KILLING 2-FORMS IN DIMENSION 4

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Abstract. A Killing p-form on a Riemannian manifold (M, g) is a p-form whose covariant derivative is totally antisymmetric. If M is a connected, oriented, 4-dimensional manifold admitting a non-parallel Killing 2-form ψ, we show that there exists a dense open subset of M on which one of the following three exclusive situations holds: either ψ is everywhere degenerate and g is locally conformal to a product metric, or g gives rise to an ambikähler structure of Calabi type, or, generically, g gives rise to an ambitoric structure of hyperbolic type, in particular depends locally on two functions of one variable. Compact examples of either types are provided.

Dedicated to Simon Salamon *on the occasion of his 60th birthday*

Contents

1. Introduction

On any n-dimensional Riemannian manifold (M, g), an exterior p-form ψ is called *conformal Killing* [13] if its covariant derivative $\nabla\psi$ is of the form

$$\nabla_X \psi = \alpha \wedge X^\flat + X \lrcorner \beta, \tag{1.1}$$

for some $(p-1)$-form α and some $(p+1)$-form β, which are then given by

$$\alpha = \frac{(-1)^p}{n-p+1}\, \delta\psi, \qquad \beta = \frac{1}{p+1}\, d\psi. \tag{1.2}$$

Conformal Killing forms have the following conformal invariance property: if ψ is a conformal Killing p-form with respect to the metric g, then, for any positive function f, $\tilde{\psi} := f^{p+1}\,\psi$ is conformal Killing with respect to the conformal metric $\tilde{g} := f^2\, g$. In other words, if L denotes the real line bundle $|\Lambda^n TM|^{\frac{1}{n}}$ and $\ell, \tilde{\ell}$ denote the sections of L determined by $g, \tilde{g}$, then, for

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any Weyl connection D relative to the conformal class $[g]$, the section $\boldsymbol{\psi} := \psi \otimes \ell^{p+1} = \tilde{\psi} \otimes \tilde{\ell}^{p+1}$ of $\Lambda^p T^* M \otimes L^{p+1}$ satisfies

$$D_X \boldsymbol{\psi} = \boldsymbol{\alpha} \wedge X + X \lrcorner \boldsymbol{\beta}, \tag{1.3}$$

for some section $\boldsymbol{\alpha}$ of $\Lambda^{p-1} T^* M \otimes L^{p-1}$ and some section $\boldsymbol{\beta}$ of $\Lambda^{p+1} T^* M \otimes L^{p+1}$ (depending on D), cf. *e.g.* [2, Appendix B].

The p-form ψ is called *Killing*, resp. $*$-*Killing*, with respect to g, if ψ satisfies (1.1) *and* $\alpha = 0$, resp. $\beta = 0$. In particular, Killing forms are co-closed, $*$-Killing forms are closed, and, if M is oriented and $*$ denotes the induced Hodge star operator, ψ is Killing if and only if $*\psi$ is $*$-Killing.

Although the terminology comes from the fact that Killing 1-forms are just metric duals of Killing vector fields, and thus encode infinitesimal symmetries of the metric, no geometric interpretation of Killing p-forms exists in general in terms of symmetries when $p \geq 2$, except in the case of Killing 2-forms in dimension 4, which is special for various reasons, the most important being the self-duality phenomenon.

On any oriented four-dimensional manifold (M, g), the Hodge star operator $*$, acting on 2-forms, is an involution and, therefore, induces the well known orthogonal decomposition

$$\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M, \tag{1.4}$$

where $\Lambda^2 M$ stands for the vector bundle of (real) 2-forms on M and $\Lambda^{\pm} M$ the eigen-subbundle for the eigenvalue ± 1 of $*$. Accordingly, any 2-form ψ splits as

$$\psi = \psi_+ + \psi_-, \tag{1.5}$$

where ψ_+, resp. ψ_-, is the *self-dual*, resp. the *anti-self-dual* part of ψ, defined by $\psi_{\pm} = \frac{1}{2}(\psi \pm *\psi)$. Since $*$ acting on 2-forms is conformally invariant, a 2-form ψ is conformal Killing if and only if ψ_+ and ψ_- are separately conformal Killing, meaning that

$$\nabla_X \psi_+ = (\alpha_+ \wedge X^\flat)_+, \qquad \nabla_X \psi_- = (\alpha_- \wedge X^\flat)_- \tag{1.6}$$

for some real 1-forms α_+, α_-, and ψ is Killing, resp. $*$-Killing, if, in addition,

$$\alpha_+ = -\alpha_-, \quad \text{resp.} \quad \alpha_+ = \alpha_-. \tag{1.7}$$

Throughout this paper, (M, g) will denote a connected, oriented, 4-dimensional Riemannian manifold and $\psi = \psi_+ + \psi_-$ a non-trivial $*$-Killing 2-form on M (the choice of the $*$-Killing ψ, instead of the Killing 2-form $*\psi$ is of pure convenience). We also discard the non-interesting case when ψ is parallel.

On the open set, M_0^+, resp. M_0^-, where ψ_+, resp. ψ_-, is non-zero, the associated skew-symmetric operators Ψ_+, Ψ_-, are of the form $\Psi_+ = f_+ J_+$, resp. $\Psi_- = f_- J_-$, where J_+, resp. J_-, is an almost complex structure inducing the chosen, resp. the opposite, orientation of M, and f_+, resp. f_-, is a positive function. It is then easily checked, cf. Section 2 below, that the first, resp. the second, condition in (1.6) is equivalent to the condition that the pair $(g_+ := f_+^{-2}\, g, J_+)$, resp. the pair $(g_- := f_-^{-2}\, g, J_-)$, is *Kähler*. On the open set $M_0 = M_0^+ \cap M_0^-$, which is actually dense in M, cf. Lemma 2.1 below, we thus get *two* Kähler structures, whose metrics belong to the same conformal class and whose complex structures induce opposite orientations (in particular, commute), hence an *ambikähler structure*, as defined in [2]. This actually holds if ψ is simply conformal Killing and had been observed in the twistorial setting by M. Pontecorvo in [12], cf. also [2, Appendix B2]. The additional coupling condition (1.7), which, on M_0, reads $J_+ df_+ = J_- df_-$, cf. Section 2, then has strong consequences, that we now explain.

A first main observation, cf. Proposition 3.3, is that the open subset, M_S, where ψ is of maximal rank, hence a symplectic 2-form, is either empty or dense in M.

The case when M_S is empty is the case when ψ is *decomposable*, *i.e.* $\psi \wedge \psi = 0$ everywhere; equivalently, $|\psi_+| = |\psi_-|$ everywhere; on M_0, we then have $f_+ = f_-$, hence $g_+ = g_- =: g_K$, and (M_0, g_K) is locally a product of two (real) Kähler surfaces $(\Sigma, g_\Sigma, \omega_\Sigma)$ and $(\tilde{\Sigma}, g_{\tilde{\Sigma}}, \omega_{\tilde{\Sigma}})$, with $f_+ = f_-$ constant on $\tilde{\Sigma}$, cf. Section 6. In this case, no non-trivial Killing vector field shows up in general, but a number of compact examples involving Killing vector fields are provided, coming from [9].

The case when M_S is dense is first handled in Proposition 2.4, where we show that the vector field $K_1 := -\frac{1}{2}\,\alpha^\sharp$ is then Killing with respect to g — the chosen normalization is for further convenience — and that each eigenvalue of the Ricci tensor, Ric, of g is of multiplicity at least 2; moreover, on the (dense) open set $M_1 = M_S \cap M_0$, K_1 is Killing with respect to g_+, g_- and Hamiltonian with respect to the Kähler forms $\omega_+ := g_+(J_+\cdot, \cdot)$ and $\omega_- := g_-(J_-\cdot, \cdot)$; also, Ric is both J_+- and J_--invariant, cf. Proposition 2.4 below. On M_1, the ambikähler structure (g_+, J_+, ω_+), (g_-, J_-, ω_-) is then of the type described in Proposition 11 (iii) of [2].

In Section 3, we set the stage for a *separation of variables* by introducing new functions x, y, defined by $x = \frac{1}{2}(f_+ + f_-)$ and $y = \frac{1}{2}(f_+ - f_-)$, which, up to a factor 2, are the "eigenvalues" of ψ, and whose gradients are easily shown to be orthogonal. In Proposition 3.1, we show that $|dx|^2 = A(x)$ and $|dy|^2 = B(y)$, for some positive functions A and B of one variable. In terms of the new functions x, y, the dual 1-form of K_1 with respect to g is simply $J_+dx + J_+dy$. Furthermore, in Proposition 3.2 a second Killing vector field, K_2, shows up, whose dual 1-form is $y^2\, J_+dx + x^2\, J_+dy$ and which turns out to coincide, up to a constant factor, with the Killing vector field constructed by W. Jelonek in [8, Lemma B], cf. also the proof of Proposition 11 in [2], namely the image of K_1 by the *Killing symmetric endomorphism* $S = \Psi_+ \circ \Psi_- + \frac{(f_+^2 + f_-^2)}{2}\mathrm{I}$, cf. Remark 3.1.

In Proposition 3.3, we then show that either K_2 is a (positive) constant multiple of K_1, and we end up with an ambikähler structure *of Calabi type*, according to Definition 5.1 taken from [1], or K_1, K_2 are independent on a dense open subset of M, determining an *ambitoric structure*, as defined in [2], [3].

The Calabi case is considered in Section 5, where it is shown that, conversely, any ambikähler structure of Calabi type gives rise, up to scaling, to a 1-parameter family of pairs $(g^{(k)}, \psi^{(k)})$, where $g^{(k)}$ is a Riemannian metric in the conformal class and $\psi^{(k)}$ a $*$-Killing 2-form with respect to $g^{(k)}$, cf. Theorem 5.1 and Remark 5.1. The example of *Hirzebruch-like* ruled surfaces is described in Section 8.

The ambitoric case is the case when dx and dy are independent on a dense open subset of M. In Section 4, we show that x, y can be locally completed into a full system of coordinates by the addition of two "angular coordinates", s, t, in such a way that $K_1 = \frac{\partial}{\partial s}$ and $K_2 = \frac{\partial}{\partial t}$ and giving rise to a general Ansatz, described in Theorem 4.1. As an Ansatz for the underlying ambikähler structure, this turns out to be the same as the ambitoric Ansatz of Proposition 13 in [2] for the "quadratic" polynomial $q(z) = 2z$, hence in the *hyperbolic* normal form of [2, Section 5.4], when the functions x, y are identified with the *adapted coordinates* x, y in [2].

The main observation at this point is that, while the adapted coordinates in [2] are obtained via a quadratic transformation, cf. [2, Section 4.3], the functions x, y are here naturally attached to the $*$-Killing 2-form ψ which determines the ambitoric structure. This is quite reminiscent of the *orthotoric* situation, described in [1] in dimension 4 and in [4] in all dimensions, where the separation of variables — and the corresponding Ansatz — are similarly obtained via the "eigenvalues" of a *Hamiltonian* 2*-form*, which share the same properties as the "eigenvalues" x, y of the $*$-Killing 2-form ψ.

In spite of this, the $*$-Killing 2-forms considered in this paper are *not* Hamiltonian 2-forms in general — for a general discussion about Killing or $*$-Killing 2-forms versus Hamiltonian 2-forms, cf. [10], in particular Theorem 4.5 and Proposition 4.8, and, also, [4, Appendix A] — but, in many respects, at least in dimension 4, the role played by Hamiltonian 2-forms in the orthotoric case is played by $*$-Killing 2-forms in the (hyperbolic) ambitoric case.

The three situations described above, namely the decomposable, the Calabi ambikähler and the ambitoric case, cf. Proposition 3.3, are nicely illustrated in the example of the round 4-sphere described in Section 7, on which every $*$-Killing form can be written as the restriction of a constant 2-form $\mathsf{a} \in \mathfrak{so}(5) \simeq \Lambda^2\mathbb{R}^5$, which is also the 2-form associated to the covariant derivative of the Killing vector field induced by a. If a has rank 2, the same holds for its restriction on a dense open subset of the sphere, so this corresponds to the decomposable case. Otherwise, a can be expressed as $\lambda\, e_1 \wedge e_2 + \mu\, e_3 \wedge e_4$ — cf. Section 7 for the notation — with λ, μ both positive, and, depending on whether λ and μ are equal or not, we obtain on a dense subset of the sphere an ambikähler structure of Calabi type or a hyperbolic ambitoric structure respectively. By using the hyperbolic ambitoric Ansatz of Section 4, it is eventually shown that the resulting $*$-Killing 2-forms are actually $*$-Killing with respect to infinitely many non-isometric Riemannian metrics on S^4, cf. Remark 7.2.

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2. Killing 2-forms and ambikähler structures

In what follows, (M, g) denotes a connected, oriented, 4-dimensional Riemannian manifold admitting a non-parallel Killing 2-form φ, and $\psi := *\varphi$ denotes the corresponding $*$-Killing 2-form; we then have

$$\nabla_X \psi = \alpha \wedge X^\flat, \tag{2.1}$$

for some real, non-zero, 1-form α, where ∇ denotes the Levi-Civita connection of g and $X^\flat$ the dual 1-form of X with respect to g, cf. [13]. By anti-symmetrizing and by contracting (2.1), it is easily checked that ψ is closed and that

$$\delta\psi = 3\alpha, \tag{2.2}$$

where δ denotes the codifferential with respect to g. Denote by $\psi_+ = \frac{1}{2}(\psi + *\psi)$, resp. $\psi_- = \frac{1}{2}(\psi - *\psi)$, the self-dual, resp. the anti-self-dual, part of ψ, where $*$ is the Hodge operator induced by the metric g and the chosen orientation. Then, (2.1) is equivalent to the following two conditions

$$\begin{aligned}\nabla_X \psi_+ &= \big(\alpha \wedge X^\flat\big)_+ = \frac{1}{2}\alpha \wedge X^\flat + \frac{1}{2} X \lrcorner * \alpha,\\ \nabla_X \psi_- &= \big(\alpha \wedge X^\flat\big)_- = \frac{1}{2}\alpha \wedge X^\flat - \frac{1}{2} X \lrcorner * \alpha.\end{aligned} \tag{2.3}$$

Here, we used the general identity:

$$*(X^\flat \wedge \phi) = (-1)^p \, X \lrcorner * \phi, \tag{2.4}$$

for any vector field X and any p-form ϕ on any oriented Riemannian manifold. In particular, ψ_+ and ψ_- are *conformally Killing*, cf. [13]. The datum of a (non-parallel) $*$-Killing 2-form ψ on (M,g) is then equivalent to the datum of a pair (ψ_+,ψ_-) consisting of a self-dual 2-form ψ_+ and an anti-self-dual 2-form ψ_-, both conformally Killing and linked together by

$$d\psi_+ + d\psi_- = 0, \tag{2.5}$$

or, equivalently, by

$$\delta\psi_+ = \delta\psi_-. \tag{2.6}$$

We denote by Ψ, Ψ_+, Ψ_- the anti-symmetric endomorphisms of TM associated to ψ, ψ_+, ψ_- respectively via the metric g, so that $g(\Psi(X),Y) = \psi(X,Y)$, $g(\Psi_+(X),Y) = \psi_+(X,Y)$, $g(\Psi_-(X),Y) = \psi_-(X,Y)$. On the open set, M_0, of M where Ψ_+ and Ψ_- have no zero, denote by J_+, J_- the corresponding almost complex structures:

$$J_+ := \frac{\Psi_+}{f_+}, \qquad J_- := \frac{\Psi_-}{f_-}, \tag{2.7}$$

where the positive functions f_+, f_- are defined by

$$f_+ := \frac{|\Psi_+|}{\sqrt{2}}, \qquad f_- := \frac{|\Psi_-|}{\sqrt{2}} \tag{2.8}$$

(here, the norms $|\Psi_+|$, $|\Psi_-|$, are relative to the conformally invariant inner product defined on the space of anti-symmetric endomorphisms of TM by $(A,B) := -\frac{1}{2}\mathrm{tr}(A\circ B)$); the open set M_0 is then defined by the condition

$$f_+ > 0, \qquad f_- > 0. \tag{2.9}$$

Notice that J_+ and J_- induce opposite orientations, hence commute to each other, so that the endomorphism

$$\tau := -J_+J_- = -J_-J_+, \tag{2.10}$$

is an involution of the tangent bundle of M_0.

