Spin^c Manifolds and Complex Contact Structures

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Abstract - *In this paper we extend our notion of projectable spinors ([9], Ch.1) to the framework of Spin^c manifolds and deduce the basic formulas relating spinors on the base and the total space of Riemannian submersions with totally geodesic one-dimensional fibres. Some geometric applications concerning positive Kähler-Einstein complex contact manifolds (e.g. their characterisation as twistor spaces over positive quaternionic Kähler manifolds) are also given.*

1 Introduction

Projectable spinors for Riemannian submersions of spin manifolds arose in a quite natural way ([9], Ch.1) and have led to important geometric applications, as the classification of Kähler manifolds admitting Kählerian Killing spinors ([8]) or results on the spectrum of the Dirac operator for certain classes of Riemannian manifolds ([10]).

In this paper we introduce projectable spinors for Riemannian submersions of Spin^c manifolds, motivated by the following facts: K.-D. Kirchberg and U. Semmelmann discovered that every complex contact manifold of complex dimension $4l+3$ admitting a Kähler–Einstein metric of positive scalar curvature carries a canonical spin structure with Kählerian Killing spinors [4]. Using this together with the results in [8], we were able to prove the following characterisation of twistor spaces over positive quaternionic Kähler manifolds in half of the possible dimensions:

Theorem A. (cf. [12]) *Let M be a compact spin Kähler manifold of positive scalar curvature and complex dimension $4l+3$. Then the following statements are equivalent:*

(i) *M is the twistor space of some quaternionic Kähler manifold;*

(ii) *M is Kähler-Einstein and admits a complex contact structure;*

(iii) *M admits Kählerian Killing spinors.*

By different methods, C. LeBrun independently obtained the following

Theorem B. (cf. [7]) *Let Z be a Fano contact manifold. Then Z is a twistor space iff it admits a Kähler–Einstein metric.*

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In complex dimensions $4l+3$, this is a direct corollary of Theorem A. The reasons for which our Theorem A fails to hold in complex dimensions $4l+1$ are of a topological nature: neither the twistor spaces, nor the complex contact manifolds of complex dimensions $4l+1$ are spin (with one exception: the complex projective space). On the other hand, each Kähler manifold admits a natural Spin^c structure; it is thus natural to try to extend the above notions to the framework of Spin^c structures, and to generalise the results in [12] to this case.

In order to keep the computations as simple as possible, we do not construct here the whole theory of projectable spinors on Spin^c manifolds, and restrict ourselves to a particular situation which is of special interest to us. Generalisations of the constructions described below can be easily obtained.

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2 Preliminaries

Definition 2.1 *A Spin^c structure on an oriented Riemannian manifold (M^n, g) is given by a $U(1)$ principal bundle $P_{U(1)}M$ and a Spin$_n^c$ principal bundle $P_{Spin_n^c}M$ together with a projection $\theta : P_{Spin_n^c}M \to P_{SO(n)}M \times P_{U(1)}M$ ($P_{SO(n)}M$ means the $SO(n)$ principal bundle of oriented orthonormal frames on M), satisfying*

$$\theta(\tilde{u}\tilde{a}) = \theta(\tilde{u})\xi(\tilde{a}),$$

for every $\tilde{u} \in P_{Spin_n^c}M$ and $\tilde{a} \in Spin_n^c$, where ξ is the canonical 2–fold covering of $Spin_n^c$ over $SO(n) \times U(1)$.

Recall that $\mathrm{Spin}_n^c = \mathrm{Spin}_n \times_{\mathbf{Z}_2} \mathrm{U}(1)$, and that ξ is given by $\xi([u,a]) = (\phi(u), a^2)$, where $\phi : \mathrm{Spin}_n \to \mathrm{SO}(n)$ is the canonical 2–fold covering.

If M has a Spin^c structure, we denote by ΣM the associated complex spinor bundle and by L the complex line bundle associated to $P_{U(1)}M$, which is called the auxiliary bundle. On ΣM there is a canonical Hermitian product $(.,.)$, with respect to which the Clifford multiplication by vectors is skew–Hermitian:

$$(X \cdot \psi, \varphi) = -(\psi, X \cdot \varphi), \quad \forall X \in TM,\ \psi, \varphi \in \Sigma M. \tag{1}$$

Every connection form A on $P_{U(1)}M$ defines, together with the Levi-Civita connection of M, a covariant derivative on ΣM denoted by ∇^A. Correspondingly, we define the Dirac operator as the composition $\gamma \circ \nabla^A$, where γ denotes the Clifford contraction. The Dirac operator can be expressed using a local orthonormal frame $\{e_1, \cdots, e_n\}$ as

$$D = \sum_{i=1}^{n} e_i \cdot \nabla^A_{e_i}.$$

Suppose now that (M^{2m}, g, J) is a Kähler manifold. We define the twisted Dirac operator $\tilde{D}$ by

$$\tilde{D} = \sum_{i=1}^{2m} J(e_i) \cdot \nabla_{e_i} = - \sum_{i=1}^{2m} e_i \cdot \nabla_{J(e_i)},$$

which satisfies

$$\tilde{D}^2 = D^2 \quad \text{and} \quad \tilde{D}D + D\tilde{D} = 0. \tag{2}$$

We also define the complex Dirac operators $D_\pm := \frac{1}{2}(D \mp i\tilde{D})$, and (2) becomes

$$D_-^2 = D_+^2 = 0 \quad \text{and} \quad D^2 = D_+ D_- + D_- D_+. \tag{3}$$

Consider a local orthonormal frame $\{X_\alpha, Y_\alpha\}$ such that $Y_\alpha = J(X_\alpha)$. Then $Z_\alpha = \frac{1}{2}(X_\alpha - iY_\alpha)$ and $Z_{\bar{\alpha}} = \frac{1}{2}(X_\alpha + iY_\alpha)$ are local frames of $T^{1,0}(M)$ and $T^{0,1}(M)$, and $D_\pm$ can be expressed as

$$D_+ = 2 \sum_{\alpha=1}^{m} Z_\alpha \cdot \nabla^A_{Z_{\bar{\alpha}}} \quad , \quad D_- = 2 \sum_{\alpha=1}^{m} Z_{\bar{\alpha}} \cdot \nabla^A_{Z_\alpha}. \tag{4}$$

