Weyl–Einstein structures in conformal geometry

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I. Introduction

Smooth manifolds

- On a smooth manifold M one can define smooth functions ($C^\infty(M)$), vector fields ($\mathfrak{X}(M)$), their flows, tensors, exterior forms, etc.
- A vector field can be seen as a derivation of the algebra of functions:

$$\mathfrak{X}(M) \times C^\infty(M) \to C^\infty(M), \qquad (X, f) \mapsto X(f).$$

- One can define the bracket of two vector fields:

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \qquad (X, Y) \mapsto [X, Y],$$

but one cannot differentiate a vector field with respect to another vector field.

- In order to do this, one needs a linear connection.

Connections and curvature

- A linear connection is a map

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \qquad (X, Y) \mapsto \nabla_X Y,$$

  which is:
  - $C^\infty(M)$-linear in the first argument,
  - $\mathbb{R}$-linear and satisfies the Leibniz rule with respect to the multiplication by functions in the second argument.
- Example: the usual directional derivative in $\mathbb{R}^n$.
- To a linear connection one can associate two tensors: a $(2,1)$-tensor called the torsion

$$T^\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y],$$

  and a $(3,1)$-tensor called the curvature

$$R^\nabla(X, Y, Z) := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

  (the "commutation failure" of ∇).

Riemannian metrics

A Riemannian metric on M is a symmetric $(2,0)$-tensor g which is positive definite at each point.

Theorem

A Riemannian metric g determines a unique linear connection ∇^g with the following properties:

- $T^{\nabla^g} = 0$,
- $\nabla^g g = 0$.

In fact ∇^g is explicitly given by Koszul's formula:

$$\begin{aligned} 2g(\nabla^g_X Y, Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &\quad + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y). \end{aligned}$$

The connection ∇^g is called the Levi-Civita connection.

The Levi-Civita connection

Proof.

Since $\nabla^g g = 0$, we can write:

$$X(g(Y,Z)) = g(\nabla^g_X Y, Z) + g(Y, \nabla^g_X Z) \quad (1)$$

$$Y(g(Z,X)) = g(\nabla^g_Y Z, X) + g(Z, \nabla^g_Y X) \quad (2)$$

$$Z(g(X,Y)) = g(\nabla^g_Z X, Y) + g(X, \nabla^g_Z Y) \quad (3)$$

The combination (1)+(2)-(3) gives the Koszul formula. □

The trace of the Riemannian curvature R^g is the Ricci tensor

$$\mathrm{Ric}^g(X,Y) := \mathrm{tr}(V \mapsto R^g(V,X,Y)).$$

The Ricci tensor is symmetric. M is called Einstein if $\mathrm{Ric}^g_0 = 0$ (i.e. if $\exists\ \lambda$ such that $\mathrm{Ric}^g = \lambda g$. If $n \geq 3$, λ is a constant.

Conformal structures on manifolds

A conformal structure on a smooth manifold M is an equivalence class of Riemannian metrics with respect to:

$$g \sim \tilde{g} \qquad \Longleftrightarrow \qquad \exists \varphi \in C^\infty(M),\ \tilde{g} = e^{2\varphi} g.$$

We denote by $c = [g]$ the conformal structure determined by g.

Definition

A Weyl connection on (M, c) is a torsion-free connection ∇ which preserves the conformal class:

$$\forall g \in c, \qquad \exists \theta^g \in \Omega^1(M), \qquad \nabla_X g = -2\theta^g(X) g,\ \forall X \in TM.$$

The 1-form θ^g is called the Lee form of ∇ wrt g.

Properties of Weyl connections

- If $\tilde{g} = e^{2\varphi} g$, then $\theta^{\tilde{g}} = \theta^g - d\varphi$.
- $\theta^g = 0 \iff \nabla = \nabla^g$.
- θ^g is exact $\iff$ ∇ is the Levi-Civita connection of some metric in c.

Definition

A Weyl connection ∇ on (M, c) is closed or exact if its Lee forms are closed or exact (this condition is independent of the metric).

Theorem (Fundamental theorem of conformal geometry)

For a fixed metric in the conformal class, there is a $1-1$ correspondence between 1-forms on M and Weyl connections.

The conformal Koszul formula

Proof of the fundamental theorem.