From (2.1), we get

$$\nabla_X \Psi = \alpha \wedge X, \tag{2.11}$$

with the following general convention: for any 1-form α and any vector field X, $\alpha\wedge X$ denotes the anti-symmetric endomorphism of TM defined by $(\alpha\wedge X)(Y) = \alpha(Y)X - g(X,Y)\alpha^\sharp$, where $\alpha^\sharp$ is the dual vector field to α relative to g (notice that the latter expression is actually independent of g in the conformal class $[g]$ of g). Equivalently:

$$\nabla_X \Psi_+ = (\alpha\wedge X)_+, \qquad \nabla_X\Psi_- = (\alpha\wedge X)_-. \tag{2.12}$$

We infer $(\nabla_X\Psi_+,\Psi_+) = \frac{1}{2}(d|\Psi_+|^2)(X) = (\Psi_+, \alpha\wedge X) = \big(\Psi_+(\alpha)\big)(X)$, hence $\Psi_+(\alpha) = \frac{1}{2}d|\Psi_+|^2$. Similarly, $\Psi_-(\alpha) = \frac{1}{2}d|\Psi_-|^2$. By using (2.7), we then get

$$\begin{aligned} \alpha &= -2\Psi_+\left(\frac{d|\Psi_+|}{|\Psi_+|}\right) = -2J_+df_+ \\ &= -2\Psi_-\left(\frac{d|\Psi_-|}{|\Psi_-|}\right) = -2J_-df_-. \end{aligned} \tag{2.13}$$

In particular,

$$J_+ df_+ = J_- df_-. \tag{2.14}$$

Remark 2.1. For any $*$-Killing 2-form ψ as above, denote by $\Phi = \Psi_+ - \Psi_-$ the skew-symmetric endomorphism associated to the Killing 2-form $\varphi = *\psi$ and by S the symmetric endomorphism defined by

$$S = -\frac{1}{2}\,\Phi \circ \Phi = \Psi_+ \circ \Psi_- + \frac{1}{2}(f_+^2 + f_-^2)\,\mathrm{I} = \frac{1}{2}\,\Psi \circ \Psi + (f_+^2 + f_-^2)\,\mathrm{I}, \tag{2.15}$$

where I denotes the identity of TM. Then, S is *Killing* with respect to g, meaning that the symmetric part of ∇S is zero or, equivalently, that $g((\nabla_X S)X, X) = 0$ for any vector field X, cf. [11], [2, Appendix B]. This readily follows from the fact that $\nabla_X \Phi(X) = X \lrcorner *(\alpha \wedge X) = 0$, so that $g(\nabla_X S(X), X) = -2g(\nabla_X \Phi(X), \Phi(X)) = 0$, for any vector field X.

Lemma 2.1. *The open subset M_0 defined by* (2.9) *is dense in M.*

Proof. Denote by $M_0^{\pm}$ the open set where $f_\pm \neq 0$, so that $M_0 = M_0^+ \cap M_0^-$. It is sufficient to show that each $M_0^{\pm}$ is dense. If not, $f_\pm = 0$ on some non-empty open set, V, of M, so that $\psi_\pm = 0$ on V, hence is identically zero, since $\psi_\pm$ is conformally Killing, cf. [13]; this, in turn, implies that α, hence also $\nabla\psi$, is identically zero, in contradiction to the hypothesis that ψ is non-parallel. ☐

In view of the next proposition, we recall the following definition, taken from [2]:

Definition 2.1 ([2])**.** An *ambikähler structure* on an oriented 4-manifold M consists of a pair of Kähler structures, $\big(g_+, J_+, \omega_+ = g_+(J_+\cdot, \cdot)\big)$ and $\big(g_-, J_-, \omega_- = g_-(J_-\cdot, \cdot)\big)$, where the Riemannian metrics g_+, g_- belong to the same conformal class, i.e. $g_- = f^2\, g_+$, for some positive function f, and the complex structure J_+, resp. the complex structure J_-, induces the chosen orientation, resp. the opposite orientation; equivalently, the Kähler forms ω_+ and ω_- are self-dual and anti-self-dual respectively.

We then have:

Proposition 2.1. *Let (M, g) be a connected, oriented, 4-dimensional Riemannian manifold, equipped with a non-parallel $*$-Killing 2-form $\psi = \psi_+ + \psi_-$ as above. Then, on the dense open subset, M_0, of M defined by* (2.9)*, the pair (g, ψ) gives rise to an ambikähler structure (g_+, J_+, ω_+), (g_-, J_-, ω_-), with $g_\pm = f_\pm^{-2}\, g$ and $J_\pm = f_\pm^{-1}\, \Psi_\pm$, by setting $f_\pm = |\Psi_\pm|/\sqrt{2}$. In particular, this ambikähler structure is equipped with two non-constant positive functions f_+, f_-, satisfying the two conditions*

$$f = \frac{f_+}{f_-}, \tag{2.16}$$

where $g_- = f^2\, g_+$, and

$$\tau(df_+) = df_-. \tag{2.17}$$

Conversely, any ambikähler structure (g_+, J_+, ω_+), $(g_- = f^2\, g_+, J_-, \omega_-)$ equipped with two non-constant positive functions f_+, f_- satisfying (2.16)–(2.17) *arises from a unique pair (g, ψ), where g is the Riemannian metric in the conformal class $[g_+] = [g_-]$ defined by*

$$g = f_+^2\, g_+ = f_-^2\, g_-, \tag{2.18}$$

and ψ is the $*$-Killing 2-form relative to g defined by

$$\psi = f_+^3\, \omega_+ + f_-^3\, \omega_-. \tag{2.19}$$

Proof. Before starting the proof, we recall the following general facts. (i) For any two Riemannian metrics, g and $\tilde{g} = \varphi^{-2}\, g$, in a same conformal class, and for any anti-symmetric endomorphism, A, of the tangent bundle with respect to the conformal class $[g] = [\tilde{g}]$, the covariant derivatives $\nabla^{\tilde{g}} A$ and $\nabla^{g} A$ are related by

$$\nabla^{\tilde{g}}_X A = \nabla^{g}_X A + \left[A, \frac{d\varphi}{\varphi} \wedge X\right] = A\left(\frac{d\varphi}{\varphi}\right) \wedge X + \frac{d\varphi}{\varphi} \wedge A(X), \tag{2.20}$$

by setting $A\left(\frac{d\varphi}{\varphi}\right) = -\frac{d\varphi}{\varphi} \circ A$. (ii) For any 1-form β and any vector field X, we have

$$\begin{aligned}(\beta \wedge X)_+ &= \frac{1}{2}\beta \wedge X - \frac{1}{2} J_+\beta \wedge J_+X - \frac{1}{2}\beta(J_+X)\, J_+ \\ &= \frac{1}{2}\beta \wedge X + \frac{1}{2} J_-\beta \wedge J_-X + \frac{1}{2}\beta(J_-X)\, J_-,\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}(\beta \wedge X)_- &= \frac{1}{2}\beta \wedge X - \frac{1}{2} J_-\beta \wedge J_-X - \frac{1}{2}\beta(J_-X)\, J_- \\ &= \frac{1}{2}\beta \wedge X + \frac{1}{2} J_+\beta \wedge J_+X + \frac{1}{2}\beta(J_+X)\, J_+,\end{aligned} \tag{2.22}$$

for *any* orthogonal (almost) complex structures J_+ and $J-$ inducing the chosen and the opposite orientation respectively.

From (2.7), (2.12), (2.13) and (2.21), we thus infer

$$\begin{aligned}\nabla_X J_+ &= -2\left(J_+\left(\frac{df_+}{|f_+|}\right) \wedge X\right)_+ - \frac{df_+}{f_+}(X)\, J_+ \\ &= -J_+\left(\frac{df_+}{f_+}\right) \wedge X - \frac{df_+}{f_+} \wedge J_+X + \frac{df_+}{f_+}(X)\, J_+ - \frac{df_+}{f_+}(X)\, J_+ \\ &= -J_+\left(\frac{df_+}{f_+}\right) \wedge X - \frac{df_+}{f_+} \wedge J_+X = \left[\frac{df_+}{f_+} \wedge X, J_+\right]\end{aligned} \tag{2.23}$$

which, by using (2.20), is equivalent to

$$\nabla^{g_+} J_+ = 0, \tag{2.24}$$

where ∇^{g_+} denotes the Levi-Civita connection of the conformal metric $g_+ = f_+^{-2}\, g$, meaning that the pair (g_+, J_+) is *Kähler*. Similarly, we have

$$\nabla_X J_- = \left[\frac{df_-}{f_-} \wedge X, J_-\right] \tag{2.25}$$

or, equivalently:

$$\nabla^{g_-} J_- = 0, \tag{2.26}$$

where ∇^{g_-} denotes the Levi-Civita connection of the conformal metric $g_- = f_-^{-2}\, g$, meaning that the pair (g_-, J_-) is Kähler as well. We thus get on M_0 an *ambikähler structure* in the sense of Definition 2.1. Moreover, because of (2.14), f_+ and f_- evidently satisfy (2.16)–(2.17).

For the converse, define g by

$$g = f_+^2\, g_+ = f_-^2\, g_- \tag{2.27}$$

and denote by ∇ the Levi-Civita connection of g. By defining $\Psi_+ = f_+ J_+$, $\Psi_- = f_- J_-$ and $\Psi = \Psi_+ + \Psi_-$, we get

$$\begin{aligned}\nabla_X \Psi_+ &= \nabla_X (f_+ J_+) \\ &= \nabla^{g_+}_X (f_+ J_+) + \left[\frac{df_+}{f_+} \wedge X, f_+ J_+\right] \\ &= df_+(X) J_+ - J_+ df_+ \wedge X - df_+ \wedge J_+ X \\ &= -2(J_+ df_+ \wedge X)_+.\end{aligned} \tag{2.28}$$

Similarly,

$$\nabla_X \Psi_- = -2 (J_- df_- \wedge X)_-. \tag{2.29}$$

By using (2.14), we obtain

$$\nabla_X \Psi = \alpha \wedge X, \tag{2.30}$$

with $\alpha := -2 J_+ df_+ = -2 J_- df_-$, meaning that the associated 2-form $\psi(X, Y) := g(\Psi(X), Y)$, is $*$-Killing. Finally $\psi = f_+ g(J_+ \cdot, \cdot) + f_- g(J_- \cdot, \cdot) = f_+^3 \omega_+ + f_-^3 \omega_-$. □

Remark 2.2. The fact that the pair $(g_+ = f_+^{-2} g, J_+)$, resp. the pair $(g_- = f_-^{-2} g, J_-)$, is Kähler only depends on, in fact is equivalent to, $\Psi_+ = f_+ J_+$, resp. $\Psi_- = f_- J_-$, being conformal Killing, i.e. ψ being conformally Killing. This was observed in a twistorial setting by M. Pontecorvo in [12], cf. also Appendix B2 in [2].

We now explain under which circumstances an ambikähler structure satisfies the conditions (2.16)–(2.17).

Proposition 2.2. *Let M be an oriented 4-manifold equipped with an ambikähler structure (g_+, J_+, ω_+), $(g_- = f^2 g_+, J_-, \omega_-)$. Assume moreover that f is not constant. Then, on the open set where $f \neq 1$, there exist non-constant positive functions f_+, f_- satisfying* (2.16)–(2.17) *of* Proposition 2.1 *if and only if the 1-form*

$$\kappa := \frac{\tau(df)}{1 - f^2} \tag{2.31}$$

is exact.

Proof. For any ambikähler structure (g_+, J_+, ω_+), $(g_- = f^2 g_+, J_-, \omega_-)$ and any positive functions f_+, f_- satisfying (2.16)–(2.17), we have

$$\begin{aligned}(1 - f^2) \frac{df_+}{f_+} &= \frac{df}{f} + \tau(df), \\ (1 - f^2) \frac{df_-}{f_-} &= f\, df + \tau(df).\end{aligned} \tag{2.32}$$

On the open set where $f \neq 1$, this can be rewritten as

$$\begin{aligned}\frac{df_+}{f_+} &= \frac{df}{f(1 - f^2)} + \frac{\tau(df)}{(1 - f^2)}, \\ \frac{df_-}{f_-} &= \frac{f df}{(1 - f^2)} + \frac{\tau(df)}{(1 - f^2)};\end{aligned} \tag{2.33}$$

in particular, κ is exact on this open set. Conversely, if κ is exact, but not identically zero, then $\kappa = \frac{d\varphi}{\varphi}$, for some, non-constant, positive function, φ, and we then *define* f_+, f_- by

$\frac{df_+}{f_+} = \frac{d\varphi}{\varphi} + \frac{df}{f(1-f^2)}$ and $\frac{df_-}{f_-} = \frac{d\varphi}{\varphi} + \frac{f\,df}{(1-f^2)}$, hence by $f_+ := \frac{f\,\varphi}{|1-f^2|^{\frac{1}{2}}}$ and $f_- := \frac{\varphi}{|1-f^2|^{\frac{1}{2}}}$, which clearly satisfy (2.16)–(2.17). □

Remark 2.3. It follows from (2.32) that if $f = k$, where k is a constant different from 1, then f_+ and f_- are constant and the corresponding $*$-Killing 2-form ψ is then parallel. More generally, the existence of a pair (g, ψ) inducing an ambikähler structure depends on the chosen relative scaling of the Kähler metrics. More precisely, if the ambikähler structure (g_+, J_+, ω_+), $(g_- = f^2\, g_+, J_-, \omega_-)$ arises from a $*$-Killing 2-form in the conformal class, in the sense of Proposition 2.1, then for any positive constant $k \neq 1$, the ambikähler structure (g_+, J_+, ω_+), $(\tilde{g}_- = k^2\, g_-, J_-, k^2\omega_-)$ *does not* arise from a $*$-Killing 2-form, unless $\tau(df) = \pm df$. This is because the 1-forms $\frac{\tau(df)}{(1-f^2)}$ and $\frac{\tau(df)}{(1-k^2 f^2)}$ would then be both closed, implying that $\tau(df) = \phi\, df$ for some function ϕ; since $|\tau(df)| = |df|$, we would then have $\phi = \pm 1$.

The 1-form κ in Proposition 2.2 is clearly exact on the open set where $f \neq 1$ whenever $\tau(df) = df$ or $\tau(df) = -df$, and it readily follows from (2.33) that f_+, f_- are then given by

$$f_+ = \frac{c\,f}{|1-f|}, \quad f_- = \frac{c}{|1-f|} = \pm c + f_+, \tag{2.34}$$

if $\tau(df) = df$, or by

$$f_+ = \frac{c\,f}{1+f}, \quad f_- = \frac{c}{1+f} = c - f_+, \tag{2.35}$$

if $\tau(df) = -df$, for some positive constant c. If

$$TM_0 = T^+ \oplus T^-, \tag{2.36}$$

denotes the orthogonal splitting determined by τ, where τ is the identity on T^+ and minus the identity on T^- — equivalently, J_+, J_- coincide on T^+ and are opposite on T^- — then $\tau(df) = \pm df$ if and only if $df_{|T^{\mp}} = 0$ and we also have:

Proposition 2.3. *The distribution $T^{\pm}$ is involutive if and only if $\tau(df) = \pm df$.*

Proof. For a general ambikähler structure (g_+, J_+, ω_+) and $(g_- = f^2\, g_+, J_-, \omega_-)$, with $g_- = f^2\, g_+$, we have

$$\frac{df(Z)}{f}\,\omega_+(X,Y) = -\omega_+([X,Y],Z), \quad \frac{df(Z)}{f}\,\omega_-(X,Y) = \omega_-([X,Y],Z), \tag{2.37}$$

for any X, Y in T^+ and any Z in T^-, and

$$\frac{df(Z)}{f}\,\omega_+(X,Y) = \omega_+([X,Y],Z), \quad \frac{df(Z)}{f}\,\omega_-(X,Y) = -\omega_-([X,Y],Z), \tag{2.38}$$

for any X, Y in T^- and any Z in T^+. This can be shown as follows. Suppose that X, Y are in T^+ and Z is in T^-. Then, since the Kähler form $\omega_+(\cdot,\cdot) = g_+(J_+\cdot,\cdot)$ and $\omega_-(\cdot,\cdot) = g(J_-\cdot,\cdot)$ are closed and T^+, T^- are ω_+- and ω_--orthogonal, we have

$$Z \cdot \omega_+(X,Y) = \omega_+([X,Y],Z) + \omega_+([Y,Z],X) + \omega_+([Z,X],Y), \tag{2.39}$$

and

$$Z \cdot \omega_-(X,Y) = \omega_-([X,Y],Z) + \omega_-([Y,Z],X) + \omega_-([Z,X],Y), \tag{2.40}$$

which can be rewritten as

$$Z \cdot \big(f^2\omega_+(X,Y)\big) = -f^2\,\omega_+([X,Y],Z) + f^2\,\omega_+([Y,Z],X) + f^2\,\omega_+([Z,X],Y), \tag{2.41}$$

or else:

$$2\frac{df(Z)}{f}\,\omega_+(X,Y)+Z\cdot\omega_+(X,Y)=\\ -\omega_+([X,Y],Z)+\omega_+([Y,Z],X)+\omega_+([Z,X],Y). \tag{2.42}$$

Comparing (2.39) and (2.42), we readily deduce the first identity in (2.37); the other three identities are checked similarly. Proposition 2.3 then readily follows from (2.37)–(2.38). ☐

In the following statement, M_0 stills denotes the (dense) open subset of M defined by (2.9); we also denote by M_S the open subset of M defined by

$$f_+\neq f_-, \tag{2.43}$$

on which ψ is a symplectic 2-form, and by M_1 the intersection $M_1:=M_0\cap M_S$.

