HIGHER RANK HOMOGENEOUS CLIFFORD STRUCTURES

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Abstract. We give an upper bound for the rank r of homogeneous (even) Clifford structures on compact manifolds of non-vanishing Euler characteristic. More precisely, we show that if $r = 2^a \cdot b$ with b odd, then $r \leq 9$ for $a = 0$, $r \leq 10$ for $a = 1$, $r \leq 12$ for $a = 2$ and $r \leq 16$ for $a \geq 3$. Moreover, we describe the four limiting cases and show that there is exactly one solution in each case.

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1. Introduction

The notion of (even) Clifford structures on Riemannian manifolds was introduced in [6], motivated by the study of Riemannian manifolds with non-trivial curvature constancy (cf. [3]). They generalize almost Hermitian and quaternionic-Hermitian structures and are in some sense dual to spin structures. More precisely:

Definition 1.1 ([6])**.** A rank r *Clifford structure* ($r \geq 1$) on a Riemannian manifold (M^n, g) is an oriented rank r Euclidean bundle (E, h) over M together with an algebra bundle morphism $\varphi : \mathrm{Cl}(E, h) \to \mathrm{End}\,(TM)$ which maps E into the bundle of skew-symmetric endomorphisms $\mathrm{End}^-(TM)$.

A rank r *even Clifford structure* ($r \geq 2$) on (M^n, g) is an oriented rank r Euclidean bundle (E, h) over M together with an algebra bundle morphism $\varphi : \mathrm{Cl}^0(E, h) \to \mathrm{End}\,(TM)$ which maps $\Lambda^2 E$ into the bundle of skew-symmetric endomorphisms $\mathrm{End}^-(TM)$.

It is easy to see that every rank r Clifford structure is in particular a rank r even Clifford structure, so the latter notion is more flexible.

In general, there exists no upper bound for the rank of a Clifford structure. In fact, Joyce provided a method (cf. [4]) to construct non-compact manifolds with arbitrarily large Clifford structures. However, in many cases the rank is bounded by above. For instance, the Riemannian manifolds carrying *parallel* (even) Clifford structures (in the sense that (E, h) has a metric connection making the Clifford morphism φ parallel) were classified in [6] and it turns out that the rank of a parallel Clifford structure is bounded by above if the manifold is nonflat: Every parallel Clifford structure has rank $r \leq 7$ and every parallel even Clifford structure

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has rank $r \leq 16$ (cf. [6, Thm. 2.14 and 2.15]). The list of manifolds with parallel even Clifford structure of rank $r \geq 9$ only contains four entries, the so-called Rosenfeld's elliptic projective planes $\mathbb{O}\mathbb{P}^2$, $(\mathbb{C}\otimes\mathbb{O})\mathbb{P}^2$, $(\mathbb{H}\otimes\mathbb{O})\mathbb{P}^2$ and $(\mathbb{O}\otimes\mathbb{O})\mathbb{P}^2$, which are inner symmetric spaces associated to the exceptional simple Lie groups F_4, E_6, E_7 and E_8 (cf. [7]) and have Clifford rank $r = 9,\ 10,\ 12$ and 16, respectively.

A natural related question is then to look for *homogeneous* (instead of parallel) even Clifford structures on homogeneous spaces $M = G/H$. We need to make some restrictions on M in order to obtain relevant results. On the first hand, we assume M to be compact (and thus G and H are compact, too). On the other hand, we need to assume that H is not too small. For example, in the degenerate case when H is just the identity of G, the tangent bundle of $M = G$ is trivial, and the unique obstruction for the existence of a rank r (even) Clifford structure is that the dimension of G has to be a multiple of the dimension of the irreducible representation of the Clifford algebra Cl_r or Cl^0_r. At the other extreme, we might look for homogeneous spaces $M = G/H$ with $\mathrm{rk}(H) = \mathrm{rk}(G)$, or, equivalently, $\chi(M) \neq 0$. The main advantage of this assumption is that we can choose a common maximal torus of H and G and identify the root system of H with a subset of the root system of G.

In this setting, the system of roots of G is made up of the system of roots of H and the weights of the (complexified) isotropy representation, which are themselves related to the weights of some spinorial representation if G/H carries a homogeneous even Clifford structure. We then show that the very special configuration of the weights of the spinorial representation Σ_r is not compatible with the usual integrality conditions of root systems, provided that r is large enough.

The main results of this paper are Theorem 4.1, where we obtain upper bounds on r depending on its 2-valuation, and Theorem 4.2, where we study the limiting cases $r = 9,\ 10,\ 12$ and 16 and show that they correspond to the symmetric spaces $\mathrm{F}_4/\mathrm{Spin}(9)$, $\mathrm{E}_6/(\mathrm{Spin}(10)\times\mathrm{U}(1)/\mathbb{Z}_4)$, $\mathrm{E}_7/\mathrm{Spin}(12)\cdot\mathrm{SU}(2)$ and $\mathrm{E}_8/(\mathrm{Spin}(16)/\mathbb{Z}_2)$.

We believe that our methods could lead to a complete classification of homogeneous Clifford structures of rank $r \geq 3$ on compact manifolds with non-vanishing Euler characteristic, eventually showing that they are all symmetric, thus parallel (cf. [6, Table 2]), but a significantly larger amount of work is needed, especially for lower ranks.

2. Preliminaries on Lie algebras and root systems

For the basic theory of root systems we refer to [1] and [8].

Definition 2.1. A set $\mathcal{R}$ of vectors in a Euclidean space $(V, \langle\cdot,\cdot\rangle)$ is called a *system of roots* if it satisfies the following conditions:

- **R1:** $\mathcal{R}$ is finite, $\mathrm{span}(\mathcal{R}) = V$, $0 \notin \mathcal{R}$.
- **R2:** If $\alpha \in \mathcal{R}$, then the only multiplies of α in $\mathcal{R}$ are $\pm\alpha$.
- **R3:** $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$, for all $\alpha, \beta \in \mathcal{R}$.
- **R4:** $s_\alpha : \mathcal{R} \to \mathcal{R}$, for all $\alpha \in \mathcal{R}$ (s_α is the reflection $s_\alpha : V \to V$, $s_\alpha(v) := v - \frac{2\langle\alpha,v\rangle}{\langle\alpha,\alpha\rangle}\alpha$).

Remark 2.2 (Properties of root systems)**.** Let $\mathcal{R}$ be a system of roots. If $\alpha, \beta \in \mathcal{R}$ such that $\beta \neq \pm\alpha$ and $\|\beta\|^2 \geq \|\alpha\|^2$, then

$$\frac{2\langle\, \alpha,\, \beta\,\rangle}{\langle\, \beta,\, \beta\,\rangle} \in \{0, \pm 1\}. \tag{1}$$

If $\langle\, \alpha,\, \beta\,\rangle \neq 0$, then the following values are possible:

$$\left(\frac{\|\beta\|^2}{\|\alpha\|^2}, \frac{2\langle\, \alpha,\, \beta\,\rangle}{\langle\, \alpha,\, \alpha\,\rangle}\right) \in \{(1, \pm 1), (2, \pm 2), (3, \pm 3)\}. \tag{2}$$

Moreover, in this case, it follows that

$$\beta - \operatorname{sgn}\left(\frac{2\langle\, \alpha,\, \beta\,\rangle}{\langle\, \alpha,\, \alpha\,\rangle}\right) k\alpha \in \mathcal{R}, \qquad \text{for } k \in \mathbb{Z}, 1 \leq k \leq \left|\frac{2\langle\, \alpha,\, \beta\,\rangle}{\langle\, \alpha,\, \alpha\,\rangle}\right|. \tag{3}$$

We shall be interested in special subsets of systems of roots and consider the following notions.

Definition 2.3. A set $\mathcal{P}$ of vectors in a Euclidean space $(V, \langle\,\cdot,\,\cdot\,\rangle)$ is called a *subsystem of roots* if it generates V and is contained in a system of roots of $(V, \langle\cdot, \cdot\rangle)$.

It is clear that any subsystem of roots $\mathcal{P}$ is included into a minimal system of roots (obtained by taking all possible reflections), which we denote by $\overline{\mathcal{P}}$.

Let $\mathcal{P}$ be a subsystem of roots of $(V, \langle\,\cdot,\,\cdot\,\rangle)$. An *irreducible component* of $\mathcal{P}$ is a minimal non-empty subset $\mathcal{P}' \subset \mathcal{P}$ such that $\mathcal{P}' \perp (\mathcal{P} \setminus \mathcal{P}')$. By rescaling the scalar product $\langle\,\cdot,\,\cdot\,\rangle$ on the subspaces generated by the irreducible components of V one can always assume that the root of maximal length of each irreducible component of $\mathcal{P}$ has norm equal to 1.

Definition 2.4. A subsystem of roots $\mathcal{P}$ in a Euclidean space $(V, \langle\,\cdot,\,\cdot\,\rangle)$ is called *admissible* if $\overline{\mathcal{P}} \setminus \mathcal{P}$ is a system of roots.

For any $q \in \mathbb{Z}$, $q \geq 1$, let $\mathcal{E}_q$ denote the set of all q-tuples $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_q)$ with $\varepsilon_j \in \{\pm 1\}$, $1 \leq j \leq q$. The following result will be used several times in the next section.

Lemma 2.5. *Let $q \in \mathbb{Z}$, $q \geq 1$ and $\{\beta_j\}_{j=\overline{1,q}}$ be a set of linearly independent vectors in a Euclidean space $(V, \langle\,\cdot,\,\cdot\,\rangle)$. If $\mathcal{P} \subset \{\sum\limits_{j=1}^{q} \varepsilon_j \beta_j\}_{\varepsilon \in \mathcal{E}_q}$ is an admissible subsystem of roots, then any two vectors in $\mathcal{P}$ of different norms must be orthogonal.*

Proof. Assuming the existence of two non-orthogonal vectors of different norms, we construct two roots in $\overline{\mathcal{P}} \setminus \mathcal{P}$ whose difference is a root in $\mathcal{P}$, thus contradicting the assumption on $\mathcal{P}$ to be admissible. More precisely, suppose that $\alpha, \alpha' \in \mathcal{P}$, $\alpha' \neq \alpha$, such that $\langle\, \alpha,\, \alpha'\,\rangle > 0$ (a similar argument works if $\langle\, \alpha,\, \alpha'\,\rangle < 0$) and $\|\alpha'\|^2 > \|\alpha\|^2$. From (2), it follows that either $\|\alpha'\|^2 = 2\|\alpha\|^2$ and $\langle\, \alpha,\, \alpha'\,\rangle = \|\alpha\|^2$ or $\|\alpha'\|^2 = 3\|\alpha\|^2$ and $\langle\, \alpha,\, \alpha'\,\rangle = \frac{3}{2}\|\alpha\|^2$. In both cases $\left|\frac{2\langle\, \alpha',\, \alpha\,\rangle}{\langle\, \alpha,\, \alpha\,\rangle}\right| \geq 2$ and (3) implies that $\alpha' - \alpha, \alpha' - 2\alpha \in \overline{\mathcal{P}}$.

We first check that $\alpha' - \alpha, \alpha' - 2\alpha \notin \mathcal{P}$. The coefficients of β_j in $\alpha' - \alpha$ and in $\alpha' - 2\alpha$ may take the values $\{0, \pm 2\}$, respectively $\{\pm 1, \pm 3\}$, for all $j = 1, \ldots, q$. Since $\{\beta_j\}_{j=\overline{1,q}}$ are linearly independent, it follows that $\alpha' - \alpha \notin \{\sum_{j=1}^{q} \varepsilon_j \beta_j\}_{\varepsilon \in \mathcal{E}_q}$. Moreover, $\alpha' - 2\alpha \in \{\sum_{j=1}^{q} \varepsilon_j \beta_j\}_{\varepsilon \in \mathcal{E}_q}$ if and only if the coefficients of each β_j in α' and α are equal, *i.e.* $\alpha' = \alpha$, which is not possible.

On the other hand, $\langle \alpha' - \alpha, \alpha' - 2\alpha \rangle \in \{\|\alpha\|^2, \frac{1}{2}\|\alpha\|^2\}$ and from (3) and the admissibility of $\mathcal{P}$, it follows that $\alpha \in \overline{\mathcal{P}} \setminus \mathcal{P}$, yielding a contradiction and finishing the proof. $\square$

Let G be a compact semi-simple Lie group with Lie algebra $\mathfrak{g}$ endowed with an $\mathrm{ad}_{\mathfrak{g}}$-invariant scalar product. Fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ and let $\mathcal{R}(\mathfrak{g}) \subset \mathfrak{t}^*$ denote its system of roots. It is well known that $\mathcal{R}(\mathfrak{g})$ satisfies the conditions in Definition 2.1. Conversely, every set of vectors satisfying the conditions in Definition 2.1 is the root system of a unique semi-simple Lie algebra of compact type.