A k-form ω acts on ΣM by

$$\omega \cdot \Psi := \sum_{i_1 < \cdots < i_k} \omega(e_{i_1}, \cdots, e_{i_k})\, e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot \Psi \ ,$$

where $\{e_i\}$ is a local orthonormal frame on M. With respect to this action, the Kähler form Ω (defined by $\Omega(X, Y) = g(X, JY)$) satisfies

$$\Omega = \frac{1}{2} \sum_{i=1}^{2m} J(e_i) \cdot e_i = -\frac{1}{2} \sum_{i=1}^{2m} e_i \cdot J(e_i). \tag{5}$$

For later use let us note that

$$\sum_{\alpha=1}^{m} Z_\alpha \cdot Z_{\bar{\alpha}} = -\frac{i}{2}\Omega - \frac{m}{2} \quad , \quad \sum_{\alpha=1}^{m} Z_{\bar{\alpha}} \cdot Z_\alpha = \frac{i}{2}\Omega - \frac{m}{2}, \tag{6}$$

where Z_α and $Z_{\bar{\alpha}}$ are local frames of $T^{1,0}(M)$ and $T^{0,1}(M)$ as before.

The action of Ω on ΣM yields an orthogonal decomposition

$$\Sigma M = \oplus_{r=0}^{m} \Sigma_r M,$$

where $\Sigma_r M$ is the eigenbundle associated to the eigenvalue $i\,\mu_r = i\,(m - 2\,r)$ of Ω. If we define $\Sigma_{-1} M = \Sigma_{m+1} M = \{0\}$, then

$$D_\pm \Gamma(\Sigma_r M) \subset \Gamma(\Sigma_{r\pm1} M). \tag{7}$$

The complex volume element

$$\omega_{\mathbf{C}} := i^m e_1 \cdot \ldots \cdot e_{2m}$$

acts on ΣM by Clifford multiplication and its square is the identity. We denote by $\Sigma^\pm M$ the eigenbundles corresponding to the eigenvalues ± 1, and it is easy to see that $\Sigma_r M \subset \Sigma^+ M$ ($\Sigma_r M \subset \Sigma^- M$) iff r is even (respectively odd). If, with respect to the decomposition $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$, a spinor ψ is written as $\psi = \psi_+ + \psi_-$, then we define its conjugate $\bar{\psi} := \psi_+ - \psi_-$.

3 Projectable spinors to Spin^c-Manifolds

As in the case of spin manifolds, projectable spinors may be defined for arbitrary Riemannian submersions of Spin^c manifolds with 1–dimensional totally geodesic fibres, but for the sake of simplicity we treat only a particular case here.

Let $P_{U(1)}M$ be the principal U(1) bundle associated to a Spin^c structure on a Riemannian manifold (M^n, g) of even dimension and suppose that on $P_{U(1)}M$ a connection form A is given. Denote by $\bar{M} := P_{U(1)}M$ and by π the canonical bundle projection. It is well-known that there exists an unique 2-form α on M whose pull-back is just i times the curvature form dA of the connection A (note that A and dA are imaginary-valued forms on $\bar{M}$). Let T be the (1,1) tensor on M associated to α (defined by $\alpha(X, Y) = g(X, TY)$).

The manifold $\bar{M}$ carries a canonical 1–parameter family of Riemannian metrics g_t which make the bundle projection $\pi : \bar{M} \to M$ into a Riemannian submersion with totally geodesic fibres. These metrics are given by

$$g_t(X, Y) = g(\pi_* X, \pi_* Y) - t^2 A(X)A(Y), \quad \forall x \in \bar{M}, \; X, Y \in T_x\bar{M},$$

and we denote by ∇^t the covariant derivative of the Levi-Civita connection of g_t and by V the unit vertical vector field on $(\bar{M}, \bar{g})$ satisfying $A(V) = i/t$. This choice of V fixes an orientation for $\bar{M}$.

Before proceeding, we mention here a simple result relating spin and Spin^c structures, that will be used in a moment.

Lemma 3.1 *A Spin^c structure with trivial auxiliary bundle is canonically identified with a spin structure. Moreover, if the connection A of the auxiliary bundle L is flat, then by this identification ∇^A corresponds to ∇ on the spinor bundles.*

PROOF. One first remarks that since the U(1) bundle associated to L is trivial, we can exhibit a global section of it, that we will call σ. Denote by $P_{Spin_n}M$ the inverse image by θ of $P_{SO(n)}M \times \sigma$. It is straightforward to check that this defines a spin structure on M, and that the connection on $P_{Spin_n^c}M$ restricts to the Levi-Civita connection on $P_{Spin_n}M$ if σ can be chosen to be parallel, i.e. if A defines a flat connection.

Q.E.D.