Recall the strategy in the proof of the Koszul formula:

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - 2\theta^g(X)g(Y,Z) \quad (4)$$

$$Y(g(Z,X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X) - 2\theta^g(Y)g(Z,X) \quad (5)$$

$$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) - 2\theta^g(Z)g(X,Y) \quad (6)$$

Then (4)+(5)-(6) gives

$$\nabla_X Y = \nabla^g_X Y + \theta^g(X)Y + \theta^g(Y)X - g(X,Y)(\theta^g)^\sharp.$$



The Gauduchon metric

Remark

One can see a Weyl structure as an equivalence class of pairs (g, θ) for the equivalence relation

$$(g, \theta) \sim (\tilde{g}, \tilde{\theta}) \iff \exists \varphi \in C^\infty(M),\ \tilde{g} = e^{2\varphi} g,\ \tilde{\theta} = \theta - d\varphi.$$

Theorem (Gauduchon, 1977)

If M is compact, then for every Weyl structure on (M, c), there exists a metric $g \in c$, unique up to constant rescaling, such that $\delta^g \theta^g = 0$.

This result has important applications in Hermitian geometry:

The Gauduchon metric

Example

Let (M, g, J) be a compact Hermitian manifold of complex dimension n with fundamental 2-form ω. By the isomorphism $\omega^{n-1}\wedge : \Lambda^1(M) \to \Lambda^{2n-1}(M)$, there exists a 1-form θ with $d(\omega^{n-1}) = -2(n-1)\theta \wedge \omega^{n-1}$. For $\tilde{g} := e^{2\varphi}g$, we have $\tilde{\omega} = e^{2\varphi}\omega$.

$$d(\tilde{\omega}^{n-1}) = 2(n-1)d\varphi \wedge \tilde{\omega}^{n-1} + e^{2(n-1)\varphi}d(\omega^{n-1}), \text{ so}$$

$$\tilde{\theta} \wedge \tilde{\omega}^{n-1} = -\frac{1}{2(n-1)}d\tilde{\omega}^{n-1} = (\theta - d\varphi) \wedge \tilde{\omega}^{n-1}.$$

Thus, $\tilde{\theta} = \theta - d\varphi$, so (g, θ) defines a Weyl structure. By Gauduchon, one can find a metric in the conformal class such that $\delta^{\tilde{g}}\tilde{\theta} = 0$, which is equivalent to $\partial\bar{\partial}(\tilde{\omega}^{n-1}) = 0$.

The Weyl-Ricci tensor

The Ricci tensor of a Weyl structure ∇ is

$$\mathrm{Ric}^{\nabla}(X,Y) := \mathrm{tr}(V \mapsto R^{\nabla}(V,X)Y), \qquad \text{for any } X, Y \in \mathfrak{X}(M).$$

Unlike the Riemannian case, Ric^{∇} is not symmetric. In fact:

$$\mathrm{Ric}^{\nabla} = \mathrm{Ric}^{g} - (n-2)(\nabla^{g}\theta^{g} - \theta^{g} \otimes \theta^{g}) + (\delta^{g}\theta^{g} - (n-2)\|\theta^{g}\|_{g}^{2})g,$$

so its skew-symmetric part is $-\frac{n-2}{2}d\theta^{g}$.

Weyl-Einstein structures

Definition

A Weyl connection ∇ is called a Weyl-Einstein structure if the trace-free symmetric component of Ric^{∇} vanishes.

By the above formula, for every metric g in c, this is equivalent to

$$\mathrm{Ric}^g = (n-2)((\nabla^g \theta^g)^{sym} - \theta^g \otimes \theta^g) + \alpha g,$$

for some function $\alpha \in C^\infty(M)$.
If $T := (\theta^g)^\sharp$ and $V := (n-2)T$, this can be written as

$$\mathrm{Ric}^g = \frac{1}{2}\mathcal{L}_V g - (n-2)\theta^g \otimes \theta^g + \alpha g$$

for some function α. Compare with Ricci solitons:

