GENERALIZED KILLING SPINORS AND CONFORMAL EIGENVALUE ESTIMATES FOR Spin^c MANIFOLDS

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Spin^c geometry has become recently a field of active research with the advent of Seiberg-Witten theory, whose applications to 4-dimensional geometry and topology are already notorious (see [16] for example).

In the same time, the shift from classical spin geometry to Spin^c-geometry has led to many new questions and several results have now been proved (*e.g.* [13, 21]).

The first part of this paper is devoted to such a problem arising from spin geometry: the study of *generalized real Killing spinors* on Spin^c manifolds. These are natural generalizations of real Killing spinors, which are useful tools in Riemannian geometry (see for example [4],[11],[19], [20]). It is well-known that generalized Killing spinors cannot exist in the usual spin context without being in fact Killing spinors ([14]), whereas the argument breaks down in the Spin^c setting. We shall prove in this paper that such spinors cannot exist on a Spin^c manifold of dimension $n \geq 4$ either. Surprisingly, it turns out that generalized Killing spinors *do* exist on low-dimensional Spin^c manifolds. We shall construct here explicit examples in small dimensions, and make a few steps towards a complete classification in dimension 3.

In the second part of the paper, we present some applications of this result. We focus on eigenvalues of the Spin^c Dirac operator and we look at inequalities of the type already considered by Th. Friedrich [10], O. Hijazi [14], J. Lott [18], Ch. Bär [3] and H. Baum [6]. Our main contribution is a study of their equality cases, with special attention to low dimensions.

The basic inequalities only involve the geometry of the associated line bundle L of the Spin^c-structure and the conformal structure of the base manifold, *via* the *perturbed conformal Laplace operator*

$$L^g_\omega = 4\frac{n-1}{n-2}\Delta_g + \mathrm{Scal}^g - c_n|\omega|_g, \quad c_n = 2\left[\frac{n}{2}\right]^{\frac{1}{2}} \tag{1}$$

(where g is the Riemannian metric, Scal^g its scalar curvature, Δ_g its Laplace operator and ω is the curvature form of L). The study of such an operator has some interesting consequences on Einstein metrics and Yamabe invariants of 4-dimensional manifolds, as shown by M. Gursky and C. LeBrun [13].

As a corollary, we get a lower bound for the first eigenvalue of the Spin^c Dirac operator for Spin^c structures whose associated line bundles have self-dual curvature. This lower bound only involves the conformal geometry of the manifold (*via* its Yamabe number) and the topology of the associated line bundle of the $\mathrm{Spin^c}$-structure (*via* its Chern numbers).

1. Preliminaries and statement of results

Consider an oriented compact Riemannian manifold M and denote by $P_{SO}M$ the oriented frame bundle of M. A $\mathrm{Spin^c}$-structure on M is given by the frame bundle $P_{S^1}M$ of some Hermitian line bundle L and a $\mathrm{Spin^c_n}$-principal bundle $P_{\mathrm{Spin^c}}M$ which is a 2-fold covering of the bundle $P_{SO}M \times P_{S^1}M$ compatible with the group covering

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin^c_n} = \mathrm{Spin}_n \times_{\mathbb{Z}_2} S^1 \longrightarrow SO_n \times S^1 \longrightarrow 0.$$

The bundle L is usually called the *associated* line bundle of the $\mathrm{Spin^c}$-structure.

We shall denote by ∇ (or by ∇^g if reference to the metric is necessary) the covariant derivative of the Levi-Civita connection of g, and, if A is a Hermitian connection on L, its (imaginary-valued) curvature will be denoted by F_A, whereas we shall define a real 2-form Ω by $F_A = i\Omega$.

From now on, by a $\mathrm{Spin^c}$ manifold, we will understand a set (M, g, L, A, σ) where M, g, L, A are as above and σ is some $\mathrm{Spin^c}$ structure.

If such data are given, one can canonically define a connection on any spinor bundle Σ (*i.e.* any vector bundle associated to $P_{\mathrm{Spin^c}}M$ with respect to some complex representation of $\mathrm{Spin^c_n}$), which will be denoted by ∇ (or $\nabla^{g,A}$ if full reference to metric and connection is needed). We also get Dirac and Penrose operators

$$\mathcal{D}^{g,A}\psi = \sum_i e_i \cdot \nabla^{g,A}_{e_i}\psi, \quad \mathcal{P}^{g,A}_X\psi = \nabla^{g,A}_X\psi + \frac{1}{n}X \cdot \mathcal{D}^{g,A}\psi,$$

acting on sections of Σ. We shall only consider here the standard spinor bundle associated with the fundamental representation of $\mathrm{Spin^c_n}$ on $\mathbb{C}^{2^{[n/2]}}$.

Recall that a *spin* structure can be seen as a Spin^c structure with trivial line bundle L and trivial connection A. Consequently, all statements or definitions concerning Spin^c manifolds have automatically a bearing on spin manifolds.

For later use, let us make the following definitions. We call *normalized Sasaki manifold* any odd-dimensional manifold (M^{2k+1}, g) whose cone $(M \times \mathbb{R}_+, \bar{g} = t^2 g + dt^2)$ is Kähler. Equivalently, M is a normalized Sasaki manifold iff there exists a Killing vector field ξ of unit length on M, such that the tensor field $\phi := \nabla\xi$ satisfies the condition $(\nabla_X\phi)(Y) = g(\xi, X)Y - g(X, Y)\xi$ for all tangent vectors X, Y on M. In this paper, a *Sasaki manifold* is any manifold homothetic to a normalized Sasaki manifold. With this definition, every Sasaki manifold admits an unique *normalized* metric.

A *generalized Killing spinor* on a Spin or Spin^c manifold is a spinor ψ satisfying

$$\nabla_X\psi = fX \cdot \psi \quad \forall X \in TM,$$

for some real function f. If f is a non-zero constant constant, ψ is called a Killing spinor. Simply connected complete manifolds M admitting Killing spinors were classified by C. Bär [4] for M spin and by the second author [21] for M Spin^c. Next, one can easily show that generalized Killing spinors do not exist on Spin manifolds (see [14]), therefore it is natural to consider the same problem in the Spin^c context.

The first part of the paper is devoted to the proof of the following result :

THEOREM 1.1. *There are no generalized Killing spinors on a* Spin^c *manifold of dimension* $n \geq 4$, *except the usual Killing spinors.*

In the same section we give examples of compact manifolds admitting generalized Killing spinors in dimensions $n \leq 3$, and describe those admitting generalized Killing spinors with *never-vanishing* Killing function f in dimension 3.

The second part of the paper consists in applications of this result to the study of equality cases of conformal inequalities for the first eigenvalue of the Dirac operator on Spin^c manifolds. The basic inequality (similar to the classical *Hijazi inequality* [14] and studied also by Ch. Bär [3]) is the following :

THEOREM 1.2. *Let* (M, g, L, A, σ) *be a compact Riemannian* Spin^c *manifold of dimension* $n \geq 3$ *and let* $i\Omega$ *denote the curvature form of* A. *Then the first eigenvalue* λ_1 *of the Dirac operator on the spinor bundle satisfies*

$$\lambda_1^2 \geq \frac{n}{4(n-1)}\, \mu_1\,, \tag{2}$$

where μ_1 *is the first eigenvalue of the perturbed scalar curvature operator* L_Ω^g *defined by (1).*

Using this we then obtain

COROLLARY 1.3. *Let* (M, g, L, A, σ) *be a compact Riemannian* Spin^c *manifold of dimension* 4 *with self-dual curvature* F_A. *Then the first eigenvalue* λ_1 *of the Dirac operator on the spinor bundle satisfies*

$$\mathrm{vol}(M,g)^{1/2}\ \lambda_1^2 \geq \frac{1}{3}\left(Y(M,[g]) - 4\pi\sqrt{2}\ \sqrt{c_1(L)^2[M]}\right),$$

where $Y(M,[g])$ *is the Yamabe number of the conformal structure of* g *and* $c_1(L)$ *is the first Chern class of the associated line bundle.*

Note that this inequality is similar to the one proved by H. Baum [6] for the Dirac operator of a twisted spin bundle but ours extends to the case where λ_1 vanishes, thus providing a link between the Yamabe invariant of the conformal structure and the first Chern class of the complex line bundle.

These inequalities are proved in Section 3 of the paper.

Equality cases are considered in Section 4. A large part of the equality case of Theorem 1.2 relies on the non-existence of *generalized Killing spinors*. We will only give some local results in the general case and the main part of this section is devoted to the study of the equality cases in dimensions 3 and 4.

Since the precise statements of the equality cases are lengthy and somewhat technical, we will only present below a rough view of our main findings. The reader is referred to Section 4 for more precise results.

In dimension 3, equality is attained in Theorem 1.2 if and only if one of the following happens :
– (M, g) is isometric to (S^3, can) or to some of its quotients (precisely known) ;
– (M, g) is conformally equivalent to an Euclidean space-form ;
– (M, g) is conformally equivalent to a S^2-bundle over S^1 ;
– (M, g) is a Sasaki manifold and satisfies to some curvature conditions.

In dimension 4, the situation is somewhat simpler and the main allowable behaviors are the following :
– (M, g) is isometric to (S^4, can) ;
– (M, g) is conformally equivalent to a complex quotient of a flat complex torus or of a K3 surface ;
– (M, g) is conformally equivalent to a Kähler-Einstein complex surface with $c_1 > 0$.

An interesting feature of the equality case of Corollary 1.3 is that it offers a characterization of the complex projective plane with its standard Fubini-Study metric among manifolds sharing the same Euler characteristic.

As a conclusion, we note that, in contrast to the spin case where the situation was quite rigid, the equality case of (2) characterizes well the local geometry of the limit, but, due to the freedom added by the auxiliary bundle in the Spin^c case, many quotients of a limiting manifold are themselves (in most cases) limiting manifolds.

2. Generalized Killing spinors on Spin^c manifolds

Let (M^n, g, L, A, σ) be a Spin^c manifold and suppose there exists a non-zero spinor ψ on M^n satisfying

$$\nabla^{g,A}_X \psi = f X \cdot \psi \qquad \forall X \in TM, \tag{3}$$

for some real function f on M.

We shall prove Theorem 1.1 in two steps. We first settle the case of dimensions $n > 4$. Then, after establishing in the Spin^c case the analog of a classical relation between the the Killing function f and the curvatures of the base manifold and the line bundle, we present in a second step a proof for $n = 4$. Note that our proofs have a local character, so that they apply to any Spin^c manifold, complete or not.