If H is a closed subgroup of G with $\mathrm{rk}(H) = \mathrm{rk}(G)$, then one may assume that its Lie algebra $\mathfrak{h}$ contains $\mathfrak{t}$, so the system of roots $\mathcal{R}(\mathfrak{g})$ is the disjoint union of the root system $\mathcal{R}(\mathfrak{h})$ and the set $\mathcal{W}$ of weights of the complexified isotropy representation of the homogeneous space G/H. This follows from the fact that the isotropy representation is given by the restriction to H of the adjoint representation of $\mathfrak{g}$.

Lemma 2.6. *The set $\mathcal{W} \subset \mathfrak{t}^*$ is an admissible subsystem of roots.*

Proof. Indeed, $\overline{\mathcal{W}} \setminus \mathcal{W} = \overline{\mathcal{W}} \cap \mathcal{R}(\mathfrak{h})$, whence $\overline{\overline{\mathcal{W}} \setminus \mathcal{W}} \subset \overline{\mathcal{W}} \cap \overline{\mathcal{R}(\mathfrak{h})} = \overline{\mathcal{W}} \cap \mathcal{R}(\mathfrak{h}) = \overline{\mathcal{W}} \setminus \mathcal{W}$. $\square$

We will now prove a few general results about Lie algebras which will be needed later on.

Lemma 2.7. *Let $\mathfrak{h}_1$ be a Lie subalgebra of a Lie algebra $\mathfrak{h}_2$ of compact type having the same rank. If $\alpha, \beta \in \mathcal{R}(\mathfrak{h}_1)$ such that $\alpha + \beta \in \mathcal{R}(\mathfrak{h}_2)$, then $\alpha + \beta \in \mathcal{R}(\mathfrak{h}_1)$.*

Proof. We first recall a general result about roots. Let $\mathfrak{h}$ be a Lie algebra of compact type and $\mathfrak{t}$ a fixed Cartan subalgebra in $\mathfrak{h}$. For any $\alpha \in \mathfrak{t}^*$, let $(\mathfrak{h})_\alpha$ denote the intersection of the nilspaces of the operators $\mathrm{ad}(A) - \alpha(A)$ acting on $\mathfrak{h}$, with A running over $\mathfrak{t}$. By definition, α is a root of $\mathfrak{h}$ if and only if $(\mathfrak{h})_\alpha \neq \{0\}$. Moreover, the Jacobi identity shows that $[(\mathfrak{h})_\alpha, (\mathfrak{h})_\beta] \subseteq (\mathfrak{h})_{\alpha+\beta}$. It is well known that in this case the space $(\mathfrak{h})_\alpha$ is 1-dimensional. Moreover, by [8, Theorem A, p. 48], there exist generators X_α of $(\mathfrak{h})_\alpha$ such that for any $\alpha, \beta \in \mathcal{R}(\mathfrak{h})$ with $\alpha + \beta \in \mathcal{R}(\mathfrak{h})$, the following relation holds: $[X_\alpha, X_\beta] = \pm(q+1) X_{\alpha+\beta}$, where q is the largest integer k such that $\beta - k\alpha$ is a root. In particular, if $\alpha + \beta \in \mathcal{R}(\mathfrak{h})$, then $[(\mathfrak{h})_\alpha, (\mathfrak{h})_\beta] = (\mathfrak{h})_{\alpha+\beta}$.

Let now $\mathfrak{t}$ be a fixed Cartan subalgebra in both $\mathfrak{h}_1$ and $\mathfrak{h}_2$ (this is possible because $\mathrm{rk}(\mathfrak{h}_1) = \mathrm{rk}(\mathfrak{h}_2)$) and let $\alpha, \beta \in \mathcal{R}(\mathfrak{h}_1) \subseteq \mathcal{R}(\mathfrak{h}_2)$, such that $\alpha + \beta \in \mathcal{R}(\mathfrak{h}_2)$. The above result applied to $\mathfrak{h}_2$ implies: $\{0\} \neq (\mathfrak{h}_2)_{\alpha+\beta} = [(\mathfrak{h}_2)_\alpha, (\mathfrak{h}_2)_\beta] = [(\mathfrak{h}_1)_\alpha, (\mathfrak{h}_1)_\beta] \subseteq (\mathfrak{h}_1)_{\alpha+\beta}$, where we use that $(\mathfrak{h}_1)_\alpha = (\mathfrak{h}_2)_\alpha$ for any $\alpha \in \mathcal{R}(\mathfrak{h}_1) \subseteq \mathcal{R}(\mathfrak{h}_2)$. Thus, $(\mathfrak{h}_1)_{\alpha+\beta} \neq \{0\}$, *i.e.* $\alpha + \beta \in \mathcal{R}(\mathfrak{h}_1)$. $\square$

We will also need the following result, whose proof is straightforward.

Lemma 2.8. *(i) Let $k \geq 2$ and let $\mathfrak{h}$ be a Lie algebra of compact type written as an orthogonal direct sum of k Lie algebras: $\mathfrak{h} = \bigoplus_{i=1}^{k} \mathfrak{h}_i$ with respect to some $\mathrm{ad}_{\mathfrak{h}}$-invariant scalar product $\langle \cdot, \cdot \rangle$*

on $\mathfrak{h}$. Then, identifying each Lie algebra $\mathfrak{h}_i$ with its dual using $\langle\cdot,\cdot\rangle$ we have $\mathcal{R}(\mathfrak{h})=\bigcup_{i=1}^{k}\mathcal{R}(\mathfrak{h}_i)$. In particular, every root of $\mathfrak{h}$ lies in one component $\mathfrak{h}_i$.

(ii) Let α and β be two roots of $\mathfrak{h}$. If there exists a sequence of roots $\alpha_0:=\alpha,\alpha_1,\dots,\alpha_n:=\beta$ $(n\geq 1)$ such that $\langle\alpha_i,\alpha_{i+1}\rangle\neq 0$ for $0\leq i\leq n-1$, then α and β belong to the same component $\mathfrak{h}_i$.

3. The isotropy representation of homogeneous manifolds with Clifford structure

Let $M=G/H$ be a compact homogeneous space. Denote by $\mathfrak{h}$ and $\mathfrak{g}$ the Lie algebras of H and G and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to some $\mathrm{ad}_{\mathfrak{g}}$-invariant scalar product on $\mathfrak{g}$. The restriction to $\mathfrak{m}$ of this scalar product defines a homogeneous Riemannian metric g on M. Since from now on we will exclusively consider even Clifford structures, and in order to simplify the terminology, we will no longer use the word "even" and make the following:

Definition 3.1. A *homogeneous Clifford structure* of rank $r\geq 2$ on a Riemannian homogeneous space $(G/H,g)$ is an orthogonal representation $\rho: H\to \mathrm{SO}(r)$ and an H-equivariant representation $\varphi:\mathfrak{so}(r)\to\mathrm{End}^-(\mathfrak{m})$ extending to an algebra representation of the even real Clifford algebra Cl^0_r on $\mathfrak{m}$.

Any homogeneous Clifford structure defines in a tautological way an *even Clifford structure* on (M,g) in the sense of Definition 1.1, by taking E to be the vector bundle associated to the H-principal bundle G over M via the representation ρ:

$$E=G\times_\rho\mathbb{R}^r.$$

In order to describe the isotropy representation of a homogeneous Clifford structure we need to recall some facts about Clifford algebras, for which we refer to [5].

The even real Clifford algebra Cl^0_r is isomorphic to a matrix algebra $\mathbb{K}(n_r)$ for $r\not\equiv 0 \mod 4$ and to a direct sum $\mathbb{K}(n_r)\oplus\mathbb{K}(n_r)$ when r is multiple of 4. The field $\mathbb{K}$ ($=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) and the dimension n_r depend on r according to a certain 8-periodicity rule. More precisely, $\mathbb{K}=\mathbb{R}$ for $r\equiv 0,\ 1,\ 7 \mod 8$, $\mathbb{K}=\mathbb{C}$ for $r\equiv 2,\ 6 \mod 8$ and $\mathbb{K}=\mathbb{H}$ for $r\equiv 3,\ 4,\ 5 \mod 8$, and if we write $r=8k+q$, $1\leq q\leq 8$, then $n_r=2^{4k}$ for $1\leq q\leq 4$, $n_r=2^{4k+1}$ for $q=5$, $n_r=2^{4k+2}$ for $q=6$ and $n_r=2^{4k+3}$ for $q=7$ or $q=8$.

Let Σ_r and $\Sigma_r^{\pm}$ denote the irreducible representations of Cl^0_r for $r\not\equiv 0 \mod 4$ and $r\equiv 0 \mod 4$ respectively. From the above, it is clear that Σ_r (or $\Sigma_r^{\pm}$) have dimension n_r over $\mathbb{K}$.

Lemma 3.2. *Assume that $M=G/H$ carries a rank r homogeneous Clifford structure and let $\iota: H\to\mathrm{Aut}(\mathfrak{m})$ denote the isotropy representation of H.*

(i) If r is not a multiple of 4, we denote by ξ the spin representation of $\mathfrak{so}(r)=\mathfrak{spin}(r)$ on the spin module Σ_r and by $\mu=\xi\circ\rho_*$ its composition with ρ_*. Then the infinitesimal isotropy representation ι_* on $\mathfrak{m}$ is isomorphic to $\mu\otimes_{\mathbb{K}}\lambda$ for some representation λ of $\mathfrak{h}$ over $\mathbb{K}$.

(ii) If r is multiple of 4, we denote by $\xi^\pm$ the half-spin representations of $\mathfrak{so}(r)=\mathfrak{spin}(r)$ on the half-spin modules $\Sigma_r^\pm$ and by $\mu_\pm=\xi^\pm\circ\rho_*$ their compositions with ρ_*. Then the infinitesimal isotropy representation ι_* on $\mathfrak{m}$ is isomorphic to $\mu_+\otimes_{\mathbb{K}}\lambda_+\oplus\mu_-\otimes_{\mathbb{K}}\lambda_-$ for some representations $\lambda_\pm$ of $\mathfrak{h}$ over $\mathbb{K}$.

Proof. (i) Consider first the case when r is not a multiple of 4. By definition, the H-equivariant representation $\varphi:\mathfrak{so}(r)\to\mathrm{End}^-(\mathfrak{m})$ extends to an algebra representation of the even Clifford algebra $\mathrm{Cl}_r^0\simeq\mathbb{K}(n_r)$ on $\mathfrak{m}$. Since every algebra representation of the matrix algebra $\mathbb{K}(n)$ decomposes in a direct sum of irreducible representations, each of them isomorphic to the standard representation on $\mathbb{K}^n$, we deduce that φ is a direct sum of several copies of Σ_r. In other words, $\mathfrak{m}$ is isomorphic to $\Sigma_r\otimes_{\mathbb{K}}\mathbb{K}^p$ for some p, and φ is given by $\varphi(A)(\psi\otimes v)=(\xi(A)\psi)\otimes v$. We now study the isotropy representation ι_* on $\mathfrak{m}=\Sigma_r\otimes_{\mathbb{K}}\mathbb{K}^p$. Note that when $\mathbb{K}=\mathbb{H}$ is non-Abelian, some care is required in order to define the tensor product of representations over $\mathbb{K}$.

The H-equivariance of φ is equivalent to:

$$\iota(h)\circ\varphi(A)\circ\iota(h)^{-1}=\varphi(\rho(h)A),\qquad\forall A\in\mathfrak{so}(r),\forall h\in H.$$

Differentiating this relation at $h=1$ yields

$$\iota_*(X)\circ\varphi(A)-\varphi(A)\circ\iota_*(X)=\varphi(\rho_*(X)A),\qquad\forall A\in\mathfrak{so}(r),\forall X\in\mathfrak{h}.$$

On the other hand,

$$\varphi(\rho_*(X)A)=\varphi([\rho_*(X),A])=[\varphi(\rho_*(X)),\varphi(A)]=[\mu(X),\varphi(A)],$$

so

$$[\iota_*(X)-\mu(X),\varphi(A)]=0,\qquad\forall A\in\mathfrak{so}(r),\forall X\in\mathfrak{h}.\tag{4}$$

We denote by $\lambda:=\iota_*-\mu$. If $\{v_i\}$ denotes the standard basis of $\mathbb{K}^p$ we introduce the maps $\lambda_{ij}:\mathfrak{h}\to\mathrm{End}_{\mathbb{K}}(\Sigma_r)$ by

$$\lambda(X)(\psi\otimes v_i)=\sum_{j=1}^p\lambda_{ji}(X)(\psi)\otimes v_j.$$

The previous relation shows that $\lambda_{ij}(X)$ commutes with the Clifford action $\xi(A)$ on Σ_r for every $A\in\mathfrak{so}(r)$, so it belongs to $\mathbb{K}$. The matrix with entries $\lambda_{ij}(X)$ thus defines a Lie algebra representation $\lambda:\mathfrak{h}\to\mathrm{End}_{\mathbb{K}}(\mathbb{K}^p)$ such that

$$\iota_*(X)(\psi\otimes v)=\mu(X)(\psi)\otimes v+\psi\otimes\lambda(X)(v),\qquad\forall X\in h,\forall\psi\in\Sigma_r,\forall v\in\mathbb{K}^p.$$

This proves the lemma in this case.