$$\mathrm{Ric}^g = \frac{1}{2}\mathcal{L}_V g + \alpha g.$$

Examples of Weyl-Einstein structures

- Einstein manifolds (M, g) with $\nabla = \nabla^g$.
- If (N, h) is Einstein with positive Einstein constant $\lambda > 0$, and $M = N \times S^1$ with the metric $g = h + dt^2$, then the Weyl structure whose Lee form with respect to g is $\theta^g = \sqrt{\frac{\lambda}{n-2}} dt$, is Weyl-Einstein.
- Jones-Tod construction: In three dimensions, Weyl-Einstein structures are deeply tied to four-dimensional hyper-Kähler manifolds that admit a tri-holomorphic vector field ξ. The space of orbits of ξ naturally inherits a Weyl-Einstein structure.
- Berger spheres, and more generally S^1-bundles over positive Kähler-Einstein manifolds.
- Orbits of cohomogeneity one actions on $\mathbb{S}^{n+1}$ or $\mathbb{S}^2 \times \mathbb{S}^{n-1}$ (Pedersen, Poon, Swann, 95).

The Gauduchon metric of Weyl-Einstein structures

Theorem (Tod, 1992)

If g is the Gauduchon metric of a Weyl-Einstein structure ∇, then the metric dual T of the Lee form θ^g is Killing with respect to g.

Corollary

If M is compact and ∇ is a closed Weyl-Einstein structure with Gauduchon metric g, then $T = (\theta^g)^\sharp$ is ∇^g-parallel, and M is a mapping torus.

Sketch of the proof.

$\ker(\theta^g)$ is totally geodesic; its integral leaves are positive Einstein and complete, so by Myers' Theorem they are compact. The universal cover of M is then isometric by de Rham to a Riemannian product $N \times \mathbb{R}$, so M is the mapping torus of an isometry of N. □

II. Weyl-Einstein structures on Lie groups

Riemannian Lie groups

Definition

Let G be a Lie group. A conformal structure c on G is called left-invariant if it contains a left-invariant metric g. A Weyl structure on (M, c) is left-invariant if θ^g is left-invariant.

If $\Gamma \subset G$ is a co-compact lattice, $M := \Gamma \backslash G$ inherits a Riemannian metric (denoted $\bar{g}$) and a left-invariant Weyl structure on G determines a Weyl structure on $(M, [\bar{g}])$.

Remark

The Gauduchon metric in this case is $\bar{g}$. Indeed, $\delta^g \theta^g = 0$ since G is unimodular.

Consider first the case of solvmanifolds (compact quotients of solvable Lie groups by co-compact lattices).

Weyl-Einstein structures on solvmanifolds

Theorem (del Barco, –, Schichl, 2023)

Let $(M, \bar{c})$ be a conformal solvmanifold defined by (G, g). Then $(M, \bar{c})$ admits a Weyl-Einstein structure ∇ if and only if (G, g) is flat.No left-invariance is assumed for ∇.

Sketch of the proof: If G is non-flat, the scalar curvature of g is a negative constant. Using the maximum principle one shows that the Weyl structure ∇ is exact, so its Gauduchon metric g_0 is Einstein. Is it left-invariant? Writing $\bar{g} = e^{2f} g_0$, we have

$$\mathrm{Scal}^{\bar{g}} = e^{-2f} \left(\mathrm{Scal}^{g_0} + 2(n-1)\Delta^{g_0} f - (n-1)(n-2) \|df\|^2_{g_0} \right).$$

Considering the points where f attains its minimum and maximum shows that f is constant. Thus $\bar{g}$ is Einstein, so flat. □

Invariant Weyl-Einstein structures

We now remove compactness, but replace it by left-invariance.

Theorem (del Barco, –, Schichl, 2023)

For any left-invariant conformal structure c on a nilpotent Lie group G, there exists a left-invariant Weyl-Einstein structure if and only if G is abelian.

Proof. Assume that G is non-abelian. We can work on the level of the Lie algebra $\mathfrak{g}$. Let $T \in \mathfrak{g}$ denote the metric dual of θ^g. The Weyl-Einstein condition reads

$$\begin{aligned}\mathrm{Ric}^g(x,x) =&(n-2)\left(g([x,T],x) - g(T,x)^2\right)\\ &+\frac{1}{n}\left(\mathrm{Scal}^g + (n-2)\|T\|_g^2\right)\|x\|_g^2.\end{aligned} \tag{7}$$

Since $\dim(\mathfrak{g}')^\perp \geq 2$, there exists a unit vector $x \in (\mathfrak{g}' + \mathbb{R})^\perp$, for which (7) becomes:

