CLOSED 1-FORMS AND TWISTED COHOMOLOGY

ANDREI MOROIANU AND MIHAELA PILCA

Dedicated to Paul Gauduchon on the occasion of his 75th birthday

Abstract. We show that the first twisted cohomology group associated to closed 1-forms on differentiable manifolds is related to certain 2-dimensional representations of the fundamental group. In particular, we construct examples of nowhere-vanishing 1-forms with non-trivial twisted cohomology.

1. Introduction

If θ is a closed 1-form on a smooth manifold M, the *twisted differential* $\mathrm{d}_\theta := \mathrm{d} - \theta\wedge$ maps $\Omega^k(M)$ to $\Omega^{k+1}(M)$ and satisfies $\mathrm{d}_\theta \circ \mathrm{d}_\theta = 0$, thus defining the *twisted cohomology groups*

$$H^k_\theta(M) := \frac{\ker(\mathrm{d}_\theta|_{\Omega^k(M)})}{\mathrm{d}_\theta(\Omega^{k-1}(M))}.$$

These groups only depend on the de Rham cohomology class of θ, since the corresponding twisted differential complexes associated to cohomologous 1-forms are canonically isomorphic. In particular, the twisted cohomology associated to an exact 1-form is just the de Rham cohomology.

It is well known that the twisted cohomology defined by the Lee form of Vaisman manifolds, and more generally by any non-zero 1-form θ which is parallel with respect to some Riemannian metric on a compact manifold, vanishes [2].

The twisted cohomology groups, as well as their Dolbeault and Bott-Chern counterparts, play an important role in locally conformally Kähler geometry (*cf.* [1] or [5], where the twisted cohomology is called Morse-Novikov cohomology).

Twisted cohomology was also used by A. Pajitnov [6], who shows that if θ is a closed 1-form with non-degenerate zeros, then for large t the dimension of $H^k_{t\theta}(M)$ gives a lower bound for the number of the zeros of θ of index k. This is an analog of Witten's approach to Morse theory, in the more general situation of closed 1-forms.

2010 *Mathematics Subject Classification.* 53A30, 53B35, 53C25, 53C29, 53C55.
Key words and phrases. Twisted cohomology, Morse-Novikov cohomology.

On the other hand, in [7], A. Pajitnov defined a different *twisted Novikov homology* theory associated to closed 1-forms θ with integral cohomology class $[\theta] \in H^1(M,\mathbb{Z})$, and shows that the twisted Novikov homology vanishes whenever $[\theta]$ admits a nowhere-vanishing representative ([7], Theorem 1.3). We will see in Example 4.2 below that the corresponding result fails for the standard twisted cohomology theory considered here.

Our main result (Theorem 2.3) relates the non-zero elements in the first twisted cohomology group associated to a closed 1-form θ with some set of non-decomposable 2-dimensional representations of the first fundamental group of M which contain a trivial subrepresentation, and whose determinant is the character of $\pi_1(M)$ canonically associated to θ.

In Section 3 we derive several applications of this result, like the vanishing of the first twisted cohomology group on manifolds with nilpotent fundamental group (Corollary 3.1), the fact that if the commutator group $[\pi_1(M),\pi_1(M)]$ is finitely generated, then the set $\{[\theta] \in H^1_{\mathrm{dR}}(M) \,|\, H^1_\theta(M) \neq 0\}$ is finite (Corollary 3.2), or the non-vanishing of twisted cohomology on Riemann surfaces of genus $g \geq 2$ (Corollary 3.3). In the last section we give several examples of explicit computations of the first twisted cohomology group on mapping tori or Vaisman manifolds.

Acknowledgments. This work was supported by the Procope Project No. 57445459 (Germany) / 42513ZJ (France). We are grateful to Andrei Pajitnov and Chenji Fu for some very useful comments on earlier versions of this work.

2. The Main Result

Notation: the cohomology class of a d_θ-closed 1-form α is denoted by $[\alpha]_\theta$.

Let us recall the following well-known result and present a proof for it, whose method will be useful in the sequel.

Lemma 2.1. *Let M be a manifold. There is a bijection between*

$$H^1_{\mathrm{dR}}(M) \overset{1:1}{\longleftrightarrow} \{\rho\colon \pi_1(M) \to (\mathbb{R}^*_+,\times) \,|\, \rho \text{ is a representation}\}.$$

Proof. Let θ be a representative of a cohomology class $[\theta] \in H^1_{\mathrm{dR}}(M)$ and denote the universal cover of M by $\pi : \widetilde{M} \to M$. Then the pull-back $\widetilde{\theta} := \pi^*\theta$ of θ is an exact form, *i.e.* there exists $\varphi \in \mathcal{C}^\infty(\widetilde{M})$ such that $\widetilde{\theta} = \mathrm{d}\varphi$. Any element $\gamma \in \pi_1(M)$ acts trivially on $\widetilde{\theta}$, so $\gamma^*\mathrm{d}\varphi = \mathrm{d}\varphi$, which implies the existence of a constant $c_\gamma \in \mathbb{R}$ with $\gamma^*\varphi = \varphi + c_\gamma$. Since $\gamma_1^*\gamma_2^* = (\gamma_2\gamma_1)^*$, we see that $\gamma \mapsto c_\gamma$ is a group morphism from $\pi_1(M)$ to $(\mathbb{R},+)$. We then associate to $[\theta] \in H^1_{\mathrm{dR}}(M)$ the representation $\rho\colon \pi_1(M) \to (\mathbb{R}^*_+,\times)$ defined by $\rho(\gamma) := e^{c_\gamma}$. The representation ρ does not depend on the choice of the representative θ in its cohomology class. Indeed, if we replace θ by $\theta + \mathrm{d}h$, then φ is replaced by $\varphi + \pi^*h$, and since π^*h is invariant by $\pi_1(M)$, the constants c_γ do not change.

