GENERALIZED KILLING SPINORS ON EINSTEIN MANIFOLDS

ANDREI MOROIANU, UWE SEMMELMANN

ABSTRACT. We study generalized Killing spinors on compact Einstein manifolds with positive scalar curvature. This problem is related to the existence of compact Einstein hypersurfaces in manifolds with parallel spinors, or equivalently, in Riemannian products of flat spaces, Calabi-Yau, hyperkähler, G_2 and Spin(7) manifolds.

1. INTRODUCTION

A *generalized Killing spinor* on a spin manifold (M, g) is a non-zero spinor $\Psi \in \Gamma(\Sigma M)$ satisfying for all vector fields X the equation

$$(1) \qquad \nabla_X \Psi = A(X) \cdot \Psi,$$

where $A \in \Gamma(\mathrm{End}^+(\mathrm{T}M))$ is some symmetric endomorphism field [4, 16, 17]. If A is a non-zero multiple of the identity, Ψ is called a Killing spinor [3, 5].

The interest in generalized Killing spinors is due to the fact that they arise in a natural way as restrictions of parallel spinors to hypersurfaces. More precisely, if (M^n, g) is a hypersurface of $(\mathcal{Z}^{n+1}, g^{\mathcal{Z}})$ and Φ is parallel spinor on $\mathcal{Z}$, then its restriction to M is a generalized Killing spinor with respect to the symmetric tensor A equal to half the second fundamental form of M, cf. [4, 15]. Conversely, if Ψ is a generalized Killing spinor on (M, g) with respect to A, then there exists a metric on an open subset $\mathcal{Z}$ of $M \times \mathbb{R}$ whose restriction to $M \times \{0\}$ is g and a parallel spinor on $\mathcal{Z}$ whose restriction to $M \times \{0\}$ is Ψ in the following cases:

(1) If A is a constant multiple of the identity, i.e. Ψ is a Killing spinor [3];
(2) Slightly more generally, if A is parallel [24];
(3) Even more generally, if A is a Codazzi tensor [4];
(4) In the generic case, under the sole assumption that A and g are analytic [2].

The common feature of the first three cases is that the ambient metric can be constructed explicitly. In the last case, the existence of the ambient metric is given by the Cauchy-Kovalevskaya theorem, and this explains the analyticity assumption. It is actually shown in [2] that this assumption cannot be dropped.

Our main objective in this article is to study generalized Killing spinors on Einstein manifolds. In some sense, this problem can be seen as an analogue of the Goldberg conjecture, which states that an almost Kähler compact Einstein manifold with positive scalar curvature is Kähler (this conjecture was proved by Sekigawa [27]).

In order to understand this analogy one needs to express both problems in terms of G-structures. An almost Hermitian manifold is equivalent to a manifold with $\mathrm{U}(m)$-structure. The intrinsic torsion of such a structure has 4 irreducible components (for $m \geq 3$) and being almost Kähler is equivalent to the vanishing of 3 components out of 4. The Goldberg conjecture simply says that if the manifold is compact and Einstein with positive scalar curvature, then the fourth component has to vanish too.

On the other hand, in small dimensions $n \leq 8$ every real (half-) spinor is *pure*, in the sense that the spin group acts transitively on the unit sphere of the real spin representation Σ_n (or $\Sigma_n^\pm$ for $n = 4$ and $n = 8$). Correspondingly, a non-vanishing (half-) spinor induces a G-structure on M where G (the stabilizer of a vector of the spin representation) equals $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$, $\mathrm{G}_2 \subset \mathrm{SO}(7)$, $\mathrm{SU}(3) \subset \mathrm{SO}(6)$, $\mathrm{SU}(2) \subset \mathrm{SO}(5)$, $\mathrm{SU}(2) \subset \mathrm{SO}(4)$ and $\{1\} \subset \mathrm{SO}(3)$ for $8 \geq n \geq 3$ respectively. Being a generalized Killing spinor is equivalent to the vanishing of certain components of the intrinsic torsion of this G-structure. More precisely, it is well known that the structure reduction defined by a generalized Killing spinor is *co-calibrated* G_2 (cf. [10, 12]) for $n = 7$, *half-flat* (cf. [8, 19]) for $n = 6$ and *hypo* (cf. [10]) for $n = 5$. Note that for $n = 4$ or $n = 8$ a generalized Killing spinor Ψ is never chiral (unless it is parallel), and each chiral part Ψ^+ and Ψ^- defines a structure reduction along the open set where it is non-vanishing. The analogue of the Goldberg conjecture in this setting (which turns out to be false in general, see below) would be that a generalized Killing spinor on a compact Einstein manifold with positive scalar curvature is necessarily Killing.

Note also that in this context, the embedding result in [2] for manifolds with generalized Killing spinors can be seen as a generalization to arbitrary dimensions of similar results by Bryant, Conti, Hitchin and Salamon in small dimensions, cf. [6, 10, 11, 19].

Surprisingly, it turns out that the problem of finding all generalized Killing spinors on a given spin manifold is out of reach at the present state of our knowledge. In dimension 2 already, the fact that every generalized Killing spinor on $\mathbb{S}^2$ is a Killing spinor, is non-trivial and follows from Liebmann's theorem [21] (see Section 4.1). Moreover, on the simplest Riemannian manifold of dimension 3, the round 3-dimensional sphere, there is no classification available. However, one can show that $\mathbb{S}^3$ carries generalized Killing spinors which are not Killing spinors (Section 4.2 below).

In dimension 4, the analogue of the Goldberg conjecture holds. In Theorem 4.8 below we show that every generalized Killing spinor on a compact 4-dimensional Einstein manifold with positive scalar curvature is Killing (and thus the manifold is isometric to $\mathbb{S}^4$, cf. [3, 5, 18]).

A similar result holds in dimension 5 for the round sphere (cf. Theorem 4.9). It is presently unknown whether other 5-dimensional Einstein manifolds, e.g. the Riemannian product $\mathbb{S}^2(\frac{1}{\sqrt{2}}) \times \mathbb{S}^3$, carry generalized Killing spinors which are not Killing.

In dimensions 6 and 7 there are several examples of Einstein manifolds carrying generalized Killing spinors that are not Killing. These examples correspond to half-flat structures on the Riemannian product $\mathbb{S}^3 \times \mathbb{S}^3$ constructed by Schulte-Hengesbach [26], who actually classified all left-invariant half-flat structures on $S^3 \times S^3$ (see also Madsen and Salamon [23]) and to co-calibrated G_2-structures on any 7-dimensional 3-Sasakian manifold, including the sphere $\mathbb{S}^7$, constructed by Agricola and Friedrich [1] (see Section 4.5 below).

Finally, no examples of generalized Killing spinors on positive Einstein manifolds in dimension $n \geq 8$ are known, other than Killing spinors on spheres, Einstein-Sasakian and 3-Sasakian manifolds [3]. We believe however that it should be possible to construct examples, at least in the 3-Sasakian case, using methods similar to those in [1].

