LCK STRUCTURES WITH HOLOMORPHIC LEE VECTOR FIELD ON VAISMAN-TYPE MANIFOLDS

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Abstract. We give a complete description of all locally conformally Kähler structures with holomorphic Lee vector field on a compact complex manifold of Vaisman type. This provides in particular examples of such structures whose Lee vector field is not homothetic to the Lee vector field of a Vaisman structure. More generally, dropping the condition of being of Vaisman type, we show that on a compact complex manifold, any lcK metric with potential and with holomorphic Lee vector field admits a potential which is positive and invariant along the anti-Lee vector field.

1. Introduction

A Hermitian metric g on a complex manifold (M, J) is called locally conformally Kähler (in short, lcK) if around any point in M, the metric g can be conformally rescaled to a Kähler metric. This condition is equivalent to the existence of a closed 1-form θ such that $\mathrm{d}\Omega = \theta \wedge \Omega$, where Ω denotes the fundamental 2-form defined as $\Omega(\cdot, \cdot) := g(J\cdot, \cdot)$. The 1-form θ is called the Lee form and its metric dual is called the Lee vector field. In this paper we assume that $\theta \neq 0$, *i.e.* (J, g, Ω) is not Kähler.

A special class of lcK structures is represented by the so-called Vaisman structures, defined by the property that the Lee form is non-zero and parallel with respect to the Levi-Civita connection of g. It is known that a Vaisman structure on a complex manifold is uniquely determined, up to a positive constant, by its Lee form, via the following identity:

$$\Omega = \frac{1}{|\theta|^2}(\theta \wedge J\theta - \mathrm{d}J\theta).$$

On a compact complex manifold of Vaisman type, *i.e.* admitting at least one Vaisman metric, the Lee vector fields of all Vaisman structures are holomorphic, and coincide up to a positive multiplicative constant. This fact was originally obtained in [12], but for the reader's convenience we give below an alternative proof.

In [8], A. Moroianu, S. Moroianu and L. Ornea proved that a compact lcK manifold with holomorphic Lee vector field is Vaisman if the Lee vector field either has constant norm or is divergence-free. Moreover, they also construct examples of non-Vaisman lcK structures with holomorphic Lee vector field on a compact manifold of Vaisman type. These lcK structures, however, have the same Lee vector field as the Vaisman structures. More recently, F. Belgun

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[1] constructed examples of lcK manifolds with holomorphic Lee vector field on compact complex manifolds which are not of Vaisman type.

In this paper we describe all lcK structures with holomorphic Lee vector field on compact complex manifolds of Vaisman type. In particular, this description provides examples of such structures whose Lee vector field is not homothetic to the common Lee vector field of the Vaisman structures (see Remark 4.9) and which were not known to exist up to date. We also describe all holomorphic vector fields on Vaisman-type manifolds which can be obtained as Lee vector fields of lcK structures.

After introducing the needed notions and notation in a preliminary section, we will recall in Section 3 the description, due to K. Tsukada [12], of the space of all 1-forms which occur as Lee forms of Vaisman structures. We give here a new presentation, based on two types of deformations of Vaisman structures.

In Section 4 we prove the main results. Given an lcK structure (Ω, θ) with holomorphic Lee vector field T on a compact complex manifold (M, J) of Vaisman type, we show in a first step that there exists a Vaisman structure $(g_0, \Omega_0, \theta_0)$ on (M, J) adapted to this lcK structure, in the sense that its Lee form θ_0 is cohomologous to θ and is JT-invariant. This follows by an averaging process and using the aforementioned deformations of Vaisman structures, as well as the convexity of the space of Lee forms of Vaisman structures. In particular, the anti-Lee vector field JT is holomorphic and Killing with respect to the Vaisman metric g_0. The space of all holomorphic Killing vector fields on a compact Vaisman manifold will be described as a separate result (cf. Lemma 4.4), which allows us to deduce that JT is completely determined by a function a, via the formula $JT = aJT_0 - J\mathrm{grad}^{g_0} a$. Moreover, we prove that the function a is necessarily positive (see Theorem 4.8).

Conversely, we show that on a compact Vaisman manifold $(M, J, g_0, \Omega_0, \theta_0)$, given a holomorphic Killing vector field of the form $K = aJT_0 - J\mathrm{grad}^{g_0} a$, where a is a positive function, there exists an lcK structure with holomorphic Lee vector field equal to $-JK$. Moreover, we describe all these lcK structures by two parameters, namely a K-invariant function and a K-invariant twisted harmonic form of type $(0, 1)$, which satisfy two certain positivity conditions, thus defining an open set in some infinite-dimensional vector space (see Theorem 4.11).

As a last result, we show that on a compact lcK manifold, not necessarily of Vaisman type, if we additionally assume that the lcK structure with holomorphic Lee vector field T has a potential, then there exists a JT-invariant potential which is positive everywhere.

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2. Preliminaries on lcK and Vaisman manifolds

Let (M, J) be a compact complex manifold. An lcK structure on (M, J) is a Hermitian metric g whose fundamental form

$$\Omega := g(J\cdot, \cdot)$$

satisfies the lcK condition $\mathrm{d}\Omega = \theta \wedge \Omega$, for some closed 1-form θ called the Lee form. The metric dual T of θ will be called the Lee vector field.

The lcK condition is conformally invariant: if (Ω, θ) is an lcK structure on (M, J), then $(e^f\Omega, \theta + \mathrm{d}f)$ is an lcK structure on (M, J) for every smooth function f. An lcK structure is thus globally conformally Kähler if and only if its Lee form is exact. If this does not hold, the lcK structure is called *strict*.

On a manifold M endowed with a closed 1-form θ, one may introduce the twisted differential $\mathrm{d}_\theta := \mathrm{d} - \theta \wedge \cdot$, which satisfies $\mathrm{d}_\theta^2 = 0$, and thus defines so-called Morse-Novikov (or twisted) cohomology groups $H^*_\theta(M)$. If (M, J) is a complex manifold, the twisted differential can be decomposed as follows:

$$\mathrm{d}_\theta = \partial_\theta + \bar\partial_\theta,$$

with

$$\partial_\theta := \partial - \theta^{1,0} \wedge \cdot \ , \qquad \bar\partial_\theta := \bar\partial - \theta^{0,1} \wedge \cdot \ ,$$

where $\theta^{1,0} := \frac{1}{2}(\theta + iJ\theta)$ and $\theta^{0,1} := \frac{1}{2}(\theta - iJ\theta)$. One may also introduce the real operator

$$\mathrm{d}^c_\theta := i(\bar\partial_\theta - \partial_\theta) = [J, \mathrm{d}_\theta],$$

where J acts on exterior forms as a derivation. Since $\mathrm{d}_\theta^2 = 0$, we immediately get

$$\partial_\theta^2 = 0, \qquad \bar\partial_\theta^2 = 0, \qquad \partial_\theta\bar\partial_\theta + \bar\partial_\theta\partial_\theta = 0, \qquad \mathrm{d}_\theta\mathrm{d}^c_\theta = 2i\partial_\theta\bar\partial_\theta.$$

Note that, by definition, a Hermitian form Ω on (M, J) is lcK if and only if there exists a closed 1-form θ such that $\mathrm{d}_\theta\Omega = 0$. The lcK structure (Ω, θ) is called d_θ-exact if there exists a 1-form β such that $\Omega = \mathrm{d}_\theta\beta$.

If $\theta' := \theta + \mathrm{d}f$, then $\mathrm{d}_{\theta'} = e^f\mathrm{d}_\theta e^{-f}$. Similar conjugation relations hold for the other operators introduced above, so that their properties only depend on the cohomology class of θ. In particular, for second order operators the following relation holds:

$$\mathrm{d}_{\theta'}\mathrm{d}^c_{\theta'}(e^f h) = e^f\mathrm{d}_\theta\mathrm{d}^c_\theta h, \qquad \text{for any } h \in \mathcal{C}^\infty(M). \tag{1}$$

Definition 2.1. An lcK metric g on (M, J) is called Vaisman if its Lee vector field T is Killing with respect to g and non-zero. A complex manifold (M, J) is called of Vaisman type if it admits a Vaisman structure.