(ii) If r is multiple of 4, the even Clifford algebra Cl_r^0 has two inequivalent algebra representations $\Sigma_r^\pm$. One can write like before $\mathfrak{m}=\Sigma^+\otimes_{\mathbb{K}}\mathbb{K}^p\oplus\Sigma^-\otimes_{\mathbb{K}}\mathbb{K}^{p'}$ for some $p,p'\geq 0$, and φ is given by $\varphi(A)(\psi^+\otimes v+\psi^-\otimes v')=(\xi^+(A)\psi^+)\otimes v+(\xi^-(A)\psi^-)\otimes v'$. The rest of the proof is similar, using the fact that every endomorphism from $\Sigma_r^\pm$ to $\Sigma_r^\mp$ commuting with the Clifford action of $\mathfrak{so}(r)$ vanishes. ∎

Let us introduce the ideals $\mathfrak{h}_1 := \ker(\rho_*)$ and $\mathfrak{h}_2 := \ker(\lambda)$ of $\mathfrak{h}$. Since the isotropy representation is faithful, $\mathfrak{h}_1 \cap \mathfrak{h}_2 = 0$ and it is easy to see that $\mathfrak{h}_1$ is orthogonal to $\mathfrak{h}_2$ with respect to the restriction to $\mathfrak{h}$ of any $\mathrm{ad}_{\mathfrak{g}}$-invariant scalar product. Denoting by $\mathfrak{h}_0$ the orthogonal complement of $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ in $\mathfrak{h}$ we obtain the following orthogonal decomposition:

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \tag{5}$$

and the corresponding splitting of the Cartan subalgebra of $\mathfrak{h}$: $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1 \oplus \mathfrak{t}_2$.

Lemma 3.2 yields further a description of the weights of the isotropy representation of homogeneous spaces with Clifford structure. We assume from now on that $\mathrm{rk}(G) = \mathrm{rk}(H)$ and choose a common Cartan subalgebra $\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{g}$. The system of roots of $\mathfrak{g}$ is then the disjoint union of the system of roots of $\mathfrak{h}$ and the weights of the complexified isotropy representation. Since each weight is simple (cf. [8, p. 38]) we deduce that all weights of $\mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C}$ are simple.

If r is not multiple of 4, Lemma 3.2 (i) shows that the isotropy representation $\mathfrak{m}$ is isomorphic to $\mu \otimes_{\mathbb{K}} \lambda$ for some representations μ and λ of $\mathfrak{h}$ over $\mathbb{K}$. In order to express $\mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C}$ it will be convenient to use the following convention: If ν is a representation over $\mathbb{K}$, we denote by $\nu^{\mathbb{C}}$ the representation over $\mathbb{C}$ given by

$$\begin{cases} \nu^{\mathbb{C}} = \nu \otimes_{\mathbb{R}} \mathbb{C}, & \text{if } \mathbb{K} = \mathbb{R} \\ \nu^{\mathbb{C}} = \nu, & \text{if } \mathbb{K} = \mathbb{C} \\ \nu^{\mathbb{C}} = \nu, & \text{if } \mathbb{K} = \mathbb{H} \end{cases}$$

where in the last row ν is viewed as complex representation by fixing one of the complex structures. Using the fact that if μ and λ are quaternionic representations, then there is a natural isomorphism between $(\mu \otimes_{\mathbb{H}} \lambda)^{\mathbb{C}}$ and $\mu^{\mathbb{C}} \otimes_{\mathbb{C}} \lambda^{\mathbb{C}}$, one can then write

$$\begin{cases} \mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C} = \mu^{\mathbb{C}} \otimes_{\mathbb{C}} \lambda^{\mathbb{C}}, & \text{if } \mathbb{K} = \mathbb{R} \text{ or } \mathbb{H} \\ \mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C} = \mu^{\mathbb{C}} \otimes_{\mathbb{C}} \lambda^{\mathbb{C}} \oplus \bar{\mu}^{\mathbb{C}} \otimes_{\mathbb{C}} \bar{\lambda}^{\mathbb{C}}, & \text{if } \mathbb{K} = \mathbb{C} \end{cases} \tag{6}$$

If r is multiple of 4, then $\mathfrak{m} = \mu_+ \otimes_{\mathbb{K}} \lambda_+ \oplus \mu_- \otimes_{\mathbb{K}} \lambda_-$ by Lemma 3.2 (ii), and the field $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{H}$. Consequently,

$$\mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C} = \mu_+^{\mathbb{C}} \otimes_{\mathbb{C}} \lambda_+^{\mathbb{C}} \oplus \mu_-^{\mathbb{C}} \otimes_{\mathbb{C}} \lambda_-^{\mathbb{C}}. \tag{7}$$

Let us denote by $\mathcal{A} := \{\alpha_1, \ldots, \alpha_p\} \subset \mathfrak{t}^*$ the weights of the representation $\lambda^{\mathbb{C}}$, defined when r is not a multiple of 4. For $r = 2q+1$ $\lambda^{\mathbb{C}}$ is self-dual, so $\mathcal{A} = -\mathcal{A}$. Moreover, $\mathbb{K} = \mathbb{H}$ if $q \equiv 1$ or $2 \mod 4$, so $p = \sharp\mathcal{A}$ is even, whereas for $q \equiv 0$ or $3 \mod 4$ p might be odd, *i.e.* one of the vectors α_i may vanish.

For $r = 2q$ with q even, we denote by $\mathcal{A} := \{\alpha_1, \ldots, \alpha_p\}$ and $\mathcal{G} := \{\gamma_1, \ldots, \gamma_{p'}\}$ the weights of the representations $\lambda_\pm^{\mathbb{C}}$. Since they are both self-dual we have $\mathcal{A} = -\mathcal{A}$ and $\mathcal{G} = -\mathcal{G}$ and we note that $\mathbb{K} = \mathbb{H}$ for $q \equiv 2 \mod 4$, whence p and p' are even in this case.

Recall now that for $r = 2q+1$, the weights of the complex spin representation $\Sigma_r^{\mathbb{C}}$ are

$$\mathfrak{W}(\Sigma_r^{\mathbb{C}}) = \left\{ \sum_{j=1}^{q} \varepsilon_j e_j,\ \varepsilon_j = \pm 1 \right\},$$

where $\{e_j\}$ is some orthonormal basis of the dual of some Cartan subalgebra of $\mathfrak{so}(2q+1)$. Similarly, if $r = 2q$ with q odd, the weights of the complex spin representation $\Sigma_r^{\mathbb{C}}$ are

$$\mathfrak{W}(\Sigma_r^{\mathbb{C}}) = \left\{ \sum_{j=1}^{q} \varepsilon_j e_j,\ \varepsilon_j = \pm 1, \prod_{j=1}^{q} \varepsilon_j = 1 \right\},$$

and for $r = 2q$ with q even, the weights of the complex half-spin representations $(\Sigma_r^{\pm})^{\mathbb{C}}$ are

$$\mathfrak{W}((\Sigma_r^{+})^{\mathbb{C}}) = \left\{ \sum_{j=1}^{q} \varepsilon_j e_j,\ \varepsilon_j = \pm 1, \prod_{j=1}^{q} \varepsilon_j = 1 \right\},$$

$$\mathfrak{W}((\Sigma_r^{-})^{\mathbb{C}}) = \left\{ \sum_{j=1}^{q} \varepsilon_j e_j,\ \varepsilon_j = \pm 1, \prod_{j=1}^{q} \varepsilon_j = -1 \right\}.$$

We denote by $\beta_j \in \mathfrak{t}^*$ the pull-back through μ_* of the vectors $\frac{1}{2}e_j$, for $j = 1, \ldots, q$. Since $\mu = \xi \circ \rho_*$ (and $\mu_\pm = \xi^\pm \circ \rho_*$ for r multiple of 4), the above relations give directly the weights of $\mu^{\mathbb{C}}$ or $\mu_\pm^{\mathbb{C}}$ as linear combinations of the vectors β_j. Taking into account Lemma 2.6, Lemma 3.2, (6)-(7) and the previous discussion, we obtain the following description of the weights of the isotropy representation of a homogeneous Clifford structure:

Proposition 3.3. *If there exists a homogeneous Clifford structure of rank r on a compact homogeneous space G/H with $\mathrm{rk}(G) = \mathrm{rk}(H)$, then the set $\mathcal{W} := \mathcal{W}(\mathfrak{m})$ of weights of the isotropy representation is an admissible subsystem of roots of $\mathcal{R}(\mathfrak{g})$ and is of one of the following types:*

(I) *If $r = 2q+1$, then there exists $\mathcal{A} := \{\alpha_1, \ldots, \alpha_p\} \subset \mathfrak{t}^*$ with $\mathcal{A} = -\mathcal{A}$ such that $\mathcal{W} = \mathcal{A} + \{\sum_{j=1}^{q} \varepsilon_j \beta_j\}_{\varepsilon \in \mathcal{E}_q}$ and $\sharp\mathcal{W} = p \cdot 2^q$. Moreover, if $q \equiv 1$ or $2 \mod 4$ then p is even, so $\alpha_i \neq 0$ for all i.*

(II) *If $r = 2q$ with q odd, then $\mathcal{W} = \{(\prod_{j=1}^{q} \varepsilon_j)\alpha_i + \sum_{j=1}^{q} \varepsilon_j \beta_j\}_{i=\overline{1,p}, \varepsilon \in \mathcal{E}_q}$ and $\sharp\mathcal{W} = p \cdot 2^q$.*

(III) *If $r = 2q$ with $q \equiv 2 \mod 4$, then there exist $\mathcal{A} := \{\alpha_1, \ldots, \alpha_p\}$ and $\mathcal{G} := \{\gamma_1, \ldots, \gamma_{p'}\}$ in $\mathfrak{t}^*$ with $\mathcal{A} = -\mathcal{A}$ and $\mathcal{G} = -\mathcal{G}$ such that*

$$\mathcal{W} = \mathcal{A} + \left\{ \sum_{j=1}^{q} \varepsilon_j \beta_j \,|\, \prod_{j=1}^{q} \varepsilon_j = 1 \right\}_{\varepsilon \in \mathcal{E}_q} \bigcup \mathcal{G} + \left\{ \sum_{j=1}^{q} \varepsilon_j \beta_j \,|\, \prod_{j=1}^{q} \varepsilon_j = -1 \right\}_{\varepsilon \in \mathcal{E}_q}$$

and $\sharp\mathcal{W} = (p + p') \cdot 2^{q-1}$. In this case one of p or p' might vanish, but p and p' are even, so the vectors α_i and γ_i are all non-zero.

(IV) *If $r = 2q$ with $q \equiv 0 \mod 4$ (in this case the semi-spinorial representation is real), then there exist $\mathcal{A} := \{\alpha_1, \ldots, \alpha_p\}$ and $\mathcal{G} := \{\gamma_1, \ldots, \gamma_{p'}\}$ in $\mathfrak{t}^*$ with $\mathcal{A} = -\mathcal{A}$ and $\mathcal{G} = -\mathcal{G}$ such that*

$$\mathcal{W} = \mathcal{A} + \left\{ \sum_{j=1}^{q} \varepsilon_j \beta_j \,|\, \prod_{j=1}^{q} \varepsilon_j = 1 \right\}_{\varepsilon \in \mathcal{E}_q} \bigcup \mathcal{G} + \left\{ \sum_{j=1}^{q} \varepsilon_j \beta_j \,|\, \prod_{j=1}^{q} \varepsilon_j = -1 \right\}_{\varepsilon \in \mathcal{E}_q}$$

and $\sharp\mathcal{W} = (p + p') \cdot 2^{q-1}$. *In this case one of* p *or* p' *might vanish, as well as one of the vectors* α_i *or* γ_i.

In order to describe the homogeneous Clifford structures we shall now obtain by purely algebraic arguments several restrictions on the possible sets of weights of the isotropy representation given by Proposition 3.3.

