Parallel spinors and holonomy groups

Andrei Moroianu and Uwe Semmelmann*

November 11, 2003

Abstract

In this paper we complete the classification of spin manifolds admitting parallel spinors, in terms of the Riemannian holonomy groups. More precisely, we show that on a given n–dimensional Riemannian manifold, spin structures with parallel spinors are in one to one correspondence with lifts to Spin_n of the Riemannian holonomy group, with fixed points on the spin representation space. In particular, we obtain the first examples of compact manifolds with two different spin structures carrying parallel spinors.

I Introduction

The present study is motivated by two articles ([1], [2]) which deal with the classification of non–simply connected manifolds admitting parallel spinors. In [1], Wang uses representation–theoretic techniques as well as some nice ideas due to McInnes ([3]) in order to obtain the complete list of the possible holonomy groups of manifolds admitting parallel spinors (see Theorem 4). We shall here be concerned with the converse question, namely:

(Q) Does a spin manifold whose holonomy group appears in the list above admit a parallel spinor ?

The first natural idea that one might have is the following (cf. [2]): let M be a spin manifold and let $\tilde{M}$ its universal cover (which is automatically spin); let Γ be the fundamental group of M and let $P_{\mathrm{Spin}_n}\tilde{M} \to P_{\mathrm{SO}_n}\tilde{M}$ be the unique spin structure of $\tilde{M}$; then there is a natural Γ–action on the principal bundle $P_{\mathrm{SO}_n}\tilde{M}$ and the lifts of this action to $P_{\mathrm{Spin}_n}\tilde{M}$ are in one–to–one correspondence with the spin structures on M. This approach seems to us quite unappropriated in the given context since it is very difficult to have a good control on these lifts. Our main idea was to remark that the question (Q) above is not well–posed. Let us, indeed, consider the following slight modification of it:

(Q') If M is a Riemannian manifold whose holonomy group belongs to the list above, does M admit a spin structure with parallel spinors?

*Supported by the Deutsche Forschungsgemeinschaft

It turns out that the answer to this question is simply "yes", (see Theorem 6 below). The related question of how many such spin structures may exist on a given Riemannian manifold is also completely solved by our Theorem 7 below. In particular, we obtain the interesting result that every Riemannian manifold with holonomy group $\mathrm{SU}_m \rtimes \mathbb{Z}_2$ ($m \equiv 0(4)$), (see explicit compact examples of such manifolds in Section 5), has exactly two different spin structures with parallel spinors. The only question which remains open is the existence of compact non–simply connected manifolds with holonomy $\mathrm{Sp}_m \times \mathbb{Z}_d$ (d odd and dividing $m+1$). We remark that our results correct statements of McInnes given in [2] (see Sections 4 and 5 below).

The topic of this paper was suggested to us by K. Galicki. We acknowledge useful discussions with D. Kotschick, A. Dessai and D. Huybrechts, as well as several improvements due to the referee. The second named author would like to thank the IHES for hospitality and support.

II Preliminaries

A spin structure on an oriented Riemannian manifold (M^n, g) is a Spin_n principal bundle over M, together with an equivariant 2–fold covering $\pi : P_{\mathrm{Spin}_n}M \to P_{\mathrm{SO}_n}M$ over the oriented orthonormal frame bundle of M. Spin structures exist if and only if the second Stiefel–Whitney class $w_2(M)$ vanishes. In that case, they are in one–to–one correspondence with elements of $H^1(M, \mathbb{Z}_2)$. Spinors are sections of the complex vector bundle $\Sigma M := P_{\mathrm{Spin}_n}M \times_\rho \Sigma_n$ associated to the spin structure via the usual spin representation ρ on Σ_n. The Levi–Civita connection on $P_{\mathrm{SO}_n}M$ induces canonically a covariant derivative ∇ acting on spinors.