Let ∇ denote the Levi-Civita connection of g. Since θ is closed, $\nabla\theta$ is a symmetric bilinear form. Moreover, the Lie derivative of g with respect to T is equal to the symmetric part of $\nabla\theta$. The above condition that T is a Killing vector field is thus equivalent to the more familiar condition that $\nabla\theta = 0$.

Assume from now on that $(M, J, g_0, \Omega_0, \theta_0)$ is Vaisman. Being parallel, θ_0 has constant norm with respect to g_0. Moreover, the Lee vector field T_0 is holomorphic with respect to J, i.e. $\mathcal{L}_{T_0}J = 0$ (cf. [13] or [8, Lemma 3]). Hence, it also follows that $\mathcal{L}_{T_0}\Omega_0 = 0$, so using the Cartan formula we get

$$0 = \mathcal{L}_{T_0}\Omega_0 = \mathrm{d}(T_0 \lrcorner \Omega_0) + T_0 \lrcorner \mathrm{d}\Omega_0 = \mathrm{d}J\theta_0 + T_0 \lrcorner (\theta_0 \wedge \Omega_0) = \mathrm{d}J\theta_0 + |\theta_0|^2\Omega_0 - \theta_0 \wedge J\theta_0,$$

whence the well known formula

$$\mathrm{d}J\theta_0 = \theta_0 \wedge J\theta_0 - |\theta_0|^2\Omega_0, \tag{2}$$

which can also be written as

$$\Omega_0 = \mathrm{d}_{\theta_0}\mathrm{d}^c_{\theta_0}\left(\frac{1}{|\theta_0|^2}\right). \tag{3}$$

Definition 2.2. An lcK structure (J, Ω, θ) for which there exists a function h such that $\Omega = \mathrm{d}_\theta \mathrm{d}^c_\theta(h)$ is called an lcK structure with potential [10].

The above formula (3) shows that Vaisman metrics have potential, and by (1), if (Ω, θ) is an lcK structure with potential h, then $(e^f\Omega, \theta + \mathrm{d}f)$ is an lcK structure with potential $e^f h$.

Remark 2.3. Every lcK metric with potential, in particular every Vaisman metric, on a compact complex manifold is strict. Indeed, if $\theta = -\mathrm{d}f$ and $\Omega = \mathrm{d}_\theta \mathrm{d}^c_\theta h$, then (1) shows that $e^f\Omega = \mathrm{d}\mathrm{d}^c(e^f h)$ is a Kähler metric with global potential, which is not possible on compact manifolds.

Note that there exist lcK metrics without potential (cf. [10]), even on Vaisman-type manifolds (cf. [3]). However, N. Istrati has recently shown the following result which will be very useful for us in the sequel.

Proposition 2.4. *(cf.* [4, Theorem 6.2]*) Let $(M, J, g_0, \Omega_0, \theta_0)$ be a compact Vaisman manifold and let (Ω, θ_0) be an lcK structure on (M, J) with the same Lee form. Then there exists a function $h \in \mathcal{C}^\infty(M, \mathbb{R})$ and a form $\alpha \in \Omega^{0,1}(M)$ in the (finite dimensional) kernel of the twisted Laplacian $\Delta_{\bar\partial_{\theta_0}} := \bar\partial_{\theta_0}\bar\partial^*_{\theta_0} + \bar\partial^*_{\theta_0}\bar\partial_{\theta_0}$, such that:*

$$\Omega = \mathrm{d}_{\theta_0}\mathrm{d}^c_{\theta_0} h + \partial_{\theta_0}\alpha + \bar\partial_{\theta_0}\overline{\alpha}, \tag{4}$$

where $\bar\partial^*_{\theta_0}$ denotes the formal adjoint of $\bar\partial_{\theta_0}$ with respect to g_0.

The proof is based on standard Hodge theory, using the identification of the twisted Bott-Chern cohomology with some twisted Dolbeault cohomology, due to the vanishing of the Morse-Novikov cohomology on Vaisman manifolds (cf. Corollary 3.5 below).

3. Deformations of Vaisman structures

In this section we consider two types of deformations of Vaisman structures, which have already been introduced in [12]. However, we present them from a different point of view that is useful for the purpose of this paper.

Recall first that the space of harmonic forms on the compact Vaisman manifold $(g_0, \Omega_0, \theta_0)$ decomposes as follows:

$$\mathcal{H}^1(M, g_0) = \mathrm{span}\{\theta_0\} \oplus \mathcal{H}^1_0(M, g_0), \tag{5}$$

where $\mathcal{H}^1_0(M)$ is J-invariant and consists of harmonic 1-forms pointwise orthogonal to θ_0 and $J\theta_0$. In particular, the Cartan formula shows that $\mathcal{L}_{T_0}\alpha = 0$ and $\mathcal{L}_{JT_0}\alpha = 0$ for every $\alpha \in \mathcal{H}^1(M, g_0)$. For a proof, see for instance [7, Lemma 5.2].

If $(g_0, \Omega_0, \theta_0)$ is a Vaisman structure on (M, J) with Lee vector field T_0, then for every positive real number $t > 0$, the triple $(g_0' := tg_0, \Omega_0' := t\Omega_0, \theta_0)$ is a new Vaisman structure with the same Lee form, whose Lee vector field is $T_0' = \frac{1}{t}T_0$. The square norms of the Lee fields with respect to the respective metrics are related by

$$|T_0'|^2_{g_0'} = \theta_0(T_0') = \frac{1}{t}\theta_0(T_0) = \frac{1}{t}|T_0|^2_{g_0}. \tag{6}$$

Definition 3.1. A Vaisman metric is called *normalized* if the Lee form (or, equivalently, the Lee vector field) has norm 1.

From (6) it follows that every Vaisman metric is proportional to a normalized Vaisman metric, which is unique in its homothety class. By (3), every normalized Vaisman metric has constant potential 1.

Lemma 3.2. *Let $(M, J, g_0, \Omega_0, \theta_0)$ be a compact Vaisman manifold and let (g, Ω, θ) be any lcK structure on (M, J). If $[\theta] = t[\theta_0] + [\alpha]$, with $t \in \mathbb{R}$ and $\alpha \in \mathcal{H}^1_0(M, g_0)$ denotes the decomposition of $[\theta]$ with respect to* (5)*, then $t > 0$.*

Proof. In the conformal class of g, there exists an lcK metric with Lee form equal to $t\theta_0 + \alpha$. So without loss of generality, we assume that $\theta = t\theta_0 + \alpha$. As $\mathrm{d}J\alpha = 0$, we get from (2)

$$\mathrm{d}J\theta = t\mathrm{d}J\theta_0 + \mathrm{d}J\alpha = t(\theta_0 \wedge J\theta_0 - |\theta_0|^2_{g_0}\Omega_0),$$

Let n denote the complex dimension of M. Taking the wedge product with Ω^{n-1} and integrating over M yields

$$\int_M \mathrm{d}J\theta \wedge \Omega^{n-1} = t\int_M (\theta_0 \wedge J\theta_0 - |\theta_0|^2_{g_0}\Omega_0) \wedge \Omega^{n-1},$$

which has the opposite sign of t, since $\int_M (\theta_0 \wedge J\theta_0 - |\theta_0|^2_{g_0}\Omega_0) \wedge \Omega^{n-1} \le 0$. On the other hand, by Stokes' theorem, we have

$$\int_M \mathrm{d}J\theta \wedge \Omega^{n-1} = \int_M J\theta \wedge \mathrm{d}(\Omega^{n-1}) = -(n-1)\int_M \theta \wedge J\theta \wedge \Omega^{n-1} < 0,$$

since $\theta \neq 0$. We thus deduce that $t > 0$. □

Lemma 3.3. *Let $(M, J, g_0, \Omega_0, \theta_0)$ be a Vaisman manifold. For every positive real number $t > 0$ and harmonic 1-form $\alpha \in \mathcal{H}^1_0(M, g_0)$, the pair (Ω_0', θ_0') defined by $\theta_0' := t\theta_0 + \alpha$ and $\Omega_0' := \theta_0' \wedge J\theta_0' - \mathrm{d}J\theta_0'$ is a normalized Vaisman structure on (M, J).*