Proposition 3.4. *Let* $\mathcal{A} := \{\alpha_1, \dots, \alpha_p\}$ *and* $\mathcal{B} := \{\beta_1, \dots, \beta_q\}$ *be subsets in a Euclidean space* $(V, \langle\cdot,\cdot\rangle)$ *with* $\beta_j \neq 0, j = \overline{1,q}$. *The following restrictions for* q *hold:*

(I) *If* $\mathcal{A} = -\mathcal{A}$ *and* $\mathcal{P}_1 := \mathcal{A} + \{\sum_{j=1}^{q} \varepsilon_j\beta_j\}_{\varepsilon\in\mathcal{E}_q}$ *is a subsystem of roots, then* $q \leq 4$. *Moreover, if* $q = 4$, *then* $\alpha_i = 0$ *for all* $1 \leq i \leq p$.

(II) *If* q *is odd and* $\mathcal{P}_2 := \{(\prod_{j=1}^{q} \varepsilon_j)\alpha_i + \sum_{j=1}^{q} \varepsilon_j\beta_j\}_{i=\overline{1,p},\varepsilon\in\mathcal{E}_q}$ *is a subsystem of roots, then* $q \leq 7$. *Moreover, if* $q = 5$ *or* $q = 7$, *then* $\alpha_i \neq 0$ *for all* $1 \leq i \leq p$.

(III)-(IV) *If* q *is even,* $\mathcal{A} = -\mathcal{A}$ *and*

$$\mathcal{P}_3 := \mathcal{A} + \left\{\sum_{j=1}^{q} \varepsilon_j\beta_j \,|\, \prod_{j=1}^{q} \varepsilon_j = 1\right\}_{\varepsilon\in\mathcal{E}_q}$$

or

$$\mathcal{P}_4 := \mathcal{A} + \left\{\sum_{j=1}^{q} \varepsilon_j\beta_j \,|\, \prod_{j=1}^{q} \varepsilon_j = -1\right\}_{\varepsilon\in\mathcal{E}_q}$$

is a subsystem of roots, then $q \leq 8$. *Moreover, if* $q = 8$, *then* $\alpha_i = 0$ *for all* $1 \leq i \leq p$. *Thus, if there exists some* $\alpha_i \neq 0$, *it follows that* $q \leq 6$.

Proof. (I) If there exists $i \in \{1, \dots, p\}$ such that $\alpha_i \neq 0$, let $\beta_0 := \alpha_i$ and $\beta := \sum_{j=0}^{q} \beta_j$. By changing the signs if necessary, we may assume without loss of generality that β has the largest norm among all elements of $\mathcal{P}_1$ and $\|\beta\|^2 = 1$. From (1) it follows that

$$\langle \beta, \beta - 2\beta_j \rangle \in \left\{0, \pm\frac{1}{2}\right\}, j = 0, \dots, q.$$

We then have $\frac{1}{2}(q + 1) \geq \sum_{j=0}^{q} \langle \beta, \beta - 2\beta_j \rangle = q - 1$, which shows that $q \leq 3$. If $\alpha_i = 0$ for all $i \in \{1, \dots, p\}$, it follows by the same argument that $q \leq 4$.

(II) If there exists $i \in \{1, \dots, p\}$ such that $\alpha_i = 0$, then $\{\sum_{j=1}^{q} \varepsilon_j\beta_j\}_{\varepsilon\in\mathcal{E}_q} \subset \mathcal{P}_2$ and it follows from (I) that $q \leq 4$. In particular, this shows that if $q \in \{5, 7\}$, then $\alpha_i \neq 0$, for all $i = 1, \dots, p$.

Otherwise, if $\alpha_i \neq 0$ for all $i = 1, \dots, p$, then by denoting $\beta_0 := \alpha_1$, we have $\{(\prod_{j=1}^{q} \varepsilon_j)\alpha_1 + \sum_{j=1}^{q} \varepsilon_j\beta_j\}_{\varepsilon\in\mathcal{E}_q} = \{\sum_{j=0}^{q} \varepsilon_j\beta_j | \prod_{j=0}^{q} \varepsilon_j = 1\}_{\varepsilon\in\mathcal{E}_{q+1}} \subset \mathcal{P}_2$. This subset is of the same type as those considered in (III)–(IV) with $q+1$ even and with all $\alpha_i = 0$. It then follows from (III)–(IV) that $q+1 \leq 8$, so $q \leq 7$.

(III)–(IV): If we denote by $\beta'_j := \beta_{2j-1} + \beta_{2j}$, for $j = 1, \dots, \frac{q}{2}$, then $\mathcal{A} + \{\sum\limits_{j=1}^{q/2} \varepsilon_j \beta'_j\}_{\varepsilon \in \mathcal{E}_{q/2}} \subset \mathcal{P}_3$ is a subsystem of roots. It then follows from (I) that $q \leq 8$ and the equality is attained only if $\alpha_i = 0$, for all $i = 1, \dots, p$. The same argument holds for $\mathcal{P}_4$ if we choose $\beta'_1 = \beta_1 - \beta_2$ and $\beta'_j := \beta_{2j-1} + \beta_{2j}$, for $j = 2, \dots, \frac{q}{2}$. $\square$

We now give a more precise description of the subsystems of roots that may occur in the limiting cases of Proposition 3.4. Namely, we determine all the possible scalar products between the roots.

Lemma 3.5. (*a*) *Let* $\mathcal{P} := \{\sum\limits_{j=1}^{q} \varepsilon_j \beta_j\}_{\varepsilon \in \mathcal{E}_q}$ *be a subsystem of roots with* $\sharp\mathcal{P} = 2^q$.

(i) *If* $q = 4$, *then the Gram matrix of scalar products* $(\langle \beta_i, \beta_j \rangle)_{ij}$ *is (up to a permutation of the subscripts and sign changes) one of the following:*

$$\frac{1}{4}\mathrm{id}_4 \ \textit{or}\ M_0 := \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{1}{16} & \frac{1}{16} \\ 0 & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ 0 & \frac{1}{16} & \frac{1}{16} & \frac{1}{8} \end{pmatrix}. \tag{8}$$

Moreover, if $\mathcal{P}$ *is admissible, then only the first case can occur, the Gram matrix is* $(\langle \beta_i, \beta_j \rangle)_{ij} = \frac{1}{4}\mathrm{id}_4$ *and* $\overline{\mathcal{P}} = \mathcal{R}(\mathfrak{so}(8))$, $\overline{\mathcal{P}} \setminus \mathcal{P} = \mathcal{R}(\mathfrak{so}(4) \oplus \mathfrak{so}(4))$.

(ii) *If* $q = 3$, *then the Gram matrix of scalar products* $(\langle \beta_i, \beta_j \rangle)_{ij}$ *is (up to a permutation of the subscripts and sign changes) one of the following:*

$$M_1 := \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \ \textit{or}\ M_2 := \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & \frac{1}{4} & 0 \\ \frac{1}{6} & 0 & \frac{1}{12} \end{pmatrix} \ \textit{or}\ M_3 := \begin{pmatrix} \frac{3}{8} & \frac{1}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{8} \end{pmatrix}. \tag{9}$$

Moreover, if $\mathcal{P}$ *is admissible, then only the first two cases can occur. For the Gram matrix* $(\langle \beta_i, \beta_j \rangle)_{ij} = M_1$ *the subsystems of roots are* $\overline{\mathcal{P}} = \mathcal{R}(\mathfrak{so}(6))$ *and* $\overline{\mathcal{P}} \setminus \mathcal{P} = \mathcal{R}(\mathfrak{so}(4))$ *and for* $(\langle \beta_i, \beta_j \rangle)_{ij} = M_2$, $\overline{\mathcal{P}} = \mathcal{R}(\mathfrak{g}_2)$ *and* $\overline{\mathcal{P}} \setminus \mathcal{P} = \mathcal{R}(\mathfrak{so}(4))$.

(*b*) *Let* $\mathcal{P} := \{\sum\limits_{j=1}^{q} \varepsilon_j \beta_j \mid \prod\limits_{j=1}^{q} \varepsilon_j = 1\}_{\varepsilon \in \mathcal{E}_q}$ *be an admissible subsystem of roots with* $\sharp\mathcal{P} = 2^q$. *If* $q = 8$, *then the Gram matrix* $(\langle \beta_i, \beta_j \rangle)_{ij}$ *is (up to a permutation of the subscripts and sign changes) equal to* $\frac{1}{8}\mathrm{id}_8$.

Proof. (i) As in the proof of Proposition 3.4, we denote by $\beta := \sum\limits_{j=0}^{q} \beta_j$ and, up to sign changes, we may assume that β has the largest norm among all elements of $\mathcal{P}$ and that this norm is equal to 1. We consider the Gram matrix of scalar products $(\langle \beta_i, \beta_j \rangle)_{ij}$. Since $\langle \beta_j, \beta \rangle = \frac{1}{4}$ for all $j = \overline{1, 4}$, the sum of the elements of each of its lines is $\frac{1}{4}$.

Since $\|\beta\|^2 = 1$ is the largest norm of the roots in $\mathcal{P}$, it follows that the square norms of the other roots may take the following values: $\{1, \frac{1}{2}, \frac{1}{3}\}$, so that

$$\|\beta - 2\beta_i\|^2 = 4\|\beta_i\|^2 \Rightarrow \|\beta_i\|^2 \in \left\{\frac{1}{4}, \frac{1}{8}, \frac{1}{12}\right\}, \text{ for all } 1 \le i \le 4. \tag{10}$$

Let $i, j \in \{1, \ldots, 4\}$, $i \neq j$ and assume that $\|\beta_i\|^2 \geq \|\beta_j\|^2$. As $\langle\, \beta - 2\beta_i,\, \beta - 2\beta_j \,\rangle = 4\langle\, \beta_i,\, \beta_j \,\rangle$, it follows by (1) that $\langle\, \beta_i,\, \beta_j \,\rangle \in \{0, \pm\frac{1}{2}\|\beta_i\|^2\}$.

The case $\langle\, \beta_i,\, \beta_j \,\rangle = -\frac{1}{2}\|\beta_i\|^2$ cannot occur, because it leads to the following contradiction:

$$0 < \|\beta - 2\beta_i - 2\beta_j\|^2 = 4\|\beta_i + \beta_j\|^2 - 1 = 4\|\beta_j\|^2 - 1 \le 0.$$

Assume that there exists $i, j \in \{1, \ldots, 4\}$, $i \neq j$, such that $\langle\, \beta_i,\, \beta_j \,\rangle = \frac{1}{2}\|\beta_i\|^2$. It then follows $\|\beta_i + \beta_j\|^2 = 2\|\beta_i\|^2 + \|\beta_j\|^2 \in \{\frac{1}{2}, \frac{3}{8}, \frac{1}{3}\}$, which combined with the restrictions (10) yield the following possible values: either $\|\beta_i\|^2 = \|\beta_j\|^2 = \frac{1}{8}$ or $\|\beta_i\|^2 = \frac{1}{8}, \|\beta_j\|^2 = \frac{1}{12}$. In both cases $\langle\, \beta_i,\, \beta_j \,\rangle = \frac{1}{16}$. Hence, all non-diagonal entries are either 0 or $\frac{1}{16}$ and since the sum of the elements of each line is equal to $\frac{1}{4}$, the case $\|\beta_j\|^2 = \frac{1}{12}$ is excluded. It follows that each line of the Gram matrix $(\langle\, \beta_i,\, \beta_j \,\rangle)_{ij}$ either has all non-diagonal entries equal to 0 and the diagonal term is $\frac{1}{4}$ or two of them are $\frac{1}{16}$, one is 0 and the diagonal term is $\frac{1}{8}$. In particular, $\|\beta_i\|^2 \in \{\frac{1}{4}, \frac{1}{8}\}$, for all $1 \le i \le 4$. Up to a permutation of the subscripts we may assume that $\|\beta_i\|^2 \geq \|\beta_j\|^2$ for all $1 \le i < j \le 4$. The following two cases may occur: either $\|\beta_1\|^2 = \|\beta_2\|^2 = \frac{1}{4}$ or $\|\beta_1\|^2 = \frac{1}{4}$ and $\|\beta_2\|^2 = \frac{1}{8}$.

In the first case at most one non-diagonal term may be $\frac{1}{16}$, showing that they must all be equal to 0. This contradicts $\langle\, \beta_i,\, \beta_j \,\rangle = \frac{1}{2}\|\beta_i\|^2 \neq 0$. In the second case, since we have one non-diagonal term equal to $\frac{1}{16}$, we must have another one and then $\|\beta_3\|^2 = \|\beta_4\|^2 = \frac{1}{8}$. Hence, the corresponding Gram matrix is equal to M_0 defined by (8).