2. Preliminaries

For basic definitions and results on spin manifolds we refer to [5] and [22]. Let (M^n, g) be an n-dimensional Riemannian spin manifold with Levi-Civita connection ∇. The real spinor bundle ΣM is endowed with a connection, also denoted by ∇, and a Euclidean product $\langle ., . \rangle$ which is parallel with respect to ∇:

$$\partial_X \langle \Psi, \Phi \rangle = \langle \nabla_X \Psi, \Phi \rangle + \langle \Psi, \nabla_X \Phi \rangle, \qquad \forall \, X \in \mathrm{T}M, \ \Psi, \Phi \in \Gamma(\Sigma M).$$

The Clifford product with vectors is parallel with respect to ∇ and skew-symmetric with respect to $\langle ., . \rangle$ ([22], Cor. 5.17), whence

$$(2) \qquad \langle X \cdot Y \cdot \Psi, \Psi \rangle = -g(X, Y)\langle \Psi, \Psi \rangle, \qquad \forall \, X, Y \in \mathrm{T}M, \ \Psi \in \Sigma M.$$

The Riemannian curvature $\mathcal{R}$ and the curvature $\mathrm{R}^{\Sigma M}$ of the spinor bundle are related by

$$(3) \qquad \mathrm{R}^{\Sigma M}_{X,Y} \Psi = \tfrac{1}{2} \mathcal{R}(X \wedge Y) \cdot \Psi \qquad \forall \, X, Y \in \mathrm{T}M, \ \Psi \in \Sigma M,$$

where $\mathcal{R} : \Lambda^2 M \to \Lambda^2 M$ denotes the curvature operator defined by

$$g(\mathcal{R}(X \wedge Y), U \wedge V) = g(\mathrm{R}_{X,Y} U, V).$$

(In particular, the curvature operator on the standard sphere acts on 2-forms by minus the identity). Recall that the Clifford multiplication with 2-forms is defined via the equation

$$(4) \qquad (X \wedge Y) \cdot \Psi = X \cdot Y \cdot \Psi + g(X, Y)\Psi.$$

Throughout this article we will identify 1-forms and bilinear forms with vectors and endomorphisms respectively, by the help of the Riemannian metric. In particular it makes sense

to speak about (skew)-symmetric endomorphism fields. The corresponding spaces will be denoted by

$$\mathrm{End}^{\pm}(\mathrm{T}M)_p := \{A \in \mathrm{End}(\mathrm{T}M)_p, \mid g(AX,Y) = \pm g(X,AY) \; \forall \; X,Y \in \mathrm{T}_pM\}.$$

If $A \in \Gamma(\mathrm{End}^+(\mathrm{T}M))$ and $\{e_i\}$ is a local orthonormal frame, then

$$(5) \qquad \qquad \sum_{i=1}^{n} e_i \cdot A(e_i) \cdot \Psi = -\mathrm{tr}(A)\, \Psi.$$

Applying the first Bianchi identity the curvature relation (3) yields the well-known formula (see also [5]):

$$(6) \qquad \qquad \mathrm{Ric}(X) \cdot \Psi \; = \; -2 \sum_{i=1}^{n} e_i \cdot \mathrm{R}^{\Sigma M}_{X,e_i}\, \Psi$$

which together with (5) yields:

$$(7) \qquad \qquad \mathrm{scal}\,\Psi \; = \; -\sum_{i=1}^{n} e_i \cdot \mathrm{Ric}(e_i) \cdot \Psi.$$

3. Generalized Killing spinors

Consider now a generalized Killing spinor Ψ on (M,g), i.e. a spinor satisfying the equation $\nabla_X \Psi = A(X) \cdot \Psi$ for some symmetric endomorphism field A. Taking the scalar product with Ψ in this equation shows that the norm of Ψ is constant. By rescaling, we may assume that $|\Psi|^2 = 1$. Using (1) together with (5) shows that

$$\mathcal{D}\Psi = -\mathrm{tr}(A)\Psi,$$

where $\mathcal{D}$ denotes the Dirac operator. We thus get

$$\mathcal{D}^2\Psi = \mathrm{tr}^2(A)\Psi - d\,\mathrm{tr}(A) \cdot \Psi.$$

Moreover, taking a further covariant derivative in (1) yields

$$\nabla^*\nabla\Psi = -\sum_{i=1}^{n}(\nabla_{e_i}A)e_i \cdot \Psi - A(e_i) \cdot A(e_i) \cdot \Psi = -\sum_{i=1}^{n}(\nabla_{e_i}A)e_i \cdot \Psi + \mathrm{tr}(A^2)\Psi,$$

so the Lichnerowicz formula implies

$$(8) \qquad \tfrac{1}{4}\mathrm{scal}\Psi = \mathcal{D}^2\Psi - \nabla^*\nabla\Psi = \mathrm{tr}^2(A)\Psi - d\mathrm{tr}(A) \cdot \Psi + \sum_{i=1}^{n}(\nabla_{e_i}A)e_i \cdot \Psi - \mathrm{tr}(A^2)\Psi.$$

Let us extend the action of A to 2-forms by $A(X \wedge Y) = A(X) \wedge A(Y)$. Using (4), the generalized Killing equation (1), and the curvature relation (3), it follows that

$$(9) \qquad \tfrac{1}{2}\mathcal{R}(X \wedge Y) \cdot \Psi \; = \; \mathrm{R}^{\Sigma M}_{X,Y}\, \Psi \; = \; [(\nabla_X A)Y - (\nabla_Y A)X] \cdot \Psi \; - \; 2\,A(X \wedge Y) \cdot \Psi.$$

We introduce the notation
$$a := \operatorname{tr}(A).$$

Lemma 3.1. *If $\delta A := -\sum_{i=1}^{n}(\nabla_{e_i}A)e_i$ denotes the divergence of A, then the following relations hold.*

$$(10) \qquad \sum_{i=1}^{n} e_i \wedge (\nabla_{e_i}A)X \cdot \Psi \; = \; [\tfrac{1}{2}\operatorname{Ric}(X) + 2A^2(X) - 2aA(X)] \cdot \Psi,$$

$$(11) \qquad\qquad\qquad 0 \; = \; \delta A \; + \; da,$$

$$(12) \qquad\qquad\qquad \operatorname{scal} \; = \; 4a^2 \; - \; 4\operatorname{tr}A^2.$$

Proof. From (4), (5), (6) and (9) we get for every tangent vector X:

$$-\tfrac{1}{2}\operatorname{Ric}(X) \cdot \Psi \; = \; \sum_{i=1}^{n} e_i \cdot \operatorname{R}^{\Sigma M}_{X,e_i} \Psi = \sum_{i=1}^{n} e_i \cdot [(\nabla_X A)e_i \; - \; (\nabla_{e_i}A)X] \cdot \Psi$$

$$-2\sum_{i=1}^{n} e_i \cdot [A(X) \cdot A(e_i) + g(A(X), A(e_i))] \cdot \Psi$$

$$= \; -\operatorname{tr}(\nabla_X A)\Psi - \sum_{i=1}^{n} e_i \cdot (\nabla_{e_i}A)X \cdot \Psi$$

$$+ 4A^2(X) \cdot \Psi - 2A(X) \cdot a\Psi - 2A^2(X) \cdot \Psi$$

$$= \; [-da(X) - \delta A(X)]\Psi - \sum_{i=1}^{n} e_i \wedge (\nabla_{e_i}A)X \cdot \Psi + [2A^2(X) - 2aA(X)] \cdot \Psi.$$

Taking the scalar product with Ψ in this equation and using the fact that the Clifford product with 1- and 2-forms is skew-symmetric yields (11), and reinjecting in the same equation gives (10). Finally, (12) follows from (8) and (11). $\qquad\square$

In order to rewrite the right hand side of (10) we introduce the symmetric endomorphism
$$B \; := \; A^2 \; - \; aA \; + \; \tfrac{1}{4}\operatorname{Ric}.$$

Note that B is traceless because of (12) and B vanishes if A is a multiple of the identity. We introduce the notation

$$T^Z \; = \; \sum_{i=1}^{n} e_i \wedge (\nabla_{e_i}A)Z \quad \text{and} \quad T = \sum_{i=1}^{n} T^{e_i} \otimes e_i.$$

Then T^Z is a 2-form on M and, considering A as a 1-form on M with values in TM, we have

$$(13) \qquad T^Z(X,Y) = g((\nabla_X A)Y \; - \; (\nabla_Y A)X, Z) = g((d^{\nabla}A)(X,Y), Z).$$

Recall that a symmetric endomorphism field $A \in \Gamma(\operatorname{End}^+(TM))$ is called *Codazzi tensor* if $d^{\nabla}A = 0$ or, equivalently, if $(\nabla_X A)Y = (\nabla_Y A)X$ for all tangent vectors X, Y.