Proposition 3.1 *Every Spin^c structure on M induces a canonical spin structure on $\bar{M}$.*

PROOF. By enlargement of the structure groups, the two-fold covering

$$\theta : P_{Spin_n^c}M \to P_{SO(n)}M \times P_{U(1)}M,$$

gives a two-fold covering

$$\theta : P_{Spin^c_{n+1}}M \to P_{SO(n+1)}M \times P_{U(1)}M,$$

which, by pull-back through π, gives rise to a Spin^c structure on $\bar{M}$

$$\begin{array}{ccc} P_{Spin^c_{n+1}}\bar{M} & \stackrel{\pi}{\longrightarrow} & P_{Spin^c_{n+1}}M \\ \pi^*\theta \downarrow & & \theta \downarrow \\ P_{SO(n+1)}\bar{M} \times P_{U(1)}\bar{M} & \stackrel{\pi}{\longrightarrow} & P_{SO(n+1)}M \times P_{U(1)}M \\ \downarrow & & \downarrow \\ \bar{M} & \stackrel{\pi}{\longrightarrow} & M \end{array}$$

Using Lemma 3.1 we see that this construction actually yields a *spin* structure on $\bar{M}$. Indeed, the pull back of a G principal bundle ($P_{U(1)}M$, in our situation) with respect to its own projection map is always trivial:

$$\begin{array}{ccc} \pi^*P \simeq P \times G & \stackrel{\pi}{\longrightarrow} & P \\ \pi^*\pi \downarrow & & \pi \downarrow \\ P & \stackrel{\pi}{\longrightarrow} & M \end{array}$$

Nevertheless, we will continue to call this spin structure the induced Spin^c structure on $\bar{M}$.

Q.E.D.

The next step is to relate the covariant derivatives of spinors on M and $\bar{M}$. We point out an important detail here: since we are actually interested in $\bar{M}$ as *spin* manifold, the connection on $P_{U(1)}\bar{M}$ (which defines the covariant derivative of spinors on $\bar{M}$) that we consider, will not be the pull-back connection, but the *flat connection* on the canonically trivial bundle $P_{U(1)}\bar{M}$. The following result relates an arbitrary connection on a principal bundle $\pi : P \to M$ and the flat connection on $\pi^*P \to P$.

Lemma 3.2 *The connection form A_0 of the flat connection on π^*P can be related to an arbitrary connection A on P by*

$$A_0((\pi^*s)_*(U)) = -A(U), \tag{8}$$

$$A_0((\pi^*s)_*(X^*)) = A(s_*X), \tag{9}$$

where U is a vertical vector field on P, X^* is the horizontal lift (with respect to A) of a vector field X on M, and s is a local section of $P \to M$.

Proof. The identification $P \times \mathrm{U}(1) \simeq \pi^* P$ is given by $(u,a) \mapsto (u, ua)$, for all $(u,a) \in P \times \mathrm{U}(1)$. For some fixed $u \in P$, take a path u_t in the fiber over $x := \pi(u)$ such that $u_0 = u$ and $\dot{u}_0 = U$. We define $a_t \in \mathrm{U}(1)$ by $u_t = s(x)a_t$, so via the above identification we have

$$(\pi^* s)(u_t) = (u_t, s(x)) = (u_t, (a_t)^{-1}),$$

and thus

$$A_0((\pi^* s)_*(U)) = -a_0^{-1}\dot{a}_0 = -A(\dot{u}_0) = -A(U).$$

Similarly, for $x \in M$ and $X \in T_x M$, take a path x_t in M such that $x_0 = x$ and $\dot{x}_0 = X$. Let $u \in \pi^{-1}(x)$ and u_t the horizontal lift of x_t such that $u_0 = u$. We define $a_t \in \mathrm{U}(1)$ by $s(x_t) = u_t a_t$, which by derivation gives $s_*(X) = R_{a_0}\dot{u}_0 + u_0\dot{a}_0$. Then

$$(\pi^* s)(u_t) = (u_t, s(x_t)) = (u_t, a_t),$$

and thus, using the fact that $\dot{u}_0$ is horizontal,

$$A_0((\pi^* s)_*(X^*)) = a_0^{-1}\dot{a}_0 = A(s_*(X)).$$

Q.E.D.

Recall that the complex Clifford representation Σ_n can be made into a $Cl(n+1)$–representation by defining

$$\mu(e_j)\cdot\psi = \begin{cases} e_j \cdot \psi & \text{for } j \leq n \\ i\bar{\psi} & \text{for } j = n+1 \end{cases}$$

Corresponding to this, we obtain an identification of the pull back $\pi^*\Sigma M$ with $\Sigma\bar{M}$, and with respect to this identification, if X is a vector and ψ a spinor on M, then

$$X^* \cdot \pi^*\psi = \pi^*(X \cdot \psi), \tag{10}$$

$$V \cdot \pi^*\psi = \pi^*(i\bar{\psi}), \tag{11}$$

where V is the unit vertical vector field defined at the beginning of this section.

Definition 3.1 *The sections of $\Sigma\bar{M}$ which can be written as pull-back of sections of ΣM are called projectable spinors.*

We now compute the covariant derivative of projectable spinors on $\bar{M}$ in terms of the covariant derivative of spinors on M.

Proposition 3.2 *The covariant derivative ∇^t on $\Sigma\bar{M}$ induced by the Levi-Civita connection on $(\bar{M}, g_t)$ and the flat connection on $\pi^* P_{U(1)}M$ satisfies*

$$\nabla^t_{X^*}(\pi^*\psi) = \pi^*(\nabla^A_X\psi - i\frac{t}{4}T(X)\cdot\bar{\psi}) \qquad \forall X \in TM, \tag{12}$$

$$\nabla^t_V \pi^*\psi = -\pi^*(\frac{t}{4}\alpha\cdot\psi + \frac{i}{2t}\psi). \tag{13}$$

PROOF. Recall that the curvature form dA of the principal U(1) bundle $\bar{M} \to M$ satisfies

$$dA = -i\pi^*\alpha. \tag{14}$$

The metric g_t is given by

$$g_t(X,Y) = g(\pi_* X, \pi_* Y) - t^2 A(X)A(Y),\ \forall X, Y \in T\bar{M}. \tag{15}$$

