_{\eta^i}\left(e^{C+Y+\frac{1}{2}[C,Y]}\right) = \eta^i(C) + \eta^i(Y) + \frac{1}{2}\eta^i([C,Y]) \\ &= u_{\eta^i}(p) + \eta^i(C) - \frac{1}{2}d\eta^i(C,Y) = u_{\eta^i}(p) + \eta^i(C) - \frac{1}{2}\sum_{1\le j,k\le n-n'} c_{ijk}e^j \otimes e^k(C,Y) \\ &= u_{\eta^i}(p) + \eta^i(C) - \frac{1}{2}\sum_{1\le j,k\le n-n'} c_{ijk}e^j(C)u_{e^k}(p).\end{aligned}$$

□

Lemma A.10. *The left-invariant integral closed 2-form F_2 has the following potential*

$$\sigma_2 = \sum_i u_{e^i}\eta^i - \frac{1}{3}\sum_{i,j,k} c_{ijk}u_{e^i}u_{e^j}e^k.$$

It gives rise to functions f_γ for which $c_{\gamma_1\gamma_2} \in \mathbb{Z}$, for all $\gamma_1, \gamma_2 \in \Gamma$.

Proof. Using the definition of α^i and the cocycle condition (A.7), we obtain

$$\sum_i u_{e^i}\alpha^i = \frac{1}{2}\sum_{i,j,k} c_{ijk}u_{e^i}e^j \wedge e^k = \frac{1}{2}\sum_{i,j,k}(c_{jik} + c_{kji})u_{e^i}e^j \wedge e^k = \sum_{i,j,k} c_{ijk}u_{e^j}e^i \wedge e^k.$$

This identity and equation (A.6) imply that

$$d\left(\sum_{i,j,k} c_{ijk}u_{e^i}u_{e^j}e^k\right) = \sum_{i,j,k} c_{ijk}(u_{e^j}e^i \wedge e^k + u_{e^i}e^j \wedge e^k) = 3\sum_i u_{e^i}\alpha^i.$$

Therefore, since $\alpha_i = d\eta^i$, we have

$$d\sigma_2 = \sum_i e^i \wedge \eta^i + \sum_i u_{e^i}d\eta^i - \sum_i u_{e^i}\alpha^i = F_2,$$

which shows the first assertion.

Let $\gamma = e^C \in \Gamma$. Then, using (A.5) we obtain

$$\begin{aligned}\gamma^*\sigma_2 - \sigma_2 &= \sum_i (u_{e^i} + e^i(C))\eta^i - \frac{1}{3}\sum_{i,j,k} c_{ijk}(u_{e^i} + e^i(C))(u_{e^j} + e^j(C))e^k - \sigma_2 \\ &= \sum_i e^i(C)\eta^i - \frac{1}{3}\sum_{i,j,k} c_{ijk}(u_{e^i}e^j(C) + u_{e^j}e^i(C) + e^i(C)e^j(C))e^k.\end{aligned}$$

We claim that $\gamma^*\sigma_2 - \sigma_2 = df_\gamma$, where

$$f_\gamma \coloneqq \sum_i e^i(C)u_{\eta^i} - \frac{1}{6}\sum_{i,j,k} c_{ijk}e^j(C)u_{e^i}u_{e^k} - \frac{1}{3}\sum_{i,j,k} c_{ijk}e^i(C)e^j(C)u_{e^k}.$$

First, applying Lemma A.6 to $\eta^i \in (\mathfrak{n}')^*$ and using (A.6), we obtain

$$du_{\eta^i} = \eta^i - \frac{1}{2}\sum_{1\le j,k\le n-n'} c_{ijk}u_{e^j}e^k.$$

Using this identity, we get

$$d\left(\sum_i e^i(C)u_{\eta^i}\right) = \sum_i e^i(C)\eta^i - \frac{1}{2}\sum_{i,j,k} c_{ijk}e^i(C)u_{e^j}e^k.$$

Moreover, recalling that $du_{e^i} = e^i$ and using the cocycle condition $c_{ijk} + c_{jki} + c_{kij} = 0$, we have

$$\begin{aligned} d\left(\sum_{i,j,k} c_{ijk} e^j(C) u_{e^i} u_{e^k}\right) &= \sum_{i,j,k} c_{ijk} e^j(C)(u_{e^k} e^i + u_{e^i} e^k) \\ &= \sum_{i,j,k} c_{kji} e^j(C) u_{e^i} e^k + \sum_{i,j,k} c_{ijk} e^j(C) u_{e^i} e^k \\ &= \sum_{i,j,k} (-c_{jik} - c_{ikj}) e^j(C) u_{e^i} e^k + \sum_{i,j,k} c_{ijk} e^j(C) u_{e^i} e^k \\ &= -\sum_{i,j,k} c_{ijk} e^i(C) u_{e^j} e^k + 2\sum_{i,j,k} c_{ijk} e^j(C) u_{e^i} e^k. \end{aligned}$$

An easy computation using the expression of f_γ and the last two identities proves the claim: $df_\gamma = \gamma^*\sigma_2 - \sigma_2$.