Proposition 2.4. *Let (M,g) be an oriented Riemannian 4-dimensional manifold admitting a non-parallel $*$-Killing 2-form ψ. Denote by $(g_+=f_+^2\,g,J_+,\omega_+)$, $(g_-=f_-^2\,g,J_-,\omega_-)$ the induced ambikähler structure on M_0 as explained above. Then, on the open set M_1, the Ricci endomorphism,* Ric*, of g is J_+- and J_--invariant, hence of the form*

$$\mathrm{Ric}=a\,\mathrm{I}+b\,\tau, \tag{2.44}$$

for some functions a,b, where I *denotes the identity of TM_1 and τ is defined by* (2.10)*. Moreover, the vector field*

$$K_1:=J_+\mathrm{grad}_g f_+=J_-\mathrm{grad}_g f_-=-\frac{1}{2}\alpha^\sharp \tag{2.45}$$

is Killing with respect to g and preserves the whole ambikähler structure.

Proof. Let R be the curvature tensor of g, defined by

$$\mathrm{R}_{X,Y}Z:=\nabla_{[X,Y]}Z-[\nabla_X,\nabla_Y]Z, \tag{2.46}$$

for any vector field X,Y,Z. We denote by Scal its scalar curvature, by Ric_0 the trace-free part of Ric, by W the Weyl tensor of g, and by W^+ and W^- its self-dual and anti-self-dual part respectively. As in the previous section, Ψ denotes the skew-symmetric endomorphism of TM determined by ψ, Ψ_+ its self-dual part, Ψ_- its anti-self-dual part, with $\Psi_+=f_+\,J_+$ and $\Psi_-=f_-\,J_-$ on M_0. Since $g=f_+^2\,g_+=f_-^2\,g_-$, where g_+ and g_- are Kähler with respect to J_+ and J_- respectively, W^+ and W^- are both degenerate and $\mathrm{W}^+(\Psi_+)=\lambda_+\,\Psi_+$, $\mathrm{W}^-(\Psi_-)=\lambda_-\Psi_-$, for some functions λ_+,λ_-. For any vector fields X,Y on M, the usual decomposition of the curvature tensor reads:

$$\begin{aligned}\mathrm{R}_{X,Y}\Psi&=[\mathrm{R}(X\wedge Y),\Psi]\\ &=\frac{\mathrm{Scal}}{12}\,[X^\flat\wedge Y,\Psi]+\frac{1}{2}[\{\mathrm{Ric}_0,X^\flat\wedge Y\},\Psi]\\ &\quad+[\mathrm{W}^+(X\wedge Y),\Psi_+]+[\mathrm{W}^-(X\wedge Y),\Psi_-],\end{aligned} \tag{2.47}$$

by setting $\{\mathrm{Ric}_0,X^\flat\wedge Y\}:=\mathrm{Ric}_0\circ(X^\flat\wedge Y)+(X^\flat\wedge Y)\circ\mathrm{Ric}_0=\mathrm{Ric}_0(X)\wedge Y+X\wedge\mathrm{Ric}_0(Y)$, cf. *e.g.* [5, Chapter 1, Section G]. On M_0 we then have:

$$\frac{\mathrm{Scal}}{12}[X\wedge Y,\Psi]=-\frac{\mathrm{Scal}}{12}\big(\Psi(X)\wedge Y+X\wedge\Psi(Y)\big), \tag{2.48}$$

$$\frac{1}{2}[\{\mathrm{Ric}_0, X\wedge Y\},\Psi] = -\frac{1}{2}\Big(\Psi\big(\mathrm{Ric}_0(X)\big)\wedge Y + \mathrm{Ric}_0(X)\wedge\Psi(Y) \\ + \Psi(X)\wedge \mathrm{Ric}_0(Y) + X\wedge\Psi\big(\mathrm{Ric}_0(Y)\big)\Big), \tag{2.49}$$

and

$$\begin{aligned} \mathrm{W}^+_{X,Y}\Psi_+ &= \frac{\lambda_+}{2}\big(\Psi_+(X)\wedge Y + X\wedge\Psi_+(Y)\big),\\ \mathrm{W}^-_{X,Y}\Psi_- &= \frac{\lambda_-}{2}\big(\Psi_-(X)\wedge Y + X\wedge\Psi_-(Y)\big). \end{aligned} \tag{2.50}$$

We thus get

$$\sum_{i=1}^4 e_i\lrcorner \mathrm{R}_{e_i,Y}\Psi = \left(\lambda_+ - \frac{\mathrm{Scal}}{6}\right)\Psi_+(Y) + \left(\lambda_- - \frac{\mathrm{Scal}}{6}\right)\Psi_-(Y) \\ + \frac{1}{2}[\mathrm{Ric}_0,\Psi](Y). \tag{2.51}$$

Similarly,

$$\sum_{i=1}^4 e_i\lrcorner \mathrm{R}_{e_i,Y}\Psi_+ = \left(\lambda_+ - \frac{\mathrm{Scal}}{6}\right)\Psi_+(Y) + \frac{1}{2}[\mathrm{Ric}_0,\Psi_+](Y) \tag{2.52}$$

and

$$\sum_{i=1}^4 e_i\lrcorner \mathrm{R}_{e_i,Y}\Psi_- = \left(\lambda_- - \frac{\mathrm{Scal}}{6}\right)\Psi_-(Y) + \frac{1}{2}[\mathrm{Ric}_0,\Psi_-](Y). \tag{2.53}$$

On the other hand, from (2.11), we get

$$\mathrm{R}_{X,Y}\Psi = \nabla_Y\alpha\wedge X - \nabla_X\alpha\wedge Y, \tag{2.54}$$

hence

$$\sum_{i=1}^4 e_i\lrcorner \mathrm{R}_{e_i,Y}\Psi = -2\nabla_Y\alpha, \tag{2.55}$$

whereas, from (2.12), we obtain

$$\mathrm{R}_{X,Y}\Psi_+ = (\nabla_Y\alpha\wedge X - \nabla_X\alpha\wedge Y)_+,\quad \mathrm{R}_{X,Y}\Psi_- = (\nabla_Y\alpha\wedge X - \nabla_X\alpha\wedge Y)_-, \tag{2.56}$$

hence

$$\sum_{i=1}^4 e_i\lrcorner \mathrm{R}_{e_i,Y}\Psi_+ = -Y\lrcorner(\nabla\alpha)^s - Y\lrcorner(d\alpha)_+, \tag{2.57}$$

where $(\nabla\alpha)^s$ denotes the symmetric part of $\nabla\alpha$. Indeed, we have

$$\begin{aligned}\sum_{i=1}^4 e_i\lrcorner\big(\nabla_Y\alpha\wedge e_i-\nabla_{e_i}\alpha\wedge Y\big)_+ &= \frac12\sum_{i=1}^4 e_i\lrcorner(\nabla_Y\alpha\wedge e_i)-\frac12\sum_{i=1}^4 e_i\lrcorner(\nabla_{e_i}\alpha\wedge Y)\\ &\quad+\frac12\sum_{i=1}^4 e_i\lrcorner*(\nabla_Y\alpha\wedge e_i)-\frac12\sum_{i=1}^4 e_i\lrcorner*(\nabla_{e_i}\alpha\wedge Y)\\ &= -\nabla_Y\alpha-\frac12\sum_{i=1}^4 e_i\lrcorner*(\nabla_{e_i}\alpha\wedge Y)\\ &= -\nabla_Y\alpha-\frac12 Y\lrcorner*d\alpha=-Y\lrcorner(\nabla\alpha)^s-Y\lrcorner(d\alpha)_+,\end{aligned}\tag{2.58}$$

as $\delta\alpha=0$ and $e_i\lrcorner*(\nabla_Y\alpha\wedge e_i)$ is clearly equal to zero thanks to the general identity (2.4). We obtain similarly:

$$\sum_{i=1}^4 e_i\lrcorner\mathrm{R}_{e_i,Y}\Psi_-=-Y\lrcorner(\nabla\alpha)^s-Y\lrcorner(d\alpha)_-.\tag{2.59}$$

From the above, we infer

$$\begin{aligned}&(d\alpha)_+=\left(\frac{\mathrm{Scal}}{6}-\lambda_+\right)\psi_+,\quad (d\alpha)_-=\left(\frac{\mathrm{Scal}}{6}-\lambda_-\right)\psi_-,\\ &(\nabla\alpha)^s=-\frac12[\mathrm{Ric}_0,\Psi_+]=-\frac12[\mathrm{Ric}_0,\Psi_-].\end{aligned}\tag{2.60}$$

It follows that

$$[\mathrm{Ric},\Psi_+]=[\mathrm{Ric},\Psi_-],\tag{2.61}$$

and that the vector field $\alpha^{\sharp_g}$ is Killing with respect to g if and only if $[\mathrm{Ric},\Psi_+]=[\mathrm{Ric},\Psi_-]=0$. We now show that (2.61) actually *implies* $[\mathrm{Ric},\Psi_+]=[\mathrm{Ric},\Psi_-]=0$ at each point where $f_+\neq f_-$. Indeed, in terms of the decomposition (1.4), Ric, J_+, J_- can be written in the following matricial form

$$\mathrm{Ric}=\begin{pmatrix}P & Q\\ Q^* & R\end{pmatrix},\quad J_+=\begin{pmatrix}J & 0\\ 0 & J\end{pmatrix},\quad J_-=\begin{pmatrix}J & 0\\ 0 & -J\end{pmatrix}\tag{2.62}$$

where J denotes the restriction of J_+ on T^+ and on T^-, so that:

$$[\mathrm{Ric}_0,J_+]=\begin{pmatrix}[P,J] & [Q,J]\\ [Q^*,J] & [R,J]\end{pmatrix},\quad [\mathrm{Ric}_0,J_-]=\begin{pmatrix}[P,J] & -\{Q,J\}\\ \{Q^*,J\} & -[R,J].\end{pmatrix}\tag{2.63}$$

Then (2.61) can be expanded as

$$\begin{aligned}&(f_+-f_-)[P,J]=0,\\ &(f_++f_-)\,QJ=(f_+-f_-)\,JQ,\\ &(f_++f_-)[R,J]=0.\end{aligned}\tag{2.64}$$

Since $f_+>0$ and $f_->0$ on M_0, from (2.64) we readily infer $[R,J]=0$ and $Q=0$, meaning that

$$\mathrm{Ric}=\begin{pmatrix}P & 0\\ 0 & R\end{pmatrix}.\tag{2.65}$$

Moreover, on the open subset $M_1 = M_0 \cap M_S$, where $f_+ - f_- \neq 0$, we also infer from (2.64) that $[P, J] = 0$, hence that $[\mathrm{Ric}, J_+] = [\mathrm{Ric}, J_-] = 0$. By (2.60), $(\nabla\alpha)^s = 0$, meaning that the the vector field $K_1 := -\frac{1}{2}\alpha^\sharp = J_+ \mathrm{grad}_g f_+$ is Killing with respect to g. Notice that

$$\begin{aligned} K_1 &= J_+ \mathrm{grad}_g f_+ = J_- \mathrm{grad}_g f_- \\ &= -J_+ \mathrm{grad}_{g_+} \frac{1}{f_+} = -J_- \mathrm{grad}_{g_-} \frac{1}{f_-}. \end{aligned} \tag{2.66}$$

In particular, K_1 is also Killing with respect to g_+ and g_- and is (real) holomorphic with respect to J_+ and J_-. □

3. Separation of variables

In this section we restrict our attention to the open subset $M_1 := M_0 \cap M_S$, defined by the conditions (2.9) and (2.43). Recall that since $\psi \wedge \psi = \psi_+ \wedge \psi_+ + \psi_- \wedge \psi_- = 2(f_+ - f_-)\, v_g$, where v_g denotes the volume form of g relative to the chosen orientation, M_S is the open subset of M where ψ is non-degenerate, hence a symplectic 2-form. According to Proposition 2.4, on M_1 the Ricci tensor Ric is of the form (2.44), for some functions a, b and the vector field $\alpha^\sharp$ is Killing; we then infer from (2.60) that $\nabla\alpha^\sharp$ can be written as:

$$\nabla\alpha^\sharp = h_+ \, J_+ + h_- \, J_-, \tag{3.1}$$

with

$$h_+ := \frac{1}{2} f_+ \left(\frac{\mathrm{Scal}}{6} - \lambda_+ \right), \quad h_- := \frac{1}{2} f_- \left(\frac{\mathrm{Scal}}{6} - \lambda_- \right). \tag{3.2}$$

We then introduce the functions x, y defined by

$$\begin{aligned} x &:= \frac{f_+ + f_-}{2}, \quad & y &:= \frac{f_+ - f_-}{2}, \\ f_+ &= x + y, & f_- &= x - y. \end{aligned} \tag{3.3}$$

Notice that $(2x, 2y)$, resp. $(2x, -2y)$, are the eigenvalues of the Hermitian operator $-J_+ \circ \Psi = f_+ \,\mathrm{I} + f_- \,\tau$, resp. $-J_- \circ \Psi = f_+ \,\tau + f_- \,\mathrm{I}$, relative to the eigen-subbundle T^+ and T^- respectively. From (2.9) and (2.43) we deduce that x, y are subject to the conditions

$$x > |y| > 0, \tag{3.4}$$

whereas, from (2.14), we infer

$$\tau(dx) = dx, \qquad \tau(dy) = -dy. \tag{3.5}$$

In particular, dx, $J_+ dx = J_- dx$, dy and $J_+ dy = -J_- dy$ are *pairwise orthogonal* and

$$|dx|^2 + |dy|^2 = |df_+|^2 = |df_-|^2, \quad |dx|^2 - |dy|^2 = (df_+, df_-). \tag{3.6}$$

We then have:

Proposition 3.1. *On each connected component of the open subset of M_1 where $dx \neq 0$ and $dy \neq 0$, the square norm of dx, dy and the Laplacians of x, y relative to g are given by*

$$\begin{aligned} |dx|^2 &= \frac{A(x)}{(x^2 - y^2)}, \quad & |dy|^2 &= \frac{B(y)}{(x^2 - y^2)}, \\ \Delta x &= -\frac{A'(x)}{(x^2 - y^2)}, & \Delta y &= -\frac{B'(y)}{(x^2 - y^2)}, \end{aligned} \tag{3.7}$$

where A, B are functions of one variable.

Proof. By using (2.23) and (2.25) and setting $g_\tau(X,Y) := g(\tau(X),Y)$, we infer from (2.13) and (3.1) that

$$\begin{aligned}
\nabla df_+ &= \left(-\frac{1}{2}h_+ + \frac{|df_+|^2}{f_+}\right) g - \frac{1}{2}\, h_-\, g_\tau \\
&\quad - \frac{1}{f_+}\big(df_+ \otimes df_+ + J_+ df_+ \otimes J_+ df_+\big), \\
\nabla df_- &= \left(-\frac{1}{2}h_- + \frac{|df_-|^2}{f_-}\right) g - \frac{1}{2}\, h_+\, g_\tau \\
&\quad - \frac{1}{f_-}\big(df_- \otimes df_- + J_- df_- \otimes J_- df_-\big).
\end{aligned} \tag{3.8}$$

In terms of the functions x, y, this can be rewritten as

$$\begin{aligned}
\nabla dx &= \big(\frac{x}{(x^2-y^2)}(|dx|^2 + |dy|^2) - \frac{1}{4}(h_+ + h_-)\big)\, g - \frac{1}{4}(h_+ + h_-)\, g_\tau \\
&\quad - \frac{x}{(x^2-y^2)}\,(dx\otimes dx + dy \otimes dy) + \frac{y}{(x^2-y^2)}\,(dx \otimes dy + dy \otimes dx) \\
&\quad - \frac{x}{(x^2-y^2)}\, J_+(dx+dy)\otimes J_+(dx+dy), \\
\nabla dy &= -\big(\frac{y}{(x^2-y^2)}(|dx|^2 + |dy|^2) + \frac{1}{4}(h_+ - h_-)\big)\, g + \frac{1}{4}(h_+ - h_-)\, g_\tau \\
&\quad + \frac{y}{(x^2-y^2)}\,(dx\otimes dx + dy \otimes dy) - \frac{x}{(x^2-y^2)}\,(dx \otimes dy + dy \otimes dx) \\
&\quad + \frac{y}{(x^2-y^2)}\,\big(J_+(dx+dy)\otimes J_+(dx+dy)\big).
\end{aligned} \tag{3.9}$$

In particular:

$$\begin{aligned}
\Delta x &= (h_+ + h_-) - \frac{2x}{(x^2-y^2)}\,(|dx|^2 + |dy|^2), \\
\Delta y &= (h_+ - h_-) + \frac{2y}{(x^2-y^2)}\,(|dx|^2 + |dy|^2).
\end{aligned} \tag{3.10}$$