The end of the section is then devoted to constructions of examples of manifolds admitting generalized Killing spinors in low dimensions 2 and 3.

Lemma 2.1. *If (3) holds and $n > 4$, then f is constant.*

Proof. For $0 \le p \le n$ we define the p-form ω_p on M by

$$\omega_p(X_1, \cdots, X_p) = < X_1 \wedge \cdots \wedge X_p \cdot \psi, \psi > \tag{4}$$

It is easy to check that ω_{4l+1} and ω_{4l+2} are imaginary-valued and ω_{4l+3} and ω_{4l+4} are real-valued forms for all allowable l. We let $i\xi = \omega_1^\flat$. For later use, we note that ξ is a Killing vector field. We also note that the norm $|\psi|$ is a constant and we may thus assume $|\psi| = 1$ for the remaining of this section.

The following computations will be done at an arbitrary point x of M; let X_i denote mutually orthogonal vector fields on M which are parallel at x and σ means cyclic sum with respect to the subscripts $1, \cdots, p+1$. We first compute

$$\begin{aligned}(p+1)d\omega_p(X_1, \cdots, X_{p+1}) &= \sigma X_1(\omega_p(X_2, \cdots, X_{p+1}))\\ &= f\sigma(< X_2 \wedge \cdots \wedge X_{p+1} \cdot X_1 \cdot \psi, \psi >\\ &\quad + < X_2 \wedge \cdots \wedge X_{p+1} \cdot \psi, X_1 \cdot \psi >)\\ &= f\sigma(< X_2 \wedge \cdots \wedge X_{p+1} \wedge X_1 \cdot \psi, \psi >\\ &\quad - < X_1 \wedge X_2 \wedge \cdots \wedge X_{p+1} \cdot \psi, \psi >)\\ &= ((-1)^p - 1)f\sigma(< X_1 \wedge \cdots \wedge X_{p+1} \cdot \psi, \psi >)\\ &= (p+1)((-1)^p - 1)f\omega_{p+1}(X_1, \cdots, X_{p+1}),\end{aligned}$$

thus showing that

$$d\omega_{2k} = 0 \tag{5}$$

and

$$d\omega_{2k+1} = -2f\,\omega_{2k+2}. \tag{6}$$

Taking the exterior differential in (6) and using (5) yields

$$df \wedge \omega_{2k} = 0, \ \forall k \geq 1. \tag{7}$$

We now suppose that in a neighborhood U of some point $x \in M$ we have $df \neq 0$ and show that this implies $n \leq 4$, which will conclude the proof of the lemma. Let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal frame spanning $df^\perp$ in T_xM. From (7) it is clear that, for $2k \leq n-1$ and for each subset $\{i_1, \ldots, i_{2k}\}$ of $\{1, \ldots, n-1\}$, we have

$$\omega_{2k}(e_{i_1}, \ldots, e_{i_{2k}}) = 0, \tag{8}$$

thus showing that the spinors ψ, $e_{i_1} \cdot e_{i_2} \cdot \psi, \ldots, e_{i_1} \cdot \ldots \cdot e_{i_{2m}} \cdot \psi$ are mutually orthogonal, where m is the integral part of $(n-1)/2$. Consequently, they span a complex vector subspace of $\Sigma_x M$ of complex dimension $\begin{pmatrix} n-1 \\ 0 \end{pmatrix} + \begin{pmatrix} n-1 \\ 2 \end{pmatrix} + \cdots + \begin{pmatrix} n-1 \\ 2m \end{pmatrix}$. Since the complex dimension of $\Sigma_x M$ equals $2^{[\frac{n}{2}]}$, we obtain

$$\begin{aligned}2^{[\frac{n}{2}]} &\geq \begin{pmatrix} n-1 \\ 0 \end{pmatrix} + \begin{pmatrix} n-1 \\ 2 \end{pmatrix} + \cdots + \begin{pmatrix} n-1 \\ 2m \end{pmatrix}\\ &= \frac{1}{2}((1+1)^{n-1} + (1-1)^{n-1})\\ &= 2^{n-2}.\end{aligned}$$

From that follows $[\frac{n}{2}] \geq n-2$, so $n \leq 4$. □

We now consider the 4-dimensional case. Let us first establish a relation between the Killing function and the curvatures of the manifold and that of the line bundle, which holds in any dimension.

LEMMA 2.2. *Every generalized Killing spinor ψ satisfies*

$$\frac{1}{2}(\mathrm{Ric}(X) - iX \lrcorner \Omega) \cdot \psi = \nabla f \cdot X \cdot \psi + nX(f)\,\psi + (2n-2)f^2 X \cdot \psi. \tag{9}$$

Proof. Let $\{e_1, ...e_n\}$ be a local orthonormal frame. From (3) we easily obtain

$$\mathcal{R}^A_{X,Y}\psi = X(f)Y \cdot \psi - Y(f)X \cdot \psi + f^2(Y \cdot X - X \cdot Y) \cdot \psi. \tag{10}$$

A local computation shows that the curvature operator of the twisted connection on the spinor bundle is given by the formula

$$\mathcal{R}^A = \mathcal{R} + \frac{i}{2}\Omega, \tag{11}$$

where

$$\mathcal{R}_{X,Y} = \frac{1}{2}\sum_{j<k} R(X, Y, e_j, e_k) e_j \cdot e_k. \tag{12}$$

Using the first Bianchi identity for the curvature tensor one obtains ([7], p.16)

$$\sum_i e_i \cdot \mathcal{R}_{e_i,X} = \frac{1}{2}\,\mathrm{Ric}(X), \tag{13}$$

so, by (11) and (13),

$$\begin{aligned} \sum_j e_j \cdot \mathcal{R}^A_{e_j,X}\psi &= \sum_j e_j \cdot (\mathcal{R}_{e_j,X}\psi + \frac{i}{2}\Omega(e_j, X)\psi) \\ &= \frac{1}{2}\mathrm{Ric}(X) \cdot \psi - \frac{i}{2} X \lrcorner \Omega \cdot \psi. \end{aligned}$$

On the other hand, from (10) we obtain

$$\begin{aligned} \sum_j e_j \cdot \mathcal{R}^A_{e_j,Y}\psi &= \sum_j e_j \cdot [e_j(f)Y \cdot \psi - Y(f)e_j \cdot \psi - 2f^2(< Y, e_j > + e_j \cdot Y) \cdot \psi] \\ &= \nabla f \cdot Y \cdot \psi + nY(f)\,\psi + (2n-2)f^2 Y \cdot \psi, \end{aligned}$$

so the lemma follows. □

The next result, which will be of fundamental importance below, when considering examples of generalized Killing spinors in low dimensions, also holds in any dimension :

LEMMA 2.3. *With the previous notations, we have :*

$$\xi \lrcorner \Omega = -2(n-1)\,df. \tag{14}$$

Proof. Take the real part of the scalar product of (9) with ψ. This yields

$$\frac{1}{2} < \xi, X \lrcorner \Omega > = (n-1)X(f)\ \forall X \in TM,$$

and the result follows. □

Using the two lemmas above we can now rule out the case $n = 4$, too.

LEMMA 2.4. *There are no generalized Killing spinors in dimension* 4.

Proof. Define the real 2-form τ on M by

$$\tau(X,Y) = -i < X \wedge Y \cdot \psi, \bar{\psi} >= \Im m < X \cdot Y \cdot \psi, \bar{\psi} > . \tag{15}$$

It is straightforward to check that $\tau = *\omega_2$ ($*$ denotes the Hodge adjoint), so (7) yields

$$df \lrcorner \tau = 0. \tag{16}$$

In (9) we now take the Clifford product with df and the scalar product with $\bar{\psi}$ and consider the imaginary part to obtain (using (16))

$$< X \lrcorner \Omega, df >< \psi, \bar{\psi} >= 0.$$

Taking $X = \xi$ and using (14) yields

$$0 = \Omega(\xi, df) < \psi, \bar{\psi} >= 2(n-1)|df|^2 < \psi, \bar{\psi} >,$$

so in the neighborhood U of the point x (where $df \neq 0$, by assumption) we have $< \psi, \bar{\psi} >= 0$. Differentiating this we obtain $f < X \cdot \psi, \bar{\psi} >= 0$ for all $X \in TM$. By restricting ourselves to a smaller neighborhood if necessary, we may suppose that f does not vanish on U. Hence $< X \cdot \psi, \bar{\psi} >= 0$, which means that, at each point of U, either $\psi_+ = 0$, or ψ_- is perpendicular to the 4-dimensional real vector space $TM \cdot \psi_+$ (with respect to the Euclidean scalar product $\Re e < .,. >$), *i.e.* $\psi_- = 0$, for dimensional reasons. As the norm of ψ is constant, we deduce that $\psi_+ = 0$ or $\psi_- = 0$ on the whole of U, which implies $f = 0$ on U, a contradiction. □

We can now conclude our study of generalized Killing spinors by constructing explicit examples in the low dimensional cases.

THEOREM 2.5. *In dimensions $n = 2$ and $n = 3$ there exist compact manifolds admitting generalized Killing spinors satisfying (3) with non-constant f.*

Proof. We will first treat the case $n = 3$.

a) 3-dimensional case. Let $(M^3, g) = (S^3, can)$ be endowed with its unique spin structure and consider (see [4],[7]) a Killing spinor, say ψ, with Killing constant $1/2$ on M. As the norm of ψ is constant, we may suppose that $|\psi| = 1$. Let ξ be the Killing vector field on M defined by

$$i < \xi, X > = < X \cdot \psi, \psi > . \tag{17}$$

We compute as before

$$i\, d\xi(X,Y) = - < X \wedge Y \cdot \psi, \psi > . \tag{18}$$

Recall that the Hodge operator is defined by $< *\omega, \tau > d\,\mathrm{vol}_g = \omega \wedge \tau$. Since in odd dimensions the volume form acts as the identity on the spinor bundle, we obtain from

(18) that

$$\begin{aligned} i < d\xi, X\wedge Y > &= i\ d\xi(X,Y) \\ &= - < X\wedge Y\cdot\psi,\psi > \\ &= < X\wedge Y\cdot \frac{*(X\wedge Y)\cdot *(X\wedge Y)\cdot\psi}{|X\wedge Y|^2},\psi > \\ &= \frac{1}{|X\wedge Y|^2} < |X\wedge Y|^2 d\,\mathrm{vol}_g\cdot *\,(X\wedge Y)\cdot\psi,\psi > \\ &= < *(X\wedge Y)\cdot\psi,\psi > \\ &= i\ < \xi, *(X\wedge Y) > \\ &= i < *\xi, X\wedge Y >, \end{aligned}$$

so $d\xi = *\xi$. In particular ξ has constant length, since

$$\nabla|\xi|^2 = -2\nabla_\xi\xi = -2d\xi(\xi,.) = 0$$

and

$$d\xi\cdot\varphi = -\ \xi\cdot\varphi,\ \text{for any spinor field } \varphi. \tag{19}$$

