ON WEYL-REDUCIBLE LOCALLY CONFORMALLY KÄHLER STRUCTURES

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A Hermitian manifold (M, g, J) of complex dimension $n \geq 2$ is called locally conformally Kähler (lcK) if around every point in M the metric g can be conformally rescaled to a Kähler metric. This condition is equivalent to the existence of a closed 1-form θ, called the Lee form, such that

$$\mathrm{d}\Omega = \theta \wedge \Omega,$$

where $\Omega := g(J\cdot, \cdot)$ denotes the fundamental 2-form. Let now $\widetilde{M}$ be the universal covering of an lcK manifold (M, J, g, θ), endowed with the pull-back lcK structure $(\tilde{J}, \tilde{g}, \tilde{\theta})$. Since $\widetilde{M}$ is simply connected, $\tilde{\theta}$ is exact, *i.e.* $\tilde{\theta} = \mathrm{d}\varphi$, and by the above considerations, the metric $g^K := e^{-\varphi}\tilde{g}$ is Kähler. The group $\pi_1(M)$ acts on $(\widetilde{M}, \tilde{J}, g^K)$ by holomorphic homotheties. Furthermore, we assume that the lcK structure is strict, in the sense that $\pi_1(M)$ is not a subgroup of the isometry group of $(\widetilde{M}, g^K)$. In particular, the Levi-Civita connection of the Kähler metric g^K projects to a closed, non-exact, Weyl structure on M, called the standard Weyl structure. The lcK manifold M is called Weyl-reducible if its standard Weyl structure is reducible and non-flat.

We now assume that (M, g, J) is a compact Weyl-reducible lcK manifold. By a result of M. Kourganoff [4, Theorem 1.5], $(\widetilde{M}, g^K)$ is isometric to $\mathbb{R}^q \times N$, where $\mathbb{R}^q$ is endowed with the Euclidean metric and N is a non-complete Riemannian manifold with irreducible holonomy.

In [4, Example 1.6] (see also [5]) examples of closed reducible Weyl structures on compact manifolds are constructed using a linear map $A \in \mathrm{SL}_{q+1}(\mathbb{Z})$, such that:

- there exists an A-invariant decomposition $\mathbb{R}^{q+1} = E^s \oplus E^u$ with $\dim(E^u) = 1$ and $A|_{E^u} = \lambda^q \mathrm{Id}_{E^u}$ for some real number $\lambda > 1$;
- there exists a positive definite symmetric bilinear form b on E^s, such that $\lambda A|_{E^s}$ is orthogonal with respect to b.

Then A induces a diffeomorphism (also denoted by A) of the torus $\mathbb{T}^{q+1}$, and the mapping torus $M_A := \mathbb{T}^{q+1} \times_\Phi (0, \infty)$, where $\Phi(x, t) := (Ax, \frac{1}{\lambda}t)$, carries a reducible non-flat Weyl structure induced by the following metric on $\mathbb{T}^{q+1} \times (0, \infty)$:

$$g_\varphi := \mathrm{d}x_1^2 + \cdots \mathrm{d}x_q^2 + \varphi(t)\mathrm{d}x_{q+1}^2 + \mathrm{d}t^2,$$

where $x_1, \ldots, x_{q+1}$ are the local coordinates with respect to an orthonormal basis $(e_1, \ldots, e_{q+1})$ with $e_{q+1} \in E^u$, and $\varphi\colon (0, +\infty) \to (0, +\infty)$ is any smooth function satisfying $\varphi(\lambda t) = \lambda^{2q+2}\varphi(t)$.

If we furthermore assume that N is 2-dimensional, then, according to [4, Theorem 1.7], M is isometric to one of the Riemannian manifolds (M_A, g_φ).