The subsystem of roots corresponding to the matrix M_0 is not admissible, since the necessary criterion given by Lemma 2.5 is not fulfilled: the set $\{\beta_i\}_{i=\overline{1,4}}$ is linearly independent (since the Gram matrix is invertible) and β, $\beta - 2\beta_2$ are two roots of the subsystem, which have different norms $\|\beta - 2\beta_2\|^2 = \frac{1}{2} = \frac{1}{2}\|\beta\|^2$ and are not orthogonal: $\langle\, \beta,\, \beta - 2\beta_2 \,\rangle = \frac{1}{2}$.

The only case left is when $\{\beta_i\}_{i=\overline{1,4}}$ is an orthonormal system. In this case the new roots obtained by considering all possible reflections are the roots $\{\pm 2\beta_i\}_{i=\overline{1,4}}$, which are of the same norm and orthogonal to each other and thus build the system of roots of $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Then the minimal set of roots is $\overline{\mathcal{P}} = \{\pm \sum_{j=1}^{4} \varepsilon_j \beta_j\} \cup \{\pm 2\beta_i\}_{i=\overline{1,4}}$, which is the system of roots of $\mathfrak{so}(8)$, proving (i).

(ii) If $q = 3$, we assume as above that $\beta := \beta_1 + \beta_2 + \beta_3$ is the element of $\mathcal{P}$ of maximal norm and $\|\beta\|^2 = 1$. From (1) it follows that $\langle\, \beta,\, \beta - 2\beta_i \,\rangle \in \{0, \pm\frac{1}{2}\}$, for all $i = \overline{1,3}$, yielding that $\langle\, \beta,\, \beta_i \,\rangle \in \{\frac{1}{2}, \frac{1}{4}, \frac{3}{4}\}$. Since $1 = \|\beta\|^2 = \sum_{i=1}^{3}\langle\, \beta_i,\, \beta \,\rangle$, the only possible values (up to a permutation of the subscripts) are $\langle\, \beta_1,\, \beta \,\rangle = \frac{1}{2}$ and $\langle\, \beta_2,\, \beta \,\rangle = \langle\, \beta_3,\, \beta \,\rangle = \frac{1}{4}$. Then, from

$\|\beta - 2\beta_i\|^2 \in \{1, \frac{1}{2}, \frac{1}{3}\}$, it follows that

$$\|\beta_1\|^2 \in \left\{\frac{1}{2}, \frac{3}{8}, \frac{1}{3}\right\}, \quad \|\beta_2\|^2, \|\beta_3\|^2 \in \left\{\frac{1}{4}, \frac{1}{8}, \frac{1}{12}\right\}. \tag{11}$$

Since $\langle\, \beta,\, \beta - 2\beta_1 \,\rangle = 0$ and $\langle\, \beta,\, \beta - 2\beta_2 \,\rangle = \langle\, \beta,\, \beta - 2\beta_3 \,\rangle = \frac{1}{2}$, we obtain the following expressions for the scalar products:

$$2\langle\, \beta_2,\, \beta_3 \,\rangle = \|\beta_1\|^2 - \|\beta_2\|^2 - \|\beta_3\|^2, \tag{12}$$

$$2\langle\, \beta_1,\, \beta_2 \,\rangle = \frac{1}{2} + \|\beta_3\|^2 - \|\beta_1\|^2 - \|\beta_2\|^2, \tag{13}$$

$$2\langle\, \beta_1,\, \beta_3 \,\rangle = \frac{1}{2} + \|\beta_2\|^2 - \|\beta_1\|^2 - \|\beta_3\|^2. \tag{14}$$

The other conditions for the scalar products of roots in $\mathcal{P}$ obtained from (1) are: $\langle\, \beta - 2\beta_i,\, \beta - 2\beta_j \,\rangle \in \{0, \pm\frac{1}{2}\max(\|\beta - 2\beta_i\|^2, \|\beta - 2\beta_j\|^2)\}$, for all $1 \leq i < j \leq 3$, which imply

$$\langle\, \beta_2,\, \beta_3 \,\rangle \in \left\{0, \pm\frac{1}{2}\max(\|\beta_2\|^2, \|\beta_3\|^2)\right\}, \tag{15}$$

$$\langle\, \beta_1,\, \beta_i \,\rangle \in \left\{\frac{1}{8}, \frac{1}{8} \pm \frac{1}{2}\max(\|\beta_1\|^2 - \frac{1}{4}, \|\beta_i\|^2)\right\}, i = 2, 3. \tag{16}$$

We may assume (up to a permutation) that $\|\beta_2\|^2 \geq \|\beta_3\|^2$. By substituting (12)–(14) in (15)–(16) we obtain the following conditions: $\|\beta_1\|^2 - \|\beta_2\|^2 - \|\beta_3\|^2 \in \{0, \pm\|\beta_2\|^2\}$ and $\|\beta_2\|^2 - \|\beta_1\|^2 - \|\beta_3\|^2 + \frac{1}{4} \in \{0, \pm\|\beta_3\|^2\}$, which together with the restrictions (11) for the norms yield the following possible values:

$$(\|\beta_1\|^2, \|\beta_2\|^2, \|\beta_3\|^2) \in \left\{\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{3}, \frac{1}{4}, \frac{1}{12}\right), \left(\frac{3}{8}, \frac{1}{8}, \frac{1}{8}\right)\right\}.$$

We thus obtain that the Gram matrix $(\langle\, \beta_i,\, \beta_j \,\rangle)_{ij}$ must be equal to one of the three matrices M_i, $i = \overline{1, 3}$, defined by (9).

By Lemma 2.5, the subsystem of roots corresponding to M_3 is not admissible, since the set $\{\beta_i\}_{i=\overline{1,3}}$ is linearly independent (its Gram matrix is invertible) and there exist two roots β, $\beta - 2\beta_2$ of different norms $\|\beta - 2\beta_2\|^2 = \frac{1}{2} = \frac{1}{2}\|\beta\|^2$, which are not orthogonal: $\langle\, \beta,\, \beta - 2\beta_2 \,\rangle = \frac{1}{2}$.

The subsystem of roots corresponding to the matrix M_1 is admissible and in this case the minimal set of roots containing $\mathcal{P}$ is obtained by adjoining the roots $\pm 2\beta_2$ and $\pm 2\beta_3$, which also have norm equal to 1, showing that $\overline{\mathcal{P}} = \mathcal{R}(\mathfrak{so}(6))$ and $\overline{\mathcal{P}} \setminus \mathcal{P} = \mathcal{R}(\mathfrak{so}(4))$.

For the Gram matrix M_2, it follows that $\beta_1 = 2\beta_3$ and $\{\beta_1, \beta_2\}$ are linearly independent. The roots in $\mathcal{P}$ have the following norms: $\|\beta\|^2 = \|\beta - 2\beta_2\|^2 = 1$ and $\|\beta - 2\beta_1\|^2 = \|\beta - \beta_1\|^2 = \frac{1}{3}$ and the roots added by all possible reflections in order to obtain the minimal system of roots $\overline{\mathcal{P}}$ are $\{\pm\beta_1, \pm 2\beta_2\}$ with $\|\beta_1\|^2 = \frac{1}{3}$, $\|2\beta_2\|^2 = 1$ and $\langle\, \beta_1,\, 2\beta_2 \,\rangle = 0$. It thus follows that $\overline{\mathcal{P}} \setminus \mathcal{P} = \mathcal{R}(\mathfrak{su}(2) \oplus \mathfrak{su}(2))$ and $\overline{\mathcal{P}} = \mathcal{R}(\mathfrak{g}_2)$ (cf. [2, p. 32]).

(b) If we denote by $\beta'_j := \beta_{2j-1}+\beta_{2j}$, for $j = 1, \ldots, 4$, then $\{\sum_{j=1}^{4}\varepsilon_j\beta'_j\}_{\varepsilon\in\mathcal{E}_4} \subset \mathcal{P}$ is a subsystem of roots. From (i) it follows that the Gram matrix $(\langle\, \beta'_i,\, \beta'_j\,\rangle)_{i,j}$ is (up to rescaling, reordering and sign change of the vectors β'_j) either $\frac{1}{4}\mathrm{id}$ or the matrix M_0 defined by (8).

We claim that the second case cannot occur. Indeed, if this were the case, then $\|\beta_1+\beta_2\|^2 = \frac{1}{4}$, $\|\beta_{2j-1} + \beta_{2j}\|^2 = \frac{1}{8}$ for $j = 2, 3, 4$, and

$$\langle\, \beta_3 + \beta_4,\, \beta_5 + \beta_6\,\rangle = \frac{1}{16}. \tag{17}$$

For every j and k with $3 \le j < k \le 8$, there exists $l \in \{2, 3, 4\}$ such that $\{2l-1, 2l\}\cap\{j, k\} = \emptyset$. Let $\{s, t\}$ denote the complement of $\{1, 2, j, k, 2l-1, 2l\}$ in $\{1, \ldots, 8\}$. The Gram matrix of $\{\beta_1 + \beta_2, \beta_{2l-1} + \beta_{2l}, \beta_j - \beta_k, \beta_s - \beta_t\}$ has at least two different values on the diagonal. Again from (i), it follows that the remaining diagonal terms $\|\beta_j - \beta_k\|^2$ and $\|\beta_s - \beta_t\|^2$ must both be equal to $\frac{1}{8}$. Thus, $\|\beta_j - \beta_k\|^2 = \frac{1}{8}$ for all $3 \le j < k \le 8$. By the same argument we also obtain $\|\beta_j + \beta_k\|^2 = \frac{1}{8}$ for all $3 \le j < k \le 8$. Thus, $\langle\, \beta_j,\, \beta_k\,\rangle = 0$ for all $3 \le j < k \le 8$, contradicting (17).

This shows that the vectors β'_j are mutually orthogonal. Applying this to different partitions of the set $\{1, \ldots, 8\}$ into four pairs we get that $\beta_j+\beta_k$ is orthogonal to $\beta_s+\beta_t$ for all mutually distinct subscripts j, k, s and t. This clearly implies that $\langle\, \beta_i,\, \beta_j\,\rangle = 0$ for all $i \ne j$. It then also follows that $\|\beta_j\|^2 = \frac{1}{8}$, for $j = 1, \ldots, 8$, proving (b). □

4. Homogeneous Clifford Structures of high rank

A direct consequence of Propositions 3.3 and 3.4 is the following upper bound for the rank of a homogeneous Clifford structure:

Theorem 4.1. *The rank r of any even homogeneous Clifford structure on a homogeneous compact manifold G/H of non-vanishing Euler characteristic is less or equal to* 16*. More precisely, the following restrictions hold for the rank depending on its* 2*-valuation:*

- (I) *If r is odd, then $r \in \{3, 5, 7, 9\}$.*
- (II) *If $r \equiv 2 \mod 4$, then $r \in \{2, 6, 10\}$.*
- (III) *If $r \equiv 4 \mod 8$, then $r \in \{4, 12\}$.*
- (IV) *If $r \equiv 0 \mod 8$, then $r \in \{8, 16\}$.*

Proof. We only need to show that in case (II), the rank r is strictly less than 14. This will be done in the proof of Theorem 4.2 (II). □

We further describe the manifolds which occur in the limiting cases for the upper bounds in Theorem 4.1.

Theorem 4.2. *The maximal rank r of an even homogeneous Clifford structure for each of the types (I)-(IV) and the corresponding compact homogeneous manifolds $M = G/H$ (with $\mathrm{rk}(G) = \mathrm{rk}(H)$) carrying such a structure are the following:*

(I) $r = 9$ *and* M *is the Cayley projective space* $\mathbb{O}\mathbb{P}^2 = \mathrm{F}_4/\mathrm{Spin}(9)$.
(II) $r = 10$ *and* $M = (\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2 = \mathrm{E}_6/(\mathrm{Spin}(10) \times \mathrm{U}(1)/\mathbb{Z}_4)$.
(III) $r = 12$ *and* $M = (\mathbb{H} \otimes \mathbb{O})\mathbb{P}^2 = \mathrm{E}_7/\mathrm{Spin}(12) \cdot \mathrm{SU}(2)$.
(IV) $r = 16$ *and* $M = (\mathbb{O} \otimes \mathbb{O})\mathbb{P}^2 = \mathrm{E}_8/(\mathrm{Spin}(16)/\mathbb{Z}_2)$.

Proof. Let $M = G/H$ ($\mathrm{rk}(G) = \mathrm{rk}(H)$) carry a homogeneous Clifford structure of rank r.

(I) If r is odd, $r = 2q + 1$, then it follows from Theorem 4.1 (I) that $r \leq 9$.