For dimensional reasons, ξ is non-zero (otherwise the real 3-dimensional vector space $T_xM\cdot\psi$ would be orthogonal to ψ with respect to the Hermitian product $< .,. >$, contradicting the fact that $\psi^\perp$ has complex dimension 1). Let $\{\xi/|\xi|, e_1, e_2\}$ be an oriented local orthonormal frame on M and let ϕ be a local section generating $\psi^\perp$. By the very definition of ξ, there exist never-vanishing complex functions a_1, a_2 such that $e_i\cdot\psi = a_i\,\phi$. Then $\xi\cdot\psi = -|\xi|\,e_1\cdot e_2\cdot\psi = |\xi|a_2/a_1\,\psi$, *i.e.* there is a complex function a with $\xi\cdot\psi = a\psi$. Then by (17) $a = i\,|\xi|^2$ (recall that we took $|\psi| = 1$) and the fundamental identity of Clifford algebras

$$\xi\cdot\xi\cdot\psi = -|\xi|^2\psi,$$

yields $|\xi|^2 = |\xi|^4$, so finally we have shown that $|\xi| = 1$, $a = i$ and

$$\xi\cdot\psi = i\,\psi. \tag{20}$$

The idea is now the following : we first change the metric by multiplying it with an arbitrary function (not depending on ξ) in directions orthogonal to ξ. We compute the covariant derivative of the Killing spinor in the new metric and obtain a spinor satisfying an equation close to that of generalized Killing spinors. Finally we "recover" the missing term by twisting the spinor bundle with a line bundle, *i.e.* by considering some non-trivial Spin^c structure.

Let h be a strictly positive non-constant function on M such that $\xi(h) = 0$. In fact, if we consider the Hopf fibration $S^3 \to S^2$ given by ξ, the above relation just means that h is the pull-back of a positive function defined on S^2. We consider the metric g^h on M given by $g^h(\xi, X) = g(\xi, X)$ for all $X\in TM$ and $g^h(X,Y) = h^{-2}g(X,Y)$ if X, Y are orthogonal to ξ. Let $Z \longmapsto Z^h$ be the isomorphism of TM defined by

$$(a\xi + X)^h = a\xi + hX \quad \forall X\perp\xi,\ a\in\mathbb{R}.$$

If X is a unit vector for g, we then have $X^h = X/\sqrt{g^h(X,X)} = hX$ if $X\perp\xi$ and $\xi^h = \xi$. Since there is no risk of confusion, from now on we identify ξ and $\xi^\sharp$. It is easy to see that

ξ is a Killing vector field with respect to g^h, too (actually $g^h = h^{-2}g + (1-h^{-2})\,\xi\otimes\xi$ so $L_\xi g^h = 0$). We choose an arbitrary point $x \in M$ and let $u = \{\xi, e_1, e_2\}$ be a positive local g-orthonormal frame defined in a neighborhood U of x as above and $u^h = \{\xi, e_1^h, e_2^h\}$. Using the Koszul formula

$$\begin{aligned} 2 < \nabla_X Y, Z > &= X < Y, Z > + Y < X, Z > - Z < X, Y > \\ &\quad + < Z, [X, Y] > - < Y, [X, Z] > - < X, [Y, Z] >, \end{aligned}$$

we compute the covariant derivative ∇^h of the metric g^h:

$$\nabla^h_\xi \xi = \nabla_\xi \xi = 0$$

$$g^h(\nabla^h_{e_i^h}\xi, e_j^h) = d\xi(e_i^h, e_j^h) = h^2 g(\nabla_{e_i}\xi, e_j)$$

$$\begin{aligned} 2g^h(\nabla^h_\xi e_i^h, e_j^h) &= g^h(e_j^h, [\xi, e_i^h]) - g^h(e_i^h, [\xi, e_j^h]) - g^h(\xi, [e_i^h, e_j^h]) \\ &= g(e_j, [\xi, e_i]) - g(e_i, [\xi, e_j]) - h^2 g(\xi, [e_i, e_j]) \\ &= 2g(\nabla_\xi e_i, e_j) + (1-h^2) g(\xi, [e_i, e_j]) \\ &= 2g(\nabla_\xi e_i, e_j) + 2(h^2-1) d\xi(e_i, e_j) \end{aligned}$$

$$\begin{aligned} g^h(\nabla^h_{e_i^h} e_i^h, e_j^h) &= -g^h(e_i^h, [e_i^h, e_j^h]) \\ &= hg(e_i, [e_i, e_j]) + e_j(h) = hg(\nabla_{e_i} e_i, e_j) + e_j(h). \end{aligned}$$

The isomorphism $P_{SO}(M, g) \to P_{SO}(M, g^h)$ given by $u \mapsto u^h$ lifts canonically to an isomorphism $P_{\mathrm{Spin}}(M, g) \to P_{\mathrm{Spin}}(M, g^h)$. We then consider a local section $\widetilde{u}$ of $P_{\mathrm{Spin}}(M, g)$ over u and the corresponding local section $\widetilde{u}^h$ of $P_{\mathrm{Spin}}(M, g^h)$ over u^h. This obviously defines an isomorphism of vector bundles $\psi = [\widetilde{u}, \phi] \mapsto \psi^h = [\widetilde{u}^h, \phi]$ satisfying

$$< \psi_1, \psi_2 > = < \psi_1^h, \psi_2^h > \text{ and } (X \cdot \psi)^h = X^h \cdot \psi^h, \quad \forall X \in TM. \tag{21}$$

Let J denote the almost complex structure of the bundle $\xi^\perp$ given by orientation (thus $Je_1 = e_2$ and $Je_2 = -e_1$). We may extend J to TM by $J\xi = 0$. Recall now that the covariant derivative of a spinor $\psi = [\widetilde{u}, \phi]$ (where $\widetilde{u}$ is a local section of the spin structure projecting to a local orthonormal frame $u = \{e_1, ..., e_n\}$) is given by

$$\nabla_X \psi = [\widetilde{u}, X(\phi)] + \frac{1}{2}\sum_{i<j} < \nabla_X e_i, e_j > e_i \cdot e_j \cdot \psi.$$

Applying this to our ψ^h and using the above formulas for the covariant derivative ∇^h on vectors, together with (19) and (20), yields

$$\begin{aligned} \nabla^h_\xi \psi^h &= [\widetilde{u}^h, \xi(\phi)] + \frac{1}{2} g(\nabla_\xi e_1, e_2)\,(e_1 \cdot e_2 \cdot \psi)^h + \frac{h^2-1}{2}\,(d\xi(e_1, e_2) e_1 \cdot e_2 \cdot \psi)^h \\ &= (\nabla_\xi \psi)^h + \frac{h^2-1}{2}\,(d\xi \cdot \psi)^h \\ &= (\nabla_\xi \psi)^h - \frac{h^2-1}{2}\,(\xi \cdot \psi)^h = (1 - \frac{h^2}{2})\xi \cdot \psi^h, \end{aligned}$$

$$\begin{aligned}
\nabla^h_{e_1^h}\psi^h &= [\widetilde{u}^h, e_1^h(\phi)] + \frac{1}{2}g^h(\nabla^h_{e_1^h}\xi, e_2^h)\xi\cdot e_2^h\cdot\psi^h + \frac{1}{2}g^h(\nabla^h_{e_1^h}e_1^h, e_2^h)e_1^h\cdot e_2^h\cdot\psi^h\\
&= h[\widetilde{u}^h, e_1(\phi)] + \frac{h^2}{2}g(\nabla_{e_1}\xi, e_2)(\xi\cdot e_2\cdot\psi)^h\\
&\quad +\frac{1}{2}(hg(\nabla_{e_1}e_1, e_2) + e_2(h))(e_1\cdot e_2\cdot\psi)^h\\
&= h(\nabla_{e_1}\psi)^h + \frac{h^2-h}{2}g(\nabla_{e_1}\xi, e_2)(\xi\cdot e_2\cdot\psi)^h + \frac{1}{2}e_2(h)(e_1\cdot e_2\cdot\psi)^h\\
&= h(\nabla_{e_1}\psi)^h + \frac{h^2-h}{2}d\xi(e_1, e_2)(\xi\cdot e_2\cdot\psi)^h - \frac{1}{2}e_2(h)\xi\cdot\psi^h\\
&= h(\nabla_{e_1}\psi)^h + \frac{h^2-h}{2}(e_1\cdot\psi)^h - \frac{i}{2}dh(J(e_1))\psi^h\\
&= \frac{h}{2}(e_1\cdot\psi)^h + \frac{h^2-h}{2}(e_1\cdot\psi)^h + \frac{i}{2}J(dh)(e_1)\psi^h\\
&= \frac{h^2}{2}(e_1\cdot\psi)^h + \frac{i}{2}J(dh)(e_1)\psi^h,
\end{aligned}$$

and similarly for the covariant derivative in direction of e_2^h. These formulas can be written in a homogeneous form as

$$\nabla^h_X\psi^h = \frac{h^2}{2}X\cdot\psi^h + i\,(\frac{1}{h}J(dh) + (1-h^2)\xi)(X)\psi. \tag{22}$$

Let α be a 1-form on M. We may view $i\alpha$ as a connection form on the trivial S^1 bundle $M\times S^1$. Let $L = M\times\mathbb{R}^2$ be the induced oriented vector bundle of rank 2 over M and ∇^0 the covariant derivative on L induced by the above connection. Let σ be a non-zero constant section of L, *i.e.* of the form $\sigma(x) = (x, c)$ with $c\in\mathbb{R}^2\setminus\{0\}$. It then satisfies

$$\nabla^0_X\sigma = i\alpha(X)\sigma,\ \forall X\in TM. \tag{23}$$

Taking $\alpha = -\frac{1}{h}J(dh) - (1-h^2)\xi$ and using (22) yields

$$\widetilde{\nabla}_X(\psi^h\otimes\sigma) = \frac{h^2}{2}X\cdot(\psi^h\otimes\sigma), \tag{24}$$

(where $\widetilde{\nabla} = \nabla^h\otimes\nabla^0$). But $\Psi = \psi^h\otimes\sigma$ is a section of $\Sigma M\otimes L$ which is, of course, the spinor bundle associated to the Spin^c structure with auxiliary line bundle L^2, so Ψ is a generalized Killing spinor with Killing function $f = \frac{h^2}{2}$.

b) 2-dimensional case. This one is somewhat similar, but the construction is more involved. The idea is roughly to take a suitable spinor on the flat torus T^2, to modify the metric in both directions by multiplication with two different functions and then to twist the spinor bundle with a trivial bundle (endowed with a non-trivial connection, as before).