To simplify the notation, we temporarily put

$$F := |dx|^2, \qquad G := |dy|^2. \tag{3.11}$$

By contracting ∇dx by dx and ∇dy by dy in (3.9), and taking (3.10) into account, we obtain:

$$\begin{aligned}
dF &= -\left(\Delta x + \frac{2x\,F}{(x^2-y^2)}\right) dx + \frac{2y\,F}{(x^2-y^2)}\, dy, \\
dG &= -\frac{2x\,G}{(x^2-y^2)}\, dx - \left(\Delta y - \frac{2y\,G}{(x^2-y^2)}\right) dy.
\end{aligned} \tag{3.12}$$

From (3.12), we get

$$\begin{aligned}
d\big((x^2-y^2)\,F\big) &= -\big((x^2-y^2)\,\Delta x\big)\, dx, \\
d\big((x^2-y^2)\,G\big) &= -\big((x^2-y^2)\,\Delta y\big)\, dy.
\end{aligned} \tag{3.13}$$

It follows that $(x^2-y^2)\,F = A(x)$, for some (smooth) function A of one variable and that $A'(x) = -(x^2-y^2)\,\Delta x$; likewise, $(x^2-y^2)\,G = B(y)$ and $B'(y) = -(x^2-y^2)\,\Delta y$. □

A simple computation using (3.10) shows that in terms of A, B, the functions h_+, h_- appearing in (3.1) and their derivatives dh_+, dh_- have the following expressions:

$$\begin{aligned} h_+ &= -\frac{A'(x)+B'(y)}{2(x^2-y^2)} + \frac{(x-y)(A(x)+B(y))}{(x^2-y^2)^2}, \\ h_- &= -\frac{A'(x)-B'(y)}{2(x^2-y^2)} + \frac{(x+y)(A(x)+B(y))}{(x^2-y^2)^2}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} dh_+ = &-\frac{A''(x)dx+B''(y)dy}{2(x^2-y^2)} \\ &+ \frac{A'(x)\big((2x-y)\,dx - y\,dy\big) + B'(y)\big(x\,dx + (x-2y)\,dy\big)}{(x^2-y^2)^2} \\ &- \frac{\big(A(x)+B(y)\big)(x-y)\big((3x-y)\,dx + (x-3y)\,dy\big)}{(x^2-y^2)^3}, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} dh_- = &-\frac{A''(x)dx-B''(y)dy}{2(x^2-y^2)} \\ &+ \frac{A'(x)\big((2x+y)\,dx - y\,dy\big) + B'(y)\big(-x\,dx + (x+2y)\,dy\big)}{(x^2-y^2)^2} \\ &- \frac{\big(A(x)+B(y)\big)(x+y)\big((3x+y)\,dx - (x+3y)\,dy\big)}{(x^2-y^2)^3}. \end{aligned} \tag{3.16}$$

In particular:

$$J_+dh_+ - J_-dh_- = \left(\frac{h_+}{f_+} - \frac{h_-}{f_-}\right). \tag{3.17}$$

Proposition 3.2. *The vector fields*

$$\begin{aligned} K_1 &:= J_+\mathrm{grad}_g(x+y) = J_-\mathrm{grad}_g(x-y) \\ &= J_+\mathrm{grad}_{g_+}\left(\frac{-1}{x+y}\right) = J_-\mathrm{grad}_{g_-}\left(\frac{-1}{x-y}\right) \end{aligned} \tag{3.18}$$

(which is equal to the vector field $K_1 = -\frac{1}{2}\alpha^\sharp$ appearing in Proposition 2.4*), and*

$$\begin{aligned} K_2 &:= y^2\,J_+\mathrm{grad}_g x + x^2\,J_+\mathrm{grad}_g y = y^2\,J_-\mathrm{grad}_g x - x^2\,J_-\mathrm{grad}_g y \\ &= J_+\mathrm{grad}_{g_+}\left(\frac{xy}{x+y}\right) = J_-\mathrm{grad}_{g_-}\left(\frac{-xy}{x-y}\right) \end{aligned} \tag{3.19}$$

are Killing with respect to g, g_+, g_- and Hamiltonian with respect to ω_+ and ω_-. The momenta, μ_1^+, μ_2^+ of K_1, K_2 with respect to ω_+, and the momenta, μ_1^-, μ_2^-, of K_1, K_2 with respect to ω_-, are given by

$$\begin{aligned} \mu_1^+ &= \frac{-1}{x+y}, \qquad \mu_2^+ = \frac{xy}{x+y}, \\ \mu_1^- &= \frac{-1}{x-y}, \qquad \mu_2^- = \frac{-xy}{x-y}, \end{aligned} \tag{3.20}$$

and Poisson commute with respect to ω_+ *and* ω_-*, meaning that* $\omega_\pm(K_1, K_2) = 0$*, so that* $[K_1, K_2] = 0$ *as well. In particular, on the open set* M_1*, the ambikähler structure* (g_+, J_+, ω_+)*,* (g_-, J_-, ω_-) *is ambitoric in the sense of* [2, Definition 3].

Proof. In terms of A, B, (3.9) can be rewritten as

$$\begin{aligned}
\nabla dx &= \frac{1}{4(x^2-y^2)^2}\Big(2x\left(A(x)+B(y)\right) + (x^2-y^2)\,A'(x)\Big)\,g \\
&- \frac{1}{4(x^2-y^2)^2}\Big(2x\left(A(x)+B(y)\right) - (x^2-y^2)\,A'(x)\Big)\,g_\tau \\
&- \frac{x}{(x^2-y^2)}\,(dx\otimes dx + dy\otimes dy) + \frac{y}{(x^2-y^2)}\,(dx\otimes dy + dy\otimes dx) \\
&- \frac{x}{(x^2-y^2)}\,J_+(dx+dy)\otimes J_+(dx+dy), \\
\nabla dy &= \frac{1}{4(x^2-y^2)^2}\Big(-2y\left(A(x)+B(y)\right) + (x^2-y^2)\,B'(y)\Big)\,g \\
&- \frac{1}{4(x^2-y^2)^2}\Big(2y\left(A(x)+B(y)\right) + (x^2-y^2)\,B'(y)\Big)\,g_\tau \\
&+ \frac{y}{(x^2-y^2)}\,(dx\otimes dx + dy\otimes dy) - \frac{x}{(x^2-y^2)}\,(dx\otimes dy + dy\otimes dx) \\
&+ \frac{y}{(x^2-y^2)}\,J_+(dx+dy)\otimes J_+(dx+dy).
\end{aligned} \tag{3.21}$$

By taking (2.23)–(2.25) into account, we infer

$$\begin{aligned}
\nabla(J_+dx) &= \frac{1}{2(x^2-y^2)}\left(\frac{(2y-x)\,A(x) + x\,B(y)}{(x^2-y^2)} + \frac{A'(x)}{2}\right)\,g(J_+\cdot,\cdot) \\
&- \frac{1}{2(x^2-y^2)}\left(\frac{xA(x)+xB(y)}{(x^2-y^2)} - \frac{A'(x)}{2}\right)\,g(J_-\cdot,\cdot) \\
&- \frac{y\,dx\wedge J_+dx + x\,dy\wedge J_+dy}{(x^2-y^2)} \\
&+ \frac{x\,(dx\otimes J_+dy + J_+dy\otimes dx) + y\,(dy\otimes J_+dx + J_+dx\otimes dy)}{(x^2-y^2)}
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
\nabla(J_+dy) &= \frac{1}{2(x^2-y^2)}\left(\frac{(-y\,A(x) + (y-2x)\,B(y)}{(x^2-y^2)} + \frac{B'(y)}{2}\right)\,g(J_+\cdot,\cdot) \\
&- \frac{1}{2(x^2-y^2)}\left(\frac{y\,A(x)+y\,B(y)}{(x^2-y^2)} + \frac{B'(y)}{2}\right)\,g(J_-\cdot,\cdot) \\
&+ \frac{y\,dx\wedge J_+dx + x\,dy\wedge J_+dy}{(x^2-y^2)} \\
&- \frac{x\,(dx\otimes J_+dy + J_+dy\otimes dx) + y\,(dy\otimes J_+dx + J_+dx\otimes dy)}{(x^2-y^2)}.
\end{aligned} \tag{3.23}$$

In particular, the symmetric parts of $\nabla(J_+dx)$ and $\nabla(J_+dy)$ are opposite and given by

$$\begin{aligned}\big(\nabla(J_+dx)\big)^s = -\big(\nabla(J_+dy)\big)^s =& \frac{x\,(dx\otimes J_+dy + J_+dy\otimes dx)}{(x^2-y^2)}\\ &+\frac{y\,(dy\otimes J_+dx + J_+dx\otimes dy)}{(x^2-y^2)}.\end{aligned} \tag{3.24}$$

The symmetric parts of $\nabla(J_+dx + J_+dy)$ and of $\nabla(y^2J_+dx + x^2J_+dy) = y^2\nabla(J_+dx) + x^2\nabla(J_+dy) + 2dy\otimes J_+dx + 2xdx\otimes J_+dy$ then clearly vanish, meaning that K_1 and K_2 are Killing with respect to g. In view of the expressions of K_1, K_2 as symplectic gradients in (3.18)–(3.19), K_1 and K_2 are Hamiltonian with respect to ω_+ and ω_-, their momenta are those given by (3.20) and their Poisson bracket with respect to $\omega_\pm$ is equal to $\omega_\pm(K_1, K_2)$, which is zero, since dx lives in the dual of T^+ and dy in the dual of T^-. This, in turn, implies that K_1 and K_2 commute. □

Remark 3.1. As already observed, the Killing vector field K_1 appearing in Proposition 3.2 is the restriction to M_1 of the smooth vector field, also denoted by K_1, appearing in Proposition 2.4, which is defined on the whole manifold M by

$$K_1 = -\frac{1}{2}\alpha^\sharp = -\frac{1}{6}\delta\Psi. \tag{3.25}$$

Similarly, it is easily checked that K_2 is the restriction to M_1 of the smooth vector field, still denoted by K_2, defined on M by

$$\begin{aligned}K_2 =& -\frac{1}{8}\delta\big((f_+^2 - f_-^2)\,(\Psi_+ - \Psi_-)\big)\\ =& \frac{1}{8}\big(\Psi_+ - \Psi_-\big)\,\big(\mathrm{grad}_g(f_+^2 - f_-^2)\big)\end{aligned} \tag{3.26}$$

(recall that the Killing 2-form $\varphi = \psi_+ - \psi_- = *\psi$ is co-closed). It is also easily checked that K_2 and K_1 are related by

$$K_2 = \frac{1}{2}\,S(K_1), \tag{3.27}$$

where, we recall, S denotes the *Killing* symmetric endomorphism defined by (2.15) in Remark 2.1; this is because, on the dense open subset M_0, S can be rewritten as

$$S = -(x^2-y^2)\,\tau + (x^2+y^2)\,\mathrm{I}, \tag{3.28}$$

whereas $K_1^\flat = J_+(dx+dy)$, so that $S(K_1^\flat) = 2y^2\,J_+dx + 2x^2\,J_+dy = 2K_2^\flat$; we thus get (3.27) on M_0, hence on M. In view of (3.27), the fact that K_2 is Killing can then be alternatively deduced from [8, Lemma B], cf. also the proof of [2, Proposition 11 (iii)].

In view of the above, we eventually get the following rough classification:

Proposition 3.3. *For any connected, oriented, 4-dimensional Riemannian manifold (M, g) admitting a non-parallel $*$-Killing 2-form ψ, the open subset M_S defined by (2.43) is either empty or dense and we have one of the following three mutually exclusive cases:*

1. *M_S is dense; the vector fields K_1, K_2 are Killing and linearly independent on a dense open set of M, or*
2. *M_S is dense; the vector fields K_1, K_2 are Killing and $K_2 = c\,K_1$, for some non-zero real number c, or*

(3) *M_S is empty, i.e. ψ is decomposable everywhere; then, K_2 is identically zero, whereas K_1 is non-identically zero and is not a Killing vector field in general.*

Proof. Being Killing on $M_0 \cap M_S$ and zero on any open set where $f_+ = f_-$, K_2 is Killing everywhere on M. We next observe that, for any x in M_S, $K_2(\mathsf{x}) = 0$ if and only if $K_1(\mathsf{x}) = 0$, as readily follows from (3.27) and from the fact that S is invertible if and only if x belongs to M_S, as the eigenvalues of S are equal to $\frac{(f_+ + f_-)^2}{2}$ and $\frac{(f_+ - f_-)^2}{2}$.

Suppose now that M_S is not dense in M, i.e. that $M \setminus M_S$ contains some non-empty open subset V; then, K_2 vanishes on V, hence vanishes identically on M, as K_2 is Killing; from (3.26), we then infer $0 = \Psi(K_2) = \frac{1}{8}(f_+^2 - f_-^2)\mathrm{grad}_g(f_+^2 - f_-^2)$, which implies that the (smooth) function $(f_+^2 - f_-^2)^2$ is *constant* on M, hence identically zero, meaning that M_S is empty. If M_S is empty, then $f_+ = f_-$ everywhere (equivalently, $\psi \wedge \psi$ is identically zero); it follows that K_2 is identically zero, whereas K_1, which is not identically zero since ψ is not parallel, is not Killing in general, cf. Section 6.

If M_S is dense, then K_1 and K_2 are both Killing vector fields on M, hence either linearly independent on some dense open subset of M or dependent everywhere and, by the above discussion, K_2 is then a constant, non-zero multiple of K_1. $\square$

In the next sections we consider in turn the three cases listed in Proposition 3.3.

4. The ambitoric Ansatz

In this section, we assume that M_S is dense and that K_1, K_2 are linearly independent on some dense open set $\mathcal{U}$. In the remainder of this section, we focus our attention on $\mathcal{U}$, i.e. we assume that dx and dy are linearly independent everywhere — equivalently, $\tau(df) \neq \pm df$ everywhere — so that $\{dx, J_+dx = J_-dx, dy, J_+dy = -J_-dy\}$ form a direct orthogonal coframe. By Proposition 3.1, the metric g and the Kähler forms ω_+, ω_- can then be written as

$$\begin{aligned} g = {} & (x^2 - y^2)\left(\frac{dx \otimes dx}{A(x)} + \frac{dy \otimes dy}{B(y)}\right) \\ & + (x^2 - y^2)\left(\frac{J_+dx \otimes J_+dx}{A(x)} + \frac{J_+dy \otimes J_+dy}{B(y)}\right), \end{aligned} \tag{4.1}$$

$$\begin{aligned} \omega_+ &= \frac{(x-y)}{(x+y)}\left(\frac{dx \wedge J_+dx}{A(x)} + \frac{dy \wedge J_+dy}{B(y)}\right), \\ \omega_- &= \frac{(x+y)}{(x-y)}\left(\frac{dx \wedge J_+dx}{A(x)} - \frac{dy \wedge J_+dy}{B(y)}\right), \end{aligned} \tag{4.2}$$

and we also have:

Proposition 4.1. *The functions* $\mathrm{Scal} = 4a$ *and* b *appearing in the expression* (2.44) *of the Ricci tensor of* g *are given by:*

$$\mathrm{Scal} = -\frac{A''(x) + B''(y)}{(x^2 - y^2)}, \tag{4.3}$$

and

$$b = -\frac{A''(x) - B''(y)}{4(x^2 - y^2)} + \frac{xA'(x) + yB'(y)}{(x^2 - y^2)^2} - \frac{A(x) + B(y)}{(x^2 - y^2)^2}. \tag{4.4}$$

Proof. Since $\alpha^\sharp$ is Killing, the Bochner formula reads:

$$\mathrm{Ric}(\alpha^\sharp) = \delta\nabla\alpha^\sharp \tag{4.5}$$

whereas, by (2.44),

$$\mathrm{Ric}(\alpha^\sharp) = a\,\alpha^\sharp + b\,\tau(\alpha^\sharp). \tag{4.6}$$

By using

$$\alpha = f_+\,\delta J_+ = f_-\,\delta J_-, \tag{4.7}$$

which easily follows from (2.23)–(2.25), we infer from (3.1) that

$$\delta\nabla\alpha^\sharp = \frac{h_+}{f_+}\,\alpha + \frac{h_-}{f_-}\,\alpha - J_+dh_+ - J_-dh_-. \tag{4.8}$$

By putting together (4.5), (4.8) and (3.17), we get

$$a\,\alpha + b\,\tau(\alpha) = 2\left(\frac{h_+}{f_+}\,\alpha - J_+dh_+\right) = 2\left(\frac{h_-}{f_-}\,\alpha - J_-dh_-\right), \tag{4.9}$$

hence

$$\begin{aligned} dh_+ &= \left(a - 2\frac{h_+}{f_+} + b\right)dx + \left(a - 2\frac{h_+}{f_+} - b\right)dy,\\ dh_- &= \left(a - 2\frac{h_-}{f_-} + b\right)dx + \left(-a + 2\frac{h_-}{f_-} + b\right)dy. \end{aligned} \tag{4.10}$$

We thus get

$$\begin{aligned} a &= \frac{1}{2}\left(\frac{\partial h_+}{\partial x} + \frac{\partial h_+}{\partial y}\right) + \frac{2h_+}{x+y} = \frac{1}{2}\left(\frac{\partial h_-}{\partial x} - \frac{\partial h_-}{\partial y}\right) + \frac{2h_-}{x+y}\\ b &= \frac{1}{2}\left(\frac{\partial h_+}{\partial x} - \frac{\partial h_+}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial h_-}{\partial x} + \frac{\partial h_-}{\partial y}\right). \end{aligned} \tag{4.11}$$

By using (3.14), we obtain (4.3) and (4.4). □

Recall that a function φ is called J_+-*pluriharmonic* if $d(J_+d\varphi) = 0$ and J_--*pluriharmonic* if $d(J_-d\varphi) = 0$.