Now, consider $\gamma_1 = \mathrm{e}^{C_1}, \gamma_2 = \mathrm{e}^{C_2} \in \Gamma$. Using equation (A.5) and Lemma A.9, we obtain

$$\begin{aligned} \gamma_2^* f_{\gamma_1} &= \sum_i e^i(C_1)\gamma_2^* u_{\eta^i} - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1)\gamma_2^* u_{e^i} \gamma_2^* u_{e^k} - \frac{1}{3}\sum_{i,j,k} c_{ijk} e^i(C_1) e^j(C_1)\gamma_2^* u_{e^k} \\ &= \sum_i e^i(C_1) u_{\eta^i} + \sum_i e^i(C_1)\eta^i(C_2) - \frac{1}{2}\sum_{i,j,k} c_{ijk} e^i(C_1) e^j(C_2) u_{e^k} \\ &\quad - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1)(u_{e^i} + e^i(C_2))(u_{e^k} + e^k(C_2)) - \frac{1}{3}\sum_{i,j,k} c_{ijk} e^i(C_1) e^j(C_1)(u_{e^k} + e^k(C_2)) \\ &= \sum_i e^i(C_1) u_{\eta^i} + \sum_i e^i(C_1)\eta^i(C_2) - \frac{1}{2}\sum_{i,j,k} c_{ijk} e^i(C_1) e^j(C_2) u_{e^k} \\ &\quad - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1) u_{e^i} u_{e^k} - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1) e^k(C_2) u_{e^i} - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1) e^i(C_2) u_{e^k} \\ &\quad - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1) e^i(C_2) e^k(C_2) - \frac{1}{3}\sum_{i,j,k} c_{ijk} e^i(C_1) e^j(C_1) u_{e^k} - \frac{1}{3}\sum_{i,j,k} c_{ijk} e^i(C_1) e^j(C_1) e^k(C_2). \end{aligned}$$

Moreover, since $\gamma_1\gamma_2 = \mathrm{e}^{C_1 + C_2 + \frac{1}{2}[C_1, C_2]}$ and $e^i|_{\mathfrak{n}'} = 0$, we obtain

$$\begin{aligned} f_{\gamma_1\gamma_2} &= \sum_i e^i(C_1 + C_2) u_{\eta^i} - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1 + C_2) u_{e^i} u_{e^k} - \frac{1}{3}\sum_{i,j,k} c_{ijk} e^i(C_1 + C_2) e^j(C_1 + C_2) u_{e^k} \\ &= \sum_i e^i(C_1) u_{\eta^i} + \sum_i e^i(C_2) u_{\eta^i} - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1) u_{e^i} u_{e^k} - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_2) u_{e^i} u_{e^k} \\ &\quad - \frac{1}{3}\sum_{i,j,k} c_{ijk}\left(e^i(C_1) e^j(C_1) + e^i(C_1) e^j(C_2) + e^i(C_2) e^j(C_1) + e^i(C_2) e^j(C_2)\right) u_{e^k}. \end{aligned}$$

Also, by definition

$$f_{\gamma_2} = \sum_i e^i(C_2) u_{\eta^i} - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_2) u_{e^i} u_{e^k} - \frac{1}{3}\sum_{i,j,k} c_{ijk} e^i(C_2) e^j(C_2) u_{e^k}$$

Combining all these expressions, we finally get

$$\begin{aligned}
c_{\gamma_1\gamma_2} &= f_{\gamma_2} + \gamma_2^* f_{\gamma_1} - f_{\gamma_1\gamma_2} \\
&= \sum_i e^i(C_1)\eta^i(C_2) - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1)e^i(C_2)e^k(C_2) - \frac{1}{3}\sum_{i,j,k} c_{ijk} e^i(C_1)e^j(C_1)e^k(C_2) \\
&\quad - \frac{1}{2}\sum_{i,j,k} c_{ijk} e^i(C_1)e^j(C_2)u_{e^k} - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1)e^k(C_2)u_{e^i} - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1)e^i(C_2)u_{e^k} \\
&\quad + \frac{1}{3}\sum_{i,j,k} c_{ijk} e^i(C_1)e^j(C_2)u_{e^k} + \frac{1}{3}\sum_{i,j,k} c_{ijk} e^i(C_2)e^j(C_1)u_{e^k} \\
&= \sum_i e^i(C_1)\eta^i(C_2) - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1)e^i(C_2)e^k(C_2) - \frac{1}{3}\sum_{i,j,k} c_{ijk} e^i(C_1)e^j(C_1)e^k(C_2) \\
&\quad - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^i(C_1)e^j(C_2)u_{e^k} - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1)e^k(C_2)u_{e^i} + \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1)e^i(C_2)u_{e^k} \\
&= \sum_i e^i(C_1)\eta^i(C_2) - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1)e^i(C_2)e^k(C_2) - \frac{1}{3}\sum_{i,j,k} c_{ijk} e^i(C_1)e^j(C_1)e^k(C_2) \\
&\quad - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^i(C_1)e^j(C_2)u_{e^k} - \frac{1}{6}\sum_{i,j,k} c_{kij} e^i(C_1)e^j(C_2)u_{e^k} - \frac{1}{6}\sum_{i,j,k} c_{jki} e^i(C_1)e^j(C_2)u_{e^k} \\
&= \sum_i e^i(C_1)\eta^i(C_2) - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1)e^i(C_2)e^k(C_2) - \frac{1}{3}\sum_{i,j,k} c_{ijk} e^i(C_1)e^j(C_1)e^k(C_2) \\
&\quad - \frac{1}{6}\sum_{i,j,k} (c_{ijk} + c_{kij} + c_{jki}) e^i(C_1)e^j(C_2)u_{e^k} \\
&= \sum_i e^i(C_1)\eta^i(C_2) - \frac{1}{6}\sum_{i,j,k} c_{ijk} e^j(C_1)e^i(C_2)e^k(C_2) - \frac{1}{3}\sum_{i,j,k} c_{ijk} e^i(C_1)e^j(C_1)e^k(C_2).
\end{aligned}$$