Parallel spinors are sections ϕ of ΣM satisfying the differential equation $\nabla\phi \equiv 0$. They obviously correspond to fixed points (in Σ_n) of the restriction of ρ to the spin holonomy group $\widetilde{\mathrm{Hol}}(M) \subset \mathrm{Spin}_n$. The importance of manifolds with parallel spinors comes from the fact that they are Ricci–flat:

Lemma 1 ([4]) *The Ricci tensor of a Riemannian spin manifold admitting a parallel spinor vanishes.*

Proof. Applying twice the covariant derivative to the parallel spinor ϕ gives that the curvature of the spin–connection ∇ vanishes in the direction of ϕ. A Clifford contraction together with the first Bianchi identity then show that $\mathrm{Ric}(X) \cdot \phi \equiv 0$ for every vector X, which proves the claim.

□

We will be concerned in this paper with irreducible Riemannian manifolds, *i.e.* manifolds whose holonomy representation is irreducible. By the de Rham decomposition theorem, a manifold is irreducible if and only if its universal cover is not a Riemannian product. Simply connected irreducible spin manifolds carrying parallel spinors are classified by their (Riemannian) holonomy group in the following way:

Theorem 2 ([4], [5]) *Let (M^n, g) be a simply connected irreducible spin manifold ($n \geq 2$). Then M carries a parallel spinor if and only if the Riemannian holonomy group $Hol(M, g)$ is one of the following : G_2 ($n = 7$); Spin_7 ($n = 8$); SU_m ($n = 2m$); Sp_k ($n = 4k$).*

PROOF. If M carries a parallel spinor, it cannot be locally symmetric. Indeed, M is Ricci–flat by the Lemma above, and Ricci–flat locally symmetric manifolds are flat. This would contradict the irreducibility hypothesis. One may thus use the Berger–Simons theorem which states that the holonomy group of M belongs to the following list: G_2 ($n = 7$); Spin_7 ($n = 8$); SU_m ($n = 2m$); Sp_k ($n = 4k$); U_m ($n = 2m$); $\mathrm{Sp}_1 \cdot \mathrm{Sp}_k$ ($n = 4k$); SO_n. On the other hand, if M carries a parallel spinor then there exists a fixed point in Σ_n of $\widetilde{\mathrm{Hol}}(M)$ and hence a vector $\xi \in \Sigma_n$ on which the Lie algebra $\widetilde{\mathfrak{hol}}(M) = \mathfrak{hol}(M)$ of $\widetilde{\mathrm{Hol}}(M)$ acts trivially. It is easy to see that the spin representation of the Lie algebras of the last three groups from the Berger–Simons list has no fixed points, thus proving the first part of the theorem (cf. [5]).

Conversely, suppose that $\mathrm{Hol}(M)$ is one of G_2, Spin_7, SU_m or Sp_k. In particular, it is simply connected. Let π denote the universal covering $\mathrm{Spin}_n \to \mathrm{SO}_n$. Since $\mathrm{Hol}(M)$ is simply connected, $\pi^{-1}\mathrm{Hol}(M)$ has two connected components, H_0 (containing the unit element) and H_1, each of them being mapped bijectively onto $\mathrm{Hol}(M)$ by π. Now, it is known that $\pi : \widetilde{\mathrm{Hol}}(M) \to \mathrm{Hol}(M)$ is onto ([6], Ch. 2, Prop. 6.1). Moreover, $\widetilde{\mathrm{Hol}}(M)$ is connected ([6], Ch. 2, Thm. 4.2) and contains the unit in Spin_n, so finally $\widetilde{\mathrm{Hol}}(M) = H_0$. The spin representation of the Lie algebra of H_0 acts trivially on some vector $\xi \in \Sigma_n$, which implies that $h(\xi)$ is constant for $h \in H_0$. In particular $h(\xi) = 1(\xi) = \xi$ for all $h \in H_0$, and one deduces that ξ is a fixed point of the spin representation of H_0.