The tensor $T = d^{\nabla} A$ can also be considered as a map $T : \Lambda^2 M \to \mathrm{T}M$ by defining

$$g(T(X \wedge Y), Z) = T^Z(X, Y).$$

Let σ be an arbitrary 2-form and Z any vector field on M. Then (9) and (10) can be rewritten as

$$(14) \qquad T(\sigma) \cdot \Psi \quad = \quad \left[\tfrac{1}{2} \mathcal{R}(\sigma) \; + \; 2\, A(\sigma) \right] \cdot \Psi, \qquad\qquad \forall\, \sigma \in \Lambda^2 M$$

$$(15) \qquad T^Z \cdot \Psi \quad = \quad 2\, B(Z) \cdot \Psi \qquad\qquad\qquad \forall\, Z \in \mathrm{T}M.$$

As a corollary of the above formulas we will now generalize a classical rigidity result by Fialkow [13] which states that every Einstein hypersurface with positive scalar curvature in $\mathbb{R}^{n+1}$ for $n \geq 3$ is umbilic and locally isometric to the sphere $\mathbb{S}^n$.

Theorem 3.2. *Let (M, g) be an Einstein hypersurface with positive scalar curvature in a spin manifold $\mathcal{Z}$ with parallel spinors. Assume moreover that the Weingarten tensor W of the embedding is a Codazzi tensor, i.e. $d^{\nabla} W = 0$ on M (this condition is automatically satisfied for $\mathcal{Z} = \mathbb{R}^{n+1}$ because of the Codazzi equation). Then $\mathcal{Z}$ is locally isometric to the Riemannian cone over (M, g) and M is umbilic in $\mathcal{Z}$.*

Proof. As explained in the introduction, M carries a generalized Killing spinor Ψ with associated tensor $A := \tfrac{1}{2} W$. Since A is Codazzi, (15) implies that the tensor B vanishes, so $A^2 - aA + \tfrac{\lambda}{4} = 0$, where λ denotes the Einstein constant. We claim that A is a multiple of the identity. If this were not the case, then on some open subset of M the symmetric tensor A would have exactly two distinct eigenvalues α and β with constant multiplicities $p \geq 1$ and $q \geq 1$. Of course, α and β are the roots of the polynomial $X^2 - aX + \tfrac{\lambda}{4}$. We thus have $\alpha + \beta = a$. On the other hand, $a = \mathrm{tr}A = p\alpha + q\beta$ so subtracting these last two equations yields $(p-1)\alpha + (q-1)\beta = 0$, which is impossible since $p + q = n \geq 3$ and $\alpha\beta = \tfrac{\lambda}{4} > 0$. This proves the claim, so $A = \tfrac{a}{n}\mathrm{id}$ and Ψ satisfies the equation $\nabla_X \psi = \tfrac{a}{n} X \cdot \Psi$. It is well known that a has then to be constant, so Ψ is a real Killing spinor.

From Bär's classification of manifolds with Killing spinors [3] and from the uniqueness part of the embedding theorem in [2], it follows that $\mathcal{Z}$ is locally isometric to the Riemannian cone over M. $\qquad\square$

4. Generalized Killing spinors on low-dimensional Einstein manifolds

4.1. **The case of dimension** 2. Any 2-dimensional Einstein spin manifold of positive scalar curvature is homothetic to the standard sphere $\mathbb{S}^2$. Using classical rigidity results it it easy to show that every generalized Killing spinor on $\mathbb{S}^2$ is a Killing spinor.

Indeed, if Ψ is a spinor satisfying (1) then every point of $\mathbb{S}^2$ has a neighborhood which embeds isometrically in $\mathbb{R}^3$ with second fundamental form $2A$. In particular, the determinant

of A is constant equal to $\frac{1}{4}$ by Gauss' Theorema Egregium. If A is not equal to $-\frac{\mathrm{id}}{2}$, then there exists a point in $\mathbb{S}^2$ where one of its eigenvalues attains its maximum which is strictly larger than $\frac{1}{2}$ and where the other eigenvalue attains its minimum, which is strictly smaller than $\frac{1}{2}$.

On the other hand, Liebmann's Theorem [21] states that if at a non-umbilic point of a surface S in $\mathbb{R}^3$ one of the principal curvatures has a local maximum and the other one has a local minimum, then the Gaussian curvature of S is non-positive at that point. This contradiction shows that A has to be scalar.

4.2. **The case of dimension** 3. Any 3-dimensional Einstein manifold of positive scalar curvature is locally homothetic to the standard sphere $\mathbb{S}^3$. We will show that $\mathbb{S}^3$ carries generalized Killing spinors which are not Killing.

Recall that, by convention, in dimension $4m + 3$ the Clifford action of the volume form on the spin bundle is the identity ([22], p.34). In dimension 3 this readily implies

$$(16) \qquad \omega \cdot \Psi = - * \omega \cdot \Psi, \qquad \forall\, \omega \in \Lambda^2 \mathbb{S}^3,\ \Psi \in \Sigma \mathbb{S}^3.$$

The standard sphere $\mathbb{S}^3$ can be identified with the Lie group $\mathrm{SU}(2)$, endowed with the bi-invariant metric defined by $-\frac{1}{4}$ times the Killing form. As such, $\mathbb{S}^3$ carries an orthonormal frame of left-invariant Killing vector fields $\{\xi_1, \xi_2, \xi_3\}$ satisfying

$$(17) \qquad \nabla_{\xi_1}\xi_2 = -\nabla_{\xi_2}\xi_1 = \xi_3, \qquad \nabla_{\xi_2}\xi_3 = -\nabla_{\xi_3}\xi_2 = \xi_1, \qquad \nabla_{\xi_3}\xi_1 = -\nabla_{\xi_1}\xi_3 = \xi_2.$$

By choosing the orientation defined by the frame $\{\xi_1, \xi_2, \xi_3\}$, one can express the above relations in a more concise way by saying that any left-invariant vector field ξ on $\mathbb{S}^3 \simeq \mathrm{SU}(2)$ satisfies

$$(18) \qquad \nabla_X \xi = *(X \wedge \xi), \qquad \forall\, X \in \mathrm{T}\mathbb{S}^3.$$

Let Φ be a Killing spinor on $\mathbb{S}^3$ with Killing constant $\frac{1}{2}$:

$$(19) \qquad \nabla_X \Phi = \tfrac{1}{2} X \cdot \Phi, \qquad \forall\, X \in \mathrm{T}\mathbb{S}^3,$$

and let ξ be a unit left-invariant Killing vector field. Using (4), (16) and (18) we compute the covariant derivative of the spinor $\Psi := \xi \cdot \Phi$:

$$
\begin{aligned}
\nabla_X \Psi &= (\nabla_X \xi) \cdot \Phi + \tfrac{1}{2} \xi \cdot X \cdot \Phi \\
&= -(X \wedge \xi) \cdot \Phi - \tfrac{1}{2} X \cdot \xi \cdot \Phi - g(X, \xi)\Phi \\
&= -X \cdot \xi \cdot \Phi - g(X, \xi)\Phi - \tfrac{1}{2} X \cdot \xi \cdot \Phi - g(X, \xi)\Phi \\
&= -\tfrac{3}{2} X \cdot \Psi + 2 g(X, \xi) \xi \cdot \Psi.
\end{aligned}
$$

This shows that Ψ is a generalized Killing spinor corresponding to the symmetric endomorphism field

$$X \mapsto A(X) := -\tfrac{3}{2} X + 2 g(X, \xi)\xi.$$

As a matter of fact, note that A is not a Codazzi tensor.