Proof. Since θ_0, $\mathcal{H}^1_0(M, g_0)$, and the expression of Ω_0' do not change if Ω_0 is rescaled by a constant, one can assume that $(M, J, g_0, \Omega_0, \theta_0)$ is normalized. Then (2) shows that $\mathrm{d}J\theta_0 = \Omega_0 - \theta_0 \wedge J\theta_0$. Moreover $\mathrm{d}\alpha = \mathrm{d}J\alpha = 0$, whence

$$\Omega_0' = (t\theta_0 + \alpha) \wedge J(t\theta_0 + \alpha) + t(\Omega_0 - \theta_0 \wedge J\theta_0). \tag{7}$$

Denoting as before by $T_0 := \theta_0^\sharp$ the metric dual of θ_0 with respect to g_0, we have that the kernel of $\Omega_0 - \theta_0 \wedge J\theta_0$ is the span of T_0 and JT_0. On the other hand,

$$[(t\theta_0 + \alpha) \wedge J(t\theta_0 + \alpha)](aT_0 + bJT_0, J(aT_0 + bJT_0)) = t^2(a^2 + b^2)$$

since α vanishes pointwise on T_0 and JT_0. This shows that the right-hand side of (7) is positive definite since it is the sum of two positive semi-definite (1,1)-forms which have no common kernel. Clearly $\mathrm{d}\Omega_0' = \theta_0' \wedge \Omega_0'$, so (Ω_0', θ_0') is lcK.

Moreover, $T_0 \lrcorner \Omega_0' = tJ\theta_0'$, so $T_0' := \frac{1}{t}T_0$ is the Lee vector field of (Ω_0', θ_0'). In particular, T_0' is holomorphic, and furthermore $\mathcal{L}_{T_0'}\Omega_0' = \frac{1}{t}\mathcal{L}_{T_0}\Omega_0' = 0$ since $\mathcal{L}_{T_0}\alpha = 0$, $\mathcal{L}_{T_0}J = 0$, and $\mathcal{L}_{T_0}\theta_0 = 0$. Hence T_0' is Killing with respect to g_0', and thus (Ω_0', θ_0') is Vaisman. Finally, $\theta_0'(T_0') = t\theta_0 + \alpha(\frac{1}{t}T_0) = \theta_0(T_0) = 1$, so (Ω_0', θ_0') is normalized. □

On a complex manifold (M, J), the Vaisman structure (Ω_0', θ_0') defined as in Lemma 3.3 will be called a *deformation of type I* of (Ω_0, θ_0).

A direct consequence of Lemmas 3.2 and 3.3 is the following result:

Proposition 3.4. *The Lee form of any lcK structure on a Vaisman manifold is cohomologous to the Lee form of a Vaisman structure which is a deformation of type I of the initial Vaisman structure.*

Thus, if we denote by $H^1_{\mathrm{lcK}}(M)$ and by $H^1_{\mathrm{Vaisman}}(M)$ the set of all de Rham cohomology classes that are represented by the Lee form of an lcK structure, respectively of a Vaisman structure, then on a compact complex manifold of Vaisman type these sets are equal and are described as follows with respect to an arbitrarily fixed Vaisman structure $(g_0, \Omega_0, \theta_0)$:

$$H^1_{\mathrm{lcK}}(M) = H^1_{\mathrm{Vaisman}}(M) = \{t[\theta_0] + [\alpha] \in H^1_{\mathrm{dR}}(M, \mathbb{R}) \,|\, t > 0, \alpha \in \mathcal{H}^1_0(M, g_0)\}.$$

Corollary 3.5. *If θ is the Lee form of some lcK structure on a Vaisman-type manifold (M, J), the twisted cohomology $H^*_\theta(M)$ vanishes.*

Proof. By Proposition 3.4, θ is cohomologous to the Lee form θ_0 of a Vaisman structure on (M, J). Since the twisted cohomology only depends on the cohomology class of the 1-form, we have $H^*_\theta(M) = H^*_{\theta_0}(M)$. On the other hand, it was shown in [6] that the twisted cohomology $H^*_{\theta_0}(M)$ vanishes whenever θ_0 is non-vanishing and parallel with respect to some Riemannian metric on M. $\square$

We now consider another type of deformation of Vaisman structures, which preserves the cohomology class of the Lee form. Let $(g_0, \Omega_0, \theta_0)$ be a Vaisman structure on (M, J) with Lee vector field T_0. Let $f \in C^\infty(M)$, such that $T_0(f) = JT_0(f) = 0$. We define the closed 1-form θ_0' and the $(1,1)$-form Ω_0' by

$$\theta_0' = \theta_0 + \mathrm{d}f, \tag{8}$$

$$\Omega_0' = \mathrm{d}_{\theta_0'}(-J\theta_0') = |\theta_0|^2_{g_0}\Omega_0 + \theta_0 \wedge J\mathrm{d}f + \mathrm{d}f \wedge J\theta_0 + \mathrm{d}f \wedge J\mathrm{d}f - \mathrm{d}\mathrm{d}^c f. \tag{9}$$

When Ω_0' is a positive $(1,1)$-form (for instance if f is close to 0 in the C^2 sense), the pair (Ω_0', θ_0') is an lcK structure on (M, J).

Moreover, we have $T_0 \lrcorner \Omega_0' = |\theta_0|^2_{g_0} J\theta_0 + |\theta_0|^2_{g_0} J\mathrm{d}f = |\theta_0|^2_{g_0} J\theta_0'$, since $T_0 \lrcorner \mathrm{d}f = T_0 \lrcorner J\mathrm{d}f = 0$ and $T_0 \lrcorner \mathrm{d}\mathrm{d}^c f = \mathcal{L}_{T_0}\mathrm{d}^c f = \mathrm{d}^c \mathcal{L}_{T_0} f = 0$. Therefore, the Lee vector field of (Ω_0', θ_0') is $T_0' = \frac{1}{|\theta_0|^2_{g_0}} T_0$. Since $\mathcal{L}_{T_0} J = 0$ and $\mathcal{L}_{T_0}\Omega_0' = 0$, and $\theta_0'(T_0') = 1$, the structure (θ_0', Ω_0') is normalized Vaisman and we will call it a *deformation of type II* of (Ω_0, θ_0).

The next result is well-known (cf. [12] or [9, Prop. 2.14]); we provide here a short proof for convenience.

Proposition 3.6. *Let $(g_0, \Omega_0, \theta_0)$ and $(g_0', \Omega_0', \theta_0')$ be two Vaisman structures on a compact complex manifold (M, J), with Lee vector fields T_0, respectively T_0'. Then there exists a positive constant $\lambda > 0$, such that $T_0' = \lambda T_0$.*

Proof. Since a constant rescaling of the fundamental form induces a constant rescaling of the Lee vector field, one can assume that the two Vaisman structures are normalized. The closed 1-form θ_0' is cohomologous to a harmonic 1-form with respect to the metric g_0. By (5), there exist $t \in \mathbb{R}$ and $\alpha \in \mathcal{H}^1_0(M,g)$ such that $[\theta'_0] = [t\theta_0 + \alpha]$. Lemma 3.2 then yields that the real number t is positive, and by Lemma 3.3 it follows that $t\theta_0 + \alpha$ is the Lee form of a normalized Vaisman structure, whose Lee vector field is $\frac{1}{t}T_0$.