$\square$

Remark. In the first part of the proof one has to use some representation theory in order to show that the last three groups in the Berger–Simons list do not occur as holonomy groups of manifolds with parallel spinors. The non–trivial part concerns only U_m and Sp_k, since the spin representation of $\mathfrak{so}_n = \mathfrak{spin}_n$ has of course no fixed point. An easier argument which excludes these two groups is the remark that they do not occur as holonomy groups of Ricci–flat manifolds (see [7]). It is natural to ask in this context whether there exist any simply connected Ricci–flat manifolds with holonomy SO_n. Our feeling is that it should be possible to construct local examples but it seems to be much more difficult to construct compact examples. Related to this, it was remarked by A. Dessai that a compact irreducible Ricci–flat manifold with vanishing first Pontrjagin class must have holonomy SO_n (c.f. [8]).

III Wang's holonomy criterion

In this section we recall the results of Wang (cf. [1]) concerning the possible holonomy groups of non–simply connected, irreducible spin manifolds with parallel spinors. By Lemma 1, every such manifold M is Ricci–flat. The restricted holonomy group $\mathrm{Hol}_0(M)$ is isomorphic the full holonomy group of the universal cover $\tilde{M}$, so it belongs to the list given by Theorem 2. Using the fact that $\mathrm{Hol}_0(M)$ is normal in $\mathrm{Hol}(M)$, one can obtain the list of all possible holonomy groups of irreducible Ricci–flat manifolds (see [1]). If M is compact this list can be considerably reduced (see [3]).

The next point is the following simple observation of Wang (which we state from a slightly different point of view, more convenient for our purposes). It gives a criterion for a subgroup of SO_n to be the holonomy group of a n–dimensional manifold with parallel spinors:

Lemma 3 *Let (M^n, g) be a spin manifold admitting a parallel spinor. Then the spin holonomy group of M projects isomorphically over the Riemannian holonomy group by the canonical projection $\pi : \mathrm{Spin}_n \to \mathrm{SO}_n$. In particular, there exists an embedding $\phi : \mathrm{Hol}(M) \to Spin_n$ such that $\pi \circ \phi = \mathrm{Id}_{\mathrm{Hol}(M)}$. Moreover, the restriction of the spin representation to $\phi(\mathrm{Hol}(M))$ has a fixed point on Σ_n.*

Finally, a case by case analysis using this criterion yields

Theorem 4 ([1]) *Let (M^n, g) be a irreducible Riemannian spin manifold which is not simply connected. If M admits a non–trivial parallel spinor, then the full holonomy group* $\mathrm{Hol}(M)$ *belongs to the following table:*

$\mathrm{Hol}_\circ(M)$	$\dim(M)$	$\mathrm{Hol}(M)$	N	*conditions*
SU_m	$2m$	SU_m	*2*	
		$\mathrm{SU}_m \rtimes \mathbb{Z}_2$	*1*	$m \equiv 0(4)$
Sp_m	$4m$	Sp_m	$m+1$	
		$\mathrm{Sp}_m \times \mathbb{Z}_d$	$(m+1)/d$	$d > 1,$ *d odd, d divides* $m+1$
		$\mathrm{Sp}_m \cdot \Gamma$	*see [1]*	$m \equiv 0(2)$
Spin_7	*8*	Spin_7	*1*	
G_2	*7*	G_2	*1*	

Table 1.

where Γ is either $\mathbb{Z}_{2d}$ $(d > 1)$, or an infinite subgroup of $\mathrm{U}(1) \rtimes \mathbb{Z}_2$, or a binary dihedral, tetrahedral, octahedral or icosahedral group. Here N denotes the dimension of the space of parallel spinors. If, moreover, M is compact, then only the following possibilities may occur:

$\mathrm{Hol}_\circ(M)$	$\dim(M)$	$\mathrm{Hol}(M)$	N	*conditions*
SU_m	$2m$	SU_m	*2*	*m odd*
		$\mathrm{SU}_m \rtimes \mathbb{Z}_2$	*1*	$m \equiv 0(4)$
Sp_m	$4m$	$\mathrm{Sp}_m \times \mathbb{Z}_d$	$(m+1)/d$	$d > 1,$ *d odd, d divides* $m+1$
G_2	*7*	G_2	*1*	

Table 2.