Since g_φ reducible, and Kähler for q even, we obtain in particular that M_A is lcK and Weyl-reducible for every even integer q. However, we will show in this note that matrices in $\mathrm{SL}_{q+1}(\mathbb{Z})$ satisfying the above conditions only exist for $q = 1$ or $q = 2$. More precisely, we have the following:

Proposition 1. *Let $q \in \mathbb{N}^*$ and $A \in \mathrm{SL}_{q+1}(\mathbb{Z})$, such that there is a direct sum decomposition $\mathbb{R}^{q+1} = E^s \oplus E^u$ invariant by A, with $\dim(E^u) = 1$. If there exists a positive definite symmetric bilinear form b on E^s and a real number $\lambda > 1$, such that $\lambda A|_{E^s}$ is orthogonal with respect to b, then $q \in \{1, 2\}$.*

Proof. Let C be a symmetric positive definite matrix, such that $b = \langle C^2\cdot, \cdot\rangle$, where $\langle\cdot,\cdot\rangle$ is the standard Euclidean scalar product. Then the following equivalence holds:

$$\lambda A|_{E^s} \in \mathrm{O}(E^s, b) \Longleftrightarrow C \cdot (\lambda A|_{E^s}) \cdot C^{-1} \in \mathrm{O}(q).$$

In particular, each eigenvalue of $\mathrm{Spec}(\lambda \mathrm{A}|_{\mathrm{E^s}})$ has module 1 and the characteristic polynomial of A denoted by μ_A is given by:

$$\mu_A(X) = (X - \lambda^q) \prod_{i=1}^{q} \left(X - \frac{z_i}{\lambda}\right),$$

with $|z_i| = 1$, for all i, and $\prod_{i=1}^q z_i = 1$. Note that μ_A is irreducible in $\mathbb{Z}[X]$, since if it were a product of two non-constant polynomials with integer coefficients, one of them would have all roots of modulus less than 1, which is impossible. We distinguish the following two cases:

If q is even, then by Lemma 3.5 in [1], it follows that $q = 2$.

If q is odd, then μ_A has at least one further real root, so either $\frac{1}{\lambda}$ or $-\frac{1}{\lambda}$ is a root of μ_A. Up to reordering the subscripts one thus has $z_1 = \pm 1$. Assume that $z_1 = 1$ (the argument for $z_1 = -1$ is the same). The monic polynomial $p \in \mathbb{Z}[X]$ defined by $p(X) := X^{q+1}\mu_A(\frac{1}{X})$ satisfies $p(0) = 1$, and its roots are $\{\lambda^{-q}, \lambda, \frac{\lambda}{z_2}, \ldots, \frac{\lambda}{z_q}\}$.

Furthermore, there exists a monic polynomial $\tilde{p} \in \mathbb{Z}[X]$ with $\tilde{p}(0) = 1$, whose roots are $\{\lambda^{-q^2}, \lambda^q, (\frac{\lambda}{z_2})^q, \ldots, (\frac{\lambda}{z_q})^q\}$ (the existence of $\tilde{p}$ follows for instance from the Newton identities).

Since the monic polynomials μ_A and $\tilde{p} \in \mathbb{Z}[X]$ (of same degree) have λ^q as common root, and μ_A is irreducible, they must coincide. In particular λ^{-q^2} is a root of μ_A. On the other hand every root of μ_A has complex modulus equal to either λ^q or $\frac{1}{\lambda}$. Since $\lambda > 1$, we obtain $q = 1$. □

Due to the fact that the real dimension of an lcK manifold is even, applying the above result to the special case of a compact strict lcK manifold, whose standard Weyl structure is reducible, we obtain the following:

Proposition 2. *Let M be a compact Weyl-reducible strict lcK manifold. If the non-flat factor N in the splitting of the universal covering $(\widetilde{M}, g^K)$ as a Riemannian product $\mathbb{R}^q \times N$ is 2-dimensional, then $q = 2$ and thus M is an Inoue surface S^0, cf.* [3].

Let us remark that the restriction on the dimension q of the Euclidean factor obtained in Proposition 2 is compatible with the known results about the more general class of compact lcK manifolds constructed by Oeljeklaus and Toma, [6], as compact quotients of $\mathbb{C}^s \times \mathbb{H}^t$, where $\mathbb{H}$ denotes the upper complex half-plane. Namely, such a manifold admits a Weyl-reducible strict lcK structure if $s = 1$ and $t > 1$, but admits no such structure if $t = 1$ and $s > 1$.

Remark 3. As pointed out by V. Vuletescu, for odd q, Proposition 1 also follows from a more general result of Ferguson [2], whose proof, however, is rather involved.

References

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