By Proposition 3.3 (I) and Lemma 3.5 (i), the set $\mathcal{W} := \mathcal{W}(\mathfrak{m})$ of weights of the isotropy representation is $\mathcal{W} = \{\sum\limits_{j=1}^{q} \varepsilon_j \beta_j\}_{\varepsilon \in \mathcal{E}_q}$ with $\sharp\mathcal{W} = 2^q$ and $\mathcal{R}(\mathfrak{so}(8)) \subseteq \mathcal{R}(\mathfrak{g})$. In particular the representation λ is trivial, so $\mathfrak{h} = \mathfrak{h}_2$ and $\sharp\mathcal{R}(\mathfrak{g}) \geq 24$.

Since $\rho_* : \mathfrak{h}_2 \to \mathfrak{so}(9)$ is injective and $\mathfrak{h} = \mathfrak{h}_2$, it follows that $\mathrm{rk}(\mathfrak{h}) \leq 4$.

If $\mathrm{rk}(\mathfrak{h}) \leq 3$, then $\sharp\mathcal{R}(\mathfrak{g}) \leq 18$ by a direct check in the list of Lie algebras of rank 3. This contradicts the fact that $\sharp\mathcal{R}(\mathfrak{g}) \geq 24$.

Thus, $\mathrm{rk}(\mathfrak{h}) = 4$ and ρ_* is a bijection when restricted to a Cartan subalgebra of $\mathfrak{h}$. On the other hand, from Lemma 3.5 (i), it also follows that the Gram matrix $(\langle \beta_i, \beta_j \rangle)_{ij}$ is equal to $\frac{1}{4}\mathrm{id}_4$. The new roots obtained by reflections given by (3) are the following: $\pm 2\beta_i = \pm e_i \circ \rho_* \in \mathcal{R}(\mathfrak{h})$, for all $i = \overline{1,4}$. As $\mathrm{rk}(\mathfrak{h}) = 4 = \mathrm{rk}(\mathfrak{so}(9))$, we may apply Lemma 2.7 for $\mathfrak{h} \subseteq \mathfrak{so}(9)$ and get that $\{\pm e_i \pm e_j \,|\, 1 \leq i < j \leq 4\}$, which are roots of $\mathfrak{so}(9)$, are also roots of $\mathfrak{h}$. Thus, $\mathfrak{h} = \mathfrak{so}(9)$. The Lie algebra $\mathfrak{g}$, whose system of roots is obtained by joining the system of roots of $\mathfrak{so}(9)$ with the weights of the spinorial representation of $\mathfrak{spin}(9)$, is then exactly $\mathfrak{f}_4$ (cf. [2, p. 55]). We note that we cannot extend $\mathfrak{f}_4$, since there is no other larger Lie algebra of the same rank. Using the fact that the closed subgroup of F_4 corresponding to the above embedding of $\mathfrak{so}(9)$ in $\mathfrak{f}_4$ is $\mathrm{Spin}(9)$ (cf. [2, Thm. 6.1]), we deduce that the only homogeneous manifold carrying a homogeneous Clifford structure of rank 9 is the Cayley projective space $\mathbb{O}\mathbb{P}^2 = \mathrm{F}_4/\mathrm{Spin}(9)$.

(II) By Theorem 4.1 (II), $r \leq 14$. We first show that there exists no homogeneous Clifford structure of rank $r = 14$.

Let $r = 2q = 14$. In this case, by Proposition 3.3 (II), the set of weights of the isotropy representation is $\mathcal{W} := \mathcal{W}(\mathfrak{m}) = \left\{ (\prod\limits_{j=1}^{7} \varepsilon_j)\alpha_i + \sum\limits_{j=1}^{7} \varepsilon_j \beta_j \right\}_{i=\overline{1,p}, \varepsilon \in \mathcal{E}_7}$ with $\sharp\mathcal{W} = p \cdot 2^7$. Proposition 3.4 (II) yields that $\alpha_i \neq 0$, for all $1 \leq i \leq p$.

We claim that the following inclusion holds: $\mathcal{R}(\mathfrak{so}(16)) \subseteq \overline{\mathcal{W}} \setminus \mathcal{W}$. This can be seen as follows. Denoting by $\beta_0 := \alpha_1$ and $\beta := \beta_0 + \cdots + \beta_7$, the set $\mathcal{W}$ contains the following subsystem of roots $\{\sum\limits_{j=0}^{7} \varepsilon_j \beta_j | \prod\limits_{j=0}^{7} \varepsilon_j = 1\}_{\varepsilon \in \mathcal{E}_8}$. From Lemma 3.5 (b) it follows that the Gram matrix $(\langle \beta_i, \beta_j \rangle)_{i,j=\overline{0,7}}$ is equal to $\frac{1}{8}\mathrm{id}_8$. Then, for all $0 \leq i < j \leq 7$ we have $\langle \beta, \beta - 2\beta_i - 2\beta_j \rangle = \frac{1}{2}$, implying by (3) that there are new roots $\pm 2(\beta_i + \beta_j) \in \overline{\mathcal{W}} \setminus \mathcal{W}$. Similarly, for any

$0 \leq k \leq 7$ distinct from i and j, $\langle\, \beta - 2\beta_i - 2\beta_k,\, \beta - 2\beta_j - 2\beta_k \,\rangle = \frac{1}{2}$ yields the new roots $\pm 2(\beta_i - \beta_j) \in \overline{\mathcal{W}} \setminus \mathcal{W}$. It thus follows that $\mathcal{R}(\mathfrak{so}(16)) = \{\pm 2(\beta_i \pm \beta_j) \,|\, 0 \leq i \leq 7\} \subseteq \overline{\mathcal{W}} \setminus \mathcal{W}$.

Since $\mathcal{R}(\mathfrak{so}(16)) \subseteq \overline{\mathcal{W}} \setminus \mathcal{W} \subseteq \mathcal{R}(\mathfrak{h})$, it follows that $\mathfrak{so}(16)$ is a Lie subalgebra of $\mathfrak{h}$. Recall the splitting (5): $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_0 \oplus \mathfrak{h}_2 \subseteq \mathfrak{so}(14)$. As $\mathfrak{so}(16)$ is a simple Lie algebra, it follows that $\mathfrak{so}(16) \subseteq \mathfrak{h}_1$. In particular, this implies $p \geq 8$.

On the other hand, we show that $p = 1$. Assume that $p \geq 2$. By Lemma 3.5 (b), the Gram matrix of both subsystems of roots $\{\alpha_1, \beta_1, \dots, \beta_7\}$ and $\{\alpha_2, \beta_1, \dots, \beta_7\}$ is equal to $\frac{1}{8}\mathrm{id}_8$. Denoting by $a := \langle\, \alpha_1,\, \alpha_2 \,\rangle$, we obtain the following values for the scalar products between the roots containing α_1 and α_2: $\{a + \frac{7}{8}, a + \frac{3}{8}, a - \frac{1}{8}\}$. From (1), we know that these values must belong to $\{0, \pm\frac{1}{2}\}$. It then follows that the only possible value for $a = \langle\, \alpha_1,\, \alpha_2 \,\rangle$ is $-\frac{3}{8}$. This leads to a contradiction by computing the following norm: $\|\alpha_1 + \alpha_2\|^2 = -\frac{1}{2} < 0$.

Thus, the case $r = 14$ is not possible.

Now, for rank $r = 10 = 2q$, by Proposition 3.3 (II), the set of weights of the isotropy representation is $\mathcal{W} := \mathcal{W}(\mathfrak{m}) = \{(\prod\limits_{j=1}^{5} \varepsilon_j)\alpha_i + \sum\limits_{j=1}^{5} \varepsilon_j \beta_j\}_{i=\overline{1,p}, \varepsilon \in \mathcal{E}_5}$ with $\sharp\mathcal{W} = p \cdot 2^5$. From Proposition 3.4 (II), it then follows that $\alpha_i \neq 0$, for all $1 \leq i \leq p$. We will further show that $p = 1$ and the following inclusions hold: $\mathcal{R}(\mathfrak{spin}(10) \oplus \mathfrak{u}(1)) \subseteq \overline{\mathcal{W}} \setminus \mathcal{W}$, $\mathcal{R}(\mathfrak{e}_6) \subseteq \overline{\mathcal{W}}$.

Let $\beta_0 := \alpha_1$ and $\beta'_j := \beta_{2j} + \beta_{2j+1}$, for $j = 0, 1, 2$. Then $\{\sum_{j=0}^{2} \varepsilon_j \beta'_j\} \subset \mathcal{W}$ is an admissible subsystem of roots. By Lemma 3.5 (ii), the Gram matrix $B' := (\langle\, \beta'_i,\, \beta'_j \,\rangle)_{i,j=\overline{0,2}}$ is one of the three matrices in (9). We may assume that β'_0 is of maximal norm and equal to 1, so that the possible Gram matrices B' are normalized as follows:

$$M_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, M_2 := \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} \end{pmatrix}, M_3 := \begin{pmatrix} 1 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}. \tag{18}$$

We first note that the norm of $\beta := \sum_{j=0}^{5} \beta_j$, which is equal to the sum of all elements of the matrix B', may take the following values: $\|\beta\|^2 = 2$ for M_1, $\|\beta\|^2 = 3$ for M_2, $\|\beta\|^2 = \frac{8}{3}$ for M_3. We will show that the last two cases cannot occur.

Let us first assume that $B' = M_2$. In this case $\beta'_0 = 2\beta'_2$, *i.e.* $\beta_0 + \beta_1 = 2(\beta_4 + \beta_5)$. Considering now another pairing by permuting the subscripts $2, 3, 4, 5$, we get a Gram matrix which must also be equal to M_2, since $\|\beta\|^2$ does not change. We may thus assume that $\beta_0 + \beta_1 = 2(\beta_2 + \beta_4)$ and is furthermore equal either to $2(\beta_2 + \beta_5)$ or to $2(\beta_3 + \beta_4)$. In both cases it follows that there exists $i \neq j \in \{2, 3, 4, 5\}$, such that $\beta_i = \beta_j$. Then, for any $k \neq i, j$, the roots $\beta - 2\beta_i - 2\beta_k, \beta - 2\beta_j - 2\beta_k \in \mathcal{W}$ are equal, which contradicts $\sharp\mathcal{W} = p \cdot 2^5$. Thus, $B' \neq M_2$.

Let us now assume that the Gram matrix B' is equal to M_3. Then $\|\beta\|^2 = \frac{8}{3}$ and since this is the maximal norm, it follows that for any other possible pairing of the vectors β_j, the corresponding Gram matrix is either M_1 or M_3 (because the sum of all elements of M_2 is $3 > \frac{8}{3}$).

We consider as above other pairings by permuting the subscripts $\{2,3,4,5\}$. Again by Lemma 3.5 (ii), it follows that the corresponding Gram matrix is one of the matrices in (18) and, since $\|\beta\|^2$ does not change, it must also be equal to M_3. In particular, we have:

$$\|\beta_i + \beta_j\|^2 = \frac{1}{3}, \tag{19}$$

$$\langle\, \beta_i + \beta_j,\, \beta_k + \beta_l \,\rangle = \frac{1}{6}, \tag{20}$$

for any permutation (i,j,k,l) of $(2,3,4,5)$.

Consider now the following pairings of the vectors β_j:

$$\beta_0'' := \beta_0 + \beta_1, \beta_1'' := \pm(\beta_i - \beta_j), \beta_2'' := \pm(\beta_k - \beta_l),$$

where (i,j,k,l) is any permutation of $(2,3,4,5)$ and in each case the signs for β_1'' and β_2'' are chosen such that

$$\|\beta_0'' + \beta_1'' + \beta_2''\|^2 = \max\{\|\beta_0'' \pm \beta_1'' \pm \beta_2''\|^2\}.$$

Then $\{\sum_{j=0}^{2} \varepsilon_j \beta_j''\}$ is a subsystem of roots of $\mathcal{W}$ and by the same argument as above its Gram matrix $B'' := (\langle\, \beta_i'',\, \beta_j'' \,\rangle)_{i,j=\overline{0,2}}$ is either M_1 or M_3.

Since $\|\beta_0''\|^2 = 1$, it follows that in both cases the norms of the other two vectors are equal: $\|\beta_1''\|^2 = \|\beta_2''\|^2$, *i.e.* $\|\beta_i - \beta_j\|^2 = \|\beta_k - \beta_l\|^2 \in \{\frac{1}{2}, \frac{1}{3}\}$. By (19), we then obtain for any $i,j \in \{2,3,4,5\}$ that $\langle\, \beta_i,\, \beta_j \,\rangle = \frac{1}{4}(\|\beta_i + \beta_j\|^2 - \|\beta_i - \beta_j\|^2) \in \{0, -\frac{1}{24}\}$. It then follows that $\langle\, \beta_i + \beta_j,\, \beta_k + \beta_l \,\rangle < 0$, for any permutation (i,j,k,l) of $(2,3,4,5)$, which contradicts (20). Thus, $B' \neq M_3$.