Invariant Weyl-Einstein structures on nilpotent Lie groups

$$\mathrm{Ric}^g(x,x) = \frac{1}{n}\left(\mathrm{Scal}^g + (n-2)\|T\|_g^2\right).$$

Moreover, $\mathrm{Ric}^g(x,x) \leq 0$ because $x \perp \mathfrak{g}'$,so

$$\mathrm{Scal}^g + (n-2)\|T\|_g^2 \leq 0. \tag{8}$$

Now, for a unit vector $y \in \mathfrak{z} \cap \mathfrak{g}'$, (7) gives

$$\mathrm{Ric}^g(y,y) + (n-2)g(T,y)^2 = \frac{1}{n}\left(\mathrm{Scal}^g + (n-2)\|T\|_g^2\right).$$

But $\mathrm{Ric}^g(y,y) > 0$, so $\mathrm{Scal}^g + (n-2)\|T\|_g^2 > 0$, contradicting (8).
□

Weyl-Einstein structures on almost abelian Lie groups

Let G be an almost abelian Lie group of dimension $n \geq 3$ with codimension 1 ideal $\mathfrak{h}$. For every metric g on the Lie algebra $\mathfrak{g}$, let b be a unit vector orthogonal to $\mathfrak{h}$. The Lie algebra $\mathfrak{g} = \mathbb{R}b \oplus \mathfrak{h}$ is determined by the derivation $D = \mathrm{ad}_b$. Let A, S be the symmetric and skew-symmetric components of D and let $\beta \in \mathfrak{g}^*$ be the metric dual of b.

Theorem (del Barco, –, Schichl, 2023)

For every $\theta \in \mathfrak{g}^*$, the left-invariant Weyl structure on $(G, [g])$ with Lee form θ with respect to g is Weyl-Einstein iff one of the following exclusive conditions holds:

1. $S = k\mathrm{Id}_{\mathfrak{h}}$ *for some* $k \in \mathbb{R}$, *and* $\theta = 0$ *or* $\theta = k\beta$, *or*
2. $S \neq 0$, $(\mathrm{tr}S)^2 = \mathrm{tr}(S^2)(n-2)$, $[A, S] = 0$ *and* $\theta = \frac{\mathrm{tr}(S)}{(n-2)}\beta$.

III. Compact manifolds with two Weyl-Einstein structures

Compact manifolds with two Weyl-Einstein structures

Consider the following:

Question

When does a compact conformal manifold carry two different Einstein-Weyl structures?

One needs to rule out some trivial solutions. Recall that a Weyl structure ∇ on (M, c) with Lee form θ wrt g is Weyl-Einstein iff

$$\mathrm{Ric}^g = (n-2)((\nabla^g\theta)^{sym} - \theta \otimes \theta) + \alpha g,$$

for some function α.If g is the Gauduchon metric, this reduces to

$$\mathrm{Ric}^g = -(n-2)\theta \otimes \theta + \alpha g.$$

Thus, if ∇ is not exact (so $\theta \neq 0$), one always has a second Weyl-Einstein structure (with Lee form $-\theta$ wrt g).

First case: two exact Weyl-Einstein structures

Consider first the case where ∇ and $\tilde{\nabla}$ are two distinct exact Weyl-Einstein structures on (M, c). That means that they are the Levi-Civita connections of two non-homothetic Einstein metrics. This case is settled by the following

Proposition (–, 2019)

Assume that M^n $(n \geq 2)$ is a compact manifold carrying two conformally related metrics g and $\tilde{g} = e^{2\varphi} g$ which are both Einstein. Then, either φ is constant (i.e. the two metrics are homothetic), or (M, g) is homothetic to the round sphere $\mathbb{S}^n$.

The relation between two Weyl-Einstein structures

Assume from now on that ∇ and $\tilde{\nabla}$ are Weyl-Einstein structures on (M, c) with Gauduchon metrics g, $\tilde{g} = e^{2f} g$ and Lee forms θ, $\tilde{\theta}$. By the above computations, this is equivalent to:

$$\mathrm{Ric}^g = -(n-2)\theta \otimes \theta + \alpha g, \qquad \mathrm{Ric}^{\tilde{g}} = -(n-2)\tilde{\theta} \otimes \tilde{\theta} + \tilde{\alpha}\tilde{g}.$$

The Ricci curvatures of g and $\tilde{g}$ are related as follows:

$$\mathrm{Ric}^{\tilde{g}} = \mathrm{Ric}^g - (n-2)(\nabla^g df - df \otimes df) + (\Delta^g f - (n-2)\|df\|_g^2)g.$$

Subtracting the two equations above and using the third one, we get

$$\nabla^g df = \tilde{\theta} \otimes \tilde{\theta} - \theta \otimes \theta + df \otimes df + \beta g,$$

for some function $\beta \in C^\infty(M)$.