If V denotes as before the unit vertical vector field, then $A(V) = i/t$, and we obtain

$$\begin{aligned} g_t(\nabla^t_{X^*}V, Y^*) &= -\frac{1}{2}g_t(V,[X^*,Y^*]) = \frac{t^2}{2}A(V)A([X^*,Y^*]) \\ &= \frac{it}{2}A([X^*,Y^*]) = -\frac{it}{2}dA(X^*,Y^*) \\ &= -\frac{t}{2}\alpha(X,Y) = \frac{t}{2}g_t(T(X)^*,Y^*), \end{aligned}$$

so finally

$$\nabla^t_{X^*}V = \frac{t}{2}T(X)^*. \tag{16}$$

Consider the pull-back $\pi^*\psi$ of a spinor field $\psi = [\sigma, \xi]$, where $\xi : U \subset M \to \Sigma_n$ is a vector valued function, and σ is a local section of $P_{Spin^c_n}M$ whose projection onto $P_{SO(n)}M$ is a local orthonormal frame $(X_1, ..., X_n)$ and whose projection onto $P_{U(1)}M$ is a local section s. Then $\pi^*\psi$ can be expressed as $\pi^*\psi = [\pi^*\sigma, \pi^*\xi]$, and it is easy to see that the projection of $\pi^*\sigma$ onto $P_{SO(n+1)}\bar{M}$ is the local orthonormal frame $(X_1^*, ..., X_n^*, V)$ and its projection onto $P_{U(1)}\bar{M}$ is just π^*s.

Using Lemma 3.2 and (16) we obtain

$$\begin{aligned} \nabla^t_{X^*}\pi^*\psi &= [\pi^*\sigma, X^*(\pi^*\xi)] + \frac{1}{2}\sum_{j<k} g_t(\nabla^t_{X^*}X_j^*, X_k^*)X_j^* \cdot X_k^* \cdot \pi^*\psi \\ &\quad + \frac{1}{2}\sum_j g_t(\nabla^t_{X^*}X_j^*, V)X_j^* \cdot V \cdot \pi^*\psi + \frac{1}{2}A_0((\pi^*s)_* X^*)\pi^*\psi \\ &= [\pi^*\sigma, \pi^*(X(\xi))] + \frac{1}{2}\sum_{j<k} g(\nabla_X X_j, X_k)\pi^*(X_j \cdot X_k \cdot \psi) \\ &\quad - \frac{1}{2}\frac{t}{2}i\sum_j g(T(X), X_j)\pi^*(X_j \cdot \bar{\psi}) + \frac{1}{2}A(s_* X)\pi^*\psi \\ &= \pi^*\Big([\sigma, (X(\xi))] + \frac{1}{2}\sum_{j<k} g(\nabla_X X_j, X_k)X_j \cdot X_k \cdot \psi \\ &\quad - i\frac{t}{4}T(X) \cdot \bar{\psi} + \frac{1}{2}A(s_* X)\psi\Big) \\ &= \pi^*(\nabla^A_X\psi - i\frac{t}{4}T(X) \cdot \bar{\psi}). \end{aligned}$$

and similarly,

$$\begin{aligned}
\nabla^t_V \pi^*\psi &= [\pi^*\sigma, V(\pi^*\xi)] + \frac{1}{2}\sum_{j<k} g_t(\nabla^t_V X^*_j, X^*_k) X^*_j \cdot X^*_k \cdot \pi^*\psi \\
&\quad + \frac{1}{2}\sum_j g_t(\nabla^t_V X^*_j, V) X^*_j \cdot V \cdot \pi^*\psi + \frac{1}{2} A_0((\pi^* s)_* V)\pi^*\psi \\
&= \frac{t}{2}\frac{1}{2}\sum_{j<k} g(T(X_j), X_k)\pi^*(X_j \cdot X_k \cdot \psi) - \frac{1}{2}A(V)\pi^*\psi \\
&= -\frac{t}{4}\pi^*(\alpha\cdot\psi) - \frac{i}{2t}\pi^*\psi \\
&= -\pi^*\Big(\frac{t}{4}\alpha\cdot\psi + \frac{i}{2t}\psi\Big).
\end{aligned}$$

Q.E.D.

We now particularise the above results to the case where M is a Kähler-Einstein manifold (M^n, g, J) of positive scalar curvature, and the auxiliary bundle L of the Spin^c structure on M is a root of the canonical bundle K, i.e. $L^{\otimes r} = K$ for some $r \in \mathbf{N}^*$. The canonical connection on K, whose curvature form is just $-i\rho$ (ρ is the Ricci form), induces then a connection A on L, whose curvature form ω satisfies $\omega = -i\rho/r$. As before, we denote by $\bar{M}$ the U(1) principal bundle associated to L. By rescaling the metric on M if necessary, we can suppose that the scalar curvature of M is equal to $n(n+2)$, and thus $\rho = (n+2)\Omega$. The 2-form α on M defined at the beginning of this section is given in this case by

$$\alpha = \frac{n+2}{r}\Omega, \tag{17}$$

so the above proposition becomes

Proposition 3.3 *If M is a Spin^c Kähler-Einstein manifold of positive scalar curvature and the auxiliary bundle L of the Spin^c structure on M is a r−root of the canonical bundle K, then the covariant derivative ∇^t on $\Sigma\bar{M}$ induced by the Levi-Civita connection on $(\bar{M}, g_t)$ and the flat connection on $\pi^* P_{U(1)}M$ satisfies*

$$\nabla^t_{X^*}(\pi^*\psi) = \pi^*(\nabla^A_X\psi - i\frac{t(n+2)}{4r}J(X)\cdot\bar{\psi}) \qquad \forall X \in TM, \tag{18}$$

$$\nabla^t_V \pi^*\psi = -\pi^*(\frac{t(n+2)}{4r}\Omega\cdot\psi + \frac{i}{2t}\psi). \tag{19}$$