Consider the flat torus $(T^2, g) = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ and a global orthonormal frame $\{X, Y\}$ given by $X = \partial/\partial x$ and $Y = \partial/\partial y$, where x, y are local coordinates on T^2 coming from the standard Euclidean coordinates on $\mathbb{R}^2$. Let us fix the orientation on T^2 given by the above frame. We also consider a parallel positive half-spinor ψ_+ and define $\psi_- = X \cdot \psi_+$. Then, since the volume form acts by multiplication with i on Σ_+, we obtain $Y \cdot \psi_+ = -i\,\psi_-$.

Let $a(x)$, $b(x)$, $c(x)$ be three positive periodic functions with period 2π, which can thus be considered as functions on T^2. We define a new metric $\widetilde{g}$ on T^2 by requiring the frame $\{aX, bY\}$ be orthonormal, and let $\widetilde{\nabla}$ be the covariant derivative corresponding to $\widetilde{g}$. If we consider the linear isomorphism of tangent spaces, denoted by $Z \longmapsto \widetilde{Z}$, defined on the basis $\{X, Y\}$ by $\widetilde{X} = aX$ and $\widetilde{Y} = bY$, and $\widetilde{\psi}_\pm$ the spinors on $(T^2, \widetilde{g})$ corresponding to $\psi_\pm$ with the identification between spinor bundles given as above, then, as before, we have $\widetilde{X} \cdot \widetilde{\psi}_\pm = \widetilde{(X \cdot \psi_\pm)}$ and $\widetilde{Y} \cdot \widetilde{\psi}_\pm = \widetilde{(Y \cdot \psi_\pm)}$.

On $(T^2, \widetilde{g})$, let us compute the covariant derivative of the spinor field

$$\psi = \cos(c(x))\widetilde{\psi}_+ + \sin(c(x))\widetilde{\psi}_-.$$

Using the Koszul formula we first compute

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{X} = 0 \tag{25}$$

$$\widetilde{g}(\widetilde{\nabla}_{\widetilde{Y}}\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{Y}, [\widetilde{Y}, \widetilde{X}]) = -\frac{ab'}{b}. \tag{26}$$

Then, as before we obtain

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{\psi}_\pm = 0, \tag{27}$$

$$\widetilde{\nabla}_{\widetilde{Y}}\widetilde{\psi}_\pm = \mp i\frac{ab'}{2b}\widetilde{\psi}_\pm, \tag{28}$$

so finally

$$\widetilde{\nabla}_{\widetilde{X}}\psi = c'(-\sin(c)\widetilde{\psi}_+ + \cos(c)\widetilde{\psi}_-) = c'\widetilde{X} \cdot \psi, \tag{29}$$

$$\widetilde{\nabla}_{\widetilde{Y}}\psi = i\frac{ab'}{2b}(-\cos(c)\widetilde{\psi}_+ + \sin(c)\widetilde{\psi}_-). \tag{30}$$

We now consider the trivial complex line bundle L over T^2, with connection form given by an imaginary-valued form $i\hat{\alpha}$ satisfying $\hat{\alpha}(\widetilde{X}) = 0$ and $\hat{\alpha}(\widetilde{Y}) = \alpha$ (where α is an arbitrary function on T^2). As before, twisting with L yields a Spin^c structure on T^2, and for any constant section σ of L we obtain a spinor $\psi \otimes \sigma$ associated with this Spin^c structure. By (29), (30) we obtain

$$\widetilde{\nabla}_{\widetilde{X}}(\psi \otimes \sigma) = c'\widetilde{X} \cdot (\psi \otimes \sigma), \tag{31}$$

$$\widetilde{\nabla}_{\widetilde{Y}}(\psi \otimes \sigma) = i\frac{ab'}{2b}(-\cos(c)(\widetilde{\psi}_+ \otimes \sigma) + \sin(c)(\widetilde{\psi}_- \otimes \sigma)) + i\alpha(\psi \otimes \sigma). \tag{32}$$

We now try to solve the equation $\widetilde{\nabla}_{\widetilde{Y}}\psi \otimes \sigma = c'\widetilde{Y} \cdot \psi \otimes \sigma$, which is equivalent to the following system

$$\begin{cases} -\frac{ab'}{2b}\cos(c) + \alpha\cos(c) = -c'\sin(c) \\ \frac{ab'}{2b}\sin(c) + \alpha\sin(c) = -c'\cos(c) \end{cases}$$

We can solve it by taking for instance

$$b(x) = c(x) = \frac{\pi}{3} + \frac{\pi}{24}\cos(x), \tag{33}$$

$$a(x) = b(x)(\tan(b(x)) - \cot(b(x))), \tag{34}$$

$$\alpha(x, y) = -\frac{1}{2}b'(x)(\tan(b(x)) + \cot(b(x))). \tag{35}$$

It is then clear that a and b are positive functions (as required) and that the spinor $\psi \otimes \sigma$ is a generalized Killing spinor on $(T^2, \widetilde{g})$ with non-constant Killing function

$$f(x, y) = -\frac{\pi}{24}\sin(x),$$

which gives the desired example. □

Note that, doing all the computations of the 3-dimensional case backwards (a part which we skip here), one also proves

THEOREM 2.6. *Let M be a 3-dimensional* Spin^c *manifold admitting a generalized Killing spinor with never-vanishing Killing function. Then M is obtained from a Sasakian manifold (N, g, ξ) by multiplying the metric in directions orthogonal to ξ with a function whose gradient is orthogonal to ξ. The* Spin^c *structure on M is obtained by twisting the canonical* Spin^c *structure on N (cf.* [21]*) with a trivial line bundle with non-trivial connection, as in the proof of Theorem 2.5 a).*

3. Proof of the inequalities

The proof of Theorem 1.2 makes essential use of conformal geometry. It can be obtained by following Hijazi's proof [14] or by using an argument due to Ch. Bär [3]. We shall here present another proof, which is perhaps more appropriate to the conformal character of our problem.

Given any Spin^c manifold (M, g, L, A, σ), we may consider its conformal Spin^c frames, *i.e.* the principal bundle with structure group $\mathrm{CSpin}^c_n = \mathbb{R}_+ \times \mathrm{Spin}^c_n$ that covers $P_{CO_n^+}M \times P_{S^1}M$, where $P_{CO_n^+}M$ is the bundle of oriented conformal frames over the conformal manifold $(M, [g])$. The covering is compatible with the group covering

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{R}_+ \times \mathrm{Spin}^c_n \xrightarrow{id\times\theta} \mathbb{R}_+ \times SO_n \times S^1 \longrightarrow 0.$$

where θ is the 2-fold covering of $SO_n \times S^1$ by Spin^c_n. For any choice of $k \in \mathbb{Z}$ and any representation μ_0 of Spin^c_n on a linear space V, we get a bundle of weight k, denoted by $V^{(k)}M$, associated to $P_{\mathrm{CSpin}^c}M$ by the representation

$$\mu_k = \lambda_k \otimes \mu_0$$

where λ_k is the representation of $\mathbb{R}_+$ over $\mathbb{R}$ given by

$$\lambda_k(a).u = a^k u, \quad \forall a \in \mathbb{R}^+,\ u \in \mathbb{R}.$$

If (μ_0, V) is the standard Spin$^{\rm c}$ representation, we get spinor bundles $\Sigma^{(k)}M$ of weight k. If g is any choice of metric in $[g]$, the usual Spin$^{\rm c}$-spinor bundle Σ^g may be identified to any of the $\Sigma^{(k)}M$'s (by restricting the frame bundle of the latter to $P_{\mathrm{Spin^c}}M$). This induces a family of isomorphisms

$$\Phi^{(k)}: \Sigma^g \longrightarrow \Sigma^{\widetilde{g}} \quad , \quad [s,u] \longmapsto [s\phi, \phi^{-k}u],$$

where $\widetilde{g} = \phi^{-2}g$, $s \in P^g_{\mathrm{Spin^c}}M$ and u is the expression of a spinor field in the spinor frame s. It is easily checked that

$$|\Phi^{(k)}\psi|^2_{\widetilde{g}} = \phi^{-2k}|\psi|^2_g \text{ for any spinor } \psi. \tag{36}$$

Now, every choice of a Weyl structure on $(M,[g])$ induces Dirac and Penrose operators which act on the weighted spinor bundles and lower the weights by 1. Moreover for $k = -(n-1)/2$ the Dirac operator and for $k = 1/2$ the Penrose operator are conformally invariant (*i.e.* do not depend on the choice of the Weyl structure, cf. [12]). As a consequence, this yields

$$\mathcal{D}^{g,A} = (\Phi^{(-\frac{1}{2}(1+n))})^{-1} \circ \mathcal{D}^{\widetilde{g},\widetilde{A}} \circ \Phi^{(\frac{1}{2}(1-n))}, \quad (\mathcal{P}^{g,A})_X = (\Phi^{\frac{1}{2}})^{-1} \circ (\mathcal{P}^{\widetilde{g},\widetilde{A}})_X \circ \Phi^{\frac{1}{2}}. \tag{37}$$

Remark. The connection A on the associated line bundle is here fixed. If A were changed in the above process to $\widetilde{A} = A - c\frac{d\phi}{\phi}$ for some constant c, the conformally invariant weights have to be changed into $(-n-1-c)/2$ and $(1-c)/2$ respectively. We shall nevertheless not use this freedom since any two choices of the constant c would lead to identical results.