IV Spin structures induced by holonomy bundles

We will now show that the algebraic restrictions on the holonomy group given by Wang's theorem are actually sufficient for the existence of a spin structure carrying parallel spinors. The main tool is the following converse to Lemma 3:

Lemma 5 *Let M be a Riemannian manifold and suppose that there exists an embedding $\phi : \mathrm{Hol}(M) \to \mathrm{Spin}_n$ which makes the diagram*

$$\begin{array}{ccc} & & \mathrm{Spin}_n \\ & \overset{\phi}{\nearrow} & \downarrow \\ \mathrm{Hol}(M) & \longrightarrow & \mathrm{SO}_n \end{array}$$

commutative. Then M carries a spin structure whose holonomy group is exactly $\phi(\mathrm{Hol}(M))$, hence isomorphic to $\mathrm{Hol}(M)$.

PROOF. Let i be the inclusion of $\mathrm{Hol}(M)$ into SO_n and $\phi : \mathrm{Hol}(M) \to \mathrm{Spin}_n$ be such that $\pi \circ \phi = i$. We fix a frame $u \in P_{\mathrm{SO}_n}M$ and let $P \subset P_{\mathrm{SO}_n}M$ denote the holonomy bundle of M through u, which is a $\mathrm{Hol}(M)$ principal bundle (see [6], Ch.2). There is then a canonical bundle isomorphism $P \times_i \mathrm{SO}_n \simeq P_{\mathrm{SO}_n}M$ and it is clear that $P \times_\phi \mathrm{Spin}_n$ together with the canonical projection onto $P \times_i \mathrm{SO}_n$ defines a spin structure on M. The spin connection comes of course from the restriction to P of the Levi–Civita connection of M and hence the spin holonomy group is just $\phi(\mathrm{Hol}(M))$, as claimed. □

Remark. The hypothesis of the lemma above is obviously equivalent to the condition that the pre–image of $\mathrm{Hol}(M)$ in Spin_n is isomorphic to $\mathbb{Z}_2 \times \mathrm{Hol}(M)$. This provides a very useful criterion to check whether a given holonomy group satisfies the hypothesis of the lemma. For instance, as the pre–image of U_m in Spin_{2m} is isomorphic to U_m itself, this group does not satisfy the conditions of the lemma.

Now, recall that Table 1 was obtained in the following way: among all possible holonomy groups of non–simply connected irreducible Ricci–flat Riemannian manifolds, one selects those whose holonomy group lifts isomorphically to Spin_n and such that the spin representation has fixed points when restricted to this lift. Using the above Lemma we then deduce at once the following classification result, which contains the converse of Theorem 4.

Theorem 6 *An oriented non–simply connected irreducible Riemannian manifold has a spin structure carrying parallel spinors if and only if its Riemannian holonomy group appears in Table 1 (or, equivalently, if it satisfies the conditions in Lemma 3).*

There is still an important point to be clarified here. Let $G = \mathrm{Hol}(M)$ be the holonomy group of a manifold such that G belongs to Table 1 and suppose that there are several lifts

$\phi_i : G \to \mathrm{Spin}_n$ of the inclusion $G \to \mathrm{SO}_n$. By Lemma 5 each of these lifts gives rise to a spin structure on M carrying parallel spinors, and one may legitimately ask whether these spin structures are equivalent or not. The answer to this question is given by the following (more general) result.

Theorem 7 *Let $G \subset \mathrm{SO}_n$ and let P be a G–structure on M which is connected as topological space. Then the enlargements to Spin_n of P using two different lifts of G to Spin_n are not equivalent as spin structures.*

Note that by "different lifts" of a subgroup G of SO_n to Spin_n we simply mean two different group morphisms $\phi_i : G \to \mathrm{Spin}_n$ $(i = 1, 2)$ such that $\pi \circ \phi_i = \mathrm{Id}_G$.