The only possibility left is $B' = M_1$. Then $\|\beta\|^2 = 2$ and since it is the element of maximal norm, it follows that for any other pairing of the vectors β_j, the corresponding Gram matrix is also equal to M_1. We then have $B' = B'' = M_1$, which implies:

$$\langle\, \beta_0 + \beta_1,\, \beta_i \pm \beta_j \,\rangle = 0, \tag{21}$$

$$\|\beta_i + \beta_j\|^2 = \|\beta_i - \beta_j\|^2 = \frac{1}{2}, \tag{22}$$

$$\langle\, \beta_i + \beta_j,\, \beta_k + \beta_l \,\rangle = \langle\, \beta_i - \beta_j,\, \beta_k - \beta_l \,\rangle = 0, \tag{23}$$

for any permutation (i,j,k,l) of $(2,3,4,5)$. From (21) it follows that $\langle\, \beta_0 + \beta_1,\, \beta_i \,\rangle = 0$ for all $2 \leq i \leq 5$. From (23) it follows that $\langle\, \beta_i,\, \beta_j \,\rangle = 0$, for $2 \leq i < j \leq 5$ and then from (22) we obtain $\|\beta_i\|^2 = \frac{1}{4}$, for $2 \leq i \leq 5$.

For pairings of the following form $\beta_0 - \beta_1,\ \beta_i - \beta_j,\ \beta_k + \beta_l$, where again (i,j,k,l) is a permutation of $(2,3,4,5)$, the corresponding Gram matrix must also be equal to M_1. Since $\|\beta_i \pm \beta_j\|^2 = \frac{1}{4}$, for all $2 \leq i < j \leq 5$, it follows that $\|\beta_0 - \beta_1\|^2 = 1$, so $\langle\, \beta_0,\, \beta_1 \,\rangle = 0$. By the above argument applied to this pairing, it follows a similar relation, namely: $\langle\, \beta_0 - \beta_1,\, \beta_i \pm \beta_j \,\rangle = 0$, which together with (21) yields $\langle\, \beta_0,\, \beta_i \,\rangle = \langle\, \beta_1,\, \beta_j \,\rangle = 0$, for $2 \leq i \leq 5$. Thus, the Gram matrix $(\langle\, \beta_i,\, \beta_j \,\rangle)_{0 \leq i,j \leq 5}$ is diagonal with $\|\beta_0\|^2 + \|\beta_1\|^2 = 1$ and $\|\beta_i\|^2 = \frac{1}{4}$, for $2 \leq i \leq 5$.

The second element in the decreasing order of the set $\{\|\beta_i+\beta_j\|^2 \,|\, 0 \le i < j \le 5\}$ must be, up to a permutation of 0 and 1, of the form $\beta_0+\beta_k$, for some $2 \le k \le 5$. By taking now, for instance, a pairing with first element equal to $\beta_0+\beta_k$, it follows that its Gram matrix is equal to M_1 and by the same argument as above $\|\beta_1\|^2 = \frac{1}{4}$ and, consequently, $\|\beta_0\|^2 = \frac{3}{4}$. Since we allowed permutations, we have to consider two cases: $\|\alpha_1\|^2 \in \{\frac{3}{4}, \frac{1}{4}\}$.

If $\|\alpha_1\|^2 = \frac{1}{4}$, we may assume that $\|\beta_1\|^2 = \frac{3}{4}$ and $\|\beta_i\|^2 = \frac{1}{4}$, for all $2 \le i \le 5$. Since the Gram matrix is now completely known, we compute the scalar products between the roots in $\mathcal{W}$ and by (3) we obtain: $\{\pm 2(\alpha_1 \pm \beta_i), \pm 2(\beta_j \pm \beta_i) | \, 2 \le i < j \le 5\} \subseteq \overline{\mathcal{W}} \setminus \mathcal{W} \subseteq \mathcal{R}(\mathfrak{h})$. Considering the orthogonal decomposition (5) of $\mathfrak{h}$, it follows by Lemma 2.8 (ii) that there is a $k \in \{0,1,2\}$ such that $\{\pm 2(\alpha_1 \pm \beta_i), \pm 2(\beta_j \pm \beta_i) | \, 2 \le i < j \le 5\} \subseteq \mathcal{R}(\mathfrak{h}_k)$. Since $\alpha_1 \in \mathfrak{h}_0 \oplus \mathfrak{h}_1$ and $\beta_i \in \mathfrak{h}_0 \oplus \mathfrak{h}_2$, for $1 \le i \le 5$, the only possible value is $k = 0$. This implies that $\mathfrak{so}(10) \subseteq \mathfrak{h}_0$ and thus $p \ge 5$. We show that this is not possible.

Assuming that $p \ge 2$ and computing the scalar products between $\alpha_1 + \beta_1 + \cdots + \beta_5$ and $\alpha_2 + \beta_1 \pm (\beta_2 + \beta_3) \pm (\beta_4 + \beta_5)$, we get the following values $\{a + \frac{7}{4}, a + \frac{3}{4}, a - \frac{1}{4}\}$, where $a := \langle \alpha_1, \alpha_2 \rangle$. By (1), we know that $\{a + \frac{7}{4}, a + \frac{3}{4}, a - \frac{1}{4}\} \subseteq \{0, \pm 1\}$. Hence, $a = -\frac{3}{4}$, which implies that $\|\alpha_1 + \alpha_2\|^2 = -1$. Thus, the case $\|\alpha_1\|^2 = \frac{1}{4}$ may not occur.

We then have $\|\alpha_1\|^2 = \frac{3}{4}$ and $\|\beta_i\|^2 = \frac{1}{4}$, for $1 \le i \le 5$. Again by computing all possible scalar products, we produce by (3) the new roots $\{\pm 2(\beta_i \pm \beta_j) | \, 1 \le i < j \le 5\} \subseteq \mathcal{R}(\mathfrak{h})$. By Lemma 2.8 (i) and (ii), there exists a $k \in \{0,1,2\}$ such that $\{\pm 2(\beta_i \pm \beta_j) | \, 1 \le i < j \le 5\}$ are all roots of one of the components $\mathfrak{h}_k$ of the orthogonal splitting $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$ given by (5). As $\beta_i \in \mathfrak{h}_0 \oplus \mathfrak{h}_2$ for $1 \le i \le 5$, it follows that $k \in \{0, 2\}$. Thus, $\mathcal{R}(\mathfrak{h}_k)$ contains the whole system of roots of $\mathfrak{so}(10)$. On the other hand, $\mathfrak{h}_0 \oplus \mathfrak{h}_2 \subseteq \mathfrak{so}(10)$. Hence, there are two possibilities: either $\mathfrak{h}_0 = \mathfrak{so}(10)$ and $\mathfrak{h}_2 = 0$ or $\mathfrak{h}_0 = 0$ and $\mathfrak{h}_2 = \mathfrak{so}(10)$.

Let us first note that if $p \ge 2$, then $\langle \alpha_i, \alpha_j \rangle = -1$, for all $1 \le i < j \le p$. By computing the scalar products between the different roots containing α_i, respectively α_j, we obtain the following values: $a := \langle \alpha_i, \alpha_j \rangle \in \{a + \frac{5}{4}, a + \frac{1}{4}, a - \frac{3}{4}\}$, which by (1) must be contained into $\{0, \pm 1\}$. Hence, the only possible value is $a = -\frac{1}{4}$.

In the first case, $\mathfrak{h}_0 = \mathfrak{so}(10)$ implies that $p \ge 5$, which by the above remark leads to the following contradiction: $\|\alpha_1 + \cdots + \alpha_5\|^2 = -\frac{5}{4} < 0$.

Thus, the second case $\mathfrak{h}_2 = \mathfrak{so}(10)$ and $\mathfrak{h}_0 = 0$ must hold. We show that $p = 1$. Assuming $p \ge 2$, we compute $\langle \alpha_1 + \beta_1 + \cdots + \beta_5, \alpha_2 + \beta_1 - \beta_2 - \cdots - \beta_5 \rangle = -1$, which by (3) yields the new mixed root $\alpha_1 + \alpha_2 + 2\beta_1 \in \mathcal{R}(\mathfrak{h})$, contradicting Lemma 2.8 (i) (since $\alpha_1 + \alpha_2 \in \mathfrak{h}_1$ and $\beta_1 \in \mathfrak{h}_2$). Thus, $p = 1$ and $\mathfrak{h}_1 = \mathfrak{u}(1)$.

Concluding, it follows that $\mathfrak{h} = \mathfrak{so}(10) \oplus \mathfrak{u}(1)$. Therefore, $\mathcal{R}(\mathfrak{g}) = \mathcal{W} \cup \mathcal{R}(\mathfrak{so}(10) \oplus \mathfrak{u}(1))$, is exactly the system of roots of $\mathfrak{e}_6$ (cf. [2, p. 57]), hence $\mathfrak{g} = \mathfrak{e}_6$. From [2, Thm. 6.1], the Lie subgroup of E_6 corresponding to the above embedding of $\mathfrak{so}(10) \oplus \mathfrak{u}(1)$ in $\mathfrak{e}_6$ is $\mathrm{Spin}(10) \times \mathrm{U}(1)/\mathbb{Z}_4$, showing that the only homogeneous manifold carrying a homogeneous Clifford structure of rank $r = 10$ is the exceptional symmetric space $(\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2 = \mathrm{E}_6/(\mathrm{Spin}(10) \times \mathrm{U}(1)/\mathbb{Z}_4)$.

(III) By Theorem 4.1 (III), the maximal rank in this case is $r = 12$. For $r = 12 = 2q$, from Proposition 3.3 (III), there exist $\mathcal{A} := \{\alpha_1, \ldots, \alpha_p\}$ and $\mathcal{G} := \{\gamma_1, \ldots, \gamma_{p'}\}$ in $\mathfrak{t}^*$ with $\mathcal{A} = -\mathcal{A}$

and $\mathcal{G}=-\mathcal{G}$ such that the set of weights of the isotropy representation is given by:

$$\mathcal{W}(\mathfrak{m})=\mathcal{A}+\left\{\sum_{j=1}^{6}\varepsilon_j\beta_j|\prod_{j=1}^{6}\varepsilon_j=1\right\}_{\varepsilon\in\mathcal{E}_6}\bigcup\mathcal{G}+\left\{\sum_{j=1}^{6}\varepsilon_j\beta_j|\prod_{j=1}^{6}\varepsilon_j=-1\right\}_{\varepsilon\in\mathcal{E}_6}$$

with $\sharp\mathcal{W}=(p+p')\cdot 2^5$, where one of p or p' might vanish, but the vectors α_i and γ_i are all non-zero.

Assume $p\neq 0$ (otherwise the same argument applies for $p'\neq 0$ by changing the sign of β_1) and denote by

$$\mathcal{W}:=\mathcal{A}+\left\{\sum_{j=1}^{6}\varepsilon_j\beta_j|\prod_{j=1}^{6}\varepsilon_j=1\right\}_{\varepsilon\in\mathcal{E}_6},$$

which is an admissible subsystem of roots as in Proposition 3.4 (III), with $\sharp\mathcal{W}=p\cdot 2^5$.

We claim that the following inclusions hold: $\mathcal{R}(\mathfrak{so}(12)\oplus\mathfrak{su}(2))\subseteq\overline{\mathcal{W}}\setminus\mathcal{W}$, $\mathcal{R}(\mathfrak{e}_7)\subseteq\overline{\mathcal{W}}$.

Let us consider all the subsystems of roots of the following form $\{\sum_{j=1}^4\varepsilon_j\beta'_j\}\subset\mathcal{W}$, where $\beta'_1=\alpha_1$, $\beta'_2=\beta_1\pm\beta_2$, $\beta'_3=\beta_3\pm\beta_4$, $\beta'_4=\beta_5\pm\beta_6$ and such that the number of minus signs in β'_2,β'_3 and β'_4 is even, as well as all the other subsystems obtained by permuting the subscripts $\{1,\dots,6\}$ of the vectors β_j.