The second main tool is the Schrödinger-Lichnerowicz formula for Spin$^{\rm c}$ Dirac operators [23], [17]:

Lemma 3.1. *For any spinor field ψ,*

$$(\mathcal{D}^{g,A})^*(\mathcal{D}^{g,A})\psi = (\mathcal{D}^{g,A})^2\psi = (\nabla^{g,A})^*(\nabla^{g,A})\psi + \frac{1}{4}\operatorname{Scal}^g \psi + \frac{1}{2}F_A\cdot\psi.$$

This can be rewritten as

$$\frac{n-1}{n}(\mathcal{D}^{g,A})^*(\mathcal{D}^{g,A})\psi - \frac{1}{4}\operatorname{Scal}^g\psi - \frac{1}{2}F_A\cdot\psi = (\mathcal{P}^{g,A})^*(\mathcal{P}^{g,A})\psi.$$

Integrating this formula, we get the

Lemma 3.2. *For any complex spinor ψ on the* Spin$^{\rm c}$ *compact manifold (M,g,L,A),*

$$\int_M \left(\frac{n-1}{n}|\mathcal{D}^{g,A}\psi|^2 - \frac{1}{4}\operatorname{Scal}^g|\psi|^2 - \frac{1}{2} < F_A\cdot\psi, \psi >_g\right) d\operatorname{vol}_g = \int_M |\mathcal{P}^{g,A}(\psi)|^2_g d\operatorname{vol}_g.$$

This holds for any metric g and any choice of the connection A. It thus holds for $\widetilde{g} = \varphi^{-2}g$, $\widetilde{A} = A$ and $\widetilde{\psi} = \Phi(\psi)$, where Φ is the isomorphism $\Phi^{(\frac{1-n}{2})}$ above. We then have

$$d\operatorname{vol}_{\widetilde{g}} = \varphi^{-n}d\operatorname{vol}_g, \quad |\mathcal{D}^{\widetilde{g},\widetilde{A}}\widetilde{\psi}|^2_{\widetilde{g}} = \varphi^{n+1}|\mathcal{D}^{g,A}\psi|^2_g, \quad |\mathcal{P}^{\widetilde{g},\widetilde{A}}\widetilde{\psi}|^2_{\widetilde{g}} = \varphi|\mathcal{P}^{g,A}(\varphi^{\frac{n}{2}}\psi)|^2_g. \tag{38}$$

and

$$\operatorname{Scal}^{\widetilde{g}} = \eta^{-\frac{n+2}{n-2}}L_g\eta, \quad \text{where } \eta = \varphi^{-\frac{n-2}{2}}. \tag{39}$$

Thus,

$$\int_M \left(\frac{n-1}{n}|\mathcal{D}^{\widetilde{g},\widetilde{A}}\widetilde{\psi}|^2 - \frac{1}{4}\operatorname{Scal}^{\widetilde{g}}|\widetilde{\psi}|^2 - \frac{1}{2} < F_{\widetilde{A}}\cdot\widetilde{\psi},\widetilde{\psi} >_{\widetilde{g}}\right) d\operatorname{vol}_{\widetilde{g}} = \int_M |\mathcal{P}^{\widetilde{g},\widetilde{A}}(\widetilde{\psi})|^2_{\widetilde{g}}\, d\operatorname{vol}_{\widetilde{g}},$$

which can be rewritten as

$$\int_M \varphi\left(\frac{n-1}{n}|\mathcal{D}^{g,A}\psi|^2 - \frac{1}{4}\eta^{-1}(L^g\eta)|\psi|^2\right) d\operatorname{vol}_g - \frac{1}{2}\int_M \varphi < F_A\cdot\psi,\psi >_g d\operatorname{vol}_g$$
$$= \int_M \varphi^{1-n}\,|\mathcal{P}^{g,A}(\varphi^{\frac{n}{2}}\psi)|^2_g\, d\operatorname{vol}_g\,.$$

We now use the following "Cauchy-Schwarz"-type inequality for the Clifford action of 2-forms on spinors:

LEMMA 3.3. *For any spinor field ψ and 2-form Ω, we have the inequality*

$$< i\Omega\cdot\psi,\psi > \;\geq\; -\left[\frac{n}{2}\right]^{\frac{1}{2}} |\Omega|\,|\psi|^2, \tag{40}$$

where the norm on a 2-form λ chosen here is

$$|\lambda|^2 = \sum_{i<j}(\lambda_{ij})^2 \tag{41}$$

in any orthonormal basis. Moreover, if equality holds in (40), then

$$\Omega\cdot\psi = i\left[\frac{n}{2}\right]^{\frac{1}{2}} |\Omega|\cdot\psi, \tag{42}$$

and furthermore either Ω vanishes or Ω has maximal rank (n for n even and $n-1$ for n odd).

Proof of the lemma. Consider Ω as a skew-Hermitian operator on $TM\otimes\mathbb{C}$. Then $i\Omega$ is Hermitian, so that all its eigenvalues are real, and $TM\otimes\mathbb{C}$ splits as a direct sum of the corresponding eigenspaces. This easily shows that we may find an orthonormal basis $\{e_i\}$ of TM such that

$$\Omega = \sum_{j=1}^{[n/2]}\lambda_j\, e_{2j-1}\wedge e_{2j}. \tag{43}$$

But the Cauchy-Schwarz inequality shows that

$$< ie_{2j-1}\cdot e_{2j}\cdot\psi,\psi >^2 \leq |ie_{2j-1}\cdot e_{2j}\cdot\psi|^2|\psi|^2 = |\psi|^4, \tag{44}$$

so the triangle inequality yields

$$< i\Omega\cdot\psi,\psi > \;\geq\; -\sum_{j=1}^{[n/2]}|\lambda_j||\psi|^2 \;\geq\; -\left[\frac{n}{2}\right]^{\frac{1}{2}} \Big(\sum_{j=1}^{[n/2]}|\lambda_j|^2\Big)^{\frac{1}{2}}|\psi|^2, \tag{45}$$

and (40) follows. If equality holds, then all the above inequalities must become equalities. This yields

$$< i\lambda_j e_{2j-1}\cdot e_{2j}\cdot\psi,\psi >= -|\lambda_j||\psi|^2, \tag{46}$$

and

$$\sum_{j=1}^{[n/2]} |\lambda_j| = \left[\frac{n}{2}\right]^{\frac{1}{2}} \Big(\sum_{j=1}^{[n/2]} |\lambda_j|^2\Big)^{\frac{1}{2}}. \tag{47}$$

It is clear that (46) implies (42), and, from (47), all the λ_i's must have equal absolute values, thus showing the last statement of the lemma. □

Remark. Here our result differs from the one obtained by Ch. Bär [3], since he considers the *operator* norm $|\lambda| = \sup |x|^{-1}|\lambda(x)|$ (with λ seen as a skew-symmetric endomorphism on spinors). Here we shall consider the *Euclidean* norm defined above, having in mind the perturbed scalar curvature operator and Corollary 1.3. The reader interested in what happens with Ch. Bär's choice of norms may however have a look at Section 4.4.

We can now conclude that

$$\int_M \varphi\left(\frac{n-1}{n}|\mathcal{D}^{g,A}\psi|^2 - \frac{1}{4}\eta^{-1}(L^g_\Omega\eta)|\psi|^2\right) d\,\mathrm{vol}_g \geq \int_M \varphi^{1-n}\,|\mathcal{P}^{g,A}(\varphi^{\frac{n}{2}}\psi)|^2_g\, d\,\mathrm{vol}_g\,. \tag{48}$$

The main inequality is then proven by evaluating the last formula against a spinor ψ and function η such that

$$\mathcal{D}^{g,A}\psi = \lambda_1\,\psi, \quad L^g_\Omega\eta = \mu_1\,\eta. \tag{49}$$

Note that the positivity of η is guaranteed by the maximum principle. At the end we get

$$(\lambda_1)^2 \;\geq\; \frac{n}{4(n-1)}\,\mu_1 \tag{50}$$

as required. □

We now proceed to prove the corollary.

Proof of Corollary 1.3. It relies on the intermediate

Lemma 3.4. *Let (M, g, L, A, σ) be a compact Riemannian* Spin^c *manifold. Then the first eigenvalue λ_1 of the Dirac operator on the spinor bundle satisfies*

$$\mathrm{vol}(M,g)^{2/n}\,\lambda_1^2 \;\geq \frac{n}{4(n-1)}\;\left(Y(M,[g]) \;-\; c_n\|\Omega\|_{\frac{n}{2}}\right),$$

where $Y(M,[g])$ is the Yamabe number of the conformal structure of g and $c_n = 2\,[n/2]^{\frac{1}{2}}$.

Proof. We first recall the Rayleigh quotient definition of the first eigenvalue μ_1, namely

$$\mu_1 = \inf_{\eta\neq 0} \frac{\int_M 4\frac{n-1}{n-2}|d\eta|^2 + (\mathrm{Scal}^g - c_n|\Omega|)\eta^2}{\int_M \eta^2}\,. \tag{51}$$

Using the usual Hölder inequality

$$\int_M |\eta|^2 \leq \left(\int_M |\eta|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \mathrm{vol}(M,g)^{\frac{2}{n}}, \tag{52}$$

we obtain

$$\mu_1 \geq \inf_{\eta\neq 0} \frac{\int_M 4\frac{n-1}{n-2}|d\eta|^2 + (\mathrm{Scal}^g - c_n|\Omega|)\eta^2}{\left(\int_M |\eta|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \mathrm{vol}(M,g)^{\frac{2}{n}}} \qquad \text{if } \mu_1 \geq 0, \tag{53}$$

and finally (using Hölder again)

$$\mu_1 \cdot \mathrm{vol}(M,g)^{\frac{2}{n}} \;\geq\; \inf_{\eta\neq 0} \frac{\int_M 4\frac{n-1}{n-2}|d\eta|^2 + (\mathrm{Scal}^g)\eta^2}{\left(\int_M |\eta|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}} - c_n \left(\int_M |\Omega|^{n/2}\right)^{2/n}. \tag{54}$$

Since the Yamabe number is defined as

$$Y(M,[g]) = \inf_{\eta\neq 0} \frac{\int_M 4\frac{n-1}{n-2}|d\eta|^2 + (\mathrm{Scal}^g)\eta^2}{\left(\int_M |\eta|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}, \tag{55}$$

we get the required inequality. □

Corollary 1.3 follows directly from the lemma and the fact that, if $n = 4$ and Ω is self-dual, then

$$\int_M |\Omega|^2 = 8\pi^2\; c_1^2(L) \cup [M] \tag{56}$$

is a topological invariant.