The formula (16) allows us to compute the Ricci tensor Ric^t of the Riemannian manifold $(\bar{M}, g_t)$. If we denote by $a := \frac{t(n+2)}{2r}$, then

$$\mathrm{Ric}^t(V,V) = na^2 \quad , \quad \mathrm{Ric}^t(X^*, V) = 0, \tag{20}$$

$$\mathrm{Ric}^t(X^*, Y^*) = (n + 2 - 2a^2) g(X, Y). \tag{21}$$

Let us take $t_0 = \frac{2r}{n+2}$ and denote $\bar{g} := g_{t_0}$, $\bar{\nabla} := \nabla^{t_0}$. The vertical vector field V defines then an Einstein–Sasakian structure on the manifold $(\bar{M}, \bar{g})$ (cf. [2]). We can synthetise the above results in the following

Theorem 3.1 *Let (M^n, g, J) be a Kähler-Einstein manifold with scalar curvature $R = n(n+2)$, $L := K^{\frac{1}{r}}$ a root of the canonical bundle K and $\bar{M}$ the associated $U(1)$ principal bundle with connection form A, induced by the Levi-Civita connection on K. Then the following hold:*

(i) *There is a canonical metric $\bar{g}$ on $\bar{M}$ making the bundle projection $\pi : \bar{M} \to M$ into a Riemannian submersion with totally geodesic fibres, and satisfying $\bar{\nabla}_{X^*} V = J(X)^*$.*

(ii) *With respect to the metric $\bar{g}$, V defines a regular Einstein–Sasakian structure on $\bar{M}$. The length of the fibres of the corresponding S^1–action is constant and equal to $\frac{4\pi r}{n+2}$.*

(iii) *The Spin^c structure on M defined by (L, A) induces a canonical spin structure on $\bar{M}$ and every spinor field on M induces a projectable spinor field $\pi^*\psi$ on $\bar{M}$, satisfying*

$$\bar{\nabla}_{X^*}(\pi^*\psi) = \pi^*(\nabla^A_X \psi - \frac{i}{2} J(X) \cdot \bar{\psi}) \qquad \forall X \in TM, \tag{22}$$

$$\bar{\nabla}_V \pi^*\psi = -\frac{1}{2}\pi^*(\Omega \cdot \psi + \frac{i(n+2)}{2r}\psi). \tag{23}$$

4 Complex contact structures

Definition 4.1 (cf. [5]) *Let M^{2m} be a complex manifold of complex dimension $m = 2k+1$. A complex contact structure is a family $\mathcal{C} = \{(U_i, \omega_i)\}$ satisfying the following conditions:*

(i) *$\{U_i\}$ is an open covering of M.*

(ii) *ω_i is a holomorphic 1-form on U_i.*

(iii) *$\omega_i \wedge (\partial\omega_i)^k \in \Gamma(\Lambda^{m,0} M)$ is different from zero at every point of U_i.*

(iv) *$\omega_i = f_{ij}\omega_j$ in $U_i \cap U_j$, where f_{ij} is a holomorphic function on $U_i \cap U_j$.*

Let $\mathcal{C} = \{(U_i, \omega_i)\}$ be a complex contact structure. Then there exists an associated holomorphic line sub–bundle $L_{\mathcal{C}} \subset \Lambda^{1,0}(M)$ with transition functions $\{f_{ij}^{-1}\}$ and local sections ω_i. It is easy to see that

$$\mathcal{D} := \{Z \in T^{1,0}M \mid \omega(Z) = 0, \ \forall \omega \in L_{\mathcal{C}}\}$$

is a codimension 1 maximally non–integrable holomorphic sub–bundle of $T^{1,0}M$, and conversely, every such bundle defines a complex contact structure. From condition (iii) immediately follows the isomorphism $L_{\mathcal{C}}^{k+1} \cong K$, where $K = \Lambda^{m,0}(M)$ denotes the canonical bundle of M.

From now on, M will denote a Kähler–Einstein manifold of odd complex dimension $m = 4l+1$ with positive scalar curvature, admitting a complex contact structure $\mathcal{C}$. The manifold M is compact, by Myers' Theorem. By rescaling the metric on M if necessary, we can suppose that the scalar curvature of M is equal to $2m(2m+2)$, and thus the Ricci form ρ and the Kähler form Ω are related by $\rho = (2m+2)\Omega$. The main objective of this section is to construct the analogues of Kählerian Killing spinors ([3], [4], [8]) for a certain Spin^c structure on M, determined by $\mathcal{C}$. This is done just as in [4].

The collection $(U_i, \omega_i \wedge (\partial\omega_i)^l)$ defines a holomorphic line bundle $L_l \subset \Lambda^{2l+1,0}M$, and from the definition of $\mathcal{C}$ we easily obtain

$$L_l \cong L_{\mathcal{C}}^{l+1}. \tag{24}$$

We now fix some $(U,\omega) \in \mathcal{C}$ and define a local section $\psi_{\mathcal{C}}$ of $\Lambda^{0,2l+1}M \otimes L_{\mathcal{C}}^{l+1}$ by

$$\psi_{\mathcal{C}}|_U := |\xi_\tau|^{-2}\bar{\tau} \otimes \xi_\tau, \tag{25}$$

where $\tau := \omega \wedge (\partial\omega)^l$ and ξ_τ is the element corresponding to τ through the isomorphism (24). The fact that $\psi_{\mathcal{C}}$ does not depend of the element $(U,\omega) \in \mathcal{C}$ shows that it actually defines a global section $\psi_{\mathcal{C}}$ of $\Lambda^{0,2l+1}M \otimes L_{\mathcal{C}}^{l+1}$.

