CLOSED 1-FORMS AND TWISTED COHOMOLOGY

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Dedicated to Paul Gauduchon on the occasion of his 75th birthday

Abstract. We show that the first twisted cohomology group associated to closed 1-forms on differentiable manifolds is related to certain 2-dimensional representations of the fundamental group. In particular, we construct examples of nowhere-vanishing 1-forms with non-trivial twisted cohomology.

1. Introduction

If θ is a closed 1-form on a smooth manifold M, the *twisted differential* $\mathrm{d}_\theta := \mathrm{d} - \theta\wedge$ maps $\Omega^k(M)$ to $\Omega^{k+1}(M)$ and satisfies $\mathrm{d}_\theta \circ \mathrm{d}_\theta = 0$, thus defining the *twisted cohomology groups*

$$H^k_\theta(M) := \frac{\ker(\mathrm{d}_\theta|_{\Omega^k(M)})}{\mathrm{d}_\theta(\Omega^{k-1}(M))}.$$

These groups only depend on the de Rham cohomology class of θ, since the corresponding twisted differential complexes associated to cohomologous 1-forms are canonically isomorphic. In particular, the twisted cohomology associated to an exact 1-form is just the de Rham cohomology.

It is well known that the twisted cohomology defined by the Lee form of Vaisman manifolds, and more generally by any non-zero 1-form θ which is parallel with respect to some Riemannian metric on a compact manifold, vanishes [2].

The twisted cohomology groups, as well as their Dolbeault and Bott-Chern counterparts, play an important role in locally conformally Kähler geometry (*cf.* [1] or [5], where the twisted cohomology is called Morse-Novikov cohomology).

Twisted cohomology was also used by A. Pajitnov [6], who shows that if θ is a closed 1-form with non-degenerate zeros, then for large t the dimension of $H^k_{t\theta}(M)$ gives a lower bound for the number of the zeros of θ of index k. This is an analog of Witten's approach to Morse theory, in the more general situation of closed 1-forms.

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On the other hand, in [7], A. Pajitnov defined a different *twisted Novikov homology* theory associated to closed 1-forms θ with integral cohomology class $[\theta] \in H^1(M, \mathbb{Z})$, and shows that the twisted Novikov homology vanishes whenever $[\theta]$ admits a nowhere-vanishing representative ([7], Theorem 1.3). We will see in Example 4.2 below that the corresponding result fails for the standard twisted cohomology theory considered here.

Our main result (Theorem 2.3) relates the non-zero elements in the first twisted cohomology group associated to a closed 1-form θ with some set of non-decomposable 2-dimensional representations of the first fundamental group of M which contain a trivial subrepresentation, and whose determinant is the character of $\pi_1(M)$ canonically associated to θ.

In Section 3 we derive several applications of this result, like the vanishing of the first twisted cohomology group on manifolds with nilpotent fundamental group (Corollary 3.1), the fact that if the commutator group $[\pi_1(M), \pi_1(M)]$ is finitely generated, then the set $\{[\theta] \in H^1_{\mathrm{dR}}(M) \,|\, H^1_\theta(M) \neq 0\}$ is finite (Corollary 3.2), or the non-vanishing of twisted cohomology on Riemann surfaces of genus $g \geq 2$ (Corollary 3.3). In the last section we give several examples of explicit computations of the first twisted cohomology group on mapping tori or Vaisman manifolds.

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2. The Main Result

Notation: the cohomology class of a d_θ-closed 1-form α is denoted by $[\alpha]_\theta$.

Let us recall the following well-known result and present a proof for it, whose method will be useful in the sequel.

Lemma 2.1. *Let M be a manifold. There is a bijection between*

$$H^1_{\mathrm{dR}}(M) \xleftrightarrow{1:1} \{\rho\colon \pi_1(M) \to (\mathbb{R}^*_+, \times) \,|\, \rho \text{ is a representation}\}.$$