It is worth noting that taking the quotient by the antipodal map, the spinor Ψ above defines a generalized Killing spinor on the projective space $\mathbb{R}\mathrm{P}^3$ for the right choice of the spin structure (the one defined by the trivial lift $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(3)$).

4.3. **The case of dimension** 4. We assume that Ψ is a generalized Killing spinor on a compact oriented 4-dimensional Einstein manifold (M, g) of positive scalar curvature. We thus have $\mathrm{Ric} = \lambda g$, with $\lambda = \frac{\mathrm{scal}}{4} > 0$, and like before we may assume that Ψ is scaled to have unit length. Then (12) reads $a^2 - \mathrm{tr}A^2 = \lambda$.

In dimension 4 the spin representation splits as $\Sigma = \Sigma^+ \oplus \Sigma^-$, where $\Sigma^\pm$ are the ± 1-eigenspaces of the multiplication with the volume element and are interchanged by Clifford multiplication with vectors. Correspondingly, Ψ splits as $\Psi = \Psi^+ + \Psi^-$ with

$$(20) \qquad \nabla_X \Psi^\pm = A(X) \cdot \Psi^\mp.$$

Let M_0 denote the open set $p \in M$ with $\Psi_p^- \neq 0$. We claim that M_0 is dense. Indeed, if U were a non-empty open subset of $M \setminus M_0$, then (20) yields $A(X) \cdot \Psi^+ = 0$ for all $X \in TU$, so $A|_U = 0$. By (20) again, Ψ^+ is parallel on U, so the Ricci tensor vanishes on U, contradicting the fact that $\mathrm{scal} > 0$.

Let $h := |\Psi^-|^2$ be the length function of Ψ^- and let η be the vector field on M given by

$$(21) \qquad g(\eta, X) = \langle X \cdot \Psi^+, \Psi^- \rangle, \qquad \forall\, X \in TM.$$

Recall that $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product on the real spinor bundle. For every $p \in M_0$ the injective map $X \in T_pM \mapsto X \cdot \Psi^- \in \Sigma_p^+ M$ is bijective since $\dim T_pM = \dim \Sigma_p^+ M$. Let ξ denote the vector field on M_0 defined by $\Psi^+ = \xi \cdot \Psi^-$. Using (2) we get $g(\eta, X) = -h\, g(X, \xi)$. Moreover, since $1 = |\Psi|^2 = |\Psi^+|^2 + |\Psi^-|^2 = |\Psi^-|^2(1 + |\xi|^2)$ we infer $|\xi|^2 = \frac{1}{h} - 1$ and thus

$$(22) \qquad |\eta|^2 = h - h^2.$$

Lemma 4.1. *(i)* $\quad dh = 2\, A(\eta)$
(ii) $\quad \nabla_X \eta = (1 - 2h)\, A(X)$
(iii) $\quad d\eta = 0, \quad \delta\eta = -(1 - 2h)\, a$

Proof. (i) Using (20) we compute for every $X \in TM$:

$$d|\Psi^-|^2(X) = 2\langle \nabla_X \Psi^-, \Psi^- \rangle = 2\langle A(X) \cdot \Psi^+, \Psi^- \rangle = 2\eta(A(X)) = 2g(A(\eta), X).$$

(ii) Taking the covariant derivative in the direction of Y in (21), assuming that X is parallel at some point and using (2) and (20) yields

$$\begin{aligned}
g(\nabla_Y \eta, X) &= \langle X \cdot A(Y) \cdot \Psi^-, \Psi^- \rangle + \langle X \cdot \Psi^+, A(Y) \cdot \Psi^+ \rangle \\
&= -g(X, A(Y))\, |\Psi^-|^2 + g(X, A(Y))\, |\Psi^+|^2 \\
&= (1 - 2h) g(A(Y), X).
\end{aligned}$$

(iii) Follows immediately from *(ii)*. $\qquad\square$

Corollary 4.2. $\quad \Delta h \ = \ -2\, da(\eta) \ - \ 2\, (a^2 - \lambda)\, (1 - 2h)$

Proof. Straightforward calculation using (11):

$$\begin{aligned}
\Delta h \ = \ \delta dh \ &= \ 2\, \delta(A(\eta)) \ = \ -2 \sum_{i=1}^{n} g(e_i, (\nabla_{e_i} A)\eta + A(\nabla_{e_i}\eta)) \\
&= \ 2g(\delta A, \eta) \ - \ 2 \sum_{i=1}^{n} g(\nabla_{e_i}\eta, A(e_i)) \\
&= \ -2da(\eta) \ - \ 2(1 - 2h)\mathrm{tr}(A^2) \ = \ -2da(\eta) \ - \ 2(1 - 2h)(a^2 - \lambda).
\end{aligned}$$

$\qquad\square$

We denote by M_1 the set of points where Ψ^+ is non-vanishing. Like before, M_1 is dense, so $M' := M_0 \cap M_1$ is dense, too.

It is well known that $\Lambda_{\pm}^2 M$ acts trivially (by Clifford multiplication) on $\Sigma^{\pm} M$. Moreover, the map $\omega^+ \mapsto \omega^+ \cdot \Psi^-$ is a bijection from the space of self-dual 2-forms $\Lambda_+^2 M'$ onto the orthogonal complement $(\Psi^-)^{\perp}$ in $\Sigma^- M'$, and similarly the map $\omega^- \mapsto \omega^- \cdot \Psi^+$ is a bijection from the space of anti-self-dual 2-forms $\Lambda_-^2 M'$ onto the orthogonal complement $(\Psi^+)^{\perp}$ in $\Sigma^+ M'$. This has the following important consequence.

Lemma 4.3. *If ω is a 2-form and X is a vector field on M' such that $\omega \cdot \Psi = X \cdot \Psi$ holds, then*

$$\omega \ = \ (X \wedge \xi)^+ \ - \ \tfrac{1}{|\xi|^2}\, (X \wedge \xi)^-.$$

where $\sigma^{\pm}$ denotes the self-dual and anti-self-dual part of a 2-form σ. In particular it follows that $B(\xi) = 0$ and that X is orthogonal to ξ.

