On Kirchberg's inequality for compact Kähler manifolds of even complex dimension

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Abstract - In 1986 K.-D. Kirchberg showed that each eigenvalue of the Dirac operator on a compact Kähler manifold (M^{2m}, g) of even complex dimension satisfies the inequality $\lambda^2 \geq \frac{m}{4(m-1)} \inf_M S$, where by S we denote the scalar curvature. It is conjectured that the manifolds for the limiting case of this inequality are products $T^2 \times N$, where T^2 is a flat torus and N is the twistor space of a quaternionic Kähler manifold of positive scalar curvature. In 1990 A. Lichnerowicz announced an affirmative answer for this conjecture (cf. [11]), but his proof seems to work only when assuming that the Ricci tensor is parallel. The aim of this note is to prove several results about manifolds satisfying the limiting case of Kirchberg's inequality and to prove the above conjecture in some particular cases.

1 Introduction

The first inequality on the eigenvalues of the Dirac operator on a compact spin manifold (M, g) was obtained in 1980 by T. Friedrich ([2]), who showed that

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M S, \tag{1}$$

where S is the scalar curvature of M. Of course, this inequality is interesting only for manifolds with positive scalar curvature. In order to obtain it, Friedrich considered a modified connection and used the Lichnerowicz formula ([10]). The eigenspinors corresponding to eigenvalues in the limiting case of the inequality (1) are just the Killing spinors of M. The manifolds carrying Killing spinors were classified by C. Bär in [1].

In 1984, O. Hijazi ([5], [6]) showed that a Kähler manifold doesn't admit Killing spinors, i.e. the inequality (1) is always strict for Kähler manifolds. In the Kählerian case Friedrich's inequality was improved by K.-D. Kirchberg (cf. [8]) who showed that each eigenvalue λ of the Dirac operator on a compact spin Kähler manifold (M^{2m}, g) satisfies

$$\lambda^2 \geq \frac{m+1}{4m} \inf_M S, \quad \text{if } m \text{ is odd}, \tag{2}$$

and

$$\lambda^2 \geq \frac{m}{4(m-1)} \inf_M S, \quad \text{if } m \text{ is even}. \tag{3}$$

Kähler manifolds satisfying the limiting case in these inequalities are called limiting manifolds. In the odd case, limiting manifolds admit Kählerian Killing spinors ([7]) and they were classified by the author (cf. [12]) in 1994.

On the other hand, limiting manifolds of even complex dimension admit spinors satisfying a more complicated equation (cf. [9], [4]) and moreover the product $T^2 \times N$ of a flat torus and an odd-dimensional limiting manifold is an even-dimensional limiting manifold. It is conjectured that each even-dimensional limiting manifold can be constructed in this way.

In 1990 A. Lichnerowicz announced a solution of this conjecture (cf. [11]) but when taking a derivative at the fourth line from the bottom of page 720, he chooses an orthonormal frame adapted with respect to the eigenvalues of the Ricci tensor and parallel in a point, so it seems that his proof only works when assuming that the Ricci tensor is parallel. The aim of this paper is to deduce several results about even-dimensional limiting manifolds, the most important being the following

Theorem. *The Ricci tensor of a limiting manifold of even complex dimension has two eigenvalues, $S/(n-2)$ and 0, with multiplicities $n-2$ and 2 respectively.*

The author is very indebted to P. Gauduchon for his competent advises and his important contribution to the preparation of this work.

2 Previous results

We use the following terminology, taken from [4]. Let (M^{2m}, g, J) be an even-dimensional compact spin Kähler manifold ($m = 2\,l$) and ΣM the spinor bundle of M. On ΣM there is an action of the exterior forms on M, given by

$$\omega \cdot \Psi = \sum_{i_1 < \cdots < i_k} \omega(e_{i_1}, \cdots, e_{i_k}) e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot \Psi \, .$$