The case where one Weyl structure is closed non-exact

Theorem (–, Pilca, 2026)

If a compact conformal manifold of dimension at least 3 has two distinct Weyl-Einstein structures and one of them is closed, non-exact, then they have the same Gauduchon metric and opposite Lee forms with respect to it.

Sketch of the proof. Assume that ∇ is closed, non-exact. By Tod, the Lee vector fields T and $\tilde{T}$ are Killing with respect to g and $\tilde{g}$, and thus T is ∇^g-parallel and non-zero. Then $\tilde{T}$ is conformal wrt g. By [Gauduchon, –, 2017], $\tilde{T}$ is also Killing wrt g (so $\tilde{T}(f) = 0$), commutes with T and has constant scalar product with T. Write $\tilde{T} = cT + \xi$ with $g(\xi, T) = 0$, $[\xi, T] = 0$ and constant c. One can assume $g(T, T) = 1$ by constant rescaling of g. Write

$$g = g_0 + \theta \otimes \theta, \qquad df = T(f) + d_0 f.$$

End of proof

Then decompose the equation

$$\nabla^g df = \tilde{\theta} \otimes \tilde{\theta} - \theta \otimes \theta + df \otimes df + \beta g,$$

in terms of the splitting

$$\mathrm{Sym}^2(T^*M) = \mathbb{R}\theta \otimes \theta \oplus (\ker \theta \odot \mathbb{R}) \oplus \mathrm{Sym}^2(\ker \theta).$$

In the system obtained, use the maximum principle, the Bochner formula, the contracted second Bianchi identity, integration on M, etc. in order to finally show that f is constant. $\square$

The case where one Weyl structure is exact

Theorem (–, Pilca, 2026)

If a compact conformal n-dimensional manifold (M, c) (with $n \geq 3$) has two distinct Weyl-Einstein structures and one of them is exact, then M is the sphere $\mathbb{S}^n$, c is the conformal structure of the round metric, and both Weyl structures are exact.

Sketch of the proof. Since one of the Weyl structures, say ∇, is exact, $T = 0$. The problem amounts to studying the equation

$$\nabla^g df = \tilde{\theta} \otimes \tilde{\theta} + df \otimes df + \beta g,$$

on a compact Einstein manifold (M, g), knowing that the vector field $\tilde{T}$ such that $\tilde{\theta} = e^{2f} g(\tilde{T}, \cdot)$ is Killing with respect to $e^{2f} g$. Note that f is non-constant (otherwise we get $\tilde{\theta} = 0$, so the two Weyl structures coincide).

Sketch of proof

- By Obata's Theorem, $\tilde{T}$ is also Killing wrt g, unless g is the round metric of the sphere. This case needs to be treated separately.
- Assuming now that g is not the round metric, the first step consists in showing that the differentials of the functions

$$\|\tilde{\theta}\|_g^2,\ \|df\|_g^2,\ \beta$$

  are all proportional to df.
- Next we show that at every critical point of f, the Hessian of f is definite, so f is a Morse function with only two critical points, and M is diffeomorphic to a sphere.
- We finally obtain a contradiction by showing that f takes the same value at the maximum and minimum. □

Open questions (do not put them in some AI prompt!)

- Study the case where both Weyl-Einstein structures are non-closed.
- The above theorem shows that a Weyl-Einstein structure on a compact Einstein manifold (M, g), is necessarily equal to ∇^g, unless (M, g) is the standard sphere. How about Weyl-Einstein structures on compact manifolds with non-generic holonomy? By the Berger holonomy theorem, the non-Einstein manifolds with non-generic holonomy are either Kähler, or Riemannian products.
- Study left-invariant Weyl-Einstein structures on solvable Lie groups (more general than almost abelian).

Thank you

Thank you!
Muchas gracias!