Proposition 4.2. (i) *The space of real J_+-pluriharmonic functions, modulo additive constants, of the form $\varphi^+ = \varphi^+(x,y)$ is generated by φ_1^+, φ_2^+ defined by:*

$$\varphi_1^+(x,y) = \int^x \frac{\zeta^2 d\zeta}{A(\zeta)} - \int^y \frac{\zeta^2 d\zeta}{B(\zeta)}, \quad \varphi_2^+(x,y) = \int^x \frac{d\zeta}{A(\zeta)} - \int^y \frac{d\zeta}{B(\zeta)}, \tag{4.12}$$

where $\int^x$, resp. $\int^y$, stands for any primitive of the variable x, resp. y.

(ii) *The space of real J_--pluriharmonic functions, modulo additive constants, of the form $\varphi^- = \varphi^-(x,y)$ is generated by φ_1^-, φ_2^- defined by:*

$$\varphi_1^-(x,y) = \int^x \frac{\zeta^2 d\zeta}{A(\zeta)} + \int^y \frac{\zeta^2 d\zeta}{B(\zeta)}, \quad \varphi_2^-(x,y) = \int^x \frac{d\zeta}{A(\zeta)} + \int^y \frac{d\zeta}{B(\zeta)}. \tag{4.13}$$

Proof. From (3.22)–(3.23), we readily infer the following expression of $d(J_\pm dx)$ and $d(J_\pm dy)$:

$$\begin{aligned} d(J_+dx) = d(J_-dx) = &\left(\frac{A'(x)}{A(x)} - \frac{2x}{x^2-y^2}\right) dx \wedge J_+dx \\ &+ \frac{2y\,A(x)}{(x^2-y^2)\,B(y)}\, dy \wedge J_+dy, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} d(J_+dy) = -d(J_-dy) = &- \frac{2x\,B(y)}{(x^2-y^2)\,A(x)}\, dx \wedge J_+dx \\ &+ \left(\frac{B'(y)}{B(y)} + \frac{2y}{x^2-x^2}\right) dy \wedge J_+dy. \end{aligned} \tag{4.15}$$

Let $\varphi = \varphi(x,y)$ be any function of x, y and denote by $\varphi_x, \varphi_y, \varphi_{xx}, etc...$ its derivative with respect to x, y. Then

$$\begin{aligned} d(J_+d\varphi) &= \varphi_x\, d(J_+dx) + \varphi_y\, d(J_+dy) \\ &+ \varphi_{xx}\, dx \wedge J_+dx + \varphi_{yy}\, dy \wedge J_+dy \\ &+ \varphi_{xy}\,(dx \wedge J_+dy + dy \wedge J_+dx). \end{aligned} \tag{4.16}$$

By (4.14)–(4.15), φ is J_+-pluriharmonic if and only if $\varphi_{xy} = 0$ — meaning that φ is of the form $\varphi(x,y) = C(x) + D(y)$ — and C, D satisfy

$$\begin{aligned} C''(x) + \left(\frac{A'(x)}{A(x)} - \frac{2x}{x^2-y^2}\right) C'(x) - \frac{2x\,B(y)\,D'(y)}{(x^2-y^2)\,A(x)} = 0, \\ D''(y) + \left(\frac{B'(y)}{B(y)} + \frac{2y}{x^2-y^2}\right) D'(y) + \frac{2y\,A(x)\,C'(x)}{(x^2-y^2)\,B(y)} = 0 \end{aligned} \tag{4.17}$$

or equivalently, by multiplying the first equation by $A(x)$ and the second by $B(y)$, which are both positive, and by setting $F(x) := A(x)C'(x)$ and $G(y) := B(y)D'(y)$:

$$F'(x) - \frac{2x\big(F(x)+G(y)\big)}{x^2-y^2} = 0, \quad G'(y) + \frac{2y\big(F(x)+G(y)\big)}{x^2-y^2} = 0. \tag{4.18}$$

It is easily checked that the general solution of this system is given by:

$$F(x) = k_1\,x^2 + k_2, \qquad G(y) = -k_1\,y^2 - k_2, \tag{4.19}$$

for real constants k_1, k_2. We thus get Part (i) of Proposition 4.2. Part (ii) is obtained similarly. ☐

In view of Proposition 4.2, we (locally) define s and t, up to additive constants, by

$$J_+d\varphi_1^+ = J_-d\varphi_1^- = ds, \qquad J_+d\varphi_2^+ = J_-d\varphi_2^- = -dt. \tag{4.20}$$

Equivalently:

$$ds = \frac{x^2\,J_+dx}{A(x)} - \frac{y^2\,J_+dy}{B(y)}, \qquad dt = -\frac{J_+dx}{A(x)} + \frac{J_+dy}{B(y)}. \tag{4.21}$$

Notice that $ds \wedge dt = \frac{(x^2-y^2)}{A(x)B(y)}\, J_+dx \wedge J_+dy$; it then follows from Proposition 3.1 that

$$dx \wedge dy \wedge ds \wedge dt = \frac{v_g}{(x^2-y^2)}, \tag{4.22}$$

where v_g denotes the volume form of g with respect to the orientation induced by J_+, showing that dx, dy, ds, dt form a direct coframe. In view of (4.1), (4.2), (4.21), on the open set where x, y, s, t form a coordinate system, the metrics g, g_+, g_-, the complex structures J_+, J_-, the involution τ and the Kähler forms ω_+, ω_- have the following expressions:

$$\begin{aligned} g &= (x^2 - y^2)\left(\frac{dx \otimes dx}{A(x)} + \frac{dy \otimes dy}{B(y)}\right) \\ &\quad + \frac{A(x)}{(x^2 - y^2)}\,(ds + y^2\,dt) \otimes (ds + y^2\,dt) \\ &\quad + \frac{B(y)}{(x^2 - y^2)}\,(ds + x^2\,dt) \otimes (ds + x^2\,dt) \\ &= (x + y)^2\, g_+ = (x - y)^2\, g_- \end{aligned} \tag{4.23}$$

$$\begin{aligned} J_+dx &= J_-dx = \frac{A(x)}{(x^2 - y^2)}\,(ds + y^2\,dt) \\ J_+dy &= -J_-dy = \frac{B(y)}{(x^2 - y^2)}\,(ds + x^2\,dt) \\ J_+dt &= \frac{dx}{A(x)} - \frac{dy}{B(y)}, \quad J_-dt = \frac{dx}{A(x)} + \frac{dy}{B(y)} \\ J_+ds &= -\frac{x^2\,dx}{A(x)} + \frac{y^2\,dy}{B(y)}, \quad J_-ds = -\frac{x^2\,dx}{A(x)} - \frac{y^2\,dy}{B(y)} \end{aligned} \tag{4.24}$$

$$\begin{aligned} \tau(dx) &= dx, \qquad \tau(dy) = -dy \\ \tau(ds) &= \frac{(x^2 + y^2)}{(x^2 - y^2)}\,ds + \frac{2x^2y^2}{(x^2 - y^2)}\,dt \\ \tau(dt) &= \frac{-2}{(x^2 - y^2)}\,ds - \frac{(x^2 + y^2)}{(x^2 - y^2)}\,dt, \end{aligned} \tag{4.25}$$

$$\begin{aligned} \omega_+ &= \frac{dx \wedge (ds + y^2\,dt) + dy \wedge (ds + x^2\,dt)}{(x + y)^2} \\ \omega_- &= \frac{dx \wedge (ds + y^2\,dt) - dy \wedge (ds + x^2\,dt)}{(x - y)^2} \end{aligned}; \tag{4.26}$$

while it follows from (2.19) that the $*$-Killing 2-form ψ is given by

$$\psi = 2x\,dx \wedge (ds + y^2\,dt) + 2y\,dy \wedge (ds + x^2\,dt). \tag{4.27}$$

Notice that, in view of (4.23), the (local) vector fields $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ are Killing with respect to g and respectively coincide with the Killing vector fields K_1 and K_2 appearing in Proposition 3.2 on their domain of definition.

It turns out that the expressions of $(g_+ = (x+y)^{-2}\,g, J_+, \omega_+)$ and $(g_- = (x-y)^{-2}\,g, J_-, \omega_-)$ just obtained coincide with the *ambitoric Ansatz* described in [2, Theorem 3], in the case where the quadratic polynomial is $q(z) = 2z$, which is the *normal form* of the ambitoric Ansatz in the *hyperbolic* case considered in [2, Paragraph 5.4].

The discussion in this section can then be summarized as follows:

Theorem 4.1. *Let (M, g) be a connected, oriented, 4-dimensional manifold admitting a non-parallel, $*$-Killing 2-form $\psi = \psi_+ + \psi_-$ and assume that the open set, M_S, where $|\psi_+| \neq |\psi_-|$ is dense, cf.* Proposition 3.3. *On the open subset, $\mathcal{U}$, of M_S where ψ_+ and ψ_- have no zero and $d|\psi_+| \wedge d|\psi_-| \neq 0$, the pair (g, ψ) gives rise to an ambitoric structure of hyperbolic type, in the sense of* [2], *relative to the conformal class of g, which, on any simply-connected open subset of $\mathcal{U}$, is described by* (4.23)–(4.24)–(4.26), *where the Hermitian structures $(g_+ = (x+y)^{-2}\, g, J_+, \omega_+)$ and $(g_- = (x-y)^{-2} g, J_-, \omega_-)$ are Kähler, while ψ is described by* (4.27).

Conversely, on the open set, $\mathcal{U}$, of $\mathbb{R}^4$, of coordinates x, y, s, t, with $x > |y| > 0$, the two almost Hermitian structures $(g_+ = (x+y)^{-2}\, g, J_+, \omega_+)$, $(g_- = (x-y)^{-2} g, J_-, \omega_-)$ defined by (4.23)–(4.24)–(4.26), *with $A(x) > 0$ and $B(y) > 0$, are Kähler and, together with the Killing vector fields $K_1 = \frac{\partial}{\partial s}$ and $K_2 = \frac{\partial}{\partial t}$, constitute an ambitoric structure of hyperbolic type, while the 2-form ψ defined by* (4.27) *is $*$-Killing with respect to g.*

Proof. The first part follows from the preceding discussion. For the converse, we first observe that the 2-forms ω_+ and ω_- defined by (4.26) are clearly closed and not degenerate. To test the integrability of the almost complex structures J_+ and J_- defined by (4.24), we consider the complex 1-forms:

$$\begin{aligned} \beta_+ &= dx + i\, J_+ dx = dx + i \frac{A(x)}{(x^2 - y^2)}\, (ds + y^2\, dt), \\ \gamma_+ &= dy + i\, J_+ dy = dy + i \frac{B(y)}{(x^2 - y^2)}\, (ds + x^2\, dt), \end{aligned} \tag{4.28}$$

which generate the space of $(1,0)$-forms with respect to J_+. We then have:

$$\begin{aligned} d\beta_+ &= i \, \frac{\big((x^2 - y^2)\, A'(x) + x\, A(x)\big)}{(x^2 - y^2)}\, dx \wedge (ds + y^2\, dt) \\ &\quad + i\, \frac{2y\, A(x)}{(x^2 - y^2)}\, dy \wedge (ds + x^2\, dt) \\ &= \frac{\big(A'(x) - 2x\, A(x)\big)}{A(x)}\, dx \wedge \beta_+ + \frac{2y\, A(x)}{B(y)}\, dy \wedge \gamma_+, \\ d\gamma_+ &= i\, \frac{\big((x^2 - y^2)\, B'(y) + 2y\, B(y)\big)}{(x^2 - y^2)}\, dy \wedge (ds + x^2\, dt) \\ &\quad - i\, \frac{2x\, B(y)}{(x^2 - y^2)}\, dx \wedge (ds + y^2\, dt) \\ &= \frac{\big(B'(y) + 2y\, B(y)\big)}{B(y)}\, dy \wedge \gamma_+ - \frac{2x\, B(y)}{A(x)}\, dx \wedge \beta_+, \end{aligned} \tag{4.29}$$

which shows that J_+ is integrable. For J_-, we likewise consider the complex 1-forms:

$$\begin{aligned} \beta_- &= dx + i\, J_- dx = \beta_+ = dx + i \frac{A(x)}{(x^2 - y^2)}\, (ds + y^2\, dt), \\ \gamma_- &= dy + i\, J_- dy = dy - i \frac{B(y)}{(x^2 - y^2)}\, (ds + x^2\, dt), \end{aligned} \tag{4.30}$$

which generate the space of $(1,0)$-forms with respect to J_+. We then get

$$\begin{aligned} d\beta_- &= d\beta_+ \\ &= \frac{\big(A'(x) - 2x\,A(x)\big)}{A(x)}\,dx\wedge\beta_- - \frac{2y\,A(x)}{B(y)}\,dy\wedge\gamma_-, \\ d\gamma_- &= -d\gamma_+ \\ &= \frac{\big(B'(y) + 2y\,B(y)\big)}{B(y)}\,dy\wedge\gamma_- + \frac{2x\,B(y)}{A(x)}\,dx\wedge\beta_-, \end{aligned} \tag{4.31}$$

which, again, shows that J_- is integrable. It follows that the almost Hermitian structures $(g_+ = (x+y)^{-2}\,g, J_+, \omega_+)$ and $(g_- = (x-y)^{-2}\,g, J_-, \omega_-)$ are both *Kähler* and thus determine an *ambikähler structure* on $\mathcal{U}$. Moreover, the vector fields $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ are clearly Killing with respect to g, g_+, g_-, and satisfy:

$$\begin{aligned} &\frac{\partial}{\partial s}\lrcorner\omega_+ = -\frac{dx+dy}{(x+y)^2} = d\left(\frac{1}{x+y}\right), \quad \frac{\partial}{\partial s}\lrcorner\omega_- = \frac{-dx+dy}{(x-y)^2} = d\left(\frac{1}{x-y}\right), \\ &\frac{\partial}{\partial t}\lrcorner\omega_+ = -\frac{y^2\,dx + x^2\,dy}{(x+y)^2} = -d\left(\frac{xy}{x+y}\right), \quad \frac{\partial}{\partial t}\lrcorner\omega_- = -\frac{y^2\,dx - x^2\,dy}{(x-y)^2} = d\left(\frac{xy}{x-y}\right), \end{aligned} \tag{4.32}$$

meaning that they are both Hamiltonian with respect to ω_+ and ω_-, with momenta given by (3.20) in Proposition 3.2. This implies that $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ preserve the two Kähler structures (g_+, J_+, ω_+) and (g_-, J_-, ω_-) and actually coincide with the vector field K_1 and K_2 respectively defined in a more general context in Proposition 3.2. We thus end up with an *ambitoric structure*, as defined in [2]. According to Theorem 3 in [2], it is an *ambitoric structure of hyperbolic type*, with "quadratic polynomial" $q(z) = 2z$. To check that the 2-form ψ defined by (4.27) — which is evidently closed — is $*$-Killing with respect to g, denote by f_+, f_- the positive functions on $\mathcal{U}$ defined by $f_+ = x+y$, $f_- = x-y$, so that $g_+ = f_+^{-2}\,g$, $g_- = f_-^{-2}\,g$ and $\psi = f_+^3\,\omega_+ + f_-^3\,\omega_-$; it then follows from (4.25) that $\tau(df_+) = df_-$, hence that ψ is $*$-Killing by Proposition 2.1. □

5. Ambikähler structures of Calabi type

The second case listed in Proposition 3.3, which is considered in this section, can be made more explicit via the following proposition:

Proposition 5.1. *Let (M, g) be a connected, oriented, Riemannian 4-manifold admitting a non-parallel $*$-Killing 2-form $\psi = \psi_+ + \psi_-$. In view of* Proposition 3.3*, assume that the open set M_S — where ψ is non-degenerate — is dense in M and that the Killing vector fields K_1, K_2 defined by* (3.25)–(3.26) *are related by $K_2 = c\,K_1$, for some non-zero real number c. Then, c is positive and one of the following three cases occurs:*

(1) *$f_+(\mathsf{x}) + f_-(\mathsf{x}) = 2\sqrt{c}$, for any x in M, or*
(2) *$f_+(\mathsf{x}) - f_-(\mathsf{x}) = 2\sqrt{c}$, for any x in M, or*
(3) *$f_-(\mathsf{x}) - f_+(\mathsf{x}) = 2\sqrt{c}$, for any x in M,*

with the usual notation: $f_+ = |\psi_+|/\sqrt{2}$ and $f_- = |\psi_-|/\sqrt{2}$.