With respect to this action, the Kähler form of M, Ω, defines a decomposition

$$\Sigma M = \oplus_{q=0}^{m} \Sigma^q M,$$

where $\Sigma^q M$ is the eigenbundle of rank C_m^q associated to the eigenvalue $i\,\mu^q = i\,(m - 2\,q)$ of Ω. Via this decomposition, every spinor Ψ can be uniquely written in the form

$$\Psi = \sum_{q=0}^{m} \Psi^q.$$

For $q \in \{0, ..., m\}$ denote by c^q the restriction of the Clifford contraction c to $T^*M \otimes \Sigma^q M$. We then have

$$c^q = c_+^q + c_-^q, \tag{4}$$

where c_+^q and c_-^q, who take their values in $\Sigma^{q+1}M$, and $\Sigma^{q-1}M$ respectively, are given by

$$c_+^q(\alpha\otimes\psi) = \frac{1}{2}(\alpha - iJ\alpha)\cdot\psi, \tag{5}$$

$$c_-^q(\alpha\otimes\psi) = \frac{1}{2}(\alpha + iJ\alpha)\cdot\psi, \tag{6}$$

$\forall \alpha \in T_x^*M,\ \forall \psi \in \Sigma_x^q M,\ \forall x \in M.$

One can introduce a natural decomposition of the Dirac operator restricted to sections of $\Sigma^q M$ into

$$D = D_+ + D_-, \tag{7}$$

where D_+ and D_- are defined by

$$D_+ = c_+^q \circ \nabla \quad , \quad D_- = c_-^q \circ \nabla. \tag{8}$$

Gauduchon introduces then the Kählerian Penrose operator given by

$$Q_X^q\Psi = \nabla_X\Psi + \frac{1}{4(q+1)}(X + iJX)\cdot D_+\Psi + \frac{1}{4(m-q+1)}(X - iJX)\cdot D_-\Psi.$$

Using this operator and a simple algebraic lemma, he easily obtains Kirchberg's inequalities and the necessary conditions for the equality case to occur. Namely, we have the following

Theorem 2.1 (cf. [4], [7], [8], [11]) *Let (M,g) be a compact spin Kähler manifold of (real) dimension $n = 2\,m = 4\,l$. Then every eigenvalue λ of the Dirac operator of M satisfies the inequality (3). The equality holds iff there exists a nonzero spinor $\Psi \in \Sigma^{l+1}M$ satisfying*

$$\nabla_X\Psi = -\frac{1}{n}(X - iJX)\cdot D\Psi \tag{9}$$

and

$$D^2\Psi = D_+D\Psi = \lambda^2\Psi \tag{10}$$

One then easily obtains

$$\Omega\cdot\Psi = 2i\Psi \quad , \quad \Omega\cdot D\Psi = 0 \tag{11}$$

From now on, we will always suppose $n > 4$. The case $n = 4$ was completely solved by T. Friedrich in [3].

Straightforward computations, that we will not reproduce here yield the following

Lemma 2.1 *For $n > 4$ and M as above, the following formulas hold*

$$\nabla_X D\Psi = -\frac{1}{4}(\mathrm{Ric}(X) + iJ\mathrm{Ric}(X)) \cdot \Psi \tag{12}$$

$$K(X - iJX) \cdot \Psi = (\mathrm{Ric}(X) - iJ\mathrm{Ric}(X)) \cdot \Psi \tag{13}$$

$$K(X - iJX) \cdot D\Psi = (\mathrm{Ric}(X) - iJ\mathrm{Ric}(X)) \cdot D\Psi \tag{14}$$

$$\rho \cdot \Psi = iK\Psi \quad , \quad \rho \cdot D\Psi = -iKD\Psi \tag{15}$$

$$\nabla_X \rho \cdot \Psi = 0 \quad , \quad \nabla_X \rho \cdot D\Psi = -i(\mathrm{Ric}^2(X) - K\mathrm{Ric}(X)) \cdot \Psi, \tag{16}$$

where ρ is the Ricci form, Ric *is the Ricci tensor and $K = \frac{S}{n-2}$.*

As standard references for these formulas see [4], [9]. One should note however a slight difference between the notations of [9] and ours, due to the fact that our Ψ is just Kirchberg's $\mathbf{j}\psi^{l-1}$, $\mathbf{j}$ being the canonical $\mathbf{C}$-anti-linear quaternionic (resp. real) structure of the spinor bundle. This is why all the complex vectors appearing in [9] have to be conjugated in our notations.