Proof. Decomposing the generalized Killing spinor as $\Psi = \Psi^+ + \Psi^-$ and the 2-form as $\omega = \omega^+ + \omega^-$, the equation $X \cdot \Psi = \omega \cdot \Psi$ can be rewritten as

$$\begin{aligned}
\omega^- \cdot \Psi^+ + \omega^+ \cdot \Psi^- \ &= \ X \cdot \Psi \ = \ X \cdot (\Psi^+ + \Psi^-) \ = \ X \cdot \xi \cdot \Psi^- \ - \ \tfrac{1}{|\xi|^2} X \cdot \xi \cdot \Psi^+ \\
&= \ (X \wedge \xi)^+ \cdot \Psi^- \ - \ g(X, \xi)\, \Psi^- \ - \ \tfrac{1}{|\xi|^2}\, (X \wedge \xi)^- \cdot \Psi^+ \\
&\quad + \tfrac{1}{|\xi|^2}\, g(X, \xi)\, \Psi^+.
\end{aligned}$$

Comparing types, we find $\omega^+ = (X \wedge \xi)^+$ and $\omega^- = -\tfrac{1}{|\xi|^2}\, (X \wedge \xi)^-$. Moreover, since $\sigma^+ \cdot \Psi^-$ is orthogonal to Ψ^- for any 2-form σ, the equation immediately implies $g(X, \xi) = 0$. Finally applying this result to Equation (15) we obtain that $g(B(Z), \xi) = 0$ for any vector field Z, thus $B(\xi) = 0$.

$\qquad\square$

Lemma 4.3 applied to Equations (14) and (15) allows us to express the full curvature tensor of (M, g) in terms of the endomorphism A. Indeed we immediately obtain

$$(23) \qquad \tfrac{1}{2}\, g(\mathcal{R}(\sigma), \tau) \,+\, 2\, g(A(\sigma), \tau) \,=\, g(T(\sigma) \wedge \xi, \tau^+) \,-\, \tfrac{1}{|\xi|^2}\, g(T(\sigma) \wedge \xi, \tau^-),$$

for any 2-forms σ and τ. Here τ^+ and τ^- are self- and anti-self-dual part of τ. The T-part needs a short calculation and is given in the following

Lemma 4.4. *Let σ and τ be any 2-forms, then*

$$g(T(\sigma) \wedge \xi, \tau) \,=\, 2\, g(B(\sigma^+(\xi)), \tau(\xi)) \,-\, \tfrac{2}{|\xi|^2}\, g(B(\sigma^-(\xi)), \tau(\xi)).$$

Proof. Lemma 4.3 together with Equation (15) imply for any vector field Z the equation

$$T^Z \,=\, 2\, (B(Z) \wedge \xi)^+ \,-\, \tfrac{2}{|\xi|^2}\, (B(Z) \wedge \xi)^-.$$

Then $T(X \wedge Y) = \sum_{i=1}^n T^{e_i}(X, Y)\, e_i = \sum_{i=1}^n g(T^{e_i}, X \wedge Y) e_i$. Thus replacing $X \wedge Y$ with any 2-form σ gives

$$\begin{aligned}
T(\sigma) \,&=\, 2 \sum_{i=1}^n g(B(e_i) \wedge \xi, \sigma^+)\, e_i \,-\, \sum_{i=1}^n \tfrac{2}{|\xi|^2}\, g(B(e_i) \wedge \xi, \sigma^-)\, e_i \\
&=\, -2 B(\sigma^+(\xi)) \,+\, \tfrac{2}{|\xi|^2}\, B(\sigma^-(\xi)).
\end{aligned}$$

Taking the scalar product with $\tau(\xi) = \xi \lrcorner\, \tau$ proves the statement of the lemma. $\qquad\square$

For later use we still need an expression in the special case $\sigma = \eta \wedge Y$, where Y is an arbitrary vector field and η is defined in (21).

Corollary 4.5. *Let Y be any vector field then*

$$T(\eta, Y) \,=\, (1 - 2h) \left(A^2(Y) - aA(Y) + \tfrac{\lambda}{4} Y \right).$$

Proof. In dimension 4 we have $*(X \wedge Y) = -X \lrcorner * Y$ and $*(X \wedge Y)(\xi) = -g(*Y, X \wedge \xi)$. Hence, taking $\sigma = X \wedge Y$, we deduce from the calculations in the proof of Lemma 4.4 that

$$T(X \wedge Y) \,=\, (\tfrac{1}{|\xi|^2} - 1) B((X \wedge Y)(\xi)) \,-\, (1 + \tfrac{1}{|\xi|^2}) B(*(X \wedge Y)(\xi)).$$

Thus specializing to $X \wedge Y = \eta \wedge Y$ the second summand vanishes. Recalling that $\eta = -h\xi$, $|\xi|^2 = \tfrac{1-h}{h}$ and $B(\xi) = 0$, we get

$$T(\eta, Y) \,=\, \tfrac{1-|\xi|^2}{|\xi|^2} g(\eta, \xi) B(Y) \,=\, (1 - 2h) B(Y) \,=\, (1 - 2h)(A^2(Y) - aA(Y) + \tfrac{\lambda}{4} Y).$$

$$\square$$

In dimension 4 the Einstein condition is equivalent to having $\mathcal{R} : \Lambda_\pm^2 T \to \Lambda_\pm^2 T$, i.e. the curvature operator preserves the space of self-dual and anti-self-dual forms. In particular we have $g(\mathcal{R}(\sigma^+), \tau^-) = 0$. Let $e_1 := \tfrac{\xi}{\|\xi\|}$. Substituting $\sigma = \sigma^+$ and $\tau = \tau^-$ in (23) and using Lemma 4.4 yields

$$(24) \qquad\qquad 0 \,=\, g(A(\sigma^+), \tau^-) \,+\, g(B(\sigma^+(e_1)), \tau^-(e_1)).$$

We will use (24) to show that $A(e_1) = a_1 e_1$ for some real function a_1. Indeed the condition $B(e_1) = 0$ implies $A^2(e_1) - aA(e_1) + \frac{\lambda}{4}e_1 = 0$. Thus the space $\mathrm{span}\{e_1, A(e_1)\}$ is invariant under A and we may choose a local orthonormal frame $\{e_1, e_2, e_3, e_4\}$, with $e_1 := \frac{\xi}{\|\xi\|^2}$ and

$$Ae_1 = a_1 e_1 + a_{12} e_2, \quad Ae_2 = a_{12} e_1 + a_2 e_2, \quad Ae_3 = a_3 e_3, \quad Ae_4 = a_4 e_4.$$

Consider the 2-forms $\sigma^+ = e_1 \wedge e_3 - e_2 \wedge e_4$ and $\tau^- = e_1 \wedge e_4 - e_2 \wedge e_3$. Then

(25) $$0 = g(A(\sigma^+), \tau^-) = -a_{12}(a_4 + a_3).$$

Next consider the 2-forms $\sigma^+ = e_1 \wedge e_4 + e_2 \wedge e_3$ and $\tau^- e_1 \wedge e_4 - e_2 \wedge e_3$. Then

$$\begin{aligned} 0 &= g(A(\sigma^+), \tau^-) + g(B(e_4), e_4) = (a_1 a_4 - a_2 a_3) + (a_4^2 - a a_4 + \tfrac{\lambda}{4}) \\ &= -a_2(a_3 + a_4) - a_3 a_4 + \tfrac{\lambda}{4} \end{aligned}$$

If $a_3 + a_4 = 0$, then $a_3 a_4 = \frac{\lambda}{4} > 0$, which is impossible. Thus $a_3 + a_4 \neq 0$ and a_{12} has to vanish, because of (25). Consequently, around every point in M' we have a local orthonormal frame $e_1 := \frac{\xi}{\|\xi\|}, e_2, e_3, e_4$ with $Ae_i = a_i e_i$ and such that the eigenvalues a_i satisfy the relation $a_2 a_3 + a_2 a_4 + a_3 a_4 = \frac{\lambda}{4}$. Moreover $a_1^2 - a a_1 + \frac{\lambda}{4} = 0$ and in particular, since $\lambda > 0$, the function a_1 is nowhere zero.