Thus, without loss of generality, we may assume that $[\theta'_0] = [\theta_0]$, so there exists $f \in C^\infty(M)$, such that $\theta'_0 = \theta_0 + \mathrm{d}f$. The (n,n)-form $(-1)^n \mathrm{d}J\theta_0 \wedge (\mathrm{d}J\theta'_0)^{n-1}$ is exact and semi-positive, hence

$$\mathrm{d}J\theta_0 \wedge (\mathrm{d}J\theta'_0)^{n-1} = 0 \tag{10}$$

by Stokes' formula. The interior product of $\mathrm{d}J\theta'_0$ with T'_0 and JT'_0 vanishes, and $(\mathrm{d}J\theta'_0)^{n-1}$ is nowhere vanishing. Thus, taking the interior product with T'_0 and JT'_0 in (10) yields

$$0 = \mathrm{d}J\theta_0(T'_0, JT'_0),$$

whence $T'_0 = cT_0 + c'JT_0$, for some functions c and c', which have to be constant, since T_0 and T'_0 are holomorphic and M is compact. Moreover,

$$0 = \mathcal{L}_{T'_0}\theta'_0 = c\mathcal{L}_{T_0}\theta'_0 + c'\mathcal{L}_{JT_0}\theta'_0 = c\mathrm{d}(T_0(f)) + c'\mathrm{d}(JT_0(f)),$$
$$0 = \mathcal{L}_{JT'_0}\theta'_0 = c\mathcal{L}_{JT_0}\theta'_0 - c'\mathcal{L}_{T_0}\theta'_0 = -c'\mathrm{d}(T_0(f)) + c\mathrm{d}(JT_0(f)).$$

hence $T_0(f) = JT_0(f) = 0$. Therefore (Ω'_0, θ'_0) is obtained from (Ω_0, θ_0) by a deformation of type II, which implies that $T_0 = sT'_0$ for some positive real number s. □

Proposition 3.6 allows to identify on a compact complex manifold of Vaisman type (M,J) two naturally oriented 1-dimensional distributions, $\mathcal{T}$ spanned by T_0 and $J\mathcal{T}$ spanned by JT_0, where T_0 is the Lee vector field of some Vaisman structure on (M,J), which also determines the orientation. The 2-dimensional distribution $\mathcal{T} \oplus J\mathcal{T}$ is called the *canonical distribution.* We denote by $\mathcal{T}^+$ the subset of $\mathcal{T}$ consisting of the union of all positive half-lines in $\mathcal{T}$, namely $\mathcal{T}^+ := \bigcup_{p \in M} \mathbb{R}^+ T_0(p)$ and correspondingly we denote by $J\mathcal{T}^+ := \bigcup_{p\in M} \mathbb{R}^+ JT_0(p)$. Proposition 3.6 ensures that $\mathcal{T}^+$ and $J\mathcal{T}^+$ are well-defined, independently of the choice of the Vaisman structure.

Moreover, the proof of Proposition 3.6 actually shows the following:

Proposition 3.7. *Let $(M, J, \Omega_0, \theta_0)$ be a compact Vaisman manifold. Then any normalized Vaisman structure (Ω'_0, θ'_0) on (M,J) is obtained by deformations of type I and II starting from the given Vaisman structure (Ω_0, θ_0).*

We now introduce the subspace $\mathcal{L} \subset \Omega^1(M)$ of Lee forms of Vaisman structures, defined by

$$\mathcal{L} := \{\theta \in \Omega^1(M) \,|\, \mathrm{d}\theta = 0 \text{ and } (\Omega := \theta \wedge J\theta - \mathrm{d}(J\theta), \theta) \text{ is a Vaisman structure}\},$$

which is tautologically in bijection with the set of normalized Vaisman structures on (M,J).

Lemma 3.8. *Let (M,J) be a Vaisman-type manifold. Then $\mathcal{L}$ is a convex cone inside $\Omega^1(M)$.*

Proof. Let θ be the Lee form of a Vaisman structure (Ω, θ) on (M,J). For any $t > 0$, the 1-form $t\theta$ is the Lee form of the Vaisman structure $(t^2\theta \wedge J\theta - t\mathrm{d}J\theta, t\theta)$ obtained from (Ω, θ) by a deformation of type I. Thus, $\mathcal{L}$ is a cone inside $\Omega^1(M)$.

We now show that $\mathcal{L}$ is also a convex set. Let $\theta_1, \theta_2 \in \mathcal{L}$. Thus $(\Omega_1 := \theta_1 \wedge J\theta_1 - \mathrm{d}(J\theta_1), \theta_1)$ and $(\Omega_2 := \theta_2 \wedge J\theta_2 - \mathrm{d}(J\theta_2), \theta_2)$ are two normalized Vaisman structures on (M, J) with associated Riemannian metrics $g_1 = \Omega_1(\cdot, J\cdot)$ and $g_2 = \Omega_2(\cdot, J\cdot)$. We need to check that $\theta := \theta_1 + \theta_2$ also belongs to $\mathcal{L}$. We define:

$$\Omega := \theta \wedge J\theta - \mathrm{d}J\theta = (\theta_1 + \theta_2) \wedge (J\theta_1 + J\theta_2) - (\mathrm{d}J\theta_1 + \mathrm{d}J\theta_2) = \Omega_1 + \Omega_2 + \theta_1 \wedge J\theta_2 + \theta_2 \wedge J\theta_1$$

which is clearly a $(1,1)$-form. We claim that the symmetric tensor $g := \Omega(\cdot, J\cdot)$ is positive definite.

For any vector field X on M the following identity holds:

$$\Omega(X, JX) = \Omega_1(X, JX) + \Omega_2(X, JX) + 2\theta_1(X)\theta_2(X) + 2J\theta_1(X)J\theta_2(X). \tag{11}$$

Using some local orthonormal bases $\{\theta_1, J\theta_1, \alpha_1, \ldots, \alpha_{n-2}\}$ and $\{\theta_1, J\theta_1, \beta_1, \ldots, \beta_{n-2}\}$ of $(\Lambda^1(M), g_1)$ and $(\Lambda^1(M), g_2)$ respectively, (11) becomes:

$$\Omega(X, JX) = (\theta_1(X) + \theta_2(X))^2 + (J\theta_1(X) + J\theta_2(X))^2 + \sum_{i=1}^{n-2}(\alpha_i(X)^2 + \beta_i(X)^2). \tag{12}$$

Thus, Ω is positive semi-definite. In order to show that Ω is actually positive definite, assume that $\Omega(X, JX) = 0$ for some vector X. If we denote by T_1 and T_2 the Lee vector fields of the Vaisman structures (Ω_1, θ_1), respectively (Ω_2, θ_2), then the above equation yields that X lies in the span of $\{T_1, JT_1\}$, respectively of $\{T_2, JT_2\}$.

By Proposition 3.6, there exists a positive constant λ, such that $T_1 = \lambda T_2$. Thus, X is of the following form: $X = aT_1 + bJT_1 = \lambda(aT_2 + bJT_2)$. The assumption $\Omega(X, JX) = 0$ together with (12) also implies that $\theta_1(X) + \theta_2(X) = 0$ and $J\theta_1(X) + J\theta_2(X) = 0$. Hence, we get that $a(\lambda + 1) = 0$ and $b(\lambda + 1) = 0$. As $\lambda > 0$, it follows that $a = b = 0$ and thus $X = 0$, showing that Ω is positive definite and (Ω, θ) is a Vaisman structure on (M, J). Let us remark that the Lee vector field T of this Vaisman structure is $T = \frac{1}{1+\lambda}T_1 = \frac{\lambda}{1+\lambda}T_2$. $\square$

4. LcK structures with holomorphic Lee vector field

This section contains the main result of the paper, namely the description of all lcK structures with holomorphic Lee vector field on a compact Vaisman-type complex manifold.

Let (M, J) be a compact complex manifold of Vaisman type, and let $\mathcal{T}$ denote as before the 1-dimensional distribution spanned by the Lee vector field of any Vaisman structure on (M, J) (cf. Proposition 3.6). We consider the space $\mathcal{HL}$ of holomorphic vector fields on (M, J) of Lee type, defined as follows:

Definition 4.1. The set $\mathcal{HL}(M, J)$ is the set of all holomorphic vector fields which can be obtained as the Lee vector field of some lcK structure on (M, J).

Let us introduce the following notion, which will be used in the sequel:

Definition 4.2. A vector field X on a manifold M is called of Killing type if there exists a Riemannian metric g on M, such that X is a Killing vector field of g.

The first observation is that if $T \in \mathcal{HL}(M, J)$ then JT is holomorphic and of Killing type. More precisely we have the following result:

Lemma 4.3. *If T is the Lee vector field of some lcK structure (g, Ω, θ), then $\mathcal{L}_{JT}\Omega = 0$. If T is holomorphic, then JT is Killing for g.*

Proof. Indeed, the Cartan formula shows that the anti-Lee vector field JT of an lcK structure always preserves the fundamental 2-form:

$$\mathcal{L}_{JT}\Omega = \mathrm{d}(JT \lrcorner \Omega + JT \lrcorner (\mathrm{d}\Omega) = \mathrm{d}\theta + JT \lrcorner (\theta \wedge \Omega) = 0.$$

If, moreover, T is holomorphic, as J is integrable, JT is also holomorphic, thus $\mathcal{L}_{JT} g = \mathcal{L}_{JT}(\Omega(\cdot, J\cdot)) = 0$. □

Let us now give the description of holomorphic Killing vector fields on Vaisman manifolds, which will be used in the sequel.