Proof. Let θ be a representative of a cohomology class $[\theta] \in H^1_{\mathrm{dR}}(M)$ and denote the universal cover of M by $\pi : \widetilde{M} \to M$. Then the pull-back $\widetilde{\theta} := \pi^*\theta$ of θ is an exact form, *i.e.* there exists $\varphi \in \mathcal{C}^\infty(\widetilde{M})$ such that $\widetilde{\theta} = \mathrm{d}\varphi$. Any element $\gamma \in \pi_1(M)$ acts trivially on $\widetilde{\theta}$, so $\gamma^*\mathrm{d}\varphi = \mathrm{d}\varphi$, which implies the existence of a constant $c_\gamma \in \mathbb{R}$ with $\gamma^*\varphi = \varphi + c_\gamma$. Since $\gamma_1^*\gamma_2^* = (\gamma_2\gamma_1)^*$, we see that $\gamma \mapsto c_\gamma$ is a group morphism from $\pi_1(M)$ to $(\mathbb{R}, +)$. We then associate to $[\theta] \in H^1_{\mathrm{dR}}(M)$ the representation $\rho\colon \pi_1(M) \to (\mathbb{R}^*_+, \times)$ defined by $\rho(\gamma) := e^{c_\gamma}$. The representation ρ does not depend on the choice of the representative θ in its cohomology class. Indeed, if we replace θ by $\theta + \mathrm{d}h$, then φ is replaced by $\varphi + \pi^*h$, and since π^*h is invariant by $\pi_1(M)$, the constants c_γ do not change.

Conversely, for any representation $\rho\colon \pi_1(M) \to (\mathbb{R}_+^*, \times)$ we will construct a positive function g on $\widetilde{M}$ which is ρ-equivariant, i.e. $a^*g = \rho(a)g$ for every $a \in \pi_1(M)$. To do this, let us pick a non-negative function f on $\widetilde{M}$ satisfying the properties (i) and (ii) of Lemma 2.2 below. We introduce the function

$$g := \sum_{\gamma \in \pi_1(M)} \rho(\gamma^{-1})\gamma^* f$$

which is well-defined and smooth on $\widetilde{M}$ since the sum is finite in the neighbourhood of any point of $\widetilde{M}$ by property (ii). Moreover, g is a positive function on $\widetilde{M}$ since $f > 0$ on V and $\pi_1(M) \cdot V = \widetilde{M}$ by property (i). For any $a \in \pi_1(M)$, we have:

$$a^*g = \sum_{\gamma \in \pi_1(M)} \rho(\gamma^{-1})(\gamma a)^* f = \sum_{\delta \in \pi_1(M)} \rho(a\delta^{-1})\delta^* f = \rho(a)g.$$

This shows that $\widetilde{\theta} := \mathrm{d}(\ln g)$ is an exact 1-form on $\widetilde{M}$, which is $\pi_1(M)$-invariant, hence $\widetilde{\theta}$ descends to a closed 1-form θ on M. We associate to ρ the cohomology class of θ in $H^1_{\mathrm{dR}}(M)$. This does not depend on the choice of f. Indeed, if g_1 is any other positive function on $\widetilde{M}$ satisfying $a^*g = \rho(a)g$ for every $a \in \pi_1(M)$, then g_1/g is $\pi_1(M)$-invariant, so it is the pull-back to $\widetilde{M}$ of some function h on M. Then the closed 1-form θ_1 on M satisfying $\pi^*\theta_1 = \mathrm{d}(\ln g_1)$ is $\theta_1 = \theta + \mathrm{d}h$, so $[\theta_1] = [\theta]$.

One can easily check that the above defined maps are inverse to each other.

□

Lemma 2.2. *There exists a non-negative function $f \in \mathcal{C}^\infty(\widetilde{M}, \mathbb{R}_+)$ satisfying the following properties:*

(i) f is positive on some open set $V \subset \widetilde{M}$ with $\pi_1(M) \cdot V = \widetilde{M}$;
(ii) any point $x \in \widetilde{M}$ has an open neighborhood V_x, such that the set

$$\{\gamma \in \pi_1(M) \,|\, \gamma \cdot V_x \cap \mathrm{supp}(f) \neq \emptyset\}$$

is finite.

Proof. Denote by $\pi : \widetilde{M} \to M$ the covering map and let $(U_i)_{i\in I}$ be an open cover of M with contractible open sets. Since U_i are simply connected, there exist open sets V_i of $\widetilde{M}$ such that $\pi|_{V_i} : V_i \to U_i$ is a diffeomeorphism for each $i \in I$.