Conversely, for any representation $\rho\colon \pi_1(M) \to (\mathbb{R}^*_+, \times)$ we will construct a positive function g on $\widetilde{M}$ which is ρ-equivariant, i.e. $a^*g = \rho(a)g$ for every $a \in \pi_1(M)$. To do this, let us pick a non-negative function f on $\widetilde{M}$ satisfying the properties (i) and (ii) of Lemma 2.2 below. We introduce the function

$$g := \sum_{\gamma \in \pi_1(M)} \rho(\gamma^{-1})\gamma^* f$$

which is well-defined and smooth on $\widetilde{M}$ since the sum is finite in the neighbourhood of any point of $\widetilde{M}$ by property (ii). Moreover, g is a positive function on $\widetilde{M}$ since $f > 0$ on V and $\pi_1(M) \cdot V = \widetilde{M}$ by property (i). For any $a \in \pi_1(M)$, we have:

$$a^*g = \sum_{\gamma \in \pi_1(M)} \rho(\gamma^{-1})(\gamma a)^* f = \sum_{\delta \in \pi_1(M)} \rho(a\delta^{-1})\delta^* f = \rho(a)g.$$

This shows that $\widetilde{\theta} := \mathrm{d}(\ln g)$ is an exact 1-form on $\widetilde{M}$, which is $\pi_1(M)$-invariant, hence $\widetilde{\theta}$ descends to a closed 1-form θ on M. We associate to ρ the cohomology class of θ in $H^1_{\mathrm{dR}}(M)$. This does not depend on the choice of f. Indeed, if g_1 is any other positive function on $\widetilde{M}$ satisfying $a^*g_1 = \rho(a)g_1$ for every $a \in \pi_1(M)$, then g_1/g is $\pi_1(M)$-invariant, so it is the pull-back to $\widetilde{M}$ of some function h on M. Then the closed 1-form θ_1 on M satisfying $\pi^*\theta_1 = \mathrm{d}(\ln g_1)$ is $\theta_1 = \theta + \mathrm{d}h$, so $[\theta_1] = [\theta]$.

One can easily check that the above defined maps are inverse to each other.

$\square$

Lemma 2.2. *There exists a non-negative function $f \in \mathcal{C}^\infty(\widetilde{M}, \mathbb{R}_+)$ satisfying the following properties:*

(i) f is positive on some open set $V \subset \widetilde{M}$ with $\pi_1(M) \cdot V = \widetilde{M}$;
(ii) any point $x \in \widetilde{M}$ has an open neighborhood V_x, such that the set

$$\{\gamma \in \pi_1(M) \,|\, \gamma \cdot V_x \cap \mathrm{supp}(f) \neq \emptyset\}$$

is finite.

Proof. Denote by $\pi : \widetilde{M} \to M$ the covering map and let $(U_i)_{i \in I}$ be an open cover of M with contractible open sets. Since U_i are simply connected, there exist open sets V_i of $\widetilde{M}$ such that $\pi|_{V_i} : V_i \to U_i$ is a diffeomeorphism for each $i \in I$.

Let $(\rho_i)_{i \in I}$ be a partition of unity subordinate to the open cover $(U_i)_{i \in I}$. By definition, we have $\rho_i \geq 0$, $\mathrm{supp}(\rho_i) \subset U_i$, and every point $y \in M$ has an open neighbourhood U_y such that the set

$$I_y := \{i \in I \,|\, U_y \cap \mathrm{supp}(\rho_i) \neq \emptyset\} \tag{1}$$

is finite. We define $f_i : \widetilde{M} \to \mathbb{R}_+$ by $f_i|_{V_i} = \rho_i \circ \pi$ and $f_i|_{\widetilde{M}\setminus V_i} = 0$. Clearly f_i are smooth since $\operatorname{supp}(\rho_i) \subset U_i$. For every point $x \in \widetilde{M}$, there is only a finite number of $i \in I$ for which the open set $\pi^{-1}(U_{\pi(x)})$ meets the support of f_i, so the function $f := \sum_{i\in I} f_i$ is well-defined, smooth and non-negative on $\widetilde{M}$. We claim that it also satisfies the properties (i) and (ii).

Let $V := f^{-1}(\mathbb{R}^*_+)$ be the open set where f is positive. For every $x \in \widetilde{M}$, there exists $i \in I$ such that $\rho_i(\pi(x)) > 0$. If γ denotes the unique element in $\pi_1(M)$ for which $\gamma(x) \in V_i$, then $f_i(\gamma(x)) > 0$, so $f(\gamma(x)) > 0$, showing that $x \in \gamma^{-1}(V)$. Thus $\pi_1(M) \cdot V = \widetilde{M}$, so (i) is verified.

Let now $x \in \widetilde{M}$ be any point. We define

$$V_x := \bigcup_{i\in I_{\pi(x)}} V_i,$$

where $I_{\pi(x)}$ is the finite subset of I given by (1). Then $\{\gamma \in \pi_1(M) \,|\, \gamma \cdot V_x \cap \operatorname{supp}(f_i) \neq \emptyset\}$ is empty for every $i \in I \setminus I_{\pi(x)}$ and has exactly one element for every $i \in I_{\pi(x)}$. This shows that the set of $\gamma \in \pi_1(M)$ for which $\gamma \cdot V_x$ meets the support of f is finite, having the same cardinal as $I_{\pi(x)}$.