Lemma 4.4. *Let $(M, J, g_0, \Omega_0, \theta_0)$ be a normalized Vaisman manifold with Lee vector field T_0. Then any holomorphic Killing vector field K on (M, J, g_0) is of the following form:*

$$K = cT_0 + aJT_0 + K_0,$$

where $K_0 \in \{T_0, JT_0\}^\perp$, c is a constant, and the function $a \in \mathcal{C}^\infty(M)$, called the Hamiltonian of the holomorphic Killing vector field K, satisfies $K_0 \lrcorner \Omega_0 = \mathrm{d}a$ and $T_0(a) = JT_0(a) = 0$.

Proof. Let K be a holomorphic Killing vector field on (M, J, g_0). Then K leaves also invariant the fundamental form, $\mathcal{L}_K \Omega_0 = 0$, and thus also the Lee form, $\mathcal{L}_K \theta_0 = 0$. From the Cartan formula and the closedness of the Lee form θ_0, it follows that $c := \theta_0(K)$ is constant. Hence, the vector field K splits as:

$$K = cT_0 + aJT_0 + K_0,$$

where $K_0 \in \{T_0, JT_0\}^\perp$ and $a := J\theta_0(K)$. Furthermore, we compute again using Cartan's formula and (2):

$$\begin{aligned} 0 = \mathcal{L}_K J\theta_0 &= K \lrcorner \mathrm{d}J\theta_0 + \mathrm{d}(K \lrcorner J\theta_0) = K \lrcorner (\theta_0 \wedge J\theta_0 - \Omega_0) + \mathrm{d}a \\ &= cJ\theta_0 - a\theta_0 - K \lrcorner \Omega_0 + \mathrm{d}a = cJ\theta_0 - a\theta_0 - cJ\theta_0 + a\theta_0 - K_0 \lrcorner \Omega_0 + \mathrm{d}a \\ &= \mathrm{d}a - K_0 \lrcorner \Omega_0. \end{aligned}$$

In particular, it follows that $T_0(a) = JT_0(a) = 0$, since $\Omega_0(K_0, T_0) = \Omega_0(K_0, JT_0) = 0$. □

Lemma 4.5. *Let (M, J) be a compact manifold of Vaisman type and let X be a holomorphic vector field of Killing type on M. Then for any class $[\theta] \in H^1_{\mathrm{lcK}}(M)$ there exists a normalized Vaisman structure (Ω_0, θ_0) on (M, J) such that $[\theta_0] = [\theta]$ and $\mathcal{L}_X \theta_0 = 0$.*

Proof. By Proposition 3.4, there exists a Vaisman structure $(g_1, \Omega_1, \theta_1)$ on (M, J), such that $[\theta_1] = [\theta]$. Let g be a Riemannian metric on M for which X is a Killing vector field. Then the flow of X consists of isometries $\varphi_t \in \mathrm{Iso}(M, g)$, so its closure $G := \overline{\{\varphi_t\}}$ in $\mathrm{Iso}(M, g)$ is a compact torus with Haar measure $\mathrm{d}\mu$, normalized such that $\int_G \mathrm{d}\mu = 1$. We define

$$\theta_0 := \int_{\gamma \in G} \gamma^* \theta_1 \, \mathrm{d}\mu \in \Omega^1(M).$$

Then the following cohomology classes are equal: $[\theta_0] = [\theta_1] = [\theta]$, since the action of any connected Lie group is trivial in cohomology. Furthermore, the 1-form θ_0 is invariant under X, *i.e.* $\mathcal{L}_X \theta_0 = 0$, since $\gamma^* \theta_0 = \theta_0$, for any $\gamma \in G$. As X is holomorphic, $\gamma^* \theta_1$ is the Lee form of the Vaisman structure $\gamma^*(\Omega_1)$ on (M, J) for every $\gamma \in G$. By the convexity of the set of Lee forms of Vaisman structures (Lemma 3.8), we obtain that θ_0 is the Lee form of a normalized Vaisman structure whose fundamental 2-form is $\Omega_0 := \theta_0 \wedge J\theta_0 - \mathrm{d}J\theta_0$. □

Corollary 4.6. *Let (M, J) be a compact manifold of Vaisman type and let $T \in \mathcal{HL}(M, J)$ be a holomorphic vector field of Lee type. Then for every lcK structure (g, Ω, θ) on (M, J) whose Lee vector field is T, there exists a normalized Vaisman structure (Ω_0, θ_0) on (M, J) such that $[\theta_0] = [\theta]$ and $\theta_0(JT) = 0$.*

Proof. By Lemma 4.3, JT is Killing for g. Then Lemma 4.5 applied to $X := JT$ and to the class $[\theta]$ ensures the existence of a normalized Vaisman structure (Ω_0, θ_0) on (M, J) such that $[\theta_0] = [\theta]$ and $\mathcal{L}_{JT}\theta_0 = 0$. We only need to check that any such Vaisman structure satisfies $\theta_0(JT) = 0$.

Cartan's formula together with $\mathcal{L}_{JT}\theta_0 = 0$ already imply that $c := \theta_0(JT)$ is constant. Moreover, since θ and θ_0 are cohomologous, there exists $f \in \mathcal{C}^\infty(M, \mathbb{R})$ such that $\theta = \theta_0 + \mathrm{d}f$ and thus we compute:

$$0 = \theta(JT) = (\theta_0 + \mathrm{d}f)(JT) = c + JT(f).$$

Considering this equality at a point of extremum for f on the compact manifold M, it follows that the constant c vanishes, hence $\theta_0(JT) = 0$. □

In the sequel, we will need the following criterion to decide when a vector field is the anti-Lee vector field of a d_θ-exact lcK structure:

Lemma 4.7. *Let (M, J, Ω, θ) be a strict lcK manifold, such that $\Omega = \mathrm{d}_\theta\beta$, for some $\beta \in \Omega^1(M)$. Let K be a vector field on M such that $\mathcal{L}_K\beta = 0$ and $\theta(K) = 0$. Then K is the anti-Lee vector field of (Ω, θ) if and only if $\beta(K) + 1 = 0$. In particular, the Lee vector field of any lcK structure on such a manifold is not vanishing at any point.*

Proof. By definition, K is the anti-Lee vector field of the lcK structure (Ω, θ) if and only if $K \lrcorner \Omega = -\theta$. On the other hand, we compute as follows, using Cartan's formula and the fact that $\mathcal{L}_K\beta = 0$ and $\theta(K) = 0$:

$$K \lrcorner \Omega = K \lrcorner \mathrm{d}_\theta\beta = K \lrcorner (\mathrm{d}\beta - \theta \wedge \beta) = \mathcal{L}_K\beta - \mathrm{d}(\beta(K)) - \theta(K)\beta + \beta(K)\theta = -\mathrm{d}_\theta(\beta(K)).$$

Hence, $K \lrcorner \Omega = -\theta$ is equivalent to $\mathrm{d}_\theta(\beta(K) + 1) = 0$.

Since the lcK structure is not exact, d_θ is injective on functions (cf. [7, Lemma 2.1]), showing that K is the anti-Lee vector field of (Ω, θ) if and only if $\beta(K) + 1 = 0$. In particular, K, and thus also the Lee vector field $-JK$, is nowhere vanishing on M. □

We will now gather some further information about holomorphic vector fields of Lee type on Vaisman-type manifolds.

Theorem 4.8. *Let (M, J) be a compact Vaisman-type manifold endowed with an lcK structure (g, Ω, θ) whose Lee vector field T is holomorphic.*

(i) There exists a normalized Vaisman structure $(g_0, \Omega_0, \theta_0, T_0)$ on (M, J) such that $[\theta] = [\theta_0]$ and $\theta_0(JT) = 0$.