Let $(\rho_i)_{i\in I}$ be a partition of unity subordinate to the open cover $(U_i)_{i\in I}$. By definition, we have $\rho_i \geq 0$, $\mathrm{supp}(\rho_i) \subset U_i$, and every point $y \in M$ has an open neighbourhood U_y such that the set

$$I_y := \{i \in I \,|\, U_y \cap \mathrm{supp}(\rho_i) \neq \emptyset\} \tag{1}$$

is finite. We define $f_i : \widetilde{M} \to \mathbb{R}_+$ by $f_i|_{V_i} = \rho_i \circ \pi$ and $f_i|_{\widetilde{M}\setminus V_i} = 0$. Clearly f_i are smooth since $\mathrm{supp}(\rho_i) \subset U_i$. For every point $x \in \widetilde{M}$, there is only a finite number of $i \in I$ for which the open set $\pi^{-1}(U_{\pi(x)})$ meets the support of f_i, so the function $f := \sum_{i\in I} f_i$ is well-defined, smooth and non-negative on $\widetilde{M}$. We claim that it also satisfies the properties (i) and (ii).

Let $V := f^{-1}(\mathbb{R}^*_+)$ be the open set where f is positive. For every $x \in \widetilde{M}$, there exists $i \in I$ such that $\rho_i(\pi(x)) > 0$. If γ denotes the unique element in $\pi_1(M)$ for which $\gamma(x) \in V_i$, then $f_i(\gamma(x)) > 0$, so $f(\gamma(x)) > 0$, showing that $x \in \gamma^{-1}(V)$. Thus $\pi_1(M) \cdot V = \widetilde{M}$, so (i) is verified.

Let now $x \in \widetilde{M}$ be any point. We define

$$V_x := \bigcup_{i\in I_{\pi(x)}} V_i,$$

where $I_{\pi(x)}$ is the finite subset of I given by (1). Then $\{\gamma \in \pi_1(M) \,|\, \gamma \cdot V_x \cap \mathrm{supp}(f_i) \neq \emptyset\}$ is empty for every $i \in I \setminus I_{\pi(x)}$ and has exactly one element for every $i \in I_{\pi(x)}$. This shows that the set of $\gamma \in \pi_1(M)$ for which $\gamma \cdot V_x$ meets the support of f is finite, having the same cardinal as $I_{\pi(x)}$.

□

Theorem 2.3. *Let M be a manifold and let θ be some non-exact closed 1-form on M. Let $\rho\colon \pi_1(M) \to (\mathbb{R}^*_+, \times)$ denote the representation associated to $[\theta] \in H^1_{\mathrm{dR}}(M)$, as in Lemma 2.1. Then the following assertions hold:*

1. *If $H^1_\theta(M) \neq 0$, then there exists an indecomposable representation $\xi\colon \pi_1(M) \to \mathrm{GL}_2(\mathbb{R})$ with $\det\xi = \rho$, which fixes the vector $\begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{R}^2$.*
2. *Conversely, if there exists an indecomposable representation $\xi\colon \pi_1(M) \to \mathrm{GL}_2(\mathbb{R})$ with $\det\xi = \rho$ and which fixes the vector $\begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{R}^2$, then $H^1_\theta(M) \neq 0$.*

Proof. (1) Let α be a d_θ-closed 1-form on M whose twisted cohomology class $[\alpha]_\theta \in H^1_\theta(M)$ is non-zero: $[\alpha]_\theta \neq 0$. If $\pi\colon \widetilde{M} \to M$ denotes as before the universal cover map and φ is a primitive of $\pi^*\theta$ on $\widetilde{M}$, then

$$\pi^*\mathrm{d}_\theta = e^{\varphi}\mathrm{d}e^{-\varphi}\pi^*, \tag{2}$$

so that $\mathrm{d}_\theta\alpha = 0$ is equivalent to $\mathrm{d}(e^{-\varphi}\pi^*\alpha) = 0$ on $\widetilde{M}$. Hence, there exists a function $h \in \mathcal{C}^\infty(\widetilde{M})$, such that $e^{-\varphi}\pi^*\alpha = \mathrm{d}h$, and thus $\gamma^*(\mathrm{d}h) = e^{-c_\gamma}\mathrm{d}h = \rho(\gamma^{-