Proof of the theorem. Recall that two spin structures Q and Q' are said to be *equivalent* if there exists a bundle isomorphism $F : Q \to Q'$ such that the diagram

$$\begin{tikzcd} Q \arrow[rr, "F"] \arrow[dr] & & Q' \arrow[dl] \\ & P_{\mathrm{SO}_n} M & \end{tikzcd}$$

commutes. Let $\phi_i : G \to \mathrm{Spin}_n$ $(i = 1, 2)$ be two different lifts of G and suppose that $P \times_{\phi_i} \mathrm{Spin}_n$ are equivalent spin structures on M. Assume that there exists a bundle map F which makes the diagram

$$\begin{tikzcd} P \times_{\phi_1} \mathrm{Spin}_n \arrow[rr, "F"] \arrow[dr] & & P \times_{\phi_2} \mathrm{Spin}_n \arrow[dl] \\ & P \times_i \mathrm{SO}_n & \end{tikzcd}$$

commutative. This easily implies the existence of a smooth mapping $f : P \times \mathrm{Spin}_n \to \mathbb{Z}_2$ such that

$$F(u \times_{\phi_1} a) = u \times_{\phi_2} f(u, a)a, \ \forall u \in P, \ a \in \mathrm{Spin}_n. \tag{1}$$

As P and Spin_n are connected we deduce that f is constant, say $f \equiv \varepsilon$. Then (1) immediately implies $\phi_1 = \varepsilon\phi_2$, hence $\varepsilon = 1$ since ϕ_i are group homomorphisms (and both map the identity in G to the identity in Spin_n), so $\phi_1 = \phi_2$, which contradicts the hypothesis.

□

Using the above results, we will construct in the next section the first examples of (compact) Riemannian manifolds with several spin structures carrying parallel spinors.

Remark. We have actually proved that the spin structures with parallel spinors on a given Riemannian manifold are in one–to–one correspondence with the lifts ϕ of the Riemannian holonomy group of the manifold to the spin group such that the restriction of the spin representation to $\phi(\mathrm{Hol}(M))$ has a fixed point on Σ_n. Indeed, every such structure is nothing else but the enlargement to Spin_n of the spin holonomy bundle, which, by Wang's remark (Lemma 3), is isomorphic to the enlargement to Spin_n of the Riemannian holonomy bundle via the corresponding lift to Spin_n of the Riemannian holonomy group.

Remark. Let us also note that a simple check through the list obtained by McInnes in [3] shows that the holonomy group of a compact, orientable, irreducible, Ricci–flat manifold of non–generic holonomy and real dimension not a multiple of four is either G_2 or SU_m (m odd) (there are two other possibilities in the non–orientable case). Theorem 2 of [2] (which states that the above manifolds have a unique spin structure with parallel spinors) follows thus immediately from the remark above, because G_2 and SU_m, being simply connected, admit a unique lift to the spin group (which is known to have fixed points on Σ_n).

V Examples and further remarks

Theorem 6 is not completely satisfactory as long as we do not know whether for each group in Tables 1 or 2, Riemannian manifolds having this group as holonomy group really exist. This is why we will show in this section that most of the concerned groups have a realization as holonomy groups. We will leave as an open problem whether there exist compact non–simply connected manifolds with holonomy $\mathrm{Sp}_m \times \mathbb{Z}_d$ (d odd and $m+1$ divisible by d). We also remark that the problem which we consider here is purely Riemannian, *i.e.* does not make reference to spinors anymore.

1. M **compact.** Besides the above case which we do not treat here, it remains to construct examples of compact non–simply connected manifolds with holonomy G_2, SU_m and $\mathrm{SU}_m \rtimes \mathbb{Z}_2$ (as these are the only cases occurring in Wang's list in the compact case). The first one is obtained directly using the work of Joyce ([9]), who has constructed several families of compact non–simply connected manifolds with holonomy G_2 for which he computes explicitly the fundamental group.