(*ii*) *The anti-Lee vector field* JT *is holomorphic and Killing with respect to the Vaisman metric* g_0 *and there exists a function* $a \in \mathcal{C}^\infty(M)$*, with* $T_0(a) = JT_0(a) = 0$*, and such that* $JT = aJT_0 - J\mathrm{grad}^{g_0} a$.

(*iii*) *The function* a *is everywhere positive on* M.

Conversely, if $(g_0, \Omega_0, \theta_0, T_0)$ *is a normalized Vaisman structure on* (M, J) *and* K *is a holomorphic Killing vector field on* (M, J, g_0) *with positive Hamiltonian* a *and vanishing* T_0*-component* c*, then* $-JK$ *is the Lee vector field of an lcK structure* (Ω, θ) *on* (M, J) *such that* $[\theta] = [\theta_0]$ *and* $\theta_0(JT) = 0$.

Proof. (i) Follows directly from Corollary 4.6.

(ii) Since T is holomorphic, the same holds for JT, so $\mathcal{L}_{JT}J = 0$. Moreover $\mathcal{L}_{JT}\theta_0 = \mathrm{d}(\theta_0(JT)) = 0$, so $\mathcal{L}_{JT}\Omega_0 = \mathcal{L}_{JT}(\theta_0 \wedge J\theta_0 - \mathrm{d}J\theta_0) = 0$, whence $\mathcal{L}_{JT}g_0 = 0$.

By Lemma 4.4, there exists a real constant c and a function $a \in \mathcal{C}^\infty(M)$ (the Hamiltonian of JT), with $T_0(a) = JT_0(a) = 0$, and such that $JT = cT_0 + aJT_0 - J\mathrm{grad}^{g_0} a$. Moreover, c vanishes, since

$$0 = \theta_0(JT) = c - \theta_0(J\mathrm{grad}^{g_0} a) = c + JT_0(a) = c.$$

(iii) Corollary 3.5 ensures that every lcK structure (Ω, θ) on a Vaisman-type manifold is d_θ-exact, i.e. $\Omega = \mathrm{d}_\theta\beta$ for some 1-form β. Using an averaging argument as before, one can assume that $\mathcal{L}_{JT}\beta = 0$. By the converse of Lemma 4.7 applied to $K = JT$ we thus obtain that T is never vanishing.

Since $[\theta] = [\theta_0]$, there exists f such that $\theta = \theta_0 + \mathrm{d}f$. We now compute:

$$\begin{aligned} |T|_g^2 &= \theta(T) = \theta_0(T) + T(f) = \theta_0(aT_0 - \mathrm{grad}^{g_0} a) + aT_0(f) - \mathrm{d}f(\mathrm{grad}^{g_0} a) \\ &= a + aT_0(f) - \langle \mathrm{d}f, \mathrm{d}a \rangle_{g_0}, \end{aligned}$$

which together with the fact that T vanishes nowhere on M, yields the following inequality:

$$a + aT_0(f) - \langle \mathrm{d}f, \mathrm{d}a \rangle_{g_0} > 0 \tag{13}$$

Let us denote by $m := \min_M a$ the minimum of the function a on the compact manifold M. Applied at any $p \in a^{-1}(m)$, the above inequality yields $a(p)(1 + T_0(f)(p)) > 0$. Hence, in particular, we have $m = a(p) \neq 0$.

We assume, by contradiction, that $m < 0$. The above inequality yields $1 + T_0(f)(p) < 0$ for all $p \in a^{-1}(m)$. As T_0 is a parallel vector field of constant length 1 with respect to the Vaisman metric g_0, each integral curve of T_0 is a complete geodesic of (M, g_0). Moreover, the restriction of the function a along any integral curve of T_0 is constant, since $T_0(a) = 0$. Hence, along a complete geodesic $\gamma\colon \mathbb{R} \to M$ starting at a point $p \in a^{-1}(m)$, the following inequality holds: $T_0(f)(\gamma(t)) < -1$, for all $t \in \mathbb{R}$, which yields a contradiction, since the function f is bounded. This proves that $m = \min_M a > 0$, hence the function a is positive on M.

Conversely, let $K = aJT_0 - J\mathrm{grad}^{g_0} a$ be a holomorphic Killing vector field on (M, J, g_0), with $T_0(a) = JT_0(a) = 0$. We claim that the Lee vector field of the lcK structure

$$\left(\Omega := \frac{1}{a}\Omega_0, \theta := \theta_0 - \frac{\mathrm{d}a}{a}\right)$$

is $-JK$. Indeed,

$$(-JK)\lrcorner\Omega = (aT_0 - \mathrm{grad}^{g_0} a)\lrcorner(\frac{1}{a}\Omega_0) = J\theta_0 - \frac{1}{a}J\mathrm{d}a = J\theta.$$

$\square$

Remark 4.9. Theorem 4.8 gives rise to examples of lcK structures on compact Vaisman-type manifolds whose Lee vector field is not homothetic to the common Lee vector field of the Vaisman structures. More precisely, if there exists on a compact Vaisman manifold $(M, J, \Omega_0, \theta_0, T_0)$ a Killing vector field K which is not a linear combination of T_0 and JT_0, then, according to Lemma 4.4, it decomposes as $K = cT_0 + aJT_0 - J\mathrm{grad}^{g_0} a$, where c is a constant and the Hamiltonian function a is not constant. As T_0 is also a Killing vector field for g_0, by subtracting cT_0 and adding kJT_0 to K, for any constant $k > |\min_M a|$, we obtain a Killing vector field whose Hamiltonian function $a + k$ is positive everywhere and has no component along T_0, namely $\widetilde{K} = (a + k)JT_0 - J\mathrm{grad}^{g_0}(a + k)$. According to the converse part of Theorem 4.8, there exists an lcK structure on (M, J) whose Lee vector field equals $-J\widetilde{K} = (a + k)T_0 - \mathrm{grad}^{g_0}(a + k)$, which is clearly not homothetic to T_0, because a is not constant and $\mathrm{grad}^{g_0}(a+k)$ is orthogonal to T_0. Examples of compact Vaisman manifolds admitting Killing vector fields which are not everywhere tangent to the canonical distribution $\mathcal{T} \oplus J\mathcal{T}$ are provided for instance by compact toric Vaisman manifolds (see [5], [7], [11]) or compact homogeneous Vaisman manifolds (see [2]).

The above result is not completely satisfactory, as the description of the space $\mathcal{HL}(M, J)$ of holomorphic vector fields of Lee type on (M, J) obtained in Theorem 4.8 depends on the choice of some background Vaisman structure. However, it is possible to give a completely intrinsic description of the space $\mathcal{HL}(M, J)$ on compact Vaisman-type manifolds:

Theorem 4.10. *Let (M, J) be a compact complex manifold of Vaisman type, and let $\mathcal{T}$ denote as before the 1-dimensional distribution spanned by the Lee vector field of any Vaisman structure on (M, J). Then a holomorphic vector field T belongs to $\mathcal{HL}(M, J)$ if and only if the following conditions are satisfied:*

(i) JT is of Killing type.

(ii) For any point $p \in M$, if $T_p \in \mathcal{T}_p \oplus J\mathcal{T}_p$, then $T_p \in \mathcal{T}_p^+$.

Moreover, if $T \in \mathcal{HL}(M, J)$, then for any cohomology class $\mu \in H^1_{\mathrm{lcK}}(M)$ there exists an lcK structure whose Lee form represents this class and whose Lee vector field is T.

Proof. If $T \in \mathcal{HL}(M, J)$, there exists an lcK structure (g, Ω, θ) on (M, J) whose Lee vector field is T. By Lemma 4.3, JT is Killing for g, so (i) is satisfied.

By Theorem 4.8, we may choose a Vaisman structure $(g_0, \Omega_0, \theta_0, T_0)$ on (M, J) such that $[\theta] = [\theta_0]$, $\theta_0(JT) = 0$, and JT is a Killing vector field with respect to the metric g_0 satisfying

$$T = aT_0 - \mathrm{grad}^{g_0} a,$$

where the Hamiltonian function $a \in \mathcal{C}^\infty(M)$ is everywhere positive.

If p is any point in M such that $T_p \in \mathcal{T}_p \oplus J\mathcal{T}_p$, then $T_p = a(p)(T_0)_p$, because $\mathrm{grad}^{g_0} a \in \{T_0, JT_0\}^\perp$. In particular, as $a > 0$, it follows that $T_p \in \mathcal{T}_p^+$. Thus (ii) is satisfied too.