4. Equality cases

Using the notations of the previous section, the equality case in Theorem 1.2 is characterized by the following equations (recall that $\widetilde{g} = \varphi^{-2}g$ and $\eta = \varphi^{-\frac{n-2}{2}}$) :

$$\mathcal{D}^{\widetilde{g},\widetilde{A}}\widetilde{\psi} = (\lambda_1\varphi)\;\widetilde{\psi} \quad \text{and} \quad \mathcal{P}^{\widetilde{g},\widetilde{A}}\widetilde{\psi} = 0, \tag{57}$$

$$\eta^{-1}(L^g_\Omega \eta) = \mu_1 \tag{58}$$

$$< F_A \cdot \psi, \psi >_g \;=\; -\left[\frac{n}{2}\right]^{\frac{1}{2}} |\Omega|_g\, |\psi|_g^2. \tag{59}$$

In particular, the spinor $\widetilde{\psi}$ is a generalized Killing spinor. Moreover, by Lemma 3.3, (59) is equivalent to

$$\Omega \cdot \psi = i\left[\frac{n}{2}\right]^{\frac{1}{2}} |\Omega|_g\, \psi \tag{60}$$

which may be written in terms of $\widetilde{g}$ as

$$\Omega \cdot \widetilde{\psi} = i\left[\frac{n}{2}\right]^{\frac{1}{2}} |\Omega|_{\widetilde{g}}\, \widetilde{\psi}. \tag{61}$$

We shall consider separately the cases where λ_1 vanishes or not, studying in detail the low dimensions 3 and 4.

4.1. Non-zero first eigenvalue. The spinor field $\widetilde{\psi}$ is a generalized Killing spinor with respect to $\widetilde{g}$. Applying Theorem 1.1, we get, in dimension $n \geq 4$, that $\widetilde{\psi}$ is a Killing spinor, the (non-zero) function φ is a constant (identically equal to 1, say) and $g = \widetilde{g}$, *i.e.* finally

$$\nabla_X^{g,A}\psi = -\frac{\lambda_1}{n} X \cdot \psi. \tag{62}$$

In dimension 3, Theorem 1.1 does not apply and we have seen that there are nontrivial examples of generalized Killing spinors. We shall however prove that these spinor fields cannot induce equality in our basic inequality (2).

LEMMA 4.1. *In dimension* 3, *the generalized Killing spinor inducing equality in (2) is a* Killing *spinor.*

Proof. From Formula (14) in Lemma 2.3,

$$\Omega(\widetilde{\xi}, .) = -4\lambda_1 \, d\varphi, \tag{63}$$

where $\widetilde{\xi}$ is the Killing vector field defined by $\widetilde{\psi}$ in the usual way

$$i\,\widetilde{g}(\widetilde{\xi}, X) \ = < X \cdot \widetilde{\psi}, \widetilde{\psi} > \quad \forall X. \tag{64}$$

Mimicking the computations done in Theorem 2.5, part a), we also get

$$d\widetilde{\xi} \ = 2\lambda_1 \varphi * \widetilde{\xi}, \ \ \nabla|\widetilde{\xi}|^2 \ = 0 \ \text{ and } \ \ \widetilde{\xi} \cdot \widetilde{\psi} \ = \ i|\widetilde{\xi}|^2\, \widetilde{\psi}. \tag{65}$$

We can now write the curvature 2-form Ω as

$$\Omega = \ F * \widetilde{\xi} \ + \ \widetilde{\xi} \wedge \alpha, \tag{66}$$

for some function F and 1-form α. Actually α satisfies

$$\Omega(\widetilde{\xi}, .) \ = \ |\widetilde{\xi}|^2 \alpha \ = \ -4\lambda_1 \, d\varphi. \tag{67}$$

We shall now compare the Clifford product of $\widetilde{\psi}$ by Ω with Formula (61). Recalling that $\widetilde{\xi} \cdot \widetilde{\psi}$ and $*\widetilde{\xi} \cdot \widetilde{\psi}$ are (complex) collinear to $\widetilde{\psi}$, we deduce that

$$\begin{aligned} \Omega \cdot \widetilde{\psi} \ &= \ \left(F * \widetilde{\xi} \ + \ \widetilde{\xi} \wedge \alpha\right) \cdot \widetilde{\psi} \\ &= \ -iF\,\widetilde{\psi} \ - i|\widetilde{\xi}|^2\, \alpha \cdot \widetilde{\psi} \end{aligned}$$

has to be collinear to $\widetilde{\psi}$. But, since

$$d\varphi(\widetilde{\xi}) \ = \ -\frac{1}{4\lambda_1}\, \Omega(\widetilde{\xi}, \widetilde{\xi}) = 0, \tag{68}$$

(67) implies that $\alpha \cdot \widetilde{\psi}$ must be orthogonal to $\widetilde{\psi}$, hence must vanish if we compare to Formula (61). This implies that $\alpha = 0$, so

$$d\varphi = 0. \tag{69}$$

The function φ is then constant and the spinor field $\widetilde{\psi}$ is a Killing spinor. $\square$

Remark. The previous proof is 3-dimensional in nature, since the expression of Ω relative to $\widetilde{\xi}$ heavily depends on dimension. In higher dimensions, we still have to rely on the general non-existence result for generalized Killing spinors.

¿From the second author's classification of simply connected Spin^c manifolds carrying Killing spinors, we get that the universal Riemannian covering $(\widetilde{M}, g)$ of (M, g) is either spin and admits Killing spinors, or Sasaki with its canonical Spin^c structure (cf. [21]).

We now study the low dimensions 3 and 4 a bit further.

a) In dimension 3, if $(\widetilde{M}, g)$ is spin, it is then Einstein with positive scalar curvature [14] and then isometric to the 3-sphere. Each Killing spinor on a quotient S^3/Γ induces a parallel spinor on the flat cone $\mathbb{R}_+^* \times S^3/\Gamma$, which is itself a quotient of $\mathbb{R}^4 \setminus \{0\}$. It will be shown later (Lemma 4.5 below) that an oriented 4-dimensional Riemannian manifold carries a Spin^c structure with a parallel spinor iff it is Kähler. Consequently, the quotients of $\mathbb{R}^4 \setminus \{0\}$ by a finite subgroup Γ of SO_4 carry a parallel Spin^c spinor iff $\Gamma \subset U_2$, and we obtain that every quotient of S^3 by a finite fixed point-free subgroup of U_2 carries a Killing (Spin^c) spinor. Moreover, each of these cases may occur as equality cases of our basic inequality since the constant curvature metrics are Yamabe metrics in their conformal class.

If Ω is not identically zero, $\widetilde{M}$ is Sasaki. We remark here that M itself is Sasaki : indeed, M carries a Killing spinor, so after renormalization of the metric, the cone over M carries parallel spinors (see [21]), hence it is Kähler (Lemma 4.5 below) *i.e.* M is Sasaki. It remains to find which Sasaki manifolds are indeed limiting manifolds.

We shall normalize the metric such that the Killing vector of the Sasaki structure satisfies

$$\nabla^g \xi \;=\; \frac{1}{2} d\xi \;=\; *\xi. \tag{70}$$

It is then easily computed that the Ricci curvature obeys the following formula (see [9])

$$\mathrm{Ric}^g = \left(\frac{\mathrm{Scal}^g}{2} - 1\right) g + \left(3 - \frac{\mathrm{Scal}^g}{2}\right) \xi \otimes \xi. \tag{71}$$

Since the curvature of the associated line bundle is the Ricci form of the cone over the Sasaki manifold and is then related to the Ricci curvature of the manifold by the Gauss equation, we can compute the expression $\mathrm{Scal}^g - 2|\Omega|$ for an arbitrary Sasaki manifold. The Gauss equation implies that

$$\begin{cases} \mathrm{Scal}^g - 2\,|\Omega| = 6 & \text{at points where } \mathrm{Scal}^g \geq 6, \\ \mathrm{Scal}^g + 2\,|\Omega| = 6 & \text{otherwise.} \end{cases}$$

Since (57) and (61) are automatically satisfied on a Sasakian manifold (where $\widetilde{\psi}$ is the canonical Killing spinor [21]), M is a limiting manifold iff (58) holds, which amounts here to say that the expression

$$\mathrm{Scal}^g - 2|\Omega| \tag{72}$$

is a positive constant since η is a constant. Thus, if $\mathrm{Scal}^g \geq 6$ everywhere on M, then M is automatically a limiting manifold.

Otherwise, the following equations have to be satisfied for the manifold to be a limit case :

$$\mathrm{Scal}^g + 2|\Omega| = 6 \quad \text{and} \quad \mu_1 = \mathrm{Scal}^g - 2|\Omega| = \text{ const. } > 0. \tag{73}$$

Moreover, the manifold is a limiting one if (73) holds, hence if Scal^g is constant and lies in the interval $]3,6[$.

Let us then define a *Berger-type metric* as follows : if V is any left-invariant vector field on the sphere, a metric will be said of Berger-type if it is obtained from the round metric by distorting it only in the direction of V. Hence, any Berger-type metric g satisfies :

$$g(V,X) = k^2 \mathrm{can}(X,V) \quad \forall X \in TM,$$

for a never-zero function k and

$$g(X,Y) = \mathrm{can}(X,Y) \quad \forall X,Y \perp V.$$

This definition somewhat differs from the classical one since distortion of a Berger metric is usually allowed only in the direction of the vertical vector field of the Hopf fibration. Easy computations similar to that of the proof of Theorem 2.5 a) now yield the following

LEMMA 4.2. *Any 3-dimensional Sasaki manifold with constant (normalized) scalar curvature strictly larger than -2 is a rescaled Berger-type metric on the sphere (or a quotient).*

Proof. The main tools are the computations done in the proof of Theorem 2.5. They show that the Levi-Civita connections of g and $g^h = h^{-2}g + (1-h^{-2})\xi\otimes\xi$ with h taken constant are related by

$$\nabla^{g^h}_X = \nabla^g_X + (h^2-1)\left(\eta(X)\otimes J + \xi\wedge JX\right), \tag{74}$$

where η is the 1-form dual to ξ and J is the skew-symmetric endomorphism associated to $*\xi$ (with respect to g). Computation of the Ricci curvature of g^h shows that there is a constant h such that the metric

$$g^h = h^{-2}g + (1-h^{-2})\xi\otimes\xi \tag{75}$$

is Einstein with positive scalar curvature. Hence, such a situation can occur only on a spherical space-form. □

Remarks. In the last lemma we consider Sasaki manifolds whose Killing vector satisfies Formula (70), *i.e. normalized* Sasaki manifolds. As already noticed in the introduction, this last condition is not scale-invariant and the arguments comparing the scalar curvature with some fixed constants make sense. Note also that the output of the lemma implies a constraint on the metric of the Sasaki manifold, but also on the Killing vector since the *round* sphere admits only standard Sasaki structures (*i.e.* given by a left-invariant vector field). Moreover, the occurrence of -2 as a threshold for the scalar curvature may be explained as follows : any 3-dimensional Sasaki structure is locally obtained from a circle bundle over a Riemann surface. O'Neill formulas then yield -2 as a threshold for the base manifold to be a sphere.