Proof. First recall that $(\Psi_+ + \Psi_-) \circ (\Psi_+ - \Psi_-) = -(f_+^2 - f_-^2)\,\mathrm{I}$. From (3.26) and $K_1 = J_+ \mathrm{grad}_g f_+ = J_- \mathrm{grad}_g f_-$, we then infer

$$\begin{aligned} \Psi(K_1) &= -\frac{1}{2}\mathrm{grad}_g \left(f_+^2 + f_-^2\right), \\ \Psi(K_2) &= -\frac{1}{16}\mathrm{grad}_g \left(\left(f_+^2 - f_-^2\right)^2\right). \end{aligned} \tag{5.1}$$

On M_S, where Ψ is invertible, the identity $K_2 = c\, K_1$ then reads:

$$(f_+^2 - f_-^2) d(f_+^2 - f_-^2) = 4c\left(df_+^2 + df_-^2\right), \tag{5.2}$$

or, else:

$$(f_+^2 - f_-^2 - 4\,c)\, df_+^2 = (f_+^2 - f_-^2 + 4\,c)\, df_-^2. \tag{5.3}$$

Since $|df_+| = |df_-|$ on M_0, on $M_1 = M_0 \cap M_S$ we then get:

$$f_+^2(f_+^2 - f_-^2 - 4\,c)^2 - f_-^2(f_+^2 - f_-^2 + 4\,c)^2 = 0. \tag{5.4}$$

Since M_1 is dense this identity actually holds on the whole manifold M. It can be rewritten as

$$(f_+^2 - f_-^2)\big((f_+ + f_-)^2 - 4\,c\big)\big((f_+ - f_-)^2 - 4\,c\big) = 0; \tag{5.5}$$

this forces c to be positive — if not, $f_+^2 - f_-^2$ would be identically zero — and we eventually get the identity:

$$(f_+^2 - f_-^2)(f_+ + f_- + 2\sqrt{c})(f_+ + f_- - 2\sqrt{c})(f_+ - f_- - 2\sqrt{c})(f_+ - f_- + 2\sqrt{c}) = 0. \tag{5.6}$$

Denote by $\tilde{M}$ the open subset of M obtained by removing the zero locus $K_1^{-1}(0)$ of K_1 from M (notice that $\tilde{M}$ is a connected, dense open subset of M, as $K_1^{-1}(0)$ is a disjoint union of totally geodesic submanifolds of codimension a least 2). It readily follows from (5.6) that $\tilde{M}$ is the union of the following four *closed* subsets $\tilde{F}_0 := F_0 \cap \tilde{M}$, $\tilde{F}_+ := F_+ \cap \tilde{M}$, $\tilde{F}_- := F_- \cap \tilde{M}$ and $\tilde{F}_S := F_S \cap \tilde{M}$ of $\tilde{M}$, where F_0, F_+, F_-, F_S denote the four closed subsets of M defined by:

$$\begin{aligned} F_0 &:= \{\mathsf{x} \in M \mid f_+(\mathsf{x}) + f_-(\mathsf{x}) = 2\sqrt{c}\}, \\ F_+ &:= \{\mathsf{x} \in M \mid f_+(\mathsf{x}) - f_-(\mathsf{x}) = 2\sqrt{c}\}, \\ F_- &:= \{\mathsf{x} \in M \mid f_-(\mathsf{x}) - f_+(\mathsf{x}) = 2\sqrt{c}\}, \\ F_S &:= \{\mathsf{x} \in M \mid f_+(\mathsf{x}) - f_-(\mathsf{x}) = 0\}. \end{aligned} \tag{5.7}$$

We now show that if the interior, V, of $\tilde{F}_0$ is non-empty then $\tilde{F}_0 = \tilde{M}$ (and thus $F_0 = M$ by density); this amounts to showing that the boundary $B := \bar{V} \setminus V$ of V in $\tilde{M}$ is empty. If not, let x be any element of B; then, x belongs to $\tilde{F}_0$, as $\tilde{F}_0$ is closed, and it also belongs to $\tilde{F}_+$ or $\tilde{F}_-$: otherwise, there would exist an open neighbourhood of x disjoint from $\tilde{F}_+ \cup \tilde{F}_-$, hence contained in $\tilde{F}_0 \cup \tilde{F}_S$; as $\tilde{F}_S$ has no interior, this neighbourhood would be contained in contained in $\tilde{F}_0$, which contradicts the fact that x sits on the boundary of V. Without loss, we may thus assume that x belongs to $\tilde{F}_+$, so that $f_+(\mathsf{x}) = 2\sqrt{c}$ and $f_-(\mathsf{x}) = 0$; since $K_1(\mathsf{x}) \neq 0$ — by the very definition of $\tilde{M}$ — f_+ is regular at x, implying that the locus of $f_+ = 2\sqrt{c}$ is a smooth hypersurface, S, of $\tilde{M}$ near x; moreover, since $\tilde{F}_+$ and $\tilde{F}_-$ are disjoint, $f_- = 0$ on S, meaning that $\Psi_- = 0$ on S; for any X in $T_{\mathsf{x}}S$ we then have $\nabla_X \Psi_- = 0$. On the other hand, $\nabla_X \Psi = (\alpha(\mathsf{x}) \wedge X)_-$, for any X in $T_{\mathsf{x}}M$, cf. (2.12), and we can then choose X in $T_{\mathsf{x}}S$ in such a way that $(\alpha(\mathsf{x}) \wedge X)_-$ be non-zero, hence $\nabla_X \psi_- \neq 0$, contradicting the previous assertion. We similarly show that $M = F_+$ or $M = F_-$ whenever the interior of $\tilde{F}_+$ or of $\tilde{F}_-$ is non-empty. $\square$

A direct consequence of Proposition 5.1 is that on the (dense) open subset M_0, the associated ambikähler structure $(g_+ = f_+^{-2}\, g, J_+ = f_+^{-1}\, \Psi_+, \omega_+)$, $(g_- = f_-^{-2}\, g = f^2\, g_+, J_- = f_-^{-1}\, \Psi_-, \omega_-)$, with $f = f_+/f_-$, satisfies

$$\tau(df) = -df \tag{5.8}$$

in the first case listed in Proposition 5.1, and

$$\tau(df) = df \tag{5.9}$$

in the remaining two cases. The ambikähler structure is then *of Calabi type*, according to the following definition, taken from [1]:

Definition 5.1. An ambikähler structure (g_+, J_+, ω_+), (g_-, J_-, ω_-), with $g_+ = f^{-2}\, g_-$, is said to be *of Calabi type* if $df \neq 0$ everywhere, and if there exists a non-vanishing vector field K, Killing with respect to g_+ and g_- and Hamiltonian with respect to ω_+ and ω_-, which satisfies

$$\tau(K) = \pm\, K, \tag{5.10}$$

with $\tau = -J_+ J_- = -J_- J_+$.

By replacing the pair (J_+, J_-) by the pair $(J_+, -J_-)$ if needed, we can assume, without loss of generality, that $\tau(K) = -K$. In the following proposition, we recall some general facts concerning this class of ambikähler structures, cf. e.g. [1, Section 3]:

Proposition 5.2. *For any ambikähler structure of Calabi type, with $\tau(K) = -K$:*
(i) *The Killing vector field K is an eigenvector of the Ricci tensor,* Ric^{g_+}*, of g_+ and of the Ricci tensor,* Ric^{g_-}*, of g_-; in particular,* Ric^{g_+} *and* Ric^{g_-} *are both J_+- and J_--invariant;*
(ii) *the Killing vector field K is a constant multiple of* $J_- \mathrm{grad}_{g_-} f = J_+ \mathrm{grad}_{g_+} \frac{1}{f}$.

Proof. By hypothesis, $K = J_+ \mathrm{grad}_{g_+} z_+ = J_- \mathrm{grad}_{g_-} z_-$, for some real functions z_+ and z_-. Since $J_- K = -J_+ K$, we infer $\mathrm{grad}_{g_+} z_+ = -\mathrm{grad}_{g_-} z_-$, hence

$$dz_+ = -f^{-2}\, dz_-. \tag{5.11}$$

Since $df \neq 0$ everywhere, this, in turn, implies that

$$z_+ = F(f), \qquad z_- = G(f) \tag{5.12}$$

for some real (smooth) functions F, G defined on $\mathbb{R}^{>0}$ up to an additive constant and satisfying:

$$G'(x) = -x^2\, F'(x). \tag{5.13}$$

Moreover,

$$\tau(df) = -df. \tag{5.14}$$

Since K has no zero and satisfies $\tau(K) = -K$, we have

$$J_+ = \frac{K^\flat \wedge J_+ K}{|K|^2} + * \frac{K^\flat \wedge J_+ K}{|K|^2}, \quad J_- = -\frac{K^\flat \wedge J_+ K}{|K|^2} + * \frac{K^\flat \wedge J_+ K}{|K|^2}, \tag{5.15}$$

so that

$$J_+ - J_- = \frac{2\, K^\flat \wedge J_+ K}{|K|^2}, \tag{5.16}$$

In (5.15)–(5.16), the dual 1-form $K^\flat$ and the square norm $|K|^2$ are relative to any metric in $[g_+] = [g_-]$. For definiteness however, we agree that they are both relative to g_+. Since $g^+ = f^{-2}\, g_-$, we have:

$$\nabla^{g_+}_X J_- = J_- \frac{df}{f} \wedge X + \frac{df}{f} \wedge J_- X. \tag{5.17}$$

By using (2.17), we then infer from (5.16):

$$\begin{aligned}\nabla^{g_+}_X (J_+ - J_-) = -\nabla^{g_+}_X J_- &= J_+ \frac{df}{f} \wedge X - \frac{df}{f} \wedge J_- X \\ &= \frac{2\,\nabla^{g_+}_X K^\flat \wedge J_+ K + 2\, K^\flat \wedge J_+ \nabla^{g_+}_X K}{|K|^2} \\ &\quad - \frac{X \cdot |K|^2}{|K|^2}\,(J_+ + J_-).\end{aligned} \tag{5.18}$$

By contracting with K, and by using $K^\flat = F'\, J_+ df$ and $J_+ \nabla^{g_+}_X K = \nabla_{J_+ X} K$ (as K is $J_\pm$-holomorphic), we obtain

$$\begin{aligned}\nabla^{g_+}_X K = &-\frac{|K|^2}{2f\,F'} J_+ X + \frac{1}{2f\,F'}\left(K^\flat \wedge J_+ K\right)(X) \\ &+ \frac{1}{2}\frac{d|K|^2}{|K|^2}(X)\, K + \frac{1}{2}\frac{J_+ d|K|^2}{|K|^2}(X)\, J_+ K.\end{aligned} \tag{5.19}$$

Since K is Killing with respect to g_+, $\nabla^{g_+} K$ is anti-symmetric; in view of (5.19), this forces $|K|^2$ to be of the form

$$|K|^2 = H(f), \tag{5.20}$$

for some (smooth) function H from $\mathbb{R}^{>0}$ to $\mathbb{R}^{>0}$, hence

$$\frac{d|K|^2}{|K|^2} = \frac{H'(f)}{H(f)}\, df = -\frac{H'(f)}{H(f) F'(f)}\, J_+ K^\flat. \tag{5.21}$$

By substituting (5.21) in (5.19), we eventually get the following expression of $\nabla^{g_+} K$:

$$\nabla^{g_+} K = \Phi_+(f)\, J_+ - \Phi_-(f)\, J_-, \tag{5.22}$$

with

$$\Phi_+ = \frac{1}{4}\left(\frac{H'(f)}{F'(f)} - \frac{H(f)}{f\, F'(f)}\right), \quad \Phi_- = \frac{1}{4}\left(\frac{H'(f)}{F'(f)} + \frac{H(f)}{f\, F'(f)}\right). \tag{5.23}$$

Since K is Killing with respect to g_+, it follows from the Bochner formula that

$$\mathrm{Ric}^{g_+}(K) = \delta \nabla^{g_+} K, \tag{5.24}$$

whereas, from (5.22) we get

$$\begin{aligned}(\nabla^{g_+})^2_{X,Y} K = &\Phi'_+\, df(X)\, J_+(Y) - \Phi'_-\, df(X)\, J_-(Y) \\ &- \Phi_- \left(\nabla^{g_+}_X J_-\right)(Y),\end{aligned} \tag{5.25}$$

and, from $\nabla^{g_+}_X J_- = [J_-, \frac{df}{f} \wedge X]$:

$$\delta J_- = -\left(\sum_{i=1}^4 \nabla^{g_+}_{e_i} J_-\right)(e_i) = -2 J_+ \frac{df}{f} = -\frac{2}{f F'(f)}\, K^\flat. \tag{5.26}$$

By putting together (5.22), (5.24), (5.25) and (5.26), we get

$$\mathrm{Ric}^{g_+}(K) = \mu\, K, \tag{5.27}$$

with

$$\mu = -\frac{\big(f\,\Phi_+'(f) + f\,\Phi_-'(f) - 2\,\Phi_-(f)\big)}{f\,F'(f)}. \tag{5.28}$$

Since the metric g_+ is Kähler with respect to J_+, in particular is J_+-invariant, (5.27) implies that the two eigenspaces of Ric^{g_+} are the space $\{K, J_+K\}$ generated by K and J_+K (where $J_- = J_+$) and its orthogonal complement, $\{K, J_+K\}^\perp$ (where $J_- = -J_+$). It follows that Ric^{g_+} is both J_+- *and* J_--invariant. This establishes the part (i) of the proposition (it is similarly shown that Ric^{g_-} is J_+- and J_--invariant).

Before proving part (ii), we first recall the general transformation rules of the curvature under a conformal change of the metric. If g and $\tilde{g} = \phi^{-2}\, g$ are two Riemannian metrics in a same conformal class $[g]$ in any n-dimensional Riemannian manifold (M, g), $n > 2$, then the scalar curvature, $\mathrm{Scal}^{\tilde{g}}$, and the trace-free part, $\mathrm{Ric}_0^{\tilde{g}}$, of $\tilde{g}$ are related to the scalar curvature, Scal^g, and the trace-free part, Ric_0^g, of g by

$$\mathrm{Scal}^{\tilde{g}} = \phi^2\,\big(\mathrm{Scal}^g - 2(n-1)\,\phi\,\Delta_g\phi - n(n-1)\,|d\phi|_g^2\big), \tag{5.29}$$

and

$$\mathrm{Ric}_0^{\tilde{g}} = \mathrm{Ric}^g - (n-2)\,\frac{(\nabla^g d\phi)_0}{\phi}, \tag{5.30}$$

where $(\nabla^g d\phi)_0$ is the trace-free part of the Hessian $\nabla^g d\phi$ of ϕ with respect of g, cf. *e.g.* [5, Chapter 1, Section J]. Applying (5.30) to the conformal pair $(g_-, g_+ = f^{-2}\, g_-)$, we get

$$\mathrm{Ric}_0^{g_+} = \mathrm{Ric}_0^{g_-} - \frac{2\,(\nabla^{g_-} df)_0}{f}. \tag{5.31}$$

Since Ric^{g_+} and Ric^{g_+} are both J_+- and J_--invariant, it follows that $(\nabla^{g_-} df)_0$ is J_--invariant, as well as $\nabla^{g_-} df$, since all metrics in $[g_+] = [g_-]$ are J_+- and J_--invariant. This means that the vector field $\mathrm{grad}_{g_-} f$ is J_--holomorphic, hence that $J_-\mathrm{grad}_{g_-} f$ is Hamiltonian with respect to ω_-, hence Killing with respect to g_-; since $J_-\mathrm{grad}_{g_-} f = \frac{1}{G'(f)}\, K$, we conclude that $G'(f)$ is constant, hence, by using (5.13), that $F(f)$ and $G(f)$ are of the form

$$F(f) = \frac{a}{f} + b, \qquad G(f) = a\,f + c, \tag{5.32}$$

for a non-zero real constant a and arbitrary real constants b, c. This, together with (2.17), establishes part (ii) of the proposition. □

Theorem 5.1. *Let (M, g) be a connected, oriented 4-manifold admitting a non-parallel $*$-Killing 2-form $\psi = \psi_+ + \psi_-$, satisfying the hypothesis of* Proposition 5.1, *corresponding to* Case (2) *of* Proposition 3.3. *Then, on the dense open set $M_0 \setminus K_1^{-1}(0)$ the associated ambikähler structure is of Calabi type, with respect to the Killing vector field $K = K_1$, with $\tau(K) = -K$ in the first case of* Proposition 5.1 *and $\tau(K) = K$ in the two remaining cases.*

Conversely, let (g_+, J_+, ω_+), $(g_- = f^2\, g_+, J_-, \omega_-)$ be any ambikähler structure of Calabi type with non-vanishing Killing vector field K, defined on some oriented 4-dimensional manifold M. If $\tau(K) = -K$, there exist, up to scaling, a unique metric g in the conformal class $[g_+] = [g_-]$ and a unique non-parallel $*$-Killing 2-form ψ with respect to g, inducing the given *ambikähler structure. If $\tau(K) = K$, such a pair (g, ψ) exists and is unique outside the locus $\{f = 1\}$.*