Conversely, let T be a holomorphic vector field satisfying (i) and (ii). By Lemma 4.5, for any cohomology class in $\mu \in H^1_{\mathrm{lcK}}(M)$, there exists a normalized Vaisman structure (Ω_0, θ_0) on (M, J) such that $[\theta_0] = \alpha$, and $\mathcal{L}_{JT}\theta_0 = 0$. Hence, JT is a Killing vector field of the Vaisman metric g_0 and according to Lemma 4.4, we can write

$$JT = cT_0 + aJT_0 - J\mathrm{grad}^{g_0} a,$$

where c is a constant and $a \in \mathcal{C}^\infty(M)$ with $T_0(a) = JT_0(a) = 0$. Applying J to this equality yields $T = -cJT_0 + aT_0 - \mathrm{grad}^{g_0} a$.

Let p be a point of minimum of the function a. Then $T_p = -c(JT_0)_p + a(p)(T_0)_p \in \mathcal{T} \oplus J\mathcal{T}$. The condition (ii) from the definition of $\mathcal{HL}$ implies that $T_p \in \mathcal{T}_p^+$, whence $c = 0$ and $a(p) > 0$. This shows that $T = aT_0 - \mathrm{grad}^{g_0} a$ and the function a is positive. The converse part of Theorem 4.8 shows that T is the Lee vector field of the lcK structure $(\Omega := \frac{1}{a}\Omega_0, \theta := \theta_0 - \frac{\mathrm{d}a}{a})$, so $T \in \mathcal{HL}(M, J)$. Moreover, the Lee form of this lcK structure satisfies $[\theta] = [\theta_0] = \mu$. $\square$

Now, that we have intrinsically characterized the set $\mathcal{HL}(M, J)$ of all holomorphic vector fields which can occur as Lee vector fields of lcK structures on Vaisman-type manifolds, we would like to describe, for each fixed $T \in \mathcal{HL}(M, J)$, the set of all lcK structures admitting T as Lee vector field. In order to do so, we will fix some cohomology class $\mu \in H^1_{\mathrm{lck}}(M, J)$ and apply Corollary 4.6 in order to choose a normalized Vaisman structure (Ω_0, θ_0) whose Lee form satisfies $[\theta_0] = \mu$ and $\theta_0(JT) = 0$.

Our aim is to describe all lcK structures with Lee vector field T, and with Lee form in the cohomology class μ. If (Ω, θ) is such an lcK structure, then $\theta = \theta_0 + \mathrm{d}f$, for some function $f \in \mathcal{C}^\infty(M, \mathbb{R})$, so $(e^{-f}\Omega, \theta_0)$ is an lcK structure on (M, J) with Lee form θ_0. By Proposition 2.4, there exists a function $h \in \mathcal{C}^\infty(M, \mathbb{R})$ and a $(0,1)$-form $\alpha \in \Omega^{0,1}(M)$ in the kernel of the twisted Laplacian $\Delta_{\bar\partial_{\theta_0}}$, such that

$$e^{-f}\Omega = \mathrm{d}_{\theta_0}\mathrm{d}^c_{\theta_0} h + \partial_{\theta_0}\alpha + \bar\partial_{\theta_0}\overline{\alpha}. \tag{14}$$

Since $\mathcal{L}_{JT}\theta = 0$ and $\mathcal{L}_{JT}\theta_0 = 0$, it follows that $\mathcal{L}_{JT}\mathrm{d}f = 0$, hence $JT(f)$ is constant. As f has critical points, this constant must be zero, so $JT(f) = 0$.

Let φ_t denote the flow of the holomorphic Killing vector field JT. Since JT preserves J, Ω, θ, θ_0 and f, we obtain:

$$e^{-f}\Omega = \varphi_t^*(e^{-f}\Omega) = \mathrm{d}_{\theta_0}\mathrm{d}^c_{\theta_0}(\varphi_t^* h) + \partial_{\theta_0}(\varphi_t^*\alpha) + \overline{\partial_{\theta_0}(\varphi_t^*\alpha)}.$$

Thus, after averaging over the compact torus $G := \overline{\{\varphi_t\}}$, we may assume that both the function h and the 1-form α are invariant under JT.

Using the fact that $\bar\partial_{\theta_0}\alpha = 0 = \partial_{\theta_0}\bar\alpha$ and the relation (1) between d_θ and d_{θ_0}, the fundamental 2-form Ω can be expressed as follows:

$$\Omega = e^f(\mathrm{d}_{\theta_0}\mathrm{d}^c_{\theta_0} h + \partial_{\theta_0}\alpha + \bar\partial_{\theta_0}\overline{\alpha}) = e^f\mathrm{d}_{\theta_0}[\mathrm{d}^c_{\theta_0} h + 2\mathrm{Re}\alpha] = \mathrm{d}_\theta[e^f(\mathrm{d}^c_{\theta_0} h + 2\mathrm{Re}\,\alpha)] = \mathrm{d}_\theta\beta,$$

where

$$\beta := e^f(\mathrm{d}^c_{\theta_0} h + 2\mathrm{Re}\,\alpha) \tag{15}$$

is also JT-invariant.

Lemma 4.7 applied to $K := JT$ shows that $\beta(JT) + 1 = 0$, which by (15) is equivalent to

$$e^{-f} = -T(h) + h\theta_0(T) - 2\mathrm{Re}(\alpha(JT)). \tag{16}$$

Conversely, suppose that $h \in \mathcal{C}^\infty(M, \mathbb{R})$ is a smooth function and $\alpha \in \Omega^{0,1}(M)$ is a $(0,1)$-form in the kernel of the twisted Laplacian $\Delta_{\bar\partial_{\theta_0}}$ such that:

(1) $\mathcal{L}_{JT} h = 0$ and $\mathcal{L}_{JT}\alpha = 0$;
(2) the right hand side of (16) is positive on M;
(3) the $(1,1)$-form $\mathrm{d}_{\theta_0}\mathrm{d}^c_{\theta_0} h + \partial_{\theta_0}\alpha + \bar\partial_{\theta_0}\overline{\alpha}$ is positive definite.

Then if f is defined by (16), the pair $(\Omega := e^f(\mathrm{d}_{\theta_0}\mathrm{d}^c_{\theta_0} h + \partial_{\theta_0}\alpha + \bar\partial_{\theta_0}\overline{\alpha}), \theta := \theta_0 + \mathrm{d}f)$ is an lcK structure on (M, J) whose Lee vector field is T by Lemma 4.7.

Motivated by the above considerations, we now introduce for any holomorphic vector field of Lee type $T \in \mathcal{HL}(M, J)$ and for any normalized Vaisman structure $(g_0, \Omega_0, \theta_0, T_0)$, the sets of functions and of twisted harmonic forms that are invariant by JT:

$$\mathcal{C}^\infty_{JT}(M) := \{h \in \mathcal{C}^\infty(M) \mid JT(h) = 0\},$$
$$\mathcal{H}^{*,*}_{\bar\partial_{\theta_0}, JT}(M) := \{\alpha \in \Omega^{*,*}(M) \mid \Delta_{\bar\partial_{\theta_0}}\alpha = 0, \mathcal{L}_{JT}\alpha = 0\},$$

and we define the open subset $\mathcal{F}_{JT}$ of $\mathcal{C}^\infty_{JT}(M) \times \mathcal{H}^{0,1}_{\bar\partial_{\theta_0}, JT}(M)$ consisting of all pairs (h, α) that satisfy the following two conditions:

$$\begin{cases} \mathrm{d}_{\theta_0}\mathrm{d}^c_{\theta_0} h + \partial_{\theta_0}\alpha + \bar\partial_{\theta_0}\overline{\alpha} \text{ is a positive definite } (1,1)\text{-form} \\ -T(h) + h\theta_0(T) - 2\mathrm{Re}(\alpha(JT)) > 0. \end{cases} \tag{17}$$