Using Lemma 4.1 (i) and (iii) we get $0 = d(A(\eta) = d(a_1 \eta) = da_1 \wedge \eta$ and thus da_1 is collinear to η. The precise relation is given in the following

Proposition 4.6. $\quad da_1 = \frac{1-2h}{h(1-h)}(\frac{\lambda}{4} - 3a_1^2)\eta$

Proof. Let f be a function with $da_1 = f\eta$. Then $da_1(\eta) = f|\eta|^2 = fh(1-h)$ and $f = \frac{\eta(a_1)}{h(1-h)}$. In order to compute $\eta(a_1)$, we take the covariant derivative of $A(\eta) = a_1 \eta$ in direction of the vector field Y. Using Lemma 4.1 (ii) we get

$$\begin{aligned} (\nabla_Y A)\eta &= -A(\nabla_Y \eta) + Y(a_1)\eta + a_1(1-2h)A(Y) \\ &= -(1-2h)A^2(Y) + Y(a_1)\eta + a_1(1-2h)A(Y). \end{aligned}$$

Next we apply (13) and Corollary 4.5 to interchange Y and η. We obtain

$$\begin{aligned} (\nabla_\eta A)Y &= T(\eta, Y) - (1-2h)A^2(Y) + Y(a_1)\eta + a_1(1-2h)A(Y) \\ &= Y(a_1) + (1-2h)\left((a_1 - a)A(Y) + \tfrac{\lambda}{4}Y\right). \end{aligned}$$

Since $|A|^2 = \mathrm{tr}(A^2)$ and scal is constant, (12) implies $\eta(|A|^2) = \eta(\mathrm{tr}(A^2)) = \eta(a^2) = 2a\eta(a)$. On the other hand, computing $\eta(|A|^2)$ with $\eta(|A|^2) = \nabla_\eta|A|^2 = 2g(\nabla_\eta A, A) = 2g((\nabla_\eta A)e_i, A(e_i))$ gives

$$\begin{aligned} a\,\eta(a) &= A(\eta)(a_1) + (1-2h)\left((a_1 - a)\mathrm{tr}(A^2) + \tfrac{\lambda}{4}a\right) \\ &= a_1\,\eta(a_1) + (1-2h)\left((a_1 - a)(a^2 - \lambda) + \tfrac{\lambda}{4}a\right). \end{aligned}$$

From $B(\xi) = 0$ we have $a_1^2 - aa_1 + \frac{\lambda}{4} = 0$ and thus $a\,\eta(a) - a_1\eta(a_1) = -\frac{(a_1-a)^2}{a_1}\eta(a_1)$. Another simple calculation gives $(a_1 - a)(a^2 - \lambda) + \frac{\lambda}{4}a = (a_1 - a)^2(4a_1 - a)$. Substituting this into the equation above yields

$$\eta(a_1) \;=\; (1 - 2h)(aa_1 - 4a_1^2) \;=\; (1 - 2h)(\tfrac{\lambda}{4} - 3a_1^2).$$

$\square$

Comparing the expression of da_1 given in Proposition 4.6 and the expression of dh given in Lemma 4.1 (ii) we get

$$da_1 \;=\; \tfrac{1-2h}{2h(1-h)}\left(\tfrac{\lambda}{4a_1} - 3a_1\right) dh.$$

This shows that the function

(26)
$$C := (h(1 - h))^3 \left(a_1^2 - \tfrac{\lambda}{12}\right)$$

is constant on M'. Note that although the function a_1 is only defined on M', the function $l := h(1 - h)a_1^2$ is well-defined on the whole M. Indeed, from Lemma 4.1 (i) and (22) we get $|dh|^2 = 4a_1^2 h(1 - h) = 4l$. Using the density of M' in M, (26) shows that

(27)
$$C := (h(1 - h))^2 \left(l - \tfrac{h(1-h)\lambda}{12}\right)$$

is constant on M. Moreover this constant turns out to be zero because of

Lemma 4.7. *The function $h(1 - h)$ has a zero and in particular the constant C vanishes.*

Proof. Since M is compact h attains its absolute minimum at some point $x_0 \in M$. By (22) h takes values in $[0, 1]$. Clearly $h(x_0) < 1$, since otherwise $h \equiv 1$ on M, i.e. $\Psi^- \equiv 0$, which is impossible.

Assume that $h(x_0) \neq 0$. Then $x_0 \in M'$ so Lemma 4.1 (i) together with $dh_{x_0} = 0$ give $2a_1(x_0)\eta_{x_0} = 0$. As a_1 is nowhere zero on M', it follows $\eta_{x_0} = 0$ and thus $0 = |\eta|^2(x_0) = h(1 - h)(x_0)$, i.e. $h(x_0) = 0$. This contradiction shows that actually h vanishes at x_0. $\square$

We can now conclude:

Theorem 4.8. *Let (M, g) be a compact 4-dimensional Einstein manifold of positive scalar curvature, admitting a generalized Killing spinor Ψ. Then (M, g) is isometric to the standard sphere and Ψ is an ordinary Killing spinor.*

Proof. Since $C = 0$ and h is non-constant, (27) gives $a_1^2 = \frac{\lambda}{12}$, so $a^2 = \frac{4}{3}\lambda$. In particular, the function a is constant on M', and thus on M. From Corollary 4.2 it follows that $1 - 2h$ is a an eigenfunction for the Laplace operator for the eigenvalue $4(a^2 - \lambda) = \frac{4}{3}\lambda$. According to the Lichnerowicz-Obata Theorem, this is the lowest possible eigenvalue of the Laplace operator on compact Einstein manifolds, and it characterizes the round spheres.

Moreover, from Equation (12) we obtain

$$\mathrm{tr}\left(A - \tfrac{a}{4}\mathrm{id}\right)^2 = \mathrm{tr}(A^2) - \tfrac{a^2}{4} = (a^2 - \lambda) - \tfrac{a^2}{4} = 0.$$

This shows that the symmetric endomorphism A is a constant multiple of the identity map, so Ψ is a Killing spinor and thus the manifold is isometric to $\mathbb{S}^4$, cf. [3, 5, 18]. $\qquad\square$

4.4. **The case of dimension** 5. We start by reviewing the algebraic theory of spinors in dimension 5. Let M be a 5-dimensional spin manifold. Since the spin representation is isomorphic to the standard representation of $\mathrm{Spin}(5) = \mathrm{Sp}(2)$ on $\mathbb{R}^8 = \mathbb{H}^2$, the spin bundle ΣM carries a quaternionic structure. However, as the real Clifford algebra Cl_5 is isomorphic with $\mathbb{C}(4)$, only one of the complex structures on ΣM (the one given by Clifford multiplication with the volume element) commutes with the Clifford product with vectors. Let us call this complex structure I and denote by J and K two other complex structures on ΣM anti-commuting with I and with each other. It is easy to check that J and K anti-commute with the Clifford product with vectors.