3 The eigenvalues of the Ricci tensor of a limiting manifold

In this section we obtain our main result, which states that the Ricci tensor of a limiting manifold of even complex dimension has only two eigenvalues, 0 with multiplicity 2 and $S/(n-2)$ with multiplicity $n-2$. This is, in our opinion, a first step towards the classification of these manifolds.

Let us define the 2-forms

$$\rho_s = \frac{1}{2}\sum_{i=1}^{n} e_i \wedge J\mathrm{Ric}^s(e_i) = \frac{1}{2}\sum_{i=1}^{n} e_i \cdot J\mathrm{Ric}^s(e_i), \tag{17}$$

(where $\{e_i\}$ is a local orthonormal frame) and consider the following six propositions:

(a_s) : $tr(\mathrm{Ric}^s) = (n-2)K^s$ (where tr denotes the trace);
(b_s) : $\rho_s \cdot \Psi = iK^s\Psi$;
(c_s) : $\rho_s \cdot D\Psi = -iK^s D\Psi$;
(d_s) : $\nabla_X \rho_s \cdot \Psi = 0$;
(e_s) : $\nabla_X \rho_s \cdot D\Psi = -i(\mathrm{Ric}^{s+1}(X) - K^s\mathrm{Ric}(X)) \cdot \Psi$;
(f_s) : $\delta\rho_s = 0$.

By the above formulas, all these propositions are true for $s = 1$. We will now prove that they are actually true for all $s \in \mathbf{N}$.

Lemma 3.1 *The following implications hold :*

1. $(a_s) \Longrightarrow (b_s)\,,\,(c_s)$;
2. (b_s) *et* $(c_s) \Longrightarrow (d_s)$ *et* (e_s);
3. (a_s) *et* $(f_{s-1}) \Longrightarrow (f_s)$;
4. $(d_s)\,,\,(e_s)\,,\,(f_s) \Longrightarrow (a_{s+1})$.

Proof. Let $x \in M$ and consider an orthonormal basis $\{X_i, X_a\}$, adapted with respect to the Ricci tensor in the sense that $\mathrm{Ric}(X_a) = \mu_a X_a$, $a \in \{1, \ldots, r\}$, with $\mu_a \neq K$ and $\mathrm{Ric}(X_i) = KX_i$, $i \in \{r+1, \ldots, n\}$. From (13) and (14) we then obtain

$$(X_a - iJX_a) \cdot \Psi = 0, \tag{18}$$

$$(X_a - iJX_a) \cdot D\Psi = 0, \tag{19}$$

for all $a \in \{1, \ldots, r\}$.

1. Suppose (a_s) is true. We then have

$$\sum_{a=1}^{r} (\mu_a^s - K^s) = tr(\mathrm{Ric}^s) - nK^s = -2K^s, \tag{20}$$

so, using (18),

$$\begin{aligned}(\rho_s - K^s\Omega) \cdot \Psi &= \frac{1}{2}\sum_{a=1}^{r}(\mu_a^s - K^s)X_a \cdot J(X_a) \cdot \Psi \\ &= \frac{1}{2}\sum_{a=1}^{r}(\mu_a^s - K^s)X_a \cdot (-i)X_a \cdot \Psi \\ &= \frac{i}{2}\sum_{a=1}^{r}(\mu_a^s - K^s)\Psi = -iK^s\Psi.\end{aligned}$$

Similarly, (19) and (20) give

$$(\rho_s - K^s\Omega) \cdot D\Psi = -iK^s D\Psi.$$

We then use (11) and obtain (b_s) and (c_s).