We now recall ([6], Appendix D) that $\Lambda^{0,*}M$ is just the spinor bundle associated to the canonical Spin^c structure on M, whose auxiliary line bundle is K^{-1}, so that $\Lambda^{0,*}M \otimes L_{\mathcal{C}}^{l+1}$ is actually the spinor bundle associated to the Spin^c structure on M with auxiliary bundle $L = K^{-1} \otimes L_{\mathcal{C}}^{2(l+1)} \cong L_{\mathcal{C}}^{-(2l+1)} \otimes L_{\mathcal{C}}^{2(l+1)} \cong L_{\mathcal{C}}$. The section $\psi_{\mathcal{C}}$ is thus a spinor lying in $\Lambda^{0,2l+1}M \otimes L_{\mathcal{C}}^{l+1} \cong \Sigma_{2l+1}M$, so

$$\Omega \cdot \psi_{\mathcal{C}} = -i\psi_{\mathcal{C}}. \tag{26}$$

Proposition 4.1 *The spinor field $\psi_{\mathcal{C}}$ satisfies $\nabla_Z \psi_{\mathcal{C}} = 0$, $\forall Z \in T^{1,0}M$ (in particular $D_- \psi_{\mathcal{C}} = 0$), and*

$$D^2\psi_{\mathcal{C}} = D_- D_+ \psi_{\mathcal{C}} = \frac{l+1}{2l+1}\left(\frac{1}{2}R\,\psi_{\mathcal{C}} - i\rho \cdot \psi_{\mathcal{C}}\right), \tag{27}$$

where R is the scalar curvature of M.

Proof. This is actually a variant of Proposition 5 from [4], the only difference being that ξ_τ (Ψ_ω in their notations) is not any more a section of $K^{1/2}$, but of $K^{(l+1)/(2l+1)}$, so the coefficients $1/2$ in formulas (8) and (9) of [4] have to be replaced by $\frac{l+1}{2l+1}$.

Q.E.D.

Using (26), (27) and the fact that $\rho = \frac{1}{8l+2} R\,\Omega = (8l+4)\Omega$, we obtain

Corollary 4.1 *The spinor field ψ_C is an eigenspinor of D^2 with respect to the eigenvalue $16l(l+1)$.*

Let us introduce some notations

$$\psi_- := \psi_C \in \Gamma(\Sigma_{2l+1}M) \quad , \quad \psi_+ := \frac{1}{4l+4} D\psi_C \in \Gamma(\Sigma_{2l+2}M). \tag{28}$$

By integration over M we immediately obtain from the above Corollary

$$|\psi_-|^2_{L^2} = \frac{l+1}{l} |\psi_+|^2_{L^2}. \tag{29}$$

Proposition 4.2 *The following relations hold*

$$\nabla_Z \psi_- = 0, \ \forall Z \in T^{1,0}M, \tag{30}$$

$$\nabla_{\bar{Z}} \psi_- + \bar{Z} \cdot \psi_+ = 0, \ \forall \bar{Z} \in T^{0,1}M, \tag{31}$$

$$\nabla_{\bar{Z}} \psi_+ = 0, \ \forall \bar{Z} \in T^{0,1}M, \tag{32}$$

$$\nabla_Z \psi_+ + Z \cdot \psi_- = 0, \ \forall Z \in T^{1,0}M. \tag{33}$$

Proof. The first relation is part of Proposition 4.1. In order to prove (31), let us consider the local frames of $T^{1,0}(M)$ and $T^{0,1}(M)$ introduced in Section 2: $Z_\alpha = \frac{1}{2}(X_\alpha - iY_\alpha)$ and $Z_{\bar{\alpha}} = \frac{1}{2}(X_\alpha + iY_\alpha)$, where $Y_\alpha = J(X_\alpha)$, and $\{X_\alpha, Y_\alpha\}$ is a local orthonormal frame of TM. From (30) we find $\nabla_{Z_{\bar{\alpha}}} \psi_- = \nabla_{X_\alpha} \psi_- = i\nabla_{Y_\alpha} \psi_-$, so using (6) and (28) gives

$$\begin{aligned}
0 &\le \sum_{\alpha=1}^{m} |\nabla_{Z_{\bar{\alpha}}} \psi_- + Z_{\bar{\alpha}} \cdot \psi_+|^2 \\
&= \sum_{\alpha=1}^{m} |\nabla_{X_\alpha} \psi_-|^2 - 2\Re e \sum_{\alpha=1}^{m} (\psi_+, Z_\alpha \cdot \nabla_{Z_{\bar{\alpha}}} \psi_-) - \sum_{\alpha=1}^{m} (\psi_+, Z_\alpha \cdot Z_{\bar{\alpha}} \cdot \psi_+) \\
&= \frac{1}{2} |\nabla \psi_-|^2 - \Re e(\psi_+, D_+ \psi_-) - \frac{1}{2} (\psi_+, (-i\Omega - m)\psi_+) \\
&= \frac{1}{2} |\nabla \psi_-|^2 - (4l+4)|\psi_+|^2 + \frac{1}{2}(4l+4)|\psi_+|^2.
\end{aligned}$$

The last expression is by construction a positive function, say $|F|^2$, on M. Integrating over M and using the generalised Lichnerowicz formula ([6], Appendix