Proof. The first part of the proposition has already been discussed in the preceding part of this section. Conversely, let (g_+, J_+, ω_+), $(g_- = f^2\, g_+, J_-, \omega_-)$ be an ambikähler structure of Calabi type, with respect to some non-vanishing Killing vector field K, with $\tau(K) = -K$ or $\tau(K) = K$. Then, according to Proposition 5.2, K can be chosen equal to

$$K = J_+ \mathrm{grad}_{g_-} f = J_+ \mathrm{grad}_{g_+} \frac{1}{f}, \tag{5.33}$$

if $\tau(K) = -K$, or

$$K = J_+ \mathrm{grad}_{g_-} f = -J_+ \mathrm{grad}_{g_+} \frac{1}{f}, \tag{5.34}$$

if $\tau(K) = K$. According to Proposition 2.2 and (2.35), if $\tau(K) = -K$, hence $\tau(df) = -df$, the ambikähler structure is then induced by the metric g, in the conformal class $[g_+] = [g_-]$, defined by $g = f_+^{-2}\, g_+ = f_-^{-2}\, g_-$, with

$$f_+ = \frac{c\, f}{1+f}, \quad f_- = \frac{c}{1+f} = c - f_+, \tag{5.35}$$

for some positive constant c, and the $*$-Killing 2-form ψ defined by

$$\psi = \frac{f^3}{(1+f)^3}\, \omega_+ + \frac{1}{(1+f)^3}\, \omega_-. \tag{5.36}$$

If $\tau(K) = K$, hence $\tau(df) = df$, it similarly follows from Proposition 2.2 and (2.34) that the ambikähler structure is induced by the metric $g = f_+^2\, g_+ = f_-^2\, g_-$, with

$$f_+ = \frac{c\, f}{1-f}, \quad f_- = \frac{c}{1-f} = c + f^+, \tag{5.37}$$

for some constant c, positive if $f < 1$, negative if $f > 1$, and the $*$-Killing 2-form

$$\psi = \frac{f^3}{(1-f)^3}\, \omega_+ + \frac{1}{(1-f)^3}\, \omega_-, \tag{5.38}$$

but the pair (g, ψ) is only defined outside the locus $\{f = 1\}$. □

Remark 5.1. Any ambikähler structure (g_+, J_+, ω_+), (g_-, J_-, ω_-) generates, up to global scaling, a 1-parameter family of ambikähler structures, parametrized by a non-zero real number k, obtained by, say, fixing the first Kähler structure (g_+, J_+, ω_+) and substituting $(g_-^{(k)} = k^{-2}\, g_- = f_k^2\, g_+, J_-^{(k)} = \epsilon(k)\, J_-, \omega_-^{(k)} = \epsilon(k)\, k^{-2}\, \omega_-)$ to the second one, with $\epsilon(k) = \frac{k}{|k|}$ and $f_k = \frac{f}{|k|}$. Assume that the ambikähler structure (g_+, J_+, ω_+), (g_-, J_-, ω_-) is of Calabi type, with $\tau(df) = -df$. For any k in $\mathbb{R} \setminus \{0\}$, we then have $\tau^{(k)}(df_k) = -\epsilon(k)\, df_k$, by setting $\tau^{(k)} = -J_+ J_-^{(k)} = -J_-^{(k)} J_+ = \epsilon(k)\, \tau$, whereas, from (2.33) we infer:

$$f_+^{(k)} = \frac{f}{|k+f|}, \quad f_-^{(k)} = \frac{|k|}{|k+f|}, \tag{5.39}$$

up to global scaling; the ambikähler structure (g_+, J_+, ω_+), $(g_-^{(k)}, J_-^{(k)}, \omega_-^{(k)})$ is then induced by the pair $(g^{(k)}, \psi^{(k)})$, where $g^{(k)}$ is defined in the conformal class by

$$g^{(k)} = \frac{f^2}{(k+f)^2}\, g_+ = \frac{(1+f)^2}{(k+f)^2}\, g, \tag{5.40}$$

and $\psi^{(k)}$ is the $*$-Killing 2-form with respect to $g^{(k)}$ defined by

$$\psi^{(k)} = \frac{f^3}{|k+f|^3}\,\omega_+ + \frac{k}{|k+f|^3}\,\omega_-, \tag{5.41}$$

both defined outside the locus $\{f + k = 0\}$.

Remark 5.2. As observed in [1, Section 3.1], any ambikähler structure of Calabi type (g_+, J_+, ω_+), $(g_- = f^2\, g_+, J_-, \omega_-)$, with $\tau(df) = df$, admits a *Hamiltonian 2-form*, ϕ^+, with respect to the Kähler structure (g_+, J_+, ω_+) and a Hamiltonian 2-form, ϕ^-, with respect to the (g_-, J_-, ω_-), given by

$$\phi^+ = f^{-1}\,\omega_+ + f^{-3}\,\omega_-, \qquad \phi^- = f^3\,\omega_+ + f\,\omega_-. \tag{5.42}$$

6. The decomposable case

Assume now that (M, g, ψ) is as in Case (3) in Proposition 3.3, that is, that the $*$-Killing 2-form $\psi = \psi_+ + \psi_-$ is degenerate (or decomposable). This latter condition holds if and only if $\psi \wedge \psi = 0$, if and only if $|\psi_+| = |\psi_-|$, i.e. $f_+ = f_- =: \varphi$, or $f = 1$, meaning that $g_+ = g_- =: g_K$, whereas $g = \varphi^2\, g_K$. Denote by ∇^K the Levi-Civita connection of g_K. Then from (2.24)–(2.26) we get $\nabla^K J_+ = \nabla^K J_- = \nabla^K \tau = 0$, which implies that (M, g_K) is locally a Kähler product of two Kähler curves of the form $M = (\Sigma, g_\Sigma, J_\Sigma, \omega_\Sigma) \times (\tilde{\Sigma}, g_{\tilde{\Sigma}}, J_{\tilde{\Sigma}}, \omega_{\tilde{\Sigma}})$, with

$$\begin{aligned} g_K &= g_\Sigma + g_{\tilde{\Sigma}}, \\ J_+ &= J_\Sigma + J_{\tilde{\Sigma}}, \quad J_- = J_\Sigma - J_{\tilde{\Sigma}}, \\ \omega_+ &= \omega_\Sigma + \omega_{\tilde{\Sigma}}, \quad \omega_- = \omega_\Sigma - \omega_{\tilde{\Sigma}}. \end{aligned} \tag{6.1}$$

Moreover, from (2.14) we readily infer $\tau(d\varphi) = d\varphi$, meaning that φ is the pull-back to M of a function defined on Σ. Conversely, for any Kähler product $M = (\Sigma, g_\Sigma, J_\Sigma, \omega_\Sigma) \times (\tilde{\Sigma}, g_{\tilde{\Sigma}}, J_{\tilde{\Sigma}}, \omega_{\tilde{\Sigma}})$ as above and for *any* positive function φ defined on Σ, regarded as a function defined on M, the metric $g := \varphi^2\,(g_\Sigma + g_{\tilde{\Sigma}})$ admits a $*$-Killing 2-form ψ, given by

$$\psi = \varphi^3\,\omega_\Sigma, \tag{6.2}$$

whose corresponding Killing 2-form $*\psi$ is given by

$$*\psi = \varphi^3\,\omega_{\tilde{\Sigma}}. \tag{6.3}$$

Note that by (2.2) $\alpha = \frac{1}{3}\delta^g \psi = \frac{1}{\varphi^2} *_\Sigma d\varphi$, so $K_1 = -\frac{1}{2}\alpha^\sharp$ is not a Killing vector field in general.

The above considerations completely describe the local structure of 4-manifolds with decomposable $*$-Killing 2-forms. They also provide compact examples, simply by taking Σ and $\tilde{\Sigma}$ to be compact Riemann surfaces. We will show, however, that there are compact 4-manifolds with decomposable $*$-Killing 2-forms which are not products of Riemann surfaces (in fact not even of Kähler type). They arise as special cases (for $n = 4$) of the classification, in [9], of compact Riemannian manifolds (M^n, g) carrying a Killing vector fields with conformal Killing covariant derivative.

It turns out that if ψ is a non-trivial $*$-Killing 2-form which can be written as $\psi = d\xi^\flat$ for some Killing vector field ξ on M, then either ψ has rank 2 on M, or M is Sasakian or has positive constant sectional curvature (Proposition 4.1 and Theorem 5.1 in [9]). For $n = 4$, the Sasakian situation does not occur, and the case when M has constant sectional curvature will be treated in detail in the next section. The remaining case — when ψ is decomposable — is the one which we are interested in, and is described by cases 3. and 4. in Theorem 8.9 in [9]. We obtain the following two classes of examples:

(1) (M,g) is a warped mapping torus

$$M = (\mathbb{R}\times N)/_{(t,x)\sim(t+1,\varphi(x))}, \qquad g = \lambda^2 d\theta^2 + g_N,$$

where (N, g_N) is is a compact 3–dimensional Riemannian manifold carrying a function λ, such that $d\lambda^\sharp$ is a conformal vector field, φ is an isometry of N preserving λ, $\xi = \frac{\partial}{\partial\theta}$ and $\psi = d\xi^\flat = 2\lambda d\lambda\wedge d\theta$. One can take for instance $(N, g_N) = \mathbb{S}^3$ and λ a first spherical harmonic. Further examples of manifolds N with this property are given in Section 7 in [9].

(2) (M,g) is a Riemannian join $\mathbb{S}^2 *_{\gamma,\lambda} \mathbb{S}^1$, defined as the smooth extension to S^4 of the metric $g = ds^2 + \gamma^2(s)g_{\mathbb{S}^2} + \lambda^2(s)d\theta^2$ on $(0,l)\times S^2\times S^1$, where $l>0$ is a positive real number, $\gamma : (0,l)\to\mathbb{R}^+$ is a smooth function satisfying the boundary conditions

$$\gamma(t) = t(1 + t^2 a(t^2)) \quad \text{and} \quad \gamma(l-t) = \frac{1}{c} + t^2 b(t^2), \quad \forall\ |t|<\epsilon,$$

for some smooth functions a and b defined on some interval $(-\epsilon,\epsilon)$, $\lambda(s) := \int_s^l \gamma(t)dt$, $\xi = \frac{\partial}{\partial\theta}$ and $\psi = 2\lambda(s)\lambda'(s)ds\wedge d\theta$.

In particular, we obtain infinite-dimensional families of metrics on $S^3\times S^1$ and on S^4 carrying decomposable $*$-Killing 2-forms.

7. Example: the sphere $\mathbb{S}^4$ and its deformations

We denote by $\mathbb{S}^4 := (S^4, g)$ the 4-dimensional sphere, embedded in the standard way in the Euclidean space $\mathbb{R}^5$, equipped with the standard induced Riemannian metric, g, of constant sectional curvature 1, namely the restriction to $\mathbb{S}^4$ of the standard inner product $(\cdot,\cdot)$ of $\mathbb{R}^5$. We first recall the following well-known facts, cf. e.g. [13]. Let $\psi = \psi_+ + \psi_-$ be any $*$-Killing 2-form with respect to g, so that $\nabla_X\Psi = \alpha\wedge X$, cf. (2.1). Since g is Einstein, the vector field $\alpha^\sharp$ is Killing and it follows from (3.1)–(3.2) that $\nabla\alpha = \psi$. Conversely, for any Killing vector field Z on $\mathbb{S}^4$, it readily follows from the general *Kostant formula*

$$\nabla_X(\nabla Z) = \mathrm{R}_{Z,X}, \tag{7.1}$$

that, in the current case, $\nabla_X(\nabla Z) = Z\wedge X$, so that the 2-form $\psi := \nabla Z^\flat$ is $*$-Killing with respect to g. The map $Z\mapsto \nabla Z^\flat$ is then an isomorphism from the space of Killing vector fields on $\mathbb{S}^4$ to the space of $*$-Killing 2-forms.

It is also well-known that there is a natural $1-1$-correspondence between the Lie algebra $\mathfrak{so}(5)$ of anti-symmetric endomorphisms of $\mathbb{R}^5$ and the space of Killing vector fields on $\mathbb{S}^4$: for any a in $\mathfrak{so}(5)$, the corresponding Killing vector field, Z_{a}, is defined by

$$Z_{\mathsf{a}}(u) = \mathsf{a}(u), \tag{7.2}$$

for any u in $\mathbb{S}^4$, where $\mathsf{a}(u)$ is viewed as an element of the tangent space $T_u\mathbb{S}^4$, via the natural identification $T_u\mathbb{S}^4 = u^\perp$.

By combining the above two isomorphisms, we eventually obtained a natural identification of $\mathfrak{so}(5)$ with the space of $*$-Killing 2-forms on $\mathbb{S}^4$ and it is easy to check that, for any a in $\mathfrak{so}(5)$, the corresponding $*$-Killing 2-form, ψ_{a}, is given by

$$\psi_{\mathsf{a}}(X,Y) = (\mathsf{a}(X), Y), \tag{7.3}$$

for any u in $\mathbb{S}^4$ and any X, Y in $T_u\mathbb{S}^4 = u^\perp$; alternatively, the corresponding endomorphism Ψ_{a} is given by

$$\Psi_{\mathsf{a}}(X) = \mathsf{a}(X) - (\mathsf{a}(X), u)\, u, \tag{7.4}$$

for any X in $T_u\mathbb{S}^4 = u^\perp$.

Since, for any u in $\mathbb{S}^4$, the volume form of $\mathbb{S}^4$ is the restriction to $T_u\mathbb{S}^4$ of the 4-form $u\lrcorner \mathrm{v}_0$, where v_0 stands for the standard volume form of $\mathbb{R}^5$, namely $\mathrm{v}_0 = e_0 \wedge e_1 \wedge e_2 \wedge e_3 \wedge e_4$, for any direct frame of $\mathbb{R}^5$ (here identified with a coframe via the standard metric), we easily check that, for any a in $\mathfrak{so}(5)$, the corresponding Killing 2-form $*\psi_{\mathsf{a}}$ has the following expression

$$(*\psi_{\mathsf{a}})(X,Y) = (u\lrcorner *_5 \mathsf{a})(X,Y) = *_5(u \wedge \mathsf{a})(X,Y), \tag{7.5}$$

for any u in $\mathbb{S}^4$ and any X, Y in $T_u\mathbb{S}^4 = u^\perp$; here, $*_5$ denotes the Hodge operator on $\mathbb{R}^5$ and we keep identifying vector and covectors via the Euclidean inner product.