For the second we have to find irreducible, non–simply connected Calabi–Yau manifolds of odd complex dimension. Such examples can be constructed in arbitrary high dimensions. For instance, one can take the quotient of a hypersurface of degree p in $\mathbb{C}P^{p-1}$ by a free $\mathbb{Z}_p$ action, where $p \geq 5$ is any prime number (see [12] for details).

Finally, we use an idea of Atiyah, Hitchin ([10]) and McInnes ([3], [11]) to construct manifolds with holonomy group $\mathrm{SU}_m \rtimes \mathbb{Z}_2$. Let a_{ij}, ($i = 1, \dots, m+1,\ j = 0, \dots, 2m+1$) be (strictly) positive real numbers and M_i be the quadric in $\mathbb{C}P^{2m+1}$ given by

$$M_i = \{[z_0, \dots, z_{2m+1}] \mid \sum_j a_{ij} z_j^2 = 0\}.$$

We define M to be the intersection of the M_i's, and remark that if the a_{ij}'s are chosen generically (i.e. such that the quadrics are mutually transversal), then M is a smooth complex m–dimensional manifold realized as a complete intersection. By Lefschetz' hyperplane Theorem ([13]) M is connected and simply connected (for $m > 1$).

Moreover, M endowed with the metric inherited from $\mathbb{C}P^{2m+1}$ becomes a Kähler manifold.

The adjunction formula (see [13]) shows that $c_1(M) = 0$. Consequently M is a Calabi–Yau manifold, and there exists a Ricci–flat Kähler metric h on M whose Kähler form Ω_h lies in the same cohomology class as the Kähler form Ω_g of g. We now consider the involution σ of M given by $\sigma([z_i]) = [\bar{z}_i]$, which has no fixed points on M because of the hypothesis $a_{ij} > 0$. The following lemma as well as the next corollary can be found in [11]. Nevertheless, we include the proofs for the sake of completeness.

Lemma 8 *The involution σ is an anti–holomorphic isometry of (M, h, J).*

PROOF. It is easy to see that σ is actually an isometry of the Fubini–Study metric on $\mathbb{C}P^{2m+1}$, hence $\sigma^*\Omega_g = -\Omega_g$. On the other hand, σ^*h is a Ricci–flat Kähler metric, too, whose Kähler form is $\Omega_{\sigma^*h} = -\sigma^*\Omega_h$. At the level of cohomology classes we have thus $[\Omega_{\sigma^*h} - \Omega_g] = \sigma^*[\Omega_g - \Omega_h] = 0$ and by the uniqueness of the solution to the Calabi–Yau problem, we deduce that $\sigma^*h = h$, as claimed.

□

We now remark that the manifold M is irreducible. Indeed, from the Lefschetz hyperplane theorem also follows $b_2(M) = 1$. On the other hand, if M would be reducible, the de Rham decomposition theorem would imply that $M = M_1 \times M_2$ where M_i are simply connected compact Kähler manifolds, hence $b_2(M) = b_2(M_1) + b_2(M_2) \geq 2$, a contradiction.

Corollary 9 *The quotient M/σ is a $2m$–dimensional Riemannian manifold with holonomy $\mathrm{SU}_m \rtimes \mathbb{Z}_2$.*

Note that this manifold is oriented if and only if m is even. For $m \equiv 0(4)$, $\mathrm{SU}_m \rtimes \mathbb{Z}_2$ has exactly two different lifts to Spin_{2m}, each of them satisfying the conditions of Lemma 3.