D), Corollary 4.1 and (29), we obtain

$$
\begin{aligned}
|F|^2_{L^2} &= \frac{1}{2}(\nabla^*\nabla\psi_-,\psi_-)_{L^2} - (4l+4)|\psi_+|^2_{L^2} + \frac{1}{2}(4l+4)|\psi_+|^2_{L^2}\\
&= \frac{1}{2}(D^2\psi_- - \frac{1}{4}R\psi_- + \frac{i}{2}\frac{1}{2l+1}\rho\cdot\psi_-,\psi_-)_{L^2} - (2l+2)|\psi_+|^2_{L^2}\\
&= |\psi_-|^2_{L^2}\Big(8l(l+1) - \frac{(8l+2)(8l+4)}{8} + \frac{i}{4}\frac{-i(8l+4)}{2l+1} - 2l\Big) = 0,
\end{aligned}
$$

thus proving that $F = 0$ and consequently (31). In order to check the last two equations one has to make use of the operator $\tilde{D}$. From $D_-\psi_- = 0$ we find

$$0 = \frac{1}{4l+4}D_+^2\psi_- = D_+\psi_+, \tag{34}$$

so

$$\tilde{D}\psi_+ = -iD\psi_+. \tag{35}$$

We take a local orthonormal frame e_i and write (using (1), (5), (28) and (35))

$$
\begin{aligned}
0 &\leq \sum_{j=1}^n |\nabla_{e_j}\psi_+ + \frac{1}{2}(e_j - iJ(e_j))\cdot\psi_-|^2\\
&= |\nabla\psi_+|^2 - \Re e((D+i\tilde{D})\psi_+,\psi_-)\\
&\quad -\frac{1}{4}\sum_{j=1}^n((e_j+iJ(e_j))\cdot(e_j-iJ(e_j))\cdot\psi_-,\psi_-)\\
&= |\nabla\psi_+|^2 - 2\Re e(D\psi_+,\psi_-) + ((m-i\Omega)\cdot\psi_-,\psi_-)\\
&= |\nabla\psi_+|^2 - 8l|\psi_-|^2 + 4l|\psi_-|^2 := |G|^2
\end{aligned}
$$

Just as before, we compute the integral over M of the positive function $|G|^2$, namely

$$
\begin{aligned}
|G|^2_{L^2} &= |\nabla\psi_+|^2_{L^2} - 4l|\psi_-|^2_{L^2}\\
&= (\nabla^*\nabla\psi_+,\psi_+)_{L^2} - 4l|\psi_-|^2_{L^2}\\
&= (D^2\psi_+ - \frac{1}{4}R\psi_+ + \frac{i}{2}\frac{1}{2l+1}\rho\cdot\psi_+,\psi_+)_{L^2} - 4l|\psi_-|^2_{L^2}\\
&= |\psi_+|^2_{L^2}\Big(16l(l+1) - \frac{(8l+2)(8l+4)}{4} + \frac{i}{2}\frac{-3i(8l+4)}{2l+1} - 4(l+1)\Big) = 0,
\end{aligned}
$$

thus proving $G = 0$. Consequently $\nabla_X\psi_+ + \frac{1}{2}(X - iJ(X))\cdot\psi_- = 0,\ \forall X \in TM$, which is equivalent to (32) and (33).

Q.E.D.

The above proposition motivates the following

Definition 4.2 *A section ψ of the spinor bundle of a given Spin^c structure on a Kähler manifold (M^{8l+2}, g, J) satisfying*

$$\nabla^A_X \psi = \frac{1}{2} X \cdot \psi + \frac{i}{2} JX \cdot \bar{\psi}, \quad \forall X \in TM \tag{36}$$

is called a Kählerian Killing spinor.

Defining $\psi := \psi_+ - \psi_-$ we immediately obtain the

Corollary 4.2 *Let $\mathcal{C}$ be a complex contact structure on a Kähler–Einstein manifold (M^{8l+2}, g, J). Then the Spin^c structure on M with auxiliary bundle $L_{\mathcal{C}}$ carries a Kählerian Killing spinor $\psi \in \Gamma(\Sigma_{2l+1} M \oplus \Sigma_{2l+2} M)$.*

5 Geometric consequences

We can now state the main application of the above results:

Theorem 5.1 *Let M be a compact Kähler manifold of positive scalar curvature and complex dimension $4l + 1$. Then the following statements are equivalent:*

(i) *M is the twistor space of some quaternionic Kähler manifold;*

(ii) *M is Kähler-Einstein and admits a complex contact structure;*

(iii) *There exist a Spin^c structure on M with auxiliary bundle L and spinor bundle ΣM such that $L^{\otimes(2l+1)} \cong \Lambda^{4l+1,0} M$ and ΣM carries a Kählerian Killing spinor $\psi \in \Gamma(\Sigma_{2l+1} M \oplus \Sigma_{2l+2} M)$.*

Proof. The implications (i)$\Longrightarrow$(ii) and (ii)$\Longrightarrow$(iii) follow directly from [13] and Corollary 4.2 respectively.

Suppose now that (iii) holds. The proof of (iii)$\Longrightarrow$(i) parallels that of [8]. We first show that M is Kähler–Einstein. Let $\psi \in \Gamma(\Sigma_{2l+1} M \oplus \Sigma_{2l+2} M)$ be a spinor field on M which satisfies (36). Taking the covariant derivative with respect to an arbitrary vector field Y we obtain

$$\nabla^A_Y \nabla^A_X \psi = \frac{1}{4}(X \cdot Y + JX \cdot JY) \cdot \psi + \frac{i}{4}(X \cdot JY - JX \cdot Y) \cdot \bar{\psi} + \nabla^A_{\nabla_Y X} \psi, \tag{37}$$

which easily implies

$$\mathcal{R}^A_{Y,X} \psi = \frac{1}{2}(X \cdot Y + JX \cdot JY + 2g(X,Y)) \cdot \psi - ig(X, JY)\bar{\psi}. \tag{38}$$

A local computation shows that the curvature operator $\mathcal{R}^A$ on the spinor bundle is given by the formula

$$\mathcal{R}^A = \mathcal{R} + \frac{i}{2}\omega, \tag{39}$$

where $i\omega := -\frac{i}{2l+1}\rho$ is the curvature form of the auxiliary bundle L, and

$$\mathcal{R}_{X,Y} = \frac{1}{2}\sum_{j<k} R(X,Y,e_j,e_k)\, e_j \cdot e_k \cdot \tag{40}$$

in a local orthonormal frame $\{e_1, ..., e_n\}$. Using the first Bianchi identity for the curvature tensor one obtains ([2], p.16)

$$\sum_i e_i \cdot \mathcal{R}_{e_i,X} = \frac{1}{2}\mathrm{Ric}(X), \tag{41}$$

so, by (39) and (41),

$$\sum_j e_j \cdot \mathcal{R}^A_{e_j,X}\psi = \sum_j e_j \cdot (\mathcal{R}_{e_j,X}\psi + \frac{i}{2}\omega(e_j,X)\psi) = \frac{1}{2}\mathrm{Ric}(X)\cdot\psi - \frac{i}{2}X \lrcorner\, \omega \cdot \psi. \tag{42}$$