2. The relation

$$\rho_s \cdot X = X \cdot \rho_s + 2J\mathrm{Ric}^s(X), \tag{21}$$

gives

$$\nabla_X \rho_s \cdot \Psi = \nabla_X(\rho_s \cdot \Psi) - \rho_s \cdot \nabla_X \Psi = iK^s \nabla_X \Psi + \frac{1}{n}\rho_s \cdot (X - iJX) \cdot D\Psi$$

$$\begin{aligned}
&= \frac{-i}{n}K^s(X-iJX)\cdot D\Psi + \frac{1}{n}(X-iJX)\cdot\rho_s\cdot D\Psi \\
&\quad + \frac{2i}{n}(\mathrm{Ric}^s(X)-iJ\mathrm{Ric}^s(X))\cdot D\Psi \\
&= \frac{-2i}{n}K^s(X-iJX)\cdot D\Psi + \frac{2i}{n}(\mathrm{Ric}^s(X)-iJ\mathrm{Ric}^s(X))\cdot D\Psi \\
&= 0.
\end{aligned}$$

and similarly,

$$\begin{aligned}
\nabla_X\rho_s\cdot D\Psi &= \nabla_X(\rho_s\cdot D\Psi)-\rho_s\cdot\nabla_X D\Psi \\
&= -iK^s\nabla_X D\Psi + \frac{1}{4}\rho_s\cdot(\mathrm{Ric}(X)+iJ\mathrm{Ric}(X))\cdot\Psi \\
&= \frac{i}{4}K^s(\mathrm{Ric}(X)+iJ\mathrm{Ric}(X))\cdot\Psi \\
&\quad + \frac{1}{4}(\mathrm{Ric}(X)+iJ\mathrm{Ric}(X))\cdot\rho_s\cdot\Psi - \frac{i}{2}(\mathrm{Ric}^{s+1}(X) \\
&\quad + iJ\mathrm{Ric}^{s+1}(X))\cdot\Psi \\
&= -\frac{i}{2}(\mathrm{Ric}^{s+1}(X)-K^s\mathrm{Ric}(X) \\
&\quad + iJ\mathrm{Ric}^{s+1}(X)-iJK^s\mathrm{Ric}(X))\cdot\Psi \\
&= -i(\mathrm{Ric}^{s+1}(X)-K^s\mathrm{Ric}(X))\cdot\Psi,
\end{aligned}$$

where the last equality is justified by a repeated use of (13).

3. Let us first explicit in terms of the Ricci tensor the equality $\delta\rho_s = 0$. If X and $\{e_i\}$ denote a vector field and a local orthonormal frame parallel in x respectively, we have

$$\delta\rho_s(X) = -\nabla_{e_i}\rho_s(e_i,X) = -\nabla_{e_i}\langle J\mathrm{Ric}^s(e_i),X\rangle = \langle\nabla_{e_i}\mathrm{Ric}^s(e_i),JX\rangle, \tag{22}$$

so

$$\delta\rho_s = 0 \iff \nabla_{e_i}\mathrm{Ric}^s(e_i) = 0. \tag{23}$$

From (a_s) we obtain

$$(\nabla_X\mathrm{Ric})(e_i,\mathrm{Ric}^{s-1}e_i) = tr(\nabla_X\mathrm{Ric}\circ\ \mathrm{Ric}^{s-1}) = \frac{1}{s}\nabla_X(tr(\mathrm{Ric}^s)) = 0, \tag{24}$$

which gives

$$\begin{aligned}
0 &= d\rho(Je_i,\mathrm{Ric}^{s-1}e_i,X) = (\nabla_{Je_i}\rho)(\mathrm{Ric}^{s-1}e_i,X) \\
&\quad + (\nabla_{\mathrm{Ric}^{s-1}e_i}\rho)(X,Je_i) + (\nabla_X\rho)(Je_i,\mathrm{Ric}^{s-1}e_i) \\
&= (\nabla_{Je_i}\mathrm{Ric})(J\mathrm{Ric}^{s-1}e_i,X) + (\nabla_{\mathrm{Ric}^{s-1}e_i}\mathrm{Ric})(JX,Je_i) \\
&\quad - (\nabla_X\mathrm{Ric})(e_i,\mathrm{Ric}^{s-1}e_i) \\
&= 2(\nabla_{e_i}\mathrm{Ric})(\mathrm{Ric}^{s-1}e_i,X) \\
&= 2\langle\nabla_{e_i}\mathrm{Ric}^s(e_i),X\rangle - 2\mathrm{Ric}(\nabla_{e_i}\mathrm{Ric}^{s-1}(e_i),X),
\end{aligned}$$

thus proving that (a_s) and (f_{s-1}) imply (f_s).