The two lifts of $\mathrm{SU}_m \rtimes \mathbb{Z}_2$ can be described as follows (cf. [1]). For SU_m we take the unique lift which maps $\mathrm{Id} \in \mathrm{SU}_m$ to $1 \in \mathrm{Spin}_{2m}$ and we lift the generator $I = \mathrm{diag}(Id_m, -Id_m) \in \mathrm{O}(2m)$ of $\mathbb{Z}_2$ to $\pm e_{m+1} \cdots e_{2m} \in \mathrm{Spin}_{2m}$, where $\{e_i\}$ is an orthonormal basis of $\mathbb{R}^{2m}$ and the spin group is realized as a subset of the Clifford algebra. It is easy to see that this construction induces two well defined lifts of $\mathrm{SU}_m \rtimes \mathbb{Z}_2$ to Spin_{2m} and that the restriction of the spin representation to each of these two lifts has exactly a one–dimensional space of fixed points (see [1]).

We thus deduce (by Theorem 7 and the remark following it) that (for $m \equiv 0(4)$) the above constructed manifold M is a compact Riemannian manifold with exactly two different spin structures carrying parallel spinors.

Remark. This result is a counterexample to McInnes' Theorem 1 in [2], which asserts that a compact, irreducible Ricci–flat manifold of non–generic holonomy and real dimension $4m$ admits a parallel spinor if and only if it is simply connected. The error in McInnes' proof comes from the fact that starting from a parallel spinor on a manifold with local holonomy SU_{2m}, the 'squaring' construction does not always furnish the whole complex volume form. In some cases one only obtain its real or complex part, which is of course not sufficient to conclude that the whole holonomy group is SU_{2m}.

2. M **non–compact.** We now give, for each group in Table 1, examples of (non–compact, non–simply connected) oriented Riemannian manifolds having this group as holonomy group. Of course, we will not consider here the holonomy groups of the compact manifolds constructed above, since it suffices to remove a point from such a manifold to obtain a non compact example. All our examples for the remaining groups in Table 1 will be obtained as cones over manifolds with special geometric structures. Recall that if (M, g) is a Riemannian manifold the cone $\bar{M}$ is the product manifold $M \times \mathbb{R}_+$ equipped with the warped product metric $\bar{g} := r^2 g \oplus dr^2$. Note that $\bar{M}$ is always a non–complete manifold, and by a result of Gallot ([14]), the cone over a complete manifold is always irreducible or flat as Riemannian manifold. Using the O'Neill formulas for warped products, it is easy to relate the different geometries of a manifold and of its cone in the following way (see for example [15], or [18] for the definitions).

Theorem 10 ([15]) *Let M be a Riemannian manifold and $\bar{M}$ the cone over it. Then $\bar{M}$ is hyperkähler or has holonomy* Spin_7 *if and only if M is a 3–Sasakian manifold, or a weak G_2–manifold respectively. There is an explicit natural correspondence between the above structures on M and $\bar{M}$.*

This directly yields examples of oriented, non–simply connected Riemannian manifolds with holonomy Spin_7 and Sp_m, as cones over non–simply connected weak G_2–manifolds (cf. [16] or [17] for examples), and non–simply connected 3–Sasakian manifolds respectively (cf. [18] for examples).

Let now M be a regular simply connected 3–Sasakian manifold other than the round sphere (all known examples of such manifolds are homogeneous). It is a classical fact that M is the total space a SO_3 principal bundle over a quaternionic Kähler manifold, such that the three Killing vector fields defining the 3–Sasakian structure define a basis of the vertical fundamental vector fields of this fibration.

For $d > 1$ odd, let Γ be the image of $\mathbb{Z}_d \subset \mathrm{U}(1) \subset \mathrm{SU}_2$ through the natural homomorphism $\mathrm{SU}_2 \to \mathrm{SU}_2/\mathbb{Z}_2 \simeq \mathrm{SO}_3$. It is clear that $\Gamma \simeq \mathbb{Z}_d$ and by the above Γ acts freely on M. On the other hand, for every $x \neq \pm 1$ in SU_2, the right action of x on SO_3 preserves a one–dimensional space of left invariant vector fields and defines a non–trivial rotation on the remaining 2–dimensional space of left invariant vector fields on SO_3. This means that if $\gamma \neq 1$ is an arbitrary element of Γ, its action on M preserves exactly one Sasakian structure and defines a rotation on the circle of Sasakian structures orthogonal to the first one. The following classical result then shows that the holonomy group of the cone over M/Γ has to be $\mathrm{Sp}_m \times \mathbb{Z}_d$.