As a conclusion, we find that the limiting manifolds either have (normalized) scalar curvature always no less than 6 or there is a region where it is strictly smaller than the threshold and hence constant. But such a region must be open and closed in our manifold, hence we are left on one hand with the Sasaki manifolds with $\mathrm{Scal}^g \geq 6$ everywhere, and on the other hand with the constant scalar curvature Sasaki manifolds (which are of Berger-type).

b) In dimension 4, the Sasaki case cannot appear, hence the curvature 2-form Ω must always vanish and the universal Riemannian covering $(\widetilde{M}, g)$ is spin and Einstein with positive scalar curvature. By Theorem 6.4 of [14], we obtain that $(\widetilde{M}, g)$ is isometric to S^4. As the only quotient of S^4 is not orientable, we deduce $(M, g) = (S^4, can)$, too, and this obviously occurs as an equality case.

4.2. Vanishing first eigenvalue. In this case $\widetilde{\psi}$ is a parallel spinor for $\widetilde{g}$. Because of the conformal invariance of the Dirac operator and of the conformal Laplacian L^g_Ω, if $\mu_1 = 0$ and $\lambda_1 = 0$ for some metric g, the same relations hold for every conformally equivalent metric $\widetilde{g}$. A manifold (M, g) is thus a limiting manifold for our inequality iff it is conformally equivalent to a Spin^c manifold $(M, \widetilde{g})$ which

- carries a parallel spinor $\widetilde{\psi}$;
- satisfies (61);
- has $\mu_1 = 0$.

But the (Spin^c) Lichnerowicz formula together with (61) imply that, if $(M, \widetilde{g})$ carries parallel spinors, then $L^{\widetilde{g}}_\Omega = 4\frac{n-1}{n-2}\Delta_{\widetilde{g}}$, hence the condition $\mu_1 = 0$ is redundant, and may be dropped out. Again by Lichnerowicz formula, since $\widetilde{\psi}$ is parallel, (61) is equivalent to

$$\mathrm{Scal}^{\widetilde{g}} = 2\left[\frac{n}{2}\right]^{\frac{1}{2}} |\Omega|. \tag{76}$$

The limiting manifolds are thus, up to a conformal change of the metric, exactly the Spin^c manifolds $(M, \widetilde{g})$ with parallel spinors whose auxiliary curvature form satisfies (76).

The classification of [21] shows that the universal Riemannian covering $(\widetilde{M}, \widetilde{g})$ is a Riemannian product $S \times K$ where S is a *spin* manifold carrying parallel spinors and K is a (non-Ricci flat) Kähler manifold endowed with its canonical Spin^c structure (S or K may be, of course, reduced to a point). Moreover, the form Ω on M is just the pull-back of the Ricci form on K. From Lemma 3.3, we deduce that, if (61) holds, then only the following cases may occur:

- K is a point, so $\widetilde{M} = S$;
- S is a point, and $\widetilde{M} = K$ is Kähler-Einstein with positive scalar curvature (n even);
- $S = \mathbb{R}$ and K is Kähler-Einstein with positive scalar curvature (n odd).

Moreover, relation (76) is automatically satisfied in each of these cases, so we have obtained the

THEOREM 4.3. *A Spin^c manifold M^n is a limiting manifold for the inequality of Theorem 1.2 (with $\lambda_1 = 0$) iff it is conformally equivalent to one of the following:*

1 - *a Spin^c manifold with flat auxiliary connection carrying a parallel spinor;*
2 - *a Kähler-Einstein manifold with positive scalar curvature (n even);*
3 - *a quotient of $\mathbb{R} \times K$ by a freely acting group of isometries, where K is Kähler-Einstein with positive scalar curvature (n odd).*

We now obtain more precise informations for the small dimensions 3 and 4.

a) In dimension 3, $(\widetilde{M}, \widetilde{g})$ is either spin, hence flat, when Ω vanishes identically, or $\widetilde{M}$ is a product $\mathbb{R} \times N^2$ where N is the sphere S^2 with an *arbitrary* metric.

In the first case, $(M, \widetilde{g})$ is a quotient of the Euclidean 3-space by a group preserving at least one parallel spinor. From J. Wolf's book [25, Theorem 3.5.5 and pages 123–124] we know that compact orientable Euclidean space-forms are quotients of the flat $\mathbb{R}^3$ by a group Γ whose linear part Γ_* (quotient of the full group by its translation part) is either a cyclic group of order between 2 and 6 or the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Any γ in Γ induces an action

$$\gamma_* : T_x\mathbb{R}^3 \longrightarrow T_{\gamma(x)}\mathbb{R}^3$$

and the quotient is Spin^c if there exists a lift $\widetilde{\Gamma}$ of Γ_* acting on the principal bundle of complex spin frames of $\mathbb{R}^3$, such that any element $\widetilde{\gamma}$ of $\widetilde{\Gamma}$ projects onto the corresponding element γ_* through the standard projection

$$P_{\mathrm{Spin}^c}M \longrightarrow P_{SO}M \times P_{S^1}M \longrightarrow P_{SO}M.$$

When identifying the spinor bundle of $\mathbb{R}^3$ with a product bundle through parallel transport, the spinors that are parallel on the quotient are simply fixed points of $\widetilde{\Gamma}$ (seen now as a subgroup of $\mathrm{Spin}^c_3 = SU_2 \times_{\mathbb{Z}_2} S^1$) acting on the fiber. But it is easily seen that any of the above quoted groups has a lift in Spin^c_3 that admits a fixed point. For example, each of the cyclic subgroups $\Gamma_* = \mathbb{Z}_m$ has a (one-to-one) lift in $SU_2 \times S^1$ of the type

$$\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} \times z^{-1/2} \text{ where } z = e^{i\frac{2k\pi}{m}} \text{ with } 0 \leq k \leq m-1. \tag{77}$$

modulo the choice of a square-root. Note that this is not a subgroup of $SU_2 \times S^1$ but it *induces* a subgroup of $SU_2 \times_{\mathbb{Z}_2} S^1$ which is an isomorphic lift of the linear part of Γ and leaves the first complex-coordinate line of the standard representation of Spin^c_3 in $\mathbb{C}^2$ fixed.

Hence every compact orientable 3-dimensional Euclidean space-form is a limiting manifold.

In the second case, $(M, \widetilde{g})$ has to be a quotient of $(\mathbb{R} \times S^2, dt^2 + \hat{g})$ ($\hat{g}$ arbitrary) having a parallel spinor.

Let Γ be the fundamental group of the manifold M. Since it acts by isometries on $\mathbb{R} \times S^2$, its tangent action preserves the eigenspaces of the Ricci tensor, hence the subspaces of the tangent space of $S^2 \times \mathbb{R}$ given by the $\mathbb{R}$-leaves and the S^2-leaves. As a consequence, any element of Γ maps spheres to spheres and lines to lines. Moreover, since Γ preserves a parallel spinor, it has to respect the Kähler structure of S^2, hence the orientations of S^2 and $\mathbb{R}$. We may then write the action of any element, say γ, of Γ on a point (q,t) as

$$\gamma \cdot (q,t) = (s(q), t+v) \in S^2 \times \mathbb{R} \tag{78}$$

where s belongs to the isometry group of $(S^2, \hat{g})$ and v is a translation vector in $\mathbb{R}$.

We conclude that M is an S^2-bundle over S^1. Recall that the only orientable S^2-bundle over S^1 is the trivial $S^2 \times S^1$. But, though all these bundles are diffeomorphic, they may carry different *non-isometric* Riemannian structures (for details the reader is referred to the examples given in [8, Remarks after Theorem 6.67]).

Moreover, any such bundle is built by identifying the fibers $S^2 \times \{0\}$ and $S^2 \times \{1\}$ of the product $S^2 \times [0,1]$ through an orientation preserving isometry of the fiber. Any such transformation preserves its complex structure, hence a parallel spinor. As a consequence, any such bundle has a parallel spinor.

b) In dimension 4, $(\widetilde{M}, \widetilde{g})$ is either spin (when Ω vanishes), or Kähler-Einstein with positive scalar curvature (when Ω has maximal rank).

In the first case, $(M, \widetilde{g})$ is Ricci-flat, hence locally hyperkähler (for the bundle $\Lambda^+ M$ of self-dual 2-forms has vanishing curvature on Ricci flat Kähler manifolds), hence it is finitely covered by a K3-surface or a flat complex torus, by N. Hitchin's work on the equality case of the well-known Hitchin-Thorpe inequality [15].

If (M,g) itself is spin, we may sum up the result in the following lemma, which is of independent interest. This is probably well-known but we include it for sake of completeness and in lack of a precise reference:

LEMMA 4.4. *Let M^4 be a compact spin manifold admitting a parallel spinor ψ. Then M is isometric either to a* K3 *surface or to a flat torus.*

Proof. We begin by recalling some well-known facts on parallel spinors in dimension 4 [1]. By taking the projection of ψ onto $\Sigma^{\pm} M$ and changing the orientation of M if necessary, we may suppose that ψ is a section of $\Sigma^+ M$. The equation

$$iX \cdot \psi = I(X) \cdot \psi \tag{79}$$

defines a parallel almost complex structure I on M, *i.e.* a Kähler structure.

Recall that in dimension 4 the bundle $\Sigma^+ M$ carries a parallel quaternionic structure commuting with the Clifford product; this just means a $\mathbb{C}$-anti-linear automorphism j satisfying $\mathrm{j}^2 = -1$. Since ψ is parallel, $\mathrm{j}\psi$ is also parallel, and let J be the Kähler structure defined by $\psi + \mathrm{j}\psi$. We can compute

$$\begin{aligned} JIX \cdot (\psi + \mathrm{j}\psi) &= iIX \cdot (\psi + \mathrm{j}\psi) = -X \cdot \psi + i\mathrm{j}iX \cdot \psi \\ &= X \cdot (-\psi + \mathrm{j}\psi)\,, \end{aligned}$$

and

$$\begin{aligned} IJX\cdot(\psi+\mathrm{j}\psi) &= iJX\cdot\psi - iJ_2X\cdot\mathrm{j}\psi = iJX\cdot(\psi-\mathrm{j}\psi)\\ &= -i\mathrm{j}JX\cdot(\psi+\mathrm{j}\psi) = -i\mathrm{j}iX\cdot(\psi+\mathrm{j}\psi)\\ &= X\cdot(\psi-\mathrm{j}\psi) = -JIX\cdot(\psi+\mathrm{j}\psi)\,, \end{aligned}$$

so $IJ=-JI$. Thus $\{I,J,K\}$ defines a hyperkähler structure on M, where $K:=IJ$. Consider now the universal covering $\widetilde{M}$ of M. Since $\widetilde{M}$ is hyperkähler, too, we deduce from the Berger-Simons Holonomy Theorem that $Hol(\widetilde{M})$ is 0 or Sp_1. In the first case $\widetilde{M}=\mathbb{R}^4$ and in the second case $\widetilde{M}$ is a K3 surface. Since the only spin quotients of a K3 surface - the Enriques surfaces - have non-trivial canonical bundle [5] and thus are not hyperkähler, we deduce that if $\widetilde{M}$ is a $K3$ surface, then $M=\widetilde{M}$.