$\square$

Theorem 2.3. *Let M be a manifold and let θ be some non-exact closed 1-form on M. Let $\rho\colon \pi_1(M) \to (\mathbb{R}^*_+, \times)$ denote the representation associated to $[\theta] \in H^1_{\mathrm{dR}}(M)$, as in Lemma 2.1. Then the following assertions hold:*

(i) If $H^1_\theta(M) \neq 0$, then there exists an indecomposable representation $\xi\colon \pi_1(M) \to \mathrm{GL}_2(\mathbb{R})$ with $\det\xi = \rho$, which fixes the vector $\begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{R}^2$.

(ii) Conversely, if there exists an indecomposable representation $\xi\colon \pi_1(M) \to \mathrm{GL}_2(\mathbb{R})$ with $\det\xi = \rho$ and which fixes the vector $\begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{R}^2$, then $H^1_\theta(M) \neq 0$.

Proof. (i) Let α be a d_θ-closed 1-form on M whose twisted cohomology class $[\alpha]_\theta \in H^1_\theta(M)$ is non-zero: $[\alpha]_\theta \neq 0$. If $\pi\colon \widetilde{M} \to M$ denotes as before the universal cover map and φ is a primitive of $\pi^*\theta$ on $\widetilde{M}$, then

$$\pi^*\mathrm{d}_\theta = e^{\varphi}\mathrm{d}e^{-\varphi}\pi^*, \tag{2}$$

so that $\mathrm{d}_\theta\alpha = 0$ is equivalent to $\mathrm{d}(e^{-\varphi}\pi^*\alpha) = 0$ on $\widetilde{M}$. Hence, there exists a function $h \in \mathcal{C}^\infty(\widetilde{M})$, such that $e^{-\varphi}\pi^*\alpha = \mathrm{d}h$, and thus $\gamma^*(\mathrm{d}h) = e^{-c_\gamma}\mathrm{d}h = \rho(\gamma^{-1})\mathrm{d}h$. Therefore, there exists for each $\gamma \in \pi_1(M)$ a constant $\lambda(\gamma) \in \mathbb{R}$, such that

$$\gamma^*h = \rho(\gamma^{-1})h + \lambda(\gamma),$$

which equivalently reads

$$(\gamma^{-1})^* h = \rho(\gamma) h + \lambda(\gamma^{-1}), \qquad \forall\, \gamma \in \pi_1(M). \tag{3}$$

We claim that the map $\xi\colon \pi_1(M) \to \mathrm{GL}_2(\mathbb{R})$ defined by

$$\xi(\gamma) := \begin{pmatrix} 1 & \lambda(\gamma^{-1}) \\ 0 & \rho(\gamma) \end{pmatrix} \tag{4}$$

is a group morphism. Indeed, if $\gamma_1, \gamma_2 \in \pi_1(M)$, we have by (3):

$$((\gamma_1\gamma_2)^{-1})^* h = (\gamma_1^{-1})^*(\gamma_2^{-1})^* h = (\gamma_1^{-1})^*(\rho(\gamma_2)h + \lambda(\gamma_2^{-1})) = \rho(\gamma_2)(\rho(\gamma_1)h + \lambda(\gamma_1^{-1})) + \lambda(\gamma_2^{-1}),$$

thus showing that $\rho(\gamma_1\gamma_2) = \rho(\gamma_1)\rho(\gamma_2)$ and $\lambda((\gamma_1\gamma_2)^{-1}) = \rho(\gamma_2)\lambda(\gamma_1^{-1}) + \lambda(\gamma_2^{-1})$. Consequently,

$$\xi(\gamma_1)\xi(\gamma_2) = \begin{pmatrix} 1 & \lambda(\gamma_1^{-1}) \\ 0 & \rho(\gamma_1) \end{pmatrix} \begin{pmatrix} 1 & \lambda(\gamma_2^{-1}) \\ 0 & \rho(\gamma_2) \end{pmatrix} = \begin{pmatrix} 1 & \rho(\gamma_2)\lambda(\gamma_1^{-1}) + \lambda(\gamma_2^{-1}) \\ 0 & \rho(\gamma_1)\rho(\gamma_2) \end{pmatrix} = \xi(\gamma_1\gamma_2).$$

We clearly have that $\det(\xi) = \rho$. It remains to check that ξ is indecomposable. Assuming by contradiction that there exists a one-dimensional subrepresentation $V \subset \mathbb{R}^2$ of ξ with $V \neq \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$, then V is generated by some vector $\begin{pmatrix} c \\ 1 \end{pmatrix} \in \mathbb{R}^2$. By (4), for each $\gamma \in \pi_1(M)$ we have

$$\xi(\gamma) \begin{pmatrix} c \\ 1 \end{pmatrix} = \begin{pmatrix} c + \lambda(\gamma^{-1}) \\ \rho(\gamma) \end{pmatrix}.$$

Thus V is preserved by ξ if and only if $\lambda(\gamma^{-1}) + c = \rho(\gamma)c$ for every $\gamma \in \pi_1(M)$.

Together with (3) we obtain:

$$(\gamma^{-1})^*(h+c) = (\gamma^{-1})^* h + c = \rho(\gamma)h + \lambda(\gamma^{-1}) + c = \rho(\gamma)h + \rho(\gamma)c = \rho(\gamma)(h+c).$$

This shows that $e^{\varphi}(h+c)$ is the pull-back through π of a function on M, *i.e.* there exists $s \in \mathcal{C}^\infty(M)$ such that $h + c = e^{-\varphi}\pi^* s$. However, this yields:

$$e^{-\varphi}\pi^*\alpha = \mathrm{d}h = \mathrm{d}(h+c) = \mathrm{d}(e^{-\varphi}\pi^* s) = e^{-\varphi}\pi^* \mathrm{d}_\theta s,$$

whence $\alpha = \mathrm{d}_\theta s$, contradicting that $[\alpha]_\theta \neq 0$. We thus conclude that ξ is indecomposable.