From (7.4), we easily infer

$$|\Psi_{\mathsf{a}}|^2 = |\mathsf{a}|^2 - 2|\mathsf{a}(u)|^2, \tag{7.6}$$

at any u in $\mathbb{S}^4$, where the norm is the usual Euclidean norm of endomorphisms, whereas the Pfaffian of ψ_{a} is given by:

$$\mathrm{pf}(\psi_{\mathsf{a}}) := \frac{\psi_{\mathsf{a}} \wedge \psi_{\mathsf{a}}}{2\,\mathrm{v}_g} = \frac{(\psi_{\mathsf{a}}, *\psi_{\mathsf{a}})}{2} = \frac{u \wedge \mathsf{a} \wedge \mathsf{a}}{2\,\mathrm{v}_0}. \tag{7.7}$$

On the other hand, when f_+, f_- are defined by (2.8), we have

$$|\Psi_{\mathsf{a}}|^2 = 4(f_+^2 + f_-^2), \tag{7.8}$$

and

$$\mathrm{pf}(\psi_{\mathsf{a}}) = f_+^2 - f_-^2. \tag{7.9}$$

For any a in $\mathfrak{so}(5)$, we may choose a direct orthonormal basis e_0, e_1, e_2, e_3, e_4 of $\mathbb{R}^5$, with respect to which a has the following form

$$\mathsf{a} = \lambda\, e_1 \wedge e_2 + \mu\, e_3 \wedge e_4, \tag{7.10}$$

for some real numbers λ, μ, with $0 \leq \lambda \leq \mu$. Then,

$$\begin{aligned} &|\mathsf{a}|^2 = 2(\lambda^2 + \mu^2), \\ &\mathsf{a}(u) = \lambda(u_1 e_2 - u_2 e_1) + \mu(u_3 e_4 - u_3 e_3), \\ &|\mathsf{a}(u)|^2 = \lambda^2(u_1^2 + u_2^2) + \mu^2(u_3^2 + u_4^2), \\ &u \wedge \mathsf{a} \wedge \mathsf{a} = 2\,\lambda\,\mu\,u_0\,e_0 \wedge e_1 \wedge e_2 \wedge e_3 \wedge e_4, \end{aligned} \tag{7.11}$$

for any $u = \sum_{i=0}^4 u_i\, e_i$ in $\mathbb{S}^4$. We thus get

$$\begin{aligned} &f_+^2 + f_-^2 = \frac{1}{2}\left(\lambda^2 + \mu^2 - \lambda^2(u_1^2 + u_2^2) - \mu^2(u_3^2 + u_4^2)\right), \\ &f_+^2 - f_-^2 = \lambda\mu\, u_0, \end{aligned} \tag{7.12}$$

hence

$$
\begin{aligned}
f_+(u) &= \frac{1}{2}\left((\lambda + \mu\, u_0)^2 + (\mu^2 - \lambda^2)\,(u_1^2 + u_2^2)\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\left((\mu + \lambda\, u_0)^2 + (\lambda^2 - \mu^2)\,(u_3^2 + u_4^2)\right)^{\frac{1}{2}}, \\
f_-(u) &= \frac{1}{2}\left((\lambda - \mu\, u_0)^2 + (\mu^2 - \lambda^2)\,(u_1^2 + u_2^2)\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\left((\mu - \lambda\, u_0)^2 + (\lambda^2 - \mu^2)\,(u_3^2 + u_4^2)\right)^{\frac{1}{2}}.
\end{aligned}
\tag{7.13}
$$

From (7.12)–(7.13), we easily obtain the following three cases, corresponding, in the same order, to the three cases listed in Proposition 3.3:

Case 1 : a is of rank 4 — i.e. λ and μ are both non-zero — and $\lambda < \mu$. Then:

(i) $f_+(u) = f_-(u)$ if and only if u belongs to the equatorial sphere $\mathbb{S}^3$ defined by $u_0 = 0$;

(ii) $f_+(u) = 0$ if and only u belongs to the circle $\mathrm{C}_+ = \{u_0 = -\frac{\lambda}{\mu}, u_1 = u_2 = 0\}$, and we then have $f_-(u) = \frac{\lambda}{2}$;

(iii) $f_-(u) = 0$ if and only if u belongs to the circle $\mathrm{C}_- = \{u_0 = \frac{\lambda}{\mu}, u_1 = u_2 = 0\}$, and we then have $f_+(u) = \frac{\lambda}{2}$;

(iv) the 2-form $df_+^2 \wedge df_-^2$ is non-zero outside the 2-spheres $\mathrm{S}^2_+ = \{u_1 = u_2 = 0\}$ and $\mathrm{S}^2_- = \{u_3 = u_4 = 0\}$; this is because

$$
\begin{aligned}
df_+^2 \wedge df_-^2 &= \frac{\lambda\mu(\lambda^2 - \mu^2)}{2}\, du_0 \wedge (u_1\, du_1 + u_2\, du_2) \\
&= \frac{\lambda\mu(\mu^2 - \lambda^2)}{2}\, du_0 \wedge (u_3\, du_3 + u_4\, du_4),
\end{aligned}
\tag{7.14}
$$

which readily follows from (7.12).

Case 2 : a is of rank 4 and $\lambda = \mu$. Then

$$
f_+(u) = \frac{\lambda}{2}(1 + u_0), \quad f_-(u) = \frac{\lambda}{2}(1 - u_0);
\tag{7.15}
$$

in particular,

$$
f_+ + f_- = \lambda;
\tag{7.16}
$$

moreover, $f_+(u) = 0$ if and only if $u = -e_0$ and $f_-(u) = 0$ if and only if $u = e_0$.

Case 3 : a is of rank 2, i.e. $\lambda = 0$. Then, $f_+ - f_-$ is identically zero and $f_+(u) = f_-(u)$ vanishes if and only if u belongs to the circle $\mathrm{C}_0 = \{u_0 = u_1 = u_2 = 0\}$.

Remark 7.1. Consider the functions $x = \frac{f_+ + f_-}{2}, y = \frac{f_+ - f_-}{2}$ defined in Section 3, as well as the functions of one variable, A and B, appearing in Proposition 3.1. If a is of rank 4, with

$0 < \lambda < \mu$, corresponding to Case 1 in the above list, we easily infer from (7.12) that

$$
\begin{aligned}
u_0 &= \frac{4xy}{\lambda\mu},\\
u_1^2 + u_2^2 &= \frac{(\lambda^2 - 4x^2)(\lambda^2 - 4y^2)}{\lambda^2(\lambda^2 - \mu^2)},\\
u_3^2 + u_4^2 &= \frac{(\mu^2 - 4x^2)(\mu^2 - 4y^2)}{\mu^2(\mu^2 - \lambda^2)}.
\end{aligned} \tag{7.17}
$$

Since $x \geq |y|$, the above identities imply that the image of (x, y) in $\mathbb{R}^2$ is the rectangle $R := \left[\frac{\lambda}{2}, \frac{\mu}{2}\right] \times \left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right]$. A simple calculation then shows that A and B are given by

$$
A(z) = -B(z) = -\left(z^2 - \frac{\lambda^2}{4}\right)\left(z^2 - \frac{\mu^2}{4}\right). \tag{7.18}
$$

Notice that $A(x)$ and $B(y)$ are positive in the interior of R, corresponding to the open set of $\mathbb{S}^4$ where dx, dy are linearly independent, and vanish on its boundary. Also notice that the above expressions of A, B fit with the identities (4.3)–(4.4), with $\mathrm{Scal} = 12$ and $b = 0$.

Remark 7.2. By using the ambitoric Ansatz in Theorem 4.1, the above situation can easily be deformed in Case 1, where a is of rank 4, with $0 < \lambda < \mu$, and the 2-form ψ_{a} defined by (7.3) is $*$-Killing with respect to the round metric (We warmly thank Vestislav Apostolov for this suggestion.) On the open set $\mathcal{U} = \mathbb{S}^4 \setminus \left(S^2_+ \cup S^2_-\right)$, where $f_+ \neq 0$, $f_- \neq 0$ and $df_+ \wedge df_- \neq 0$, the round metric of $\mathbb{S}^4$ takes the form (4.23), where A and B are given by (7.18), $x \in \left(\frac{\lambda}{2}, \frac{\mu}{2}\right)$, $y \in \left(-\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ are determined by (7.17) and ds, dt are explicit exact 1-forms determined by the last two equations of (4.24). It can actually be shown that outside the 2-spheres S^2_+ and S^2_-, ds and dt are given by:

$$
\begin{aligned}
ds &= \frac{2}{\mu^2 - \lambda^2}\left(\lambda\, \frac{u_1 du_2 - u_2 du_1}{u_1^2 + u_2^2} - \mu\, \frac{u_3 du_4 - u_4 du_3}{u_3^2 + u_4^2}\right)\\
&= \frac{2}{\mu^2 - \lambda^2}\, d\left(\lambda \arctan\frac{u_2}{u_1} - \mu \arctan\frac{u_4}{u_3}\right),\\
dt &= \frac{8}{\mu^2 - \lambda^2}\left(-\frac{1}{\lambda}\, \frac{u_1 du_2 - u_2 du_1}{u_1^2 + u_2^2} + \frac{1}{\mu}\, \frac{u_3 du_4 - u_4 du_3}{u_3^2 + u_4^2}\right)\\
&= \frac{8}{\mu^2 - \lambda^2}\, d\left(-\frac{1}{\lambda} \arctan\frac{u_2}{u_1} + \frac{1}{\mu} \arctan\frac{u_4}{u_3}\right).
\end{aligned} \tag{7.19}
$$

Moreover, ψ_{a} is given by (4.27) with respect to these coordinates.

Consider now a small perturbation $\tilde{A}$, $\tilde{B}$ of the functions A and B such that $\tilde{A}(x) = A(x)$ near $x = \frac{\lambda}{2}$ and $x = \frac{\mu}{2}$ and $\tilde{B}(y) = B(y)$ near $y = \pm\frac{\lambda}{2}$. If the perturbation is small enough, the expression analogue to (4.23)

$$
\begin{aligned}
\tilde{g} := (x^2 - y^2)&\left(\frac{dx \otimes dx}{\tilde{A}(x)} + \frac{dy \otimes dy}{\tilde{B}(y)}\right)\\
&+ \frac{\tilde{A}(x)}{(x^2 - y^2)}\,(ds + y^2\, dt) \otimes (ds + y^2\, dt)\\
&+ \frac{\tilde{B}(y)}{(x^2 - y^2)}\,(ds + x^2\, dt) \otimes (ds + x^2\, dt)
\end{aligned} \tag{7.20}
$$

is still positive definite so defines a Riemannian metric on $\mathcal{U}$, which coincides with the canonical metric on an open neighbourhood of $\mathbb{S}^4 \setminus \mathcal{U} = S^2_+ \cup S^2_-$, and thus has a smooth extension to $\mathbb{S}^4$ which we still call $\tilde{g}$. Since the expression (4.27) of the $*$-Killing form in the Ansatz of Section 4 does not depend on A and B, the 2-form ψ_{a} is still $*$-Killing with respect to the new metric $\tilde{g}$. We thus get an infinite-dimensional family (depending on two functions of one variable) of Riemannian metrics on S^4 which all carry *the same* non-parallel $*$-Killing form.

8. Example: complex ruled surfaces

In general, a (geometric) *complex ruled surface* is a compact, connected, complex manifold of the form $M = \mathbb{P}(E)$, where E denotes a rank 2 holomorphic vector bundle over some (compact, connected) Riemann surface, Σ, and $\mathbb{P}(E)$ is then the corresponding projective line bundle, i.e. the holomorphic bundle over Σ, whose fiber at each point y of Σ is the complex projective line $\mathbb{P}(E_y)$, where E_y denotes the fiber of E at y. A complex ruled surface is said to be *of genus* $\mathbf{g}$ if Σ is of genus $\mathbf{g}$.

In this section, we restrict our attention to complex ruled surfaces $\mathbb{P}(E)$ as above, when $E = L \oplus \mathbb{C}$ is the Whitney sum of some holomorphic line bundle, L, over Σ and of the trivial complex line bundle $\Sigma \times \mathbb{C}$, here simply denoted $\mathbb{C}$: M is then the *compactification* of the total space of L obtained by adding the *point at infinity* $[L_y] := \mathbb{P}(L_y \oplus \{0\})$ to each fiber of M over y. The union of the points at infinity is a divisor of M, denoted by Σ_∞, whereas the (image of) the zero section of L, viewed as a divisor of M, is denoted Σ_0; both Σ_0 and Σ_∞ are identified with Σ by the natural projection, π, from M to Σ. The open set $M \setminus (\Sigma_0 \cup \Sigma_\infty)$, denoted M^0, is naturally identified with $L \setminus \Sigma_0$. We moreover assume that the degree, $\mathrm{d}(L)$, of L is *negative* and we set: $\mathrm{d}(L) = -k$, where k is a positive integer.

Complex ruled surfaces of this form will be called *Hirzebruch-like ruled surfaces.* When $\mathbf{g} = 0$, these are exactly those complex ruled surfaces introduced by F. Hirzebruch in [7]. When $\mathbf{g} \geq 2$, they were named *pseudo-Hirzebruch* in [14].

In general, the Kähler cone of a complex ruled surface $\mathbb{P}(E)$ was described by A. Fujiki in [6]. In the special case considered in this section, when $M = \mathbb{P}(L \oplus \mathbb{C})$ is a Hirzebruch-like ruled surface, if $[\Sigma_0]$, $[\Sigma_\infty]$ and $[F]$ denote the Poincaré duals of the (homology class of) Σ_0, Σ_∞ and of any fiber F of π in $\mathrm{H}^2(M, \mathbb{Z})$, the latter is freely generated by $[\Sigma_0]$ and $[F]$ or by $[\Sigma_\infty]$ and $[F]$, with $[\Sigma_0] = [\Sigma_\infty] - k\,[F]$, and the Kähler cone is the set of those elements, Ω_{a_0, a_∞}, of $\mathrm{H}(M, \mathbb{R})$ which are of the form $\Omega_{a_0, a_\infty} = 2\pi\big(-a_0\,[\Sigma_0] + a_\infty\,[\Sigma_\infty]\big)$, for any two real numbers a_0, a_∞ such that $0 < a_0 < a_\infty$.

We assume that Σ comes equipped with a Kähler metric $(g_\Sigma, \omega_\Sigma)$ polarized by L, in the sense that L is endowed with a Hermitian (fiberwise) inner product, h, in such a way that the curvature, R^∇, of the associated Chern connection, ∇, is related to the Kähler form ω_Σ by $\mathrm{R}^\nabla = i\,\omega$; in particular, $[\omega_\Sigma] = 2\pi\,c_1(L^*)$, where $[\omega_\Sigma]$ denotes the de Rham class of ω_Σ, L^* the dual line bundle to L and $c_1(L^*)$ the (de Rham) Chern class of L^*. The natural action of $\mathbb{C}^*$ extends to a holomorphic $\mathbb{C}^*$-action on M, trivial on Σ_0 and Σ_∞; we denote by K the generator of the restriction of this action on $S^1 \subset \mathbb{C}^*$. On $M^0 = L \setminus \Sigma_0$, we denote by t the function defined by

$$t = \log r, \tag{8.1}$$

where r stands for the distance to the origin in each fiber of L determined by h; on M^0, we then have

$$dd^c t = \pi^* \omega_\Sigma, \qquad d^c t(K) = 1 \tag{8.2}$$

(beware: the function t defined by (8.1) has nothing to do with the local coordinate t appearing in Section 4). Any (smooth) function $F = F(t)$ of t will be regarded as function defined on M^0, which is evidently K-invariant; moreover:

(1) $F = F(t)$ smoothly extends to Σ_0 if and only if $F(t) = \Phi_+(e^{2t})$ near $t = -\infty$, for some smooth function Φ_+ defined on some neighbourhood of 0 in $\mathbb{R}^{\geq 0}$, and
(2) $F = F(t)$ smoothly extends to Σ_∞ if and only if $F(t) = \Phi_-(e^{-2t})$ near $t = \infty$, for some smooth function Φ_- defined on some neighbourhood of 0 in $\mathbb{R}^{\geq 0}$, cf. *e.g.* [14], [1, Section 3.3].

For any (smooth) real function $\varphi = \varphi(t)$, denote by ω_φ the real, J-invariant 2-form defined on M^0 by

$$\omega_\varphi = \varphi\, dd^c t + \varphi'\, dt \wedge d^c t, \tag{8.3}$$

where φ' denotes the derivative of φ with respect to t. Then, ω_φ is a Kähler form on M^0, with respect to the natural complex structure $J = J_+$, of M, if and only if φ is positive and increasing as a function of t; moreover, ω_ϕ extends to a smooth Kähler form on M, in the Kähler class Ω_{a_0,a_∞}, if and only if φ satisfies the above asymptotic conditions (1)–(2), with $\Phi_+(0) = a_0 > 0$, $\Phi'_+(0) > 0$, $\Phi_-(0) = a_\infty > 0$, $\Phi'_-(0) < 0$. Kähler forms of this form on M, as well as the corresponding Kähler metrics

$$g_\varphi = \varphi\, \pi^* g_\Sigma + \varphi'\, (dt \otimes dt + d^c t \otimes d^c t), \tag{8.4}$$

are called *admissible*.

Denote by J_- the complex structure, first defined on the total space of L by keeping J on the horizontal distribution determined by the Chern connection and by substituting $-J$ on the fibers, then smoothly extended to M. The new complex structure induces the opposite orientation, hence commutes with $J_+ = J$.

Any admissible Kähler form ω_φ is both J_+- and J_--invariant, as well as the associated 2-form $\tilde{\omega}_\varphi$ defined by

$$\tilde{\omega}_\varphi := \frac{1}{\varphi}\, dd^c t - \frac{\varphi'}{\varphi^2}\, dt \wedge d^c t, \tag{8.5}$$

which is moreover Kähler with respect to J_-, with metric

$$\tilde{g}_\varphi = \frac{1}{\varphi^2}\, g_\varphi. \tag{8.6}$$

We thus obtain an ambikähler structure of Calabi-type, as defined in Section 5, with $f = \frac{1}{\varphi}$ and $\tau(K) = -K$. According to Theorem 5.1 and Remark 5.1, for any k in $\mathbb{R} \setminus \{0\}$, the metric $g^{(k)}$ defined, outside the locus $\{1 + k\,\varphi = 0\}$, by

$$g^{(k)}_\varphi = \frac{1}{(1 + k\,\varphi)^2}\, g_\varphi, \tag{8.7}$$

there admits a non-parallel $*$-Killing 2-form $\psi^{(k)}_\varphi$, namely

$$\begin{aligned} \psi^{(k)}_\varphi &= \frac{1}{(1 + k\,\varphi)^3}\, \omega_\phi + \frac{k\,\varphi^3}{(1 + k\,\varphi)^3}\, \tilde{\omega}_\varphi \\ &= \frac{\varphi}{(1 + k\,\varphi)^2}\, dd^c t + \frac{(1 - k\,\varphi)\varphi'}{(1 + k\,\varphi)^3}\, dt \wedge d^c t. \end{aligned} \tag{8.8}$$

Notice that the pair $(g_\varphi^{(k)}, \psi_\varphi^{(k)})$ smoothly extends to M for any $k \in \mathbb{R} \setminus [-\frac{1}{a_0}, -\frac{1}{a_\infty}]$, including $k = 0$ for which we simply get the Kähler pair (g_φ, ω_+).

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