By Lemma 3.5 (i), the Gram matrix $(\langle\beta'_i,\beta'_j\rangle)_{i,j}$ is either $\frac14\mathrm{id}_4$ or the matrix M_0 defined by (8). We further show that the second case can not occur. Indeed, if this were the case, then at least two of the norms $\|\beta'_2\|^2$, $\|\beta'_3\|^2$ and $\|\beta'_4\|^2$ are equal and after reordering, rescaling and sign change of the vectors β_j we may assume $\beta'_2=\beta_1+\beta_2$, $\beta'_3=\beta_3+\beta_4$, $\beta'_4=\beta_5+\beta_6$ and $\|\beta'_3\|^2=\|\beta'_4\|^2=\frac18$, $\langle\beta'_3,\beta'_4\rangle=\frac{1}{16}$. If we denote by $\beta''_j:=\beta'_j$, $j=1,2$ and $\beta''_3=\beta_3-\beta_4$, $\beta''_4=\beta_5-\beta_6$, then $\{\sum_{j=1}^4\varepsilon_j\beta''_j\}\subset\mathcal{W}$ is also a subsystem of roots. Since $\{\|\beta''_1\|^2,\|\beta''_2\|^2\}=\{\frac14,\frac18\}$, it follows again by Lemma 3.5 (i) that the Gram matrix $(\langle\beta''_i,\beta''_j\rangle)_{i,j}$ is equal to M_0 defined by (8), so that $\|\beta''_4\|^2=\|\beta''_3\|^2=\frac18=\|\beta'_3\|^2=\|\beta'_4\|^2$. Thus, $\|\beta_1+\beta_2\|^2=\frac14$, $\|\beta_{2j-1}+\beta_{2j}\|^2=\frac18$ for $j=2,3,4$, and

$$\langle\beta_3+\beta_4,\beta_5+\beta_6\rangle=\frac{1}{16}. \tag{24}$$

For every $3\leq j<k\leq 8$, there exists $l\in\{2,3,4\}$ such that $\{2l-1,2l\}\cap\{j,k\}=\emptyset$. Let $\{s,t\}$ denote the complement of $\{1,2,j,k,2l-1,2l\}$ in $\{1,\dots,8\}$. The Gram matrix of $\{\beta_1+\beta_2,\beta_{2l-1}+\beta_{2l},\beta_j-\beta_k,\beta_s-\beta_t\}$ has at least two different values on the diagonal. By Lemma 3.5 (i) again, the remaining diagonal terms $\|\beta_j-\beta_k\|^2$ and $\|\beta_s-\beta_t\|^2$ are both equal to $\frac18$. Thus $\|\beta_j-\beta_k\|^2=\frac18$ for $3\leq j<k\leq 8$, and similarly $\|\beta_j+\beta_k\|^2=\frac18$ for $3\leq j<k\leq 8$. Thus $\langle\beta_j,\beta_k\rangle=0$ for all $3\leq j<k\leq 8$, contradicting (24).

It then follows that the Gram matrix of any subsystem of roots $\{\sum_{j=1}^4\varepsilon_j\beta'_j\}$ as above is $(\langle\beta'_i,\beta'_j\rangle)_{i,j}=\frac14\mathrm{id}_4$. In particular, this shows that for any $1\leq i<j\leq 6$, $\|\beta_i+\beta_j\|^2=\|\beta_i-\beta_j\|^2=\frac14$, and $\langle\alpha_1,\beta_i+\beta_j\rangle=\langle\alpha,\beta_i-\beta_j\rangle=0$, implying that $\langle\beta_i,\beta_j\rangle=0$ and $\langle\beta_i,\alpha_1\rangle=0$. For any $k\neq i,j$ we also have $\|\beta_i+\beta_k\|^2=\|\beta_j+\beta_k\|^2=\frac14$, which then yields $\|\beta_i\|^2=\frac18$ for $1\leq i\leq 6$. Denoting by $\beta':=\sum_{i=1}^4\beta'_i$, we compute $\langle\beta'-2\beta'_i,\beta'\rangle=\frac12$, for

$1 \le i \le 4$. By (3) we obtain the new roots $\pm 2\beta_i' \in \overline{\mathcal{W}} \setminus \mathcal{W}$. Since the argument is true for any such subsystem of roots, we have the new roots $\{\pm 2\alpha_1, \pm 2(\beta_i \pm \beta_j) |\, 1 \le i < j \le 6\} \subseteq \overline{\mathcal{W}} \setminus \mathcal{W}$, which, by the above orthogonality relations, build the system of roots of $\mathfrak{so}(12) \oplus \mathfrak{su}(2)$. It thus follows that $\mathfrak{so}(12) \oplus \mathfrak{su}(2) \subseteq \mathfrak{h}$. Furthermore, $\mathcal{R}(\mathfrak{so}(12) \oplus \mathfrak{su}(2)) \cup \mathcal{W} = \mathcal{R}(\mathfrak{e}_7)$ (cf. [2, p. 56]), showing that $\mathfrak{e}_7 \subseteq \mathfrak{g}$.

We claim that $p = 1$. Assuming that $p \ge 2$, we consider $\alpha_2 \in \mathcal{A} \setminus \{\pm\alpha_1\}$. The previous argument also shows that $\langle \beta_i, \alpha_2 \rangle = 0$ for all $1 \le i \le 6$. Denoting by $a := \langle \alpha_1, \alpha_2 \rangle$, the scalar products between $\beta := \alpha_1 + \sum_{j=1}^6 \beta_j$ and $\alpha_2 \pm (\beta_1 + \beta_2) \pm (\beta_3 + \beta_4) \pm (\beta_5 + \beta_6)$ take four possible values: $\{a \pm \frac{3}{4}, a \pm \frac{1}{4}\}$, thus contradicting (1), which only allows 3 different values for these scalar products. Hence, $p = 1$, and similarly $p' \le 1$.

We further prove that $p' = 0$. Assuming the contrary, there exists $\gamma_1 \neq 0$ in $\mathcal{G}$. The same arguments as before show that $\langle \beta_i, \gamma_1 \rangle = 0$ for $1 \le i \le 6$ and $\|\gamma_1\|^2 = \frac{1}{4}$. Denoting by $a := \langle \alpha_1, \alpha_2 \rangle$, the set of scalar products between the unit vectors $\alpha_1 + \sum_{j=1}^6 \beta_j$ and $\gamma_1 + \beta_1 - \beta_2 \pm (\beta_3 + \beta_4) \pm (\beta_5 + \beta_6)$ equals $\{a, a \pm \frac{1}{2}\}$. By (1), one has necessarily $a = 0$, *i.e.* $\alpha_1 \perp \gamma_1$. We then denote by $\beta_7 := \frac{1}{2}(\alpha_1 + \beta_1)$ and $\beta_8 := \frac{1}{2}(\alpha_1 - \beta_1)$. The above relations show that $\langle \beta_i, \beta_j \rangle = 0$, $\|\beta_i\|^2 = \frac{1}{8}$ for $1 \le i < j \le 8$ and

$$\mathcal{W}(\mathfrak{m}) = \left\{ \sum_{j=1}^{8} \varepsilon_j \beta_j |\, \prod_{j=1}^{8} \varepsilon_j = 1 \right\}_{\varepsilon \in \mathcal{E}_8}.$$

Let $\beta := \sum_{i=1}^8 \beta_i$. Since $\langle \beta, \beta - 2\beta_i - 2\beta_j \rangle = \frac{1}{2}$ for $i \neq j$ and $\langle \beta - 2\beta_i - 2\beta_k, \beta - 2\beta_j - 2\beta_k \rangle = \frac{1}{2}$ for $i \neq j \neq k \neq i$, by (3) we obtain that $\{\pm 2(\beta_i \pm \beta_j) |\, 1 \le i < j \le 8\} \in \overline{\mathcal{W}} \setminus \mathcal{W} \subset \mathcal{R}(\mathfrak{h})$. This is exactly the system of roots of $\mathfrak{so}(16)$. Thus, $\mathfrak{so}(16) \subseteq \mathfrak{h}$. Since $\mathfrak{so}(16)$ is simple, the restriction to $\mathfrak{so}(16)$ of the Clifford morphism $\rho_* : \mathfrak{h} \to \mathfrak{so}(12)$ must vanish. Moreover, the restriction to $\mathfrak{so}(16)$ of the representations $\lambda_\pm$ from Lemma 3.2 vanish, too. Indeed, $p = p' = 1$ and $\mathbb{K} = \mathbb{H}$ so their complex dimensions equal 2. Thus the isotropy representation of G/H would vanish on $\mathfrak{so}(16) \subset \mathfrak{h}$, a contradiction.

As $p' = 0$, Lemma 3.2 shows that the isotropy representation can be written $\mathfrak{m} = \mu_+ \otimes_{\mathbb{H}} \lambda_+$, so as in (5) we can write

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \tag{25}$$

with $\mathfrak{h}_1 := \ker(\rho_*)$, $\mathfrak{h}_2 := \ker(\lambda_+)$ and $\mathfrak{h}_0 = (\mathfrak{h}_1 \oplus \mathfrak{h}_2)^\perp$. Since $p = 1$, it follows that $\mathfrak{h}_0 \oplus \mathfrak{h}_1 \subseteq \mathfrak{su}(2)$. On the other hand, $\mathfrak{h}_0 \oplus \mathfrak{h}_2 \subseteq \mathfrak{so}(12)$ and we have proved that $\mathfrak{so}(12) \oplus \mathfrak{su}(2) \subseteq \mathfrak{h}$. Hence, we obtain $\mathfrak{h}_2 = \mathfrak{so}(12)$, $\mathfrak{h}_0 = 0$ and $\mathfrak{h}_1 = \mathfrak{su}(2)$. In particular, we have $\mathfrak{h} = \mathfrak{so}(12) \oplus \mathfrak{su}(2)$, and $\mathcal{R}(\mathfrak{g}) = \mathcal{W}(\mathfrak{m}) \oplus \mathcal{R}(\mathfrak{h})$ is isometric to the root system of $\mathfrak{e}_7$ (cf. [2, p. 56]). We conclude that $M = \mathrm{E}_7/\mathrm{Spin}(12) \cdot \mathrm{SU}(2)$ by [2, Thm 6.1].

(IV) For $r = 16 = 2q$, it follows from Proposition 3.3 (IV) and Proposition 3.4 (III)-(IV) (for the extremal case $q = 8$) that (up to a sign change for one of the vectors β_i) $p = 1$, $p' = 0$ and the set of weights of the isotropy representation is given by:

$$\mathcal{W}(\mathfrak{m}) = \left\{ \sum_{j=1}^{8} \varepsilon_j \beta_j |\, \prod_{j=1}^{8} \varepsilon_j = 1 \right\}_{\varepsilon \in \mathcal{E}_8}.$$

By Lemma 3.5 (b) it follows that the Gram matrix $(\langle \beta_i, \beta_j \rangle)_{ij}$ is equal to $\frac{1}{8}\mathrm{id}_8$. Thus, all roots in $\mathcal{W}$ have norm 1 and $\langle \beta, \beta_i \rangle = \frac{1}{8}$, where $\beta := \sum_{i=1}^{8} \beta_i$. This yields the following values for the scalar products for all i, j and k mutually distinct : $\langle \beta, \beta - 2\beta_i - 2\beta_j \rangle = \langle \beta - 2\beta_i - 2\beta_j, \beta - 2\beta_i - 2\beta_k \rangle = \frac{1}{2}$ and all other scalar products of roots in $\mathcal{W}$ are 0. The new roots we obtain by (3) are then $\{\pm 2(\beta_i \pm \beta_j) | 1 \leq i < j \leq 8\}$, which build the system of roots of $\mathfrak{so}(16)$. Thus, $\mathfrak{so}(16) \subseteq \mathfrak{h}$. As $p = 1$ and $p' = 0$ and $\mathfrak{so}(16) \subseteq \mathfrak{h}$, it follows with the notation from (25) that $\mathfrak{so}(16) \subseteq \mathfrak{h}_2$. On the other hand, ρ_* maps $\mathfrak{h}_0 \oplus \mathfrak{h}_2$ one-to-one into $\mathfrak{so}(16)$ and thus we must have equality: $\mathfrak{h}_0 \oplus \mathfrak{h}_2 = \mathfrak{so}(16)$. Consequently, $\mathcal{R}(\mathfrak{g}) = \mathcal{R}(\mathfrak{so}_{16}) \cup \mathcal{W}$ is isometric to the system of roots of $\mathfrak{e}_8$ (cf. [2, p. 56]), showing that $\mathfrak{g} = \mathfrak{e}_8$ and $M = \mathrm{E}_8/(\mathrm{Spin}(16)/\mathbb{Z}_2)$ by [2, Thm 6.1]. $\square$

As already mentioned in Section 1, similar methods could be used to examine the remaining cases $r \leq 8$. However, the arguments tend to be much more intricate as fast as the rank decreases. In order to keep this paper at a reasonable length, we have thus decided to limit our study to the extremal cases of Theorem 4.1.

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