4. In (e_s) we put $X = e_i$, make the Clifford product with e_i and sum over i, to obtain

$$(d\rho_s + \delta\rho_s) \cdot D\Psi = i(tr(\mathrm{Ric}^{s+1} - K^s tr(\mathrm{Ric})))\Psi. \tag{25}$$

Taking the scalar product with Ψ in this formula and using (d_s) et (e_s) gives

$$(*) \qquad i(tr(\mathrm{Ric}^{s+1} - (n-2)K^{s+1}))\langle\Psi, \Psi\rangle = \langle(d\rho_s) \cdot D\Psi, \Psi\rangle$$

$$= \langle D\Psi, d\rho_s \cdot \Psi\rangle = \langle D\Psi, D\rho_s \cdot \Psi\rangle = 0,$$

and as the support of Ψ is dense in M, we obtain (a_{s+1}).

QED

The reader can easily convince himself that the direct recurrence $(a_s) \Longrightarrow (a_{s+1})$ doesn't work: one has to use the intermediary formulas $(b_s) - (f_s)$.

The formulas (a_s) show that the sum of the s^{th} powers of the eigenvalues of Ric equals $(n-2)K^s$ for all s, so by Newton's relations we deduce our main result:

Theorem 3.1 *The Ricci tensor of a limiting manifold of even complex dimension has two eigenvalues,* K *and* 0*, the first one with multiplicity* $n-2$ *and the second one with multiplicity* 2*.*

The above result does not imply that the Ricci tensor of a limiting manifold has to be parallel; we have nevertheless the strong conviction that this must hold, and consequently that Lichnerowicz' conjecture is true. We give here a result in favor of this conjecture:

Proposition 3.1 *Suppose that one of the following holds:*

1° *The length of* Ψ *is constant on* M*;*

2° *The two distributions* TM^0 *and* TM^K *on* M *corresponding to the two eigenvalues of the Ricci tensor are holomorphic;*

3° *The restriction of the sectional curvature to* TM^0 *vanishes.*

Then the Ricci tensor of M *is parallel.*

For a proof (using Theorem 3.1), see [13], section 9.4.

References

[1] C. BÄR, *Real Killing spinors and holonomy,* Commun. Math. Phys. **154** (1993), 509-521.

[2] Th. FRIEDRICH, *Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalar-krümmung,* Math. Nach. **97** (1980), 117-146.

[3] Th. FRIEDRICH, *The Classification of* 4*-dimensional Kähler Manifolds with Small Eigenvalue of the Dirac Operator,* Math. Ann. **295** no.3 (1993), 565-574.

[4] P. GAUDUCHON, *L'opérateur de Penrose kählérien et les inégalités de Kirchberg,* unpublished.

[5] O. HIJAZI, *Opérateurs de Dirac sur les variétés riemanniennes : Minoration des valeurs propres,* Thèse de 3ème Cycle, Ecole Polytechnique (1984).

[6] O. HIJAZI, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors,* Commun. Math. Phys. **104** (1986), 151-162.

[7] O. HIJAZI, *Eigenvalues of the Dirac operator on compact Kähler manifolds,* Commun. Math. Phys. **160** (1994), 563-579.

[8] K.-D. KIRCHBERG, *An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature,* Ann. Global Anal. Geom. **3** (1986), 291-325.

[9] K.-D. KIRCHBERG, *The first Eigenvalue of the Dirac Operator on Kähler Manifolds,* J. Geom. Phys. **7** (1990), 449-468.

[10] A. LICHNEROWICZ, *Spineurs harmoniques,* C. R. Acad. Sci. Paris, t. **257**, Série A-B (1963), 7-9.

[11] A. LICHNEROWICZ, *La première valeur propre de l'opérateur de Dirac pour une variété kählérienne et son cas limite,* C. R. Acad. Sci. Paris, t. **311**, Série **I** (1990), 717-722.

[12] A. MOROIANU, *La première valeur propre de l'opérateur de Dirac sur les variétés kählériennes compactes,* Commun. Math. Phys. **169** (1995), 373-384.

[13] A. MOROIANU, *Opérateur de Dirac et submersions riemanniennes,* Thèse de Doctorat, Ecole Polytechnique (1996).