For every nowhere vanishing spinor Ψ, the spin bundle has the following orthogonal direct sum decomposition:

$$(28) \qquad \Sigma M = \mathrm{T}M \cdot \Psi \oplus \langle \Psi \rangle \oplus \langle J\Psi \rangle \oplus \langle K\Psi \rangle.$$

Indeed, it is straightforward to check from the above properties of the complex structures J and K that all factors are mutually orthogonal. Since $I\Psi$ is orthogonal to the last three factors, we must have $I\Psi \in \mathrm{T}M \cdot \Psi$, so there exists a unit vector field ξ such that

$$(29) \qquad I\Psi = \xi \cdot \Psi.$$

We denote by $D := \xi^\perp$ the distribution orthogonal to ξ. For every $X \in D$ the spinor $X \cdot I\Psi$ is orthogonal to $\Psi, I\Psi, J\Psi$ and $K\Psi$, so there exists a vector $Y_X \in D$ such that $X \cdot I\Psi = Y_X \cdot \Psi$. We denote by L the endomorphism of $\mathrm{T}M$ which maps ξ to 0 and X to Y_X for $X \in D$. It is easy to check that L is skew-symmetric and satisfies the relations

$$(30) \qquad L^2 = -\mathrm{id} + \xi \otimes \xi,$$

(so L defines a complex structure on D), and

$$(31) \qquad X \cdot I\Psi = LX \cdot \Psi - g(X, \xi)\Psi, \qquad \forall X \in \mathrm{T}M.$$

This last relation allows us to explicit the Clifford product of 2-forms with Ψ. We decompose $\Lambda^2 M$ as

$$\Lambda^2 M = \xi \wedge \mathrm{T}M \oplus \Lambda^{(1,1)}D \oplus \Lambda^{(2,0)+(0,2)}D$$

and using (31) we get

$$(32) \qquad (\xi \wedge X) \cdot \Psi = -LX \cdot \Psi, \qquad \forall X \in \mathrm{T}M,$$

$$(33) \qquad \Lambda^{(1,1)}D \cdot \Psi = \langle I\Psi \rangle = \langle \xi \cdot \Psi \rangle,$$

$$(34) \qquad \Lambda^{(2,0)+(0,2)}D \cdot \Psi = \langle J\Psi \rangle \oplus \langle K\Psi \rangle.$$

The last relation actually yields a trivialization of $\Lambda^{(2,0)+(0,2)}D$ (and thus a SU(2) reduction of the structure group of M), but we will not need this in the sequel.

We are now ready to prove the main result of this section:

Theorem 4.9. *Any generalized Killing spinor on the 5-sphere $\mathbb{S}^5$ is a real Killing spinor.*

Proof. Let Ψ be a generalized Killing spinor on $\mathbb{S}^5$ satisfying (1). The curvature endomorphism of the sphere is minus the identity on 2-forms, so (14) reads

$$T(X \wedge Y) \cdot \Psi = \left[2AX \wedge AY - \tfrac{1}{2} X \wedge Y \right] \cdot \Psi, \qquad \forall X, Y \in \mathrm{T}M.$$

From (32)–(34) we deduce that

$$(35) \qquad\qquad AX \wedge AY - \tfrac{1}{4} X \wedge Y \in \xi \wedge \mathrm{T}M \oplus \Lambda^{(1,1)}D, \qquad \forall X, Y \in \mathrm{T}M.$$

Let $A\xi = \alpha\xi + \zeta$ with $\zeta \in D$. Taking $X = \xi$ in (35) we obtain that $\zeta \wedge AY$ belongs to $\Lambda^{(1,1)}D$ for every Y orthogonal to $A\xi$. Assume that $\zeta \neq 0$. Then AY belongs to the subspace spanned by ζ and $L\zeta$ for all Y orthogonal to $A\xi$, whence the image of A is a subset of $\langle \xi, \zeta, L\zeta \rangle$. Since A is symmetric, $\mathrm{Ker}(A)$ contains the orthogonal complement of $\langle \xi, \zeta, L\zeta \rangle$ in $\mathrm{T}M$. Let Y be a vector in this orthogonal complement. By (35) again, $X \wedge Y \in \Lambda^{(1,1)}D$ for all $X \in D$, which is a contradiction for $X = \zeta$. This shows that $\zeta = 0$, so D is left invariant by A.

Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis of eigenvectors of the restriction of A to D corresponding to the eigenvalues α_i. Thus in the basis $\{\xi, e_1, e_2, e_3, e_4\}$ the matrix of A is diagonal, with entries $\{\alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Equation (35) implies that

$$\left(\alpha_i\alpha_j - \tfrac{1}{4}\right) e_i \wedge e_j \in \Lambda^{(1,1)}D$$

for all subscripts $1 \leq i, j \leq 4$. This shows that for every subscript i, there are at least two other subscripts j and k such that $\alpha_i\alpha_j = \alpha_i\alpha_k = \tfrac{1}{4}$. Up to a permutation we can thus assume that $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4 = \tfrac{1}{4\alpha_1}$. We see that either $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$, or $\alpha_1 \neq \alpha_3$, in which case $e_1 \wedge e_2$ and $e_3 \wedge e_4$ belong to $\Lambda^{(1,1)}D$. In this last case L preserves the eigenspaces $\langle e_1, e_2 \rangle$ and $\langle e_3, e_4 \rangle$ of A, so in both cases L and A commute.

We now take a covariant derivative with respect to an arbitrary vector X in (29) and use (31) to obtain

$$
\begin{aligned}
\nabla_X \xi \cdot \Psi &= \nabla_X(\xi \cdot \Psi) - \xi \cdot \nabla_X \Psi \\
&= I(AX \cdot \Psi) - \xi \cdot AX \cdot \Psi = AX \cdot I\Psi + AX \cdot \xi \cdot \Psi + 2g(AX, \xi)\Psi \\
&= 2AX \cdot I\Psi + 2g(AX, \xi)\Psi = 2LAX \cdot \Psi,
\end{aligned}
$$

whence

$$(36) \qquad\qquad \nabla_X \xi = 2LAX, \qquad \forall X \in \mathrm{T}M.$$

Since L and A commute, LA is skew-symmetric, thus ξ is a Killing vector field on $\mathbb{S}^5$, i.e. there exists a skew-symmetric matrix $M \in \mathfrak{so}(6)$ such that $\xi_x = Mx$ for every $x \in \mathbb{S}^5 \subset \mathbb{R}^6$. Moreover, ξ has constant length 1, thus M is orthogonal, so $M^2 = -\mathrm{id}$. On the other

hand, the covariant derivative of ξ can also be computed by projecting in $T\mathbb{S}^5$ the Euclidean covariant derivative in $\mathbb{R}^6$. We thus obtain $(\nabla_X \xi)_x = \mathrm{pr}_{T_x\mathbb{S}^5}(MX) = MX - \langle x, MX \rangle x$. Let φ denote the skew-symmetric endomorphism corresponding to $\nabla \xi$. The previous relation implies

$$\varphi^2(X) = \varphi(MX - \langle x, MX \rangle x) = M^2 X - \langle x, MX \rangle Mx = -X + \langle Mx, X \rangle Mx = -X + g(X, \xi)\xi.$$

(Note that this relation could also have been obtained by saying that every unit Killing vector field on $\mathbb{S}^5$ is Sasakian). Comparing it with (36) and using (30) yields

$$-X + g(X, \xi)\xi = 4L^2 A^2 X = -4A^2 X + 4g(A^2 X, \xi)\xi.$$

Consequently, the restriction of $4A^2$ to D is the identity, thus $\mathrm{tr} A^2 = \alpha^2 + 1$. Moreover the eigenvalues α_i of $A|_D$ belong to $\{\pm\frac{1}{2}\}$. Since these eigenvalues are pairwise equal, assuming that they are not all equal then $\mathrm{tr}(A|_D) = 0$ so $a = \mathrm{tr} A = \alpha$. This contradicts Equation (12) which in our case reads $a^2 - \mathrm{tr} A^2 = \frac{1}{4}\mathrm{scal} = 5$. In the remaining cases the eigenvalues of $A|_D$ are all equal to either $\alpha_1 = \frac{1}{2}$ or $\alpha_1 = -\frac{1}{2}$, so $a = \alpha + 4\alpha_1$ and from (12) again we get $(\alpha + 4\alpha_1)^2 - (1 + \alpha^2) = 5$, which finally gives $\alpha\alpha_1 = \frac{1}{4}$ i.e. $\alpha = \alpha_1$ and A is a constant scalar matrix $\pm\frac{1}{2}\mathrm{id}$. This finishes the proof of the theorem. $\qquad\square$

4.5. **The case of dimension** 6 **and** 7. As already mentioned, generalized Killing spinors are equivalent to half-flat SU(3)-structures [8, 19] in dimension 6 and to co-calibrated G_2-structures [10, 12] in dimension 7.