Suppose now that $\widetilde{M}=\mathbb{R}^4$ and let Γ be the group of isometries of $\widetilde{M}$ such that $M=\widetilde{M}/\Gamma$. Then Γ acts freely on $\widetilde{M}$ and preserves the hyperkähler structure induced by that of M. We first remark that on $\mathbb{R}^4$ there exist only one hyperkähler structure, namely the usual one, since $\Lambda^2_+\mathbb{R}^4$ has rank 3. Recall that the usual hyperkähler structure on $\mathbb{R}^4$ is given by left multiplication with i, j and k on each tangent space $T_x\mathbb{R}^4\simeq\mathbb{R}^4\simeq\mathbb{H}$. Let us denote by Sp_1^l resp. Sp_1^r the images of Sp_1 in SO_4 acting on $\mathbb{H}$ by left, resp. right, multiplication. Now, the fact that Γ preserves the hyperkähler structure just means that the linear part of any $\gamma\in\Gamma$ commutes with the action of Sp_1^l, and thus lies in Sp_1^r (since the centralizer of Sp_1^l in SO_4 is Sp_1^r). Let $\gamma=v+a$ be an arbitrary element of Γ, where $v\in\mathbb{R}^4$, $a\in Sp_1^r$, its action on $\mathbb{R}^4\simeq\mathbb{H}$ being $\gamma(h)=ha+v$, for all h in $\mathbb{H}$. Suppose that $a\neq 1$. Then $v(1-a)^{-1}$ is a fixed point of γ, thus contradicting the fact that γ acts freely. This shows that Γ consists only of translations, so finally $M=\mathbb{R}^4/\Gamma$ is a flat torus. □

If (M,g) is not Spin but only Spin^c, we need the following result, which has already been used in the characterization of limiting manifolds in dimension 3 before.

LEMMA 4.5. *A 4-dimensional* Spin^c *manifold carries a parallel spinor iff it is Kähler.*

Proof. The necessity is given by the same argument as in the proof of the previous Lemma. Conversely, every Kähler manifold admits at least one Spin^c-structure carrying parallel spinors, namely the canonical Spin^c-structure whose spinor bundle $\Lambda^{0,*}$ obviously has the constant functions as parallel spinors. □

For $\widetilde{M}=K3$, since M has a parallel spinor, it must admit a Kähler structure, hence the only admissible quotients are the Enriques surfaces.

If $\widetilde{M}$ is Kähler-Einstein of positive scalar curvature, then $M=\widetilde{M}$ by the theorem of Kobayashi. It is then known that such a compact complex surface is $\mathbb{C}P^1\times\mathbb{C}P^1$, $\mathbb{C}P^2$ or $\mathbb{C}P^2$ with k points blown-up in general position, for $3\leq k\leq 8$ [8, 24]. Each of these has a unique Kähler-Einstein metric [2] which is also the unique Yamabe metric in its conformal class [22]. Hence, all these cases may occur.

Collecting all the results of the last two subsections gives the following

THEOREM 4.6. *Equality is attained in Theorem 1.2 if and only if :*

• *in dimension* 3,

– *either* (M, g) *is isometric to* (S^3, can) *or to any of its quotients by a finite fixed point-free subgroup of* U_2 ;

– *or* (M, g) *is conformally equivalent to a compact orientable Euclidean space-form ;*

– *or* (M, g) *is conformally equivalent to an orientable* S^2*-bundle over* S^1 *with a product-type metric ;*

– *or* (M, g) *is Sasaki and either its (normalized) scalar curvature satisfies* $\mathrm{Scal}^g \geq 6$, *or* g *is a Berger-type metric on the sphere or one of its quotients.*

• *in dimension* 4,

– *either* (M, g) *is isometric to* (S^4, can);

– *or* (M, g) *is conformally equivalent to a quotient of a complex torus, a* K3 *surface or a Enriques surface;*

– *or* (M, g) *is conformally equivalent to a Kähler-Einstein complex surface with* $c_1 > 0$.

4.3. A characterization of the complex projective plane. In view of the results of the previous section, we shall here study the equality cases of Corollary 1.3 in order to get the following:

Theorem 4.7. *Let* (M, g, L, A, σ) *a compact Riemannian* Spin^c *manifold with Euler characteristic* $\chi(M) = 3$ *and self-dual curvature on the auxiliary bundle. If its first eigenvalue* λ_1 *satisfies*

$$\mathrm{vol}(M, g)^{1/2}\ \lambda_1^2 \ = \ \frac{1}{3}\ \left(Y(M, [g])\ - 4\pi\sqrt{2}\ \sqrt{c_1(L)^2[M]}\right),$$

then $\lambda_1 = 0$ *and* (M, g) *is isometric to* $\mathbb{C}P^2$ *with its standard Fubini-Study metric.*

The proof of this last result goes through a thorough study of the equality cases of Lemma 3.4 in dimension 4.

If equality is achieved in Lemma 3.4, the equality case of the classical Hölder inequality shows that the conformal factor φ is identically constant (equal to 1, say) so that $\widetilde{g} = g$ and any conformal equivalence is an isometry.

In the case the first eigenvalue is non-zero, the round sphere is clearly a limiting case. In the case $\lambda_1 = 0$, all the proposed limiting metrics do also achieve the equality. Obata's theorem [22] shows they are the unique (up to rescaling) Yamabe metrics in their conformal classes since they are Einstein.

Hence we finally get the

Lemma 4.8. *If equality is satisfied in Lemma 3.4, in dimension* 4, *then* (M, g) *is either isometric to* (S^4, can) *or* (M, g) *is isometric to a quotient of a complex torus, to a* K3 *surface or a Enriques surface, or* (M, g) *is a Kähler-Einstein complex surface with* $c_1 > 0$.

A quick look at the possible Euler characteristics (computed for instance in [5]) implies immediately Theorem 4.7.

4.4. A remark on the choice of norms. The disappearance of the possible product structures in the equality cases in Theorem 1.2 comes from our choice of norm on 2-forms : the equality case in

$$|\lambda \cdot \psi| \leq |\lambda|\,|\psi| \tag{80}$$

is achieved only for 2-forms of maximal rank. If we consider Ch. Bär's choice of the *operator* norm on 2-forms, we may also study the equality case in dimension 4 (in dimension 3 our previous study is not altered). The Kähler case doesn't really get more intricate (it even stays exactly the same), and there is more wealth in the product case, since Equation (61) above on the scalar curvature only implies that N has the topology of a 2-sphere.

Recall first that the flat S^2-principal bundles over T^2 are classified by the morphisms $\rho : \pi_1(T^2) = \mathbb{Z}^2 \to PSO_3 = SO_3$ (we shall denote them by $T^2 \times_\rho S^2$). The flat connection induces a splitting of the tangent space of $T^2 \times_\rho S^2$ hence a canonical Riemannian metric. As above, different morphisms ρ may define non isometric manifolds since they are not isomorphic as flat bundles over T^2. We can now end the remark by proving the

LEMMA 4.9. *Let M^4 be a compact Kähler manifold whose universal covering is isometric to $(S^2, can) \times (\mathbb{R}^2, eucl)$. Then M is isometric to some $T^2 \times_\rho S^2$.*

Proof. Let Γ be the group of covering transformations, so that $M = (S^2 \times \mathbb{R}^2)/\Gamma$. Choose an arbitrary element $\gamma \in \Gamma$ different from the identity. Since γ_* commutes with the Ricci tensor, it preserves the distributions tangent to S^2 and $\mathbb{R}^2$ (where Ric $= Id$ resp. 0). Then γ preserves the integral manifolds of these distributions, in other words, γ maps spheres to spheres and planes to planes. This means that we can find transformations $a : S^2 \to S^2$ and $b : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\gamma(p, x) = (a(p), b(x)), \quad \forall (p, x) \in S^2 \times \mathbb{R}^2. \tag{81}$$

Since γ preserves the metric and complex structure, it follows that a take values in SO_3 and is a constant matrix A. The same reasoning shows that b is the semi-direct product of a matrix B in SO_2 and a vector V on $\mathbb{R}^2$. So $\gamma(p, x) = (Ap, Bx + V)$. Now, every $A \in SO_3$ has a fixed point, so there exist $p_0 \in S^2$ with $Ap_0 = p_0$. If the matrices B were different from 1, we could find $x_0 \in \mathbb{R}^2$ such that $Bx_0 + V = x_0$, so (p_0, x_0) would be a fixed point of γ, a contradiction. The same is true if V and $A - Id$ vanish at the same time. This shows that each element of Γ has the form $\gamma(p, x) = (Ap, x + V)$ with $V \neq 0$ if $A \neq Id$; we make the notation $\gamma = (A, V)$. Since the translation parts of Γ behave additively, any of its elements is of infinite order.

The next step is to show that $\Gamma \simeq \mathbb{Z}^2$. First of all, if it had only one generator, M would not be compact. Moreover, if we take $\gamma = (A, V)$ and $\kappa = (B, W)$ two elements of Γ, then

$$\gamma^{-1}\kappa^{-1}\gamma\kappa = (A^{-1}B^{-1}AB, -V - W + V + W) = (A^{-1}B^{-1}AB, 0) \tag{82}$$

and this shows that γ and κ commute. Hence $\Gamma \simeq \mathbb{Z}^k$ and if we had $k > 2$, we would have elements in Γ with vanishing second component, which is impossible; thus $\Gamma \simeq \mathbb{Z}^2$. We then define a homomorphism $\rho : \Gamma \to SO_3$ by $\rho(\gamma) = A$ for all $\gamma = (A, V) \in \Gamma$. Let Λ be the lattice of $\mathbb{R}^2$ defined by the second components of the elements of Γ. It is now obvious that $M \simeq T^2 \times_\rho S^2$, where $T^2 = \mathbb{R}^2/\Lambda$. $\square$

ACKNOWLEDGEMENTS. Both authors are grateful to Paul Gauduchon for the nice formalism leading to the proof of the basic inequalities exposed in this paper and to Olivier Biquard for useful discussions.

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