Since $c_{ijk} \in \mathbb{Z}$ and $C_1, C_2 \in \mathrm{span}_{\mathbb{Z}}(6e_1, \ldots, 6e_{n-n'}, z_1, \ldots, z_{n'})$, the thesis follows. □

Appendix B. The classification of 7-dimensional 2-step nilpotent Lie algebras

In this appendix, we recall the classification of real 7-dimensional 2-step nilpotent Lie algebras from [18]. For each Lie algebra $\mathfrak{n}$, the structure equations are written with respect to a basis $\{e^1, \ldots, e^7\}$ of the dual Lie algebra $\mathfrak{n}^*$.

The notation $\mathfrak{n}_{n,n'}$ or $\mathfrak{n}_{n,n',\bullet}$ means that the Lie algebra has dimension n and derived algebra of dimension n', while different capital letters in the third argument are used to distinguish non-isomorphic Lie algebras whose derived algebras have the same dimension. Moreover, $\mathfrak{h}_n$ denotes the Heisenberg Lie algebra of dimension n (for odd n), and $\mathfrak{h}_3^{\mathbb{C}}$ denotes the real Lie algebra underlying the complex 3-dimensional Heisenberg Lie algebra.

- 7-dimensional 2-step nilpotent Lie algebras $\mathfrak{n}$ with $\dim(\mathfrak{n}') = 1$:

$$\begin{aligned}
\mathfrak{h}_3 \oplus \mathbb{R}^4 &= \left(0,0,0,0,0,0,e^{12}\right), \\
\mathfrak{h}_5 \oplus \mathbb{R}^2 &= \left(0,0,0,0,0,0,e^{12}+e^{34}\right), \\
\mathfrak{h}_7 &= \left(0,0,0,0,0,0,e^{12}+e^{34}+e^{56}\right).
\end{aligned}$$

- 7-dimensional 2-step nilpotent Lie algebras $\mathfrak{n}$ with $\dim(\mathfrak{n}')=2$:

$$\begin{aligned}
\mathfrak{n}_{5,2}\oplus\mathbb{R}^2 &= \left(0,0,0,0,e^{12},e^{13},0\right),\\
\mathfrak{h}_3\oplus\mathfrak{h}_3\oplus\mathbb{R} &= \left(0,0,0,0,e^{12},e^{34},0\right),\\
\mathfrak{h}_3^{\mathbb{C}}\oplus\mathbb{R} &= \left(0,0,0,0,e^{13}-e^{24},e^{14}+e^{23},0\right),\\
\mathfrak{n}_{6,2}\oplus\mathbb{R} &= \left(0,0,0,0,e^{12},e^{14}+e^{23},0\right),\\
\mathfrak{n}_{7,2,A} &= \left(0,0,0,0,0,e^{12},e^{14}+e^{35}\right),\\
\mathfrak{n}_{7,2,B} &= \left(0,0,0,0,0,e^{12}+e^{34},e^{15}+e^{23}\right).
\end{aligned}$$

- 7-dimensional 2-step nilpotent Lie algebras $\mathfrak{n}$ with $\dim(\mathfrak{n}')=3$:

$$\begin{aligned}
\mathfrak{n}_{6,3}\oplus\mathbb{R} &= \left(0,0,0,0,e^{12},e^{13},e^{23}\right),\\
\mathfrak{n}_{7,3,A} &= \left(0,0,0,0,e^{12},e^{23},e^{24}\right),\\
\mathfrak{n}_{7,3,B} &= \left(0,0,0,0,e^{12},e^{23},e^{34}\right),\\
\mathfrak{n}_{7,3,B_1} &= \left(0,0,0,0,e^{12}-e^{34},e^{13}+e^{24},e^{14}\right)\\
\mathfrak{n}_{7,3,C} &= \left(0,0,0,0,e^{12}+e^{34},e^{23},e^{24}\right),\\
\mathfrak{n}_{7,3,D} &= \left(0,0,0,0,e^{12}+e^{34},e^{13},e^{24}\right),\\
\mathfrak{n}_{7,3,D_1} &= \left(0,0,0,0,e^{12}-e^{34},e^{13}+e^{24},e^{14}-e^{23}\right).
\end{aligned}$$

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