Using this correspondence, examples of Einstein metrics with generalized Killing spinors in these dimensions can be found in the recent literature. In [26], Table 3, p. 74, Schulte-Hengesbach constructs a half-flat structure on the Riemannian product $\mathbb{S}^3 \times \mathbb{S}^3$ (see also [26], Remark 1.12, p. 87). In fact Schulte-Hengesbach classifies all half-flat structures on the product of two 3-dimensional spheres, and it turns out that for a certain choice of the parameters the corresponding structure is compatible with the product metric $\mathbb{S}^3 \times \mathbb{S}^3$ (see also [23], p. 14).

The Fubini-Study metric on $\mathbb{CP}^3$ admits a half-flat structure with respect to the non-integrable almost complex structure. This example can be found in [9], Section 4.5, Prop. 4.12. Conti considers the complex projective space $\mathbb{CP}^3$ realized as a hypersurface in the total space of the vector bundle of anti-self-dual 2-forms $\Lambda^2_-\mathbb{S}^4$ equipped with the parallel G_2-structure found by Bryant and Salamon [7].

In fact, using the methods of [5], Chapter 5.4, it is easy to show that on the homogeneous spaces $\mathbb{S}^3 \times \mathbb{S}^3$, $\mathbb{CP}^3$ and the flag manifold $SU(3)/\mathbb{T}^2$ there exists a 1-parameter family of metrics with a generalized Killing spinor, given as a constant map on the respective groups. For each of the three cases, this family of metrics contains exactly two Einstein metrics. One of these Einstein metrics is compatible with a nearly Kähler structure and the corresponding spinor is a Killing spinor. The second Einstein metric is the standard Kähler-Einstein metric on the two twistor spaces $\mathbb{CP}^3$ and $SU(3)/\mathbb{T}^2$ and the product metric on $\mathbb{S}^3 \times \mathbb{S}^3$. Note that in

the first two cases, the generalized Killing spinor turns out to be a Kählerian Killing spinor [20, 25].

In dimension 7, examples of generalized Killing spinors that are not Killing were recently constructed by Agricola and Friedrich [1] on every 3-Sasakian metric (it is well known that 3-Sasakian metrics are automatically Einstein). They actually show that every 7-dimensional 3-Sasakian manifold carries a so-called canonical spinor Ψ_0, which is not a Killing spinor itself, but generates the 3-dimensional space of Killing spinors in the sense that the three Killing spinors of the 3-Sasakian structure are obtained by Clifford product of Ψ_0 with the three Sasakian Killing vector fields. Agricola and Friedrich show that the G_2-structure defined by Ψ_0 is co-calibrated not only on the original metric g but on the whole 1-parameter family of metrics g_t obtained by rescaling g along the 3-dimensional Sasakian distribution. Thus Ψ_0 is a generalized Killing spinor for each metric in this family, which contains two Einstein metrics: the 3-Sasakian metric g for $t = 1$ and a proper G_2-metric for $t = \frac{1}{5}$ (cf. [14]). For this second Einstein metric, Ψ_0 turns out to be a genuine Killing spinor.

Note in particular that this construction gives a generalized Killing spinor which is not Killing on the standard sphere $\mathbb{S}^7$.

5. Final remarks and open questions

In view of the correspondence between generalized Killing spinors and hypersurface embeddings in manifolds with parallel spinors, we obtain the following corollaries of our main results:

Corollary 5.1. *Let (M^4, g) be a compact Einstein hypersurface with positive scalar curvature in a Riemannian product $(\mathcal{Z}^5, g^{\mathcal{Z}}) = \mathbb{R} \times (N^4, h)$ where (N, h) is simply connected and hyperkähler (e.g. the flat space $\mathbb{R}^4$, a K3 surface, an Eguchi-Hanson manifold, a Taub-NUT or a Kronheimer ALE space). Then (N, h) is flat and $(M^4, g) = \mathbb{S}^4$.*

Proof. The manifold $(\mathcal{Z}, g^{\mathcal{Z}})$ is spin and has a parallel spinor. By [4], Equation (30), (M^4, g) carries a generalized Killing spinor, so $(M^4, g) = \mathbb{S}^4$ by Theorem 4.8. From the uniqueness part of Theorem 1.1 in [2] we deduce that the ambient metric $g^{\mathcal{Z}}$ is flat. □

A similar argument together with Theorem 4.9 yields:

Corollary 5.2. *Let $(\mathcal{Z}^6, g^{\mathcal{Z}})$ be a simply connected Ricci-flat Kähler threefold (e.g. the flat space $\mathbb{R}^6$, a Calabi-Yau threefold, or $\mathbb{R}^2 \times K3$). If $\mathbb{S}^5$ has an isometric embedding in $(\mathcal{Z}, g^{\mathcal{Z}})$, then $\mathcal{Z}$ is a flat space.*

We finally list some open questions which arose during the preparation of this work, which we think worth of further investigation:

- Does the sphere $\mathbb{S}^n$ carry generalized Killing spinors which are not Killing? From the above, we know that the answer is "yes" for $n = 3$ and $n = 7$, "no" for $n = 2$, $n = 4$

and $n = 5$, and unknown for $n = 6$ and $n \geq 8$. It is surprising that one does not know whether a half-flat structure on $\mathbb{S}^6$ is necessarily nearly Kähler.

- Find all generalized Killing spinors on a given spin manifold. To our knowledge, the only cases where a complete answer is available are given by Theorems 4.8 and 4.9. Even for one of the simplest possible manifolds, the standard sphere $\mathbb{S}^3$, the set of generalized Killing spinors is unknown.
- In the real analytic case, a spin manifold with generalized Killing spinors embeds as a hypersurface in a manifold with parallel spinors ([2], Theorem 1.1). What is (locally) the ambient metric corresponding to the examples above, e.g. for the spinors on $\mathbb{S}^3$ constructed in Section 4.2? This metric is interesting since it is hyperkähler, non-flat, and contains standard spheres as hypersurfaces.
- In all available examples of generalized Killing spinors on Einstein manifolds, the symmetric tensor A has constant eigenvalues. Is this a general phenomenon?
- Is it possible to construct examples of generalized Killing spinors on 3-Sasakian manifolds of dimension $4n + 3 \geq 11$ using methods similar to those in [1]?

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Andrei Moroianu, Université de Versailles-St Quentin, Laboratoire de Mathématiques, UMR 8100 du CNRS, 45 avenue des États-Unis, 78035 Versailles, France

Email address: `andrei.moroianu@math.cnrs.fr`

Uwe Semmelmann, Institut für Geometrie und Topologie, Fachbereich Mathematik, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

Email address: `uwe.semmelmann@mathematik.uni-stuttgart.de`