On the other hand, a straightforward computation using (38) and the fact that $\psi \in \Gamma(\Sigma_{2l+1}M \oplus \Sigma_{2l+2}M)$ yields

$$\begin{aligned}\sum_j e_j \cdot \mathcal{R}^A_{e_j,X}\psi &= (4l+2)X\cdot\psi + iJX\cdot\bar{\psi} + JX\cdot\Omega\cdot\psi \\ &= (4l+2)X\cdot\psi - 2iJX\cdot\psi,\end{aligned}$$

which, together with (42), gives

$$\Big(\frac{1}{2}\mathrm{Ric}(X) - (4l+2)X\Big)\cdot\psi = \frac{i}{2l+1}J\Big(\frac{1}{2}\mathrm{Ric}(X) - (4l+2)X\Big)\cdot\psi. \tag{43}$$

As ψ never vanishes, if the equality $A\cdot\psi = iB\cdot\psi$ holds for some real vectors $A,\ B$, then $|A| = |B|$. The above formula thus shows that $\mathrm{Ric}(X) = (8l+4)X,\ \forall X \in TM$, so M is Kähler–Einstein with scalar curvature $R = (8l+2)(8l+4)$.

From Theorem 3.1 we deduce that the principal U(1) bundle $\bar{M}$ associated to L admits a canonical metric $\bar{g}$ and a canonical spin structure such that the spinor $\pi^*\psi$ induced by ψ satisfies

$$\bar{\nabla}_{X^*}(\pi^*\psi) = \pi^*(\nabla^A_X\psi - \frac{i}{2}J(X)\cdot\bar{\psi}) = \pi^*(\frac{1}{2}X\cdot\psi) \qquad \forall X \in TM, \tag{44}$$

$$\bar{\nabla}_V\pi^*\psi = -\frac{1}{2}\pi^*(\Omega\cdot\psi + \frac{i(8l+4)}{2(2l+1)}\psi) = \pi^*(\frac{i}{2}\bar{\psi}), \tag{45}$$

and (10), (11) show that $\pi^*\psi$ is a Killing spinor on $\bar{M}$.

The spinor field $\pi^*\psi$ induces then a parallel spinor Ψ on the cone $C\bar{M}$ over $\bar{M}$, which is a Kähler manifold (cf. [1], [8], [11]). Moreover, using (45) we can compute the action of the Kähler form of $C\bar{M}$ on Ψ, and obtain that $\Psi \in \Sigma_{2l+3}C\bar{M}$. From C. Bär's classification [1] we know that the restricted holonomy group of $C\bar{M}$ is one of the following: $\mathrm{SU}(4l+2)$, $\mathrm{Sp}(2l+1)$ or 0. The fixed points of the spin representation of $\mathrm{SU}(4l+2)$ ly in Σ_0 and Σ_{4l+2}, so as Ψ is a parallel spinor in $\Sigma_{2l+3}C\bar{M}$, the restricted holonomy group of $C\bar{M}$ cannot be equal to $\mathrm{SU}(4l+2)$. This implies that the universal covering of $C\bar{M}$ is hyperkähler, and thus that the universal covering of $\bar{M}$ is 3–Sasakian (see [1]).

Let us denote by $\bar{M}'$ the $\mathrm{U}(1)$ bundle associated to some maximal root of L. Using the Gysin exact sequence we deduce that $\bar{M}'$ is simply connected (see [2], p.85). Moreover, there exists a canonical covering projection $\bar{M}' \to \bar{M}$, thus proving that $\bar{M}'$ is the universal covering of $\bar{M}$. Consequently, $(\bar{M}', \bar{g}')$ is a 3–Sasakian manifold, where $\bar{g}'$ is the metric induced from $\bar{g}$ via the covering projection. On the other hand, the unit vertical vector field V' on $\bar{M}'$ defines a Sasakian structure, since this is true for its projection V on $\bar{M}$. It is well known that any Sasakian structure on a 3–Sasakian manifold P^{4k-1} of non-constant sectional curvature belongs to the 2–sphere of Sasakian structures. Indeed, the cone CP over P has restricted holonomy $\mathrm{Sp}(k)$, and since the centraliser of $\mathrm{Sp}(k)$ in $\mathrm{U}(2k)$ is just $\mathrm{Sp}(1)$, every Kähler structure on CP must belong to the 2–sphere of Kähler structures of CP, which is equivalent to our statement.

Now, $\bar{M}'$ is regular in the direction of V', so an old result of Tanno implies that it is actually a regular 3–Sasakian manifold (cf. [14]). It is then well known that the quotient of $\bar{M}'$ by the corresponding $\mathrm{SO}(3)$ action is a quaternionic Kähler manifold of positive scalar curvature, say N, and that the twistor space over N is biholomorphic to the quotient of $\bar{M}'$ by each of the S^1 actions given by the Sasakian vector fields, so in particular to M, which is the quotient of $\bar{M}'$ by the S^1 action generated by V'.

Q.E.D.

From Theorem A and Theorem 5.1 we immediately obtain the result of LeBrun mentioned in Section 1:

Corollary 5.1 *Let Z be a Fano contact manifold. Then Z is a twistor space iff it admits a Kähler–Einstein metric.*

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