Note that the set $\mathcal{F}_{JT}$ is non-empty. Indeed, the pair $(1, 0)$ belongs to $\mathcal{F}_{JT}$, because the $(1,1)$-form $\mathrm{d}_{\theta_0}\mathrm{d}^c_{\theta_0} 1 = \Omega_0$ is positive definite and the second inequality is fulfilled since $\theta_0(T)$ is the Hamiltonian of the Killing vector field JT, which is positive by Theorem 4.8. We can now state our main result:

Theorem 4.11. *Let $T \in \mathcal{HL}(M, J)$ be a holomorphic vector field of Lee type on a Vaisman-type manifold, and $\mu \in H^1_{\mathrm{lck}}(M, J)$ be a fixed cohomology class of lcK type. Fix any normalized Vaisman structure (Ω_0, θ_0) whose Lee form satisfies $[\theta_0] = \mu$ and $\theta_0(JT) = 0$ (the existence of such a Vaisman structure is granted by Corollary 4.6). There is a surjective map from the set $\mathcal{F}_{JT}$ to the set of all lcK structures (Ω, θ) on (M, J) having $[\theta] = \mu$ and the Lee vector field equal to T, given by:*

$$(h, \alpha) \mapsto (\Omega := e^f(\mathrm{d}_{\theta_0}\mathrm{d}^c_{\theta_0} h + \partial_{\theta_0}\alpha + \bar\partial_{\theta_0}\overline{\alpha}), \theta := \theta_0 + \mathrm{d}f), \tag{18}$$

with $e^{-f} := -T(h) + h\theta_0(T) - 2\mathrm{Re}(\alpha(JT))$.

Moreover, two pairs $(h, \alpha), (\widetilde{h}, \widetilde{\alpha}) \in \mathcal{F}_{JT}$ are mapped to the same lcK structure if and only if there is a positive constant c, such that $\alpha = c\widetilde{\alpha}$ and $h - c\widetilde{h}_0$ is equal to the imaginary part of a function in $\mathcal{H}^{0,0}_{\bar\partial_{\theta_0}, JT}(M)$.

Proof. The first part of the theorem was already proved above. It remains to determine under which circumstances two pairs $(h, \alpha), (\widetilde{h}, \widetilde{\alpha})$ define the same lcK structure on (M, J).

Assume that $(h,\alpha),(\widetilde{h},\widetilde{\alpha})$ are two pairs in $\mathcal{F}_{JT}$ that define the same lcK structure, *i.e.* $\widetilde{\Omega}=\Omega$ and $\widetilde{\theta}=\theta$. The last equality is equivalent to $\mathrm{d}f=\mathrm{d}\widetilde{f}$, showing that there exists a positive constant c, such that $e^{-\widetilde{f}}=ce^{-f}$.Thus, $\widetilde{\Omega}=\Omega$ reads:

$$i\partial_{\theta_0}\bar{\partial}_{\theta_0}(h-c\widetilde{h})+\partial_{\theta_0}(\alpha-c\widetilde{\alpha})+\bar{\partial}_{\theta_0}\overline{(\alpha-c\widetilde{\alpha})}=0,$$

or equivalently, using the fact that α and $\widetilde{\alpha}$ are in the kernel of the twisted Laplacian Δ_{θ_0}, so in particular $\bar{\partial}_{\theta_0}\alpha=\bar{\partial}_{\theta_0}\widetilde{\alpha}=0$ and $\bar{\partial}^*_{\theta_0}\alpha=\bar{\partial}^*_{\theta_0}\widetilde{\alpha}=0$, as:

$$\mathrm{d}_{\theta_0}[i\bar{\partial}_{\theta_0}(h-c\widetilde{h})+\alpha-c\widetilde{\alpha}+\overline{\alpha}-c\overline{\widetilde{\alpha}}]=0, \tag{19}$$

By Corollary 3.5, the homology of d_{θ_0} vanishes, so (19) is fulfilled if and only if there exists a function $b\in\mathcal{C}^\infty(M,\mathbb{C})$, such that $i\bar{\partial}_{\theta_0}(h-c\widetilde{h})+\alpha-c\widetilde{\alpha}+\overline{\alpha}-c\overline{\widetilde{\alpha}}=\mathrm{d}_{\theta_0}b$, or equivalently, when considering on both sides the forms of type $(1,0)$, respectively $(0,1)$:

$$\begin{cases}\overline{\alpha}-c\overline{\widetilde{\alpha}}=\partial_{\theta_0}b,\\ \alpha-c\widetilde{\alpha}+i\bar{\partial}_{\theta_0}(h-c\widetilde{h})=\bar{\partial}_{\theta_0}b.\end{cases}\iff\begin{cases}\alpha-c\widetilde{\alpha}=\bar{\partial}_{\theta_0}\bar{b},\\ \bar{\partial}_{\theta_0}\bar{b}+i\bar{\partial}_{\theta_0}(h-c\widetilde{h})=\bar{\partial}_{\theta_0}b.\end{cases} \tag{20}$$

The last equation yields $\bar{\partial}_{\theta_0}(2\mathrm{Im}\,b-h+c\widetilde{h})=0$. On the universal cover $\widetilde{M}$ of M, this equation translates into the condition that the real-valued function $(2\mathrm{Im}\,b-h+c\widetilde{h})e^{-\varphi}$, where φ is a primitive of θ_0 on $\widetilde{M}$, is holomorphic. This is only possible if the function is constant, and because of the equivariance condition, this constant must actually vanish. Hence, we obtain that $h-c\widetilde{h}=2\mathrm{Im}\,b$.
On the other hand, since α and $\widetilde{\alpha}$ are in the kernel of Δ_{θ_0}, the first equation in (20) implies

$$\bar{\partial}^*_{\theta_0}\bar{\partial}_{\theta_0}\bar{b}=\bar{\partial}^*_{\theta_0}(\alpha-c\widetilde{\alpha})=0. \tag{21}$$

Taking the scalar product with $\bar{b}$ and integrating over the compact manifold M yields $\bar{\partial}_{\theta_0}\bar{b}=0$, which together with the first equation in (20) shows that $\alpha-c\widetilde{\alpha}=0$. We thus also obtain that $h-c\widetilde{h}=\mathrm{Im}\,(-2\bar{b})$, with $-2\bar{b}\in\mathcal{H}^{0,0}_{\bar{\partial}_{\theta_0},JT}(M)$. ☐

Our last result concerns lcK structures with holomorphic Lee vector field on complex manifolds which are not necessarily of Vaisman type, but assuming instead that the lcK structure has a potential (cf. Definition 2.2).

Proposition 4.12. *Let (M,J,g,Ω,θ) be a compact lcK manifold with potential and with holomorphic Lee vector field T. Then T is not vanishing at any point of M and the following assertions hold:*

- *(i) There exists a JT-invariant potential h.*
- *(ii) Any JT-invariant potential h satisfies the equation $T(h)-h\theta(T)+1=0$.*
- *(iii) Any JT-invariant potential h is positive.*

Proof. The fact that T is not vanishing at any point of M is a direct consequence of Lemma 4.7, as any lcK structure with potential on a compact manifold is strict (*cf.* Remark 2.3).

(i) Let h_0 be a potential of the lcK structure, *i.e.* $\Omega=\mathrm{d}_\theta\mathrm{d}^c_\theta h_0$. Since $\mathcal{L}_{JT}\theta=0$ and $\mathcal{L}_{JT}\Omega=0$, the flow φ_t of JT preserves J, θ and Ω and hence $\varphi_t^*h_0$ is also a potential of

(Ω, θ), for all t. Therefore, denoting as before by $G := \overline{\{\varphi_t\}}$ the closure of the flow of JT in $\mathrm{Iso}(M, g)$, the function $h := \int_G \varphi_t^* h_0 \, \mathrm{d}\mu$ is a JT-invariant potential.

(ii) If the potential h is JT-invariant, then we may apply Lemma 4.7 to $\beta := \mathrm{d}^c_\theta h$ and obtain that JT satisfies $\beta(JT) + 1 = 0$, which is equivalent to $T(h) - h\theta(T) + 1 = 0$.

(iii) If h be a JT-invariant potential, then by (ii), h satisfies $T(h) - h\theta(T) + 1 = 0$. If p is a minimum of h on M, then this equation implies that $h(p) = \frac{1}{\theta_p(T(p))} = \frac{1}{|T(p)|_g^2} > 0$, hence $h > 0$ on M. $\square$

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