(*ii*) We denote by M_γ the matrix of $\xi(\gamma)$ with respect to the standard basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, which is of the form $M_\gamma = \begin{pmatrix} 1 & \lambda(\gamma^{-1}) \\ 0 & \rho(\gamma) \end{pmatrix}$. Consider again the function $f \in \mathcal{C}^\infty(\widetilde{M}, \mathbb{R}_+)$ given by Lemma 2.2, and define the function $g\colon \widetilde{M} \to \mathbb{R}^2$ as follows:

$$g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} := \sum_{\gamma \in \pi_1(M)} M_{\gamma^{-1}} \cdot \gamma^* \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

As before, the function g is well-defined and smooth, since the sum is finite in the neighbourhood of any point of $\widetilde{M}$, by property (ii) in Lemma 2.2. Note that the function $g_2 = \sum_{\gamma\in\pi_1(M)} \rho(\gamma^{-1})\gamma^* f$ is positive on $\widetilde{M}$, by property (i) in Lemma 2.2. We compute for any $a \in \pi_1(M)$:

$$
\begin{aligned}
a^* g &= \sum_{\gamma\in\pi_1(M)} M_{\gamma^{-1}} \cdot a^*\gamma^* \begin{pmatrix} 0 \\ f \end{pmatrix} = \sum_{\gamma\in\pi_1(M)} M_{\gamma^{-1}} \cdot (\gamma a)^* \begin{pmatrix} 0 \\ f \end{pmatrix} \\
&= \sum_{\gamma\in\pi_1(M)} M_{a\gamma^{-1}} \cdot \gamma^* \begin{pmatrix} 0 \\ f \end{pmatrix} = M_a \cdot \sum_{\gamma\in\pi_1(M)} M_{\gamma^{-1}} \cdot \gamma^* \begin{pmatrix} 0 \\ f \end{pmatrix} = M_a \cdot g.
\end{aligned}
$$

Thus, for any $a \in \pi_1(M)$, we have:

$$
\begin{pmatrix} a^* g_1 \\ a^* g_2 \end{pmatrix} = \begin{pmatrix} g_1 + \lambda(a^{-1}) g_2 \\ \rho(a) g_2 \end{pmatrix}.
$$

Since $g_2 > 0$ on $\widetilde{M}$ and satisfies $a^* g_2 = \rho(a) g_2$, for all $a \in \pi_1(M)$, we conclude as in the proof of Lemma 2.1, that $\mathrm{d}(\ln g_2)$ is the pull-back of a closed 1-form θ' on M cohomologous to θ. Up to changing the representative, we may assume that $\pi^*\theta = \mathrm{d}(\ln g_2)$.

We define $h\colon \widetilde{M} \to \mathbb{R}$, $h := \frac{g_1}{g_2}$ and compute for every $a \in \pi_1(M)$:

$$
a^* h = \frac{a^* g_1}{a^* g_2} = \frac{g_1 + \lambda(a^{-1}) g_2}{\rho(a) g_2} = \rho(a^{-1}) h + \rho(a^{-1})\lambda(a^{-1}). \tag{5}
$$

This shows that $a^*\mathrm{d}h = \rho(a^{-1})\mathrm{d}h$ for all $a \in \pi_1(M)$, so the 1-form $g_2\mathrm{d}h$ is invariant under the action of $\pi_1(M)$. Consequently, there exists $\alpha \in \Omega^1(M)$ with $\pi^*\alpha = g_2 \mathrm{d}h$. We now check that α defines a non-trivial twisted cohomology class in $H^1_\theta(M)$. Firstly, α is d_θ closed, because

$$
\pi^*(\mathrm{d}_\theta \alpha) = e^{\varphi}\mathrm{d}e^{-\varphi}\pi^*\alpha = g_2 \mathrm{d}\left(\frac{1}{g_2}\pi^*\alpha\right) = g_2\mathrm{d}(\mathrm{d}h) = 0.
$$

We now assume that $[\alpha]_\theta = 0$ in $H^1_\theta(M)$, *i.e.* there exists $s \in \mathcal{C}^\infty(M)$ such that $\alpha = \mathrm{d}_\theta s$. Using (2), this implies

$$
g_2\mathrm{d}h = \pi^*\alpha = \pi^*\mathrm{d}_\theta s = g_2 \mathrm{d}\left(\frac{1}{g_2}\pi^* s\right),
$$

hence there exists a constant c such that $h = \frac{1}{g_2}\pi^* s + c$. We claim that the one-dimensional eigenspace spanned by the vector $\begin{pmatrix} c \\ 1 \end{pmatrix} \in \mathbb{R}^2$ is invariant under ξ. Namely, the following equality holds for all $a \in \pi_1(M)$, according to (5) and to the definition of c:

$$
\rho(a^{-1})h + \rho(a^{-1})\lambda(a^{-1}) = a^* h = a^*\left(\frac{1}{g_2}\pi^* s + c\right) = \frac{\pi^* s}{\rho(a) g_2} + c = \rho(a^{-1})(h - c) + c,
$$

which implies that $c + \lambda(a^{-1}) = \rho(a)c$. Hence, for any $a \in \pi_1(M)$, we have:

$$\xi(a)\begin{pmatrix} c \\ 1 \end{pmatrix} = M_a \begin{pmatrix} c \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda(a^{-1}) \\ 0 & \rho(a) \end{pmatrix}\begin{pmatrix} c \\ 1 \end{pmatrix} = \begin{pmatrix} c + \lambda(a^{-1}) \\ \rho(a) \end{pmatrix} = \rho(a)\begin{pmatrix} c \\ 1 \end{pmatrix}.$$

This contradicts the assumption that ξ is indecomposable, hence we conclude that $[\alpha]_\theta \neq 0$.

□

The indecomposability hypothesis in the above result can be equivalently stated as follows:

Lemma 2.4. *Let $\xi : \Gamma \to \mathrm{GL}_2(\mathbb{R})$ be a two-dimensional representation of a group Γ, which fixes the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2$ and such that $\rho := \det(\xi)$ is non-trivial. Then ξ is decomposable if and only if $[\Gamma, \Gamma] \subset \ker(\xi)$.*

Proof. If ξ is decomposable, then all matrices in $\xi(\Gamma)$ are simultaneously diagonalizable, so they commute, whence $\xi([\Gamma, \Gamma]) = \{\mathrm{I}_2\}$.

Assume, conversely, that $[\Gamma, \Gamma] \subset \ker(\xi)$. By hypothesis, there exists some $\gamma_0 \in \Gamma$ with $\rho(\gamma_0) \neq 1$. Then $\xi(\gamma_0)$ has two distinct eigenvalues, 1 and $\rho(\gamma_0)$, so it has two one-dimensional eigenspaces E_1 and E_2. For every element $\gamma \in \Gamma$, $\xi(\gamma)$ commutes with $\xi(\gamma_0)$, so $\xi(\gamma)$ preserves E_1 and E_2. Thus ξ is decomposable. □

3. Applications

We now derive some consequences of Theorem 2.3.

Corollary 3.1. *Let M be a manifold whose fundamental group $\pi_1(M)$ is nilpotent. Then for any non-trivial cohomology class $[\theta] \in H^1_{\mathrm{dR}}(M)$, we have $H^1_\theta(M) = 0$.*

Proof. Let $[\theta] \in H^1_{\mathrm{dR}}(M)$ with $[\theta] \neq 0$, and let $\rho \colon \pi_1(M) \to (\mathbb{R}^*_+, \times)$ denote the representation associated to $[\theta] \in H^1_{\mathrm{dR}}(M)$, given by Lemma 2.1. Applying Theorem 2.3, we have to show that any representation $\xi \colon \pi_1(M) \to \mathrm{GL}_2(\mathbb{R})$ with $\det \xi = \rho$ and which fixes the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is decomposable. We assume by contradiction that there exists such a representation ξ which is indecomposable.

Since $[\theta] \neq 0$, we have $\rho \neq 1$, so there exists $a \in \pi_1(M)$ such that $\det(\xi(a)) \neq 1$. Then $\xi(a)$ is diagonalizable, so there exists a basis of $\mathbb{R}^2$, such that the matrix of $\xi(a)$ with respect to this basis is given by $M_a = \begin{pmatrix} 1 & 0 \\ 0 & \rho(a) \end{pmatrix}$. Since ξ is assumed to be indecomposable, by Lemma 2.4, there exists $b_0 \in [\pi_1(M), \pi_1(M)]$ with $M_{b_0} = \begin{pmatrix} 1 & \lambda(b_0^{-1}) \\ 0 & \rho(b_0) \end{pmatrix}$ and $\lambda(b_0) \neq 0$. We then obtain for $b_1 := b_0^{-1}a^{-1}b_0a$:

$$M_{b_1} = \begin{pmatrix} 1 & -\frac{\lambda(b_0^{-1})}{\rho(b_0)} \\ 0 & \frac{1}{\rho(b_0)} \end{pmatrix} \begin{pmatrix} 1 & -\frac{\lambda(a^{-1})}{\rho(a)} \\ 0 & \frac{1}{\rho(a)} \end{pmatrix} \begin{pmatrix} 1 & \lambda(b_0^{-1}) \\ 0 & \rho(b_0) \end{pmatrix} \begin{pmatrix} 1 & \lambda(a^{-1}) \\ 0 & \rho(a) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \lambda(b_0^{-1})(\rho(a)-1) \\ 0 & 1 \end{pmatrix},$$

which shows that also $\lambda(b_1^{-1}) = \lambda(b_0^{-1})(\rho(a)-1) \neq 0$, because $\rho(a) \neq 1$ and $\lambda(b_0) \neq 0$. If we define for $i \in \mathbb{N}$ inductively $b_{i+1} := b_i^{-1}a_0^{-1}b_ia_0$, then $\lambda(b_i) \neq 0$, for all i, which contradicts the hypothesis that $\pi_1(M)$ is nilpotent.

□

Corollary 3.2. *Let M be a manifold whose commutator subgroup $G := [\pi_1(M), \pi_1(M)]$ is finitely generated. Then the set*

$$\{[\theta] \in H^1_{\mathrm{dR}}(M) \mid H^1_\theta(M) \neq 0\}$$

is finite and has at most $\mathrm{rank}(G)^{\mathrm{rank}(\pi_1(M))}$ *elements.*

Proof. Let $\{a_1, \ldots, a_m\}$ be a set of generators of $\pi_1(M)$ and let $\{b_1, \ldots, b_k\}$ be a set of generators of G. Let $[\theta] \in H^1_{\mathrm{dR}}(M)$ with $H^1_\theta(M) \neq 0$. Let $\rho\colon \pi_1(M) \to (\mathbb{R}^*_+, \times)$ denote the representation associated to $[\theta] \in H^1_{\mathrm{dR}}(M)$, given by Lemma 2.1, and let $\xi\colon \pi_1(M) \to \mathrm{GL}_2(\mathbb{R})$ be a representation associated to $[\theta]$, as in Theorem 2.3.

We denote by M_i the matrix of $\xi(b_i)$ with respect to the standard basis of $\mathbb{R}^2$. Since $b_i \in G = [\pi_1(M), \pi_1(M)]$, we have $\rho(b_i) = 1$, so the matrix M_i has the following form: $M_i = \begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix}$, for some $x_i \in \mathbb{R}$. Let us remark that at least one of the numbers x_i does not vanish, since otherwise the restriction of ξ to G would be trivial and then, by Lemma 2.4, ξ would be decomposable.

For any $1 \leq j \leq m$ and $1 \leq i \leq k$, the element $a_j^{-1}b_ia_j$ belongs to G. Therefore, there exist integers $n_{ij\ell}$, for $1 \leq \ell \leq k$, such that $a_j^{-1}b_ia_j = \prod\limits_{\ell=1}^{k} b_\ell^{n_{ij\ell}}$. On the one hand, we compute:

$$\xi(a_j^{-1})\xi(b_i)\xi(a_j) = \begin{pmatrix} 1 & -\frac{\lambda(a_j^{-1})}{\rho(a_j)} \\ 0 & \frac{1}{\rho(a_j)} \end{pmatrix} \begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda(a_j^{-1}) \\ 0 & \rho(a_j) \end{pmatrix} = \begin{pmatrix} 1 & x_i\rho(a_j) \\ 0 & 1 \end{pmatrix}.$$

On the other hand, we have:

$$\xi(a_j^{-1})\xi(b_i)\xi(a_j) = \xi(a_j^{-1}b_ia_j) = \xi\left(\prod_{\ell=1}^{k} b_\ell^{n_{ij\ell}}\right) = \prod_{\ell=1}^{k} M_\ell^{n_{ij\ell}} = \begin{pmatrix} 1 & \sum_{\ell=1}^{k} n_{ij\ell}x_\ell \\ 0 & 1 \end{pmatrix}.$$

Hence, for all $1 \leq j \leq m$ and $1 \leq i \leq k$, the following equality holds: $x_i\rho(a_j) = \sum_{\ell=1}^{k} n_{ij\ell}x_\ell$. If for each fixed $j \in \{1, \dots, m\}$, we define the $k \times k$-matrix with integer entries $N_j := (n_{ij\ell})_{i,\ell}$, then the above system of equations for j fixed can be equivalently written as:

$$(N_j - \rho(a_j)\mathrm{I}_k)\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = 0.$$

As previously noticed, at least one of the x_i's is non-zero. Thus $\rho(a_j)$ must be an eigenvalue of N_j, so each $\rho(a_j)$ can take at most k different values. Therefore, when j varies, there are overall at most k^m different possibilities for defining ρ, or, equivalently, for defining a cohomology class $[\theta] \in H^1_{\mathrm{dR}}(M)$ with $H^1_\theta(M) \neq 0$.

□

Corollary 3.3. *If S is a compact Riemann surface of genus $g \geq 2$, then $H^1_\theta(S) \neq 0$ for every closed 1-form θ on S.*

Proof. It is well known that $\pi_1(S)$ has $2g$ generators $\gamma_1, \dots, \gamma_{2g}$ subject to the relation

$$\prod_{j=1}^{g}(\gamma_{2j-1}\gamma_{2j}\gamma_{2j-1}^{-1}\gamma_{2j}^{-1}) = 1. \tag{6}$$

Any representation $\rho : \pi_1(S) \to (\mathbb{R}^*_+, \times)$ is defined by the $2g$ positive real numbers $y_i := \rho(\gamma_i)$. According to Lemma 2.4 and Theorem 2.3, we need to show for every such ρ, there exists a two-dimensional representation $\xi : \pi_1(S) \to \mathrm{GL}_2(\mathbb{R})$ with $\det(\xi) = \rho$, which fixes the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2$ and whose restriction to $[\pi_1(S), \pi_1(S)]$ is non-trivial.

We look for ξ of the form $\xi(\gamma_i) := \begin{pmatrix} 1 & x_i \\ 0 & y_i \end{pmatrix}$. The commutator of two such matrices is

$$\begin{pmatrix} 1 & x_i \\ 0 & y_i \end{pmatrix}\begin{pmatrix} 1 & x_j \\ 0 & y_j \end{pmatrix}\begin{pmatrix} 1 & x_i \\ 0 & y_i \end{pmatrix}^{-1}\begin{pmatrix} 1 & x_j \\ 0 & y_j \end{pmatrix}^{-1} = \begin{pmatrix} 1 & x_i(y_j-1) - x_j(y_i-1) \\ 0 & 1 \end{pmatrix},$$

so by (6), the condition that ξ defines a representation reads

$$\sum_{j=1}^{g} (x_{2j-1}(y_{2j}-1) - x_{2j}(y_{2j-1}-1)) = 0. \tag{7}$$

Moreover, such a representation is non-trivial on $[\pi_1(S), \pi_1(S)]$ provided that

$$\exists\, i, j \in \{1, \ldots, 2g\} \text{ such that } x_i(y_j - 1) - x_j(y_i - 1) \neq 0. \tag{8}$$

Since $g \geq 2$, for any positive real numbers y_i $(1 \leq i \leq 2g)$, one can choose the real numbers x_i such that (7) and (8) are satisfied.

□

4. Examples

Let f_A be the diffeomorphism of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ induced by a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ and let M_A be the mapping torus of f_A. In other words, M_A is the quotient of $\mathbb{T}^2 \times \mathbb{R}$ by the free $\mathbb{Z}$-action generated by the diffeomorphism $(p, t) \mapsto (f_A(p), t+1)$. The fundamental group of M_A is isomorphic to the semidirect product of $\mathbb{Z}$ acting on $\mathbb{Z}^2$: $\pi_1(M_A) \simeq \mathbb{Z}^2 \rtimes_A \mathbb{Z}$.

We pick some non-zero constant $c \in \mathbb{R}$ and denote by θ_c the closed form on M_A whose pull-back to $\mathbb{T}^2 \times \mathbb{R}$ is $c\,\mathrm{d}t$. The associated representation $\rho_c : \pi_1(M_A) \to (\mathbb{R}^*_+, \times)$ maps $\mathbb{Z}^2$ to 1 and the generator of $\mathbb{Z}$ to e^c.

Lemma 4.1. *$H^1_{\theta_c}(M_A) \neq 0$ if and only if e^c is an eigenvalue of A.*

Proof. If $H^1_{\theta_c}(M_A) \neq 0$, Theorem 2.3 shows that there exists an indecomposable representation $\xi : \pi_1(M_A) \to \mathrm{GL}_2(\mathbb{R})$ which fixes the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2$ and such that $\det(\xi) = \rho_c$. This means that for every $v \in \mathbb{Z}^2$ there exists $\lambda(v) \in \mathbb{R}$ such that $\xi(v) = \begin{pmatrix} 1 & \lambda(v) \\ 0 & 1 \end{pmatrix}$ and if a denotes the generator of the subgroup $\mathbb{Z} \subset \pi_1(M_A)$, there exists $x \in \mathbb{R}$ such that $\xi(a) = \begin{pmatrix} 1 & x \\ 0 & e^c \end{pmatrix}$.

The map λ is clearly a group morphism from $\mathbb{Z}^2$ to $(\mathbb{R}, +)$, so

$$\lambda(v_1, v_2) = \lambda_1 v_1 + \lambda_2 v_2, \qquad \forall v = (v_1, v_2) \in \mathbb{Z}^2. \tag{9}$$

Moreover, by Lemma 2.4, λ is not identically zero since $[\pi_1(M_A), \pi_1(M_A)] = \mathbb{Z}^2$.

Since $ava^{-1} = Av$, we get

$$\begin{pmatrix} 1 & \lambda(Av) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & e^c \end{pmatrix} \begin{pmatrix} 1 & \lambda(v) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & e^c \end{pmatrix}^{-1} = \begin{pmatrix} 1 & e^{-c}\lambda(v) \\ 0 & 1 \end{pmatrix},$$

whence

$$\lambda(Av) = e^{-c}\lambda(v), \qquad \forall v \in \mathbb{Z}^2. \tag{10}$$

By (9), this is equivalent to

$$ {}^tA \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = e^{-c} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}. \tag{11}$$

Thus e^{-c} is an eigenvalue of ${}^tA$, and since the spectra of A and ${}^tA$ are the same and $\det(A) = 1$, it follows that e^c is an eigenvalue of A.

Conversely, if e^c is an eigenvalue of A, then there exists $(\lambda_1, \lambda_2) \in \mathbb{R}^2 \setminus \{0\}$ such that (11) holds. Then (10) also holds for λ defined by (9).

We can then define a representation $\xi : \pi_1(M_A) \simeq \mathbb{Z}^2 \rtimes_A \mathbb{Z} \to \mathrm{GL}_2(\mathbb{R})$ by $\xi(v) := \begin{pmatrix} 1 & \lambda(v) \\ 0 & 1 \end{pmatrix}$, for $v \in \mathbb{Z}^2$ and $\xi(k) := \begin{pmatrix} 1 & 0 \\ 0 & e^{ck} \end{pmatrix}$, for $k \in \mathbb{Z}$. By Lemma 2.4, this representation is indecomposable, so by Theorem 2.3, we conclude that $H^1_{\theta_c}(M_A) \neq 0$.

$\square$

Example 4.2. Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, inducing a diffeomorphism f_A of $\mathbb{T}^2$ and let M_A denote the mapping torus of f_A as before. Since $\frac{3+\sqrt{5}}{2}$ is an eigenvalue of A, Lemma 4.1 shows that for $c := \ln \frac{3+\sqrt{5}}{2}$, the first twisted cohomology group associated to the nowhere vanishing 1-form $\theta_c := c\,\mathrm{d}t$ on M is non-zero: $H^1_{\theta_c}(M_A) \neq 0$.

By [2, Theorem 4.5], the twisted cohomology associated to a closed 1-form which is parallel with respect to some Riemannian metric, vanishes. The above example thus shows the existence of compact manifolds carrying nowhere vanishing closed 1-forms which are not parallel with respect to any Riemannian metric.

Our last example concerns the twisted cohomology on Vaisman manifolds. Recall that a Vaisman manifold is a locally conformally Kähler manifold with parallel Lee form [8]. The space of harmonic 1-forms on a compact Vaisman manifold (M, g, J) with Lee form θ decomposes as follows:

$$\mathcal{H}^1(M, g) = \mathrm{span}\{\theta\} \oplus \mathcal{H}^1_0(M, g), \tag{12}$$

where $\mathcal{H}^1_0(M, g)$ is J-invariant and consists of harmonic 1-forms pointwise orthogonal to θ and $J\theta$ (see for instance [3, Lemma 5.2]). That means that every harmonic 1-form on M can be written as $\beta = t\theta + \alpha$, with $t \in \mathbb{R}$ and $\alpha \in \mathcal{H}^1_0(M, g)$.

By [4, Lemma 3.3], every harmonic form $\beta = t\theta + \alpha$ with $t > 0$ is the Lee form of a Vaisman metric on M. In particular, for every non-vanishing t, there exists a metric on M whith respect to which β is parallel. By [2, Theorem 4.5], the twisted cohomology $H^*_{t\theta+\alpha}(M)$

vanishes for all $t \neq 0$ and $\alpha \in \mathcal{H}^1_0(M,g)$. It remains to understand the case where $t=0$, *i.e.* the twisted cohomology associated to forms $\alpha \in \mathcal{H}^1_0(M,g)$.

It turns out that there exist Vaisman manifolds (M,g) with $\mathcal{H}^1_0(M,g) \neq 0$, for which $H^*_\alpha(M)$ is non-zero for every $\alpha \in \mathcal{H}^1_0(M,g) \setminus \{0\}$.

Example 4.3. Let S be a compact oriented Riemann surface and let $\pi : N \to S$ be the principal S^1-bundle whose first Chern class is the positive generator $e \in H^2(S,\mathbb{Z})$. For every Riemannian metric g_S on S, the 3-dimensional manifold N carries a Riemannian metric g_N making π a Riemannian submersion, and which is Sasakian. Consequently, the Riemannian product $(M,g) := S^1 \times (N, g_N)$ is Vaisman. Its Lee form is just the length element of S^1, denoted by $\theta = \mathrm{d}t$.

The Gysin exact sequence associated to the fibration $\pi : N \to S$ reads

$$0 \mapsto H^1_{\mathrm{dR}}(S) \xrightarrow{\pi^*} H^1_{\mathrm{dR}}(N) \xrightarrow{\pi_*} H^0_{\mathrm{dR}}(S) \xrightarrow{c_1(N)\wedge} H^2_{\mathrm{dR}}(S) \longrightarrow \cdots .$$

By the choice of $c_1(N) = e$, the last arrow is an isomorphism, thus showing that $\pi^* : H^1_{\mathrm{dR}}(S) \to H^1_{\mathrm{dR}}(N)$ is an isomorphism too. Since $\pi : (N, g_N) \to (S, g_S)$ is a Riemannian submersion, we thus have $\pi^*(\mathcal{H}^1(S, g_S)) = \mathcal{H}^1(N, g_N)$.

Moreover, if $p_2 : M = S^1 \times N \to N$ denotes the projection on the second factor, we clearly have $\mathcal{H}^1(M,g) = \mathrm{span}\{\theta\} \oplus p_2^*(\mathcal{H}^1(N, g_N))$.

Denoting by $p := \pi \circ p_2$, the decomposition (12) becomes

$$\mathcal{H}^1(M,g) = \mathrm{span}\{\theta\} \oplus p^*(\mathcal{H}^1(S, g_S)). \tag{13}$$

Let α be a non-zero harmonic form in $\mathcal{H}^1(S, g_S)$ and let $\rho \colon \pi_1(S) \to (\mathbb{R}^*_+, \times)$ be the character of $\pi_1(S)$ associated to α, given by Lemma 2.1. Clearly, the character of $\pi_1(M)$ associated to $p^*\alpha$ is $\tilde{\rho} := \rho \circ p_*$, where $p_* : \pi_1(M) \to \pi_1(S)$ is the induced morphism of the fundamental groups. Note that, since the fibers of $p : M \to S$ are connected, the exact homotopy sequence shows that p_* is surjective.

By the proof of Corollary 3.3, there exists a two-dimensional representation $\xi : \pi_1(S) \to \mathrm{GL}_2(\mathbb{R})$ with $\det(\xi) = \rho$, which fixes the vector $\begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{R}^2$ and whose restriction to the commutator $[\pi_1(S), \pi_1(S)]$ is non-trivial.

Composing ξ with p_* yieds a two-dimensional representation $\tilde{\xi} := \xi \circ p_* : \pi_1(M) \to \mathrm{GL}_2(\mathbb{R})$ with $\det(\tilde{\xi}) = \tilde{\rho}$, which fixes the vector $\begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{R}^2$ and whose restriction to $[\pi_1(M), \pi_1(M)]$ is non-trivial (since p_* is surjective). By Theorem 2.3, the first twisted cohomology group $H^1_{p^*\alpha}(M)$ is non-vanishing.

References

[1] N. Istrati, *LCK Metrics on Toric LCS Manifolds*, J. Geom. Phys. **149** (3) (2020), article 103583.
[2] M. de León, B. López, J. C. Marrero, E. Padrón, *On the computation of the Lichnerowicz-Jacobi cohomology*, J. Geom. Phys. **44** (4) (2003), 507–522.
[3] F. Madani, A. Moroianu, M. Pilca, *On toric locally conformally Kähler manifolds*, Ann. Global Anal. Geom. **51** (4) (2017), 401–417.
[4] F. Madani, A. Moroianu, M. Pilca, *LCK structures with holomorphic Lee vector field on Vaisman-type manifolds*, to appear in Geom. Dedicata. doi:10.1007/s10711-020-00578-8.
[5] L. Ornea, M. Verbitsky, *Morse-Novikov cohomology of locally conformally Kähler manifolds*, J. Geom. Phys. **59** (3) (2009), 295–305.
[6] A. Pajitnov, *An analytic proof of the real part of Novikov's inequalities*, Soviet Math. Dokl. **35** (2) (1987), 456–457.
[7] A. Pajitnov, *Novikov Homology, twisted Alexander polynomials and Thurston cones,* St. Petersburg Math. J. **18** (2007), 809–835.
[8] I. Vaisman, *Locally conformal Kähler manifolds with parallel Lee form*, Rend. Mat. **12** (2) (1979), 263–284.

Andrei Moroianu, Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, 91405, Orsay, France

Email address: andrei.moroianu@math.cnrs.fr

Mihaela Pilca, Fakultät für Mathematik, Universität Regensburg, Universitätsstr. 31 D-93040 Regensburg, Germany

Email address: mihaela.pilca@mathematik.uni-regensburg.de