Proposition 11 *Let (M, g) be a Riemannian manifold with universal cover $\tilde{M}$. If the natural surjective homomorphism $\pi_1(M) \to$* $\mathrm{Hol}/\mathrm{Hol}_0$ *is not bijective, then there exists a subgroup $K \subset \pi_1(M)$ such that $\tilde{M}/K$ is a manifold with* $\mathrm{Hol} = \mathrm{Hol}_0$. *The group K is actually the kernel of the homomorphism above.*

Similarly one may construct examples of manifolds with holonomy $\mathrm{Sp}_m \cdot \Gamma$ for every group Γ listed in Theorem 4.

References

[1] M. Y WANG, *On Non–simply Connected Manifolds with Non–trivial Parallel Spinors,* Ann Global Anal. Geom. **13** (1995), 31–42.

[2] B. McINNES, *Existence of parallel spinors on nonsimply connected Riemannian manifolds,* J. Math. Phys. **39** (1998), 2362–2366.

[3] B. McINNES, *Methods of holonomy theory for Ricci–flat Riemannian manifolds,* J. Math. Phys. **32** (1991), 888–896.

[4] N. HITCHIN, *Harmonic spinors,* Advances in Math. **14** (1974), 1–55.

[5] M. Y. WANG, *Parallel Spinors and Parallel Forms,* Ann Global Anal. Geom. **7** (1989), 59–68.

[6] S. KOBAYASHI, K. NOMIZU, *The Foundations of Differential Geometry,* (vol. I), Interscience Publishers, New York, 1963.

[7] A. L. BESSE, *Einstein Manifolds,* Springer-Verlag, New York 1987.

[8] A. DESSAI, *On the topology of scalar–flat manifolds,* preprint, Augsburg, (1999).

[9] D. D. JOYCE, *Compact Riemannian* 7*–manifolds with holonomy* G_2*. I, II,* J. Differential Geom. **43** (1996), 291–328, 329–375.

[10] N. HITCHIN, *Compact four-dimensional Einstein manifolds,* J. Differential Geom. **9** (1974), 435–441.

[11] B. McINNES, *Examples of Einstein manifolds with all possible holonomy groups in dimensions less than seven,* J. Math. Phys. **34** (1993), 4287–4304.

[12] A. BEAUVILLE, *Variétés Kähleriennes dont la première classe de Chern est nulle,* J. Differential Geom.**18** (1983), 755–782.

[13] P. GRIFFITHS, J. HARRIS, *Principles of algebraic geometry,* Pure and Applied Mathematics. Wiley–Interscience, New York, 1978.

[14] S. GALLOT, *Equations différentielles caractéristiques de la sphère,* Ann. Sc. Ec. Norm. Sup. **12** (1979), 235–267.

[15] C. BÄR, *Real Killing spinors and holonomy,* Comm. Math. Phys. **154** (1993), 509–521.

[16] T. FRIEDRICH, I. KATH, A. MOROIANU, U. SEMMELMANN, *On nearly parallel* G_2*–structures,* J. Geom. Phys. **23** (1997), 259–286.

[17] K. GALICKI, S. SALAMON, *Betti numbers of* 3*–Sasakian manifolds,* Geom. Dedicata **63** (1996), 45–68.

[18] C. P. BOYER, K. GALICKI, B. M. MANN, *The geometry and topology of* 3*–Sasakian manifolds,* J. Reine Angew. Math **455** (1994), 183–220.

(A. Moroianu) Centre de Mathématiques, Ecole Polytechnique, UMR 7640 du CNRS, Palaiseau, France
(U. Semmelmann) Mathematisches Institut, Universität München, Theresienstr. 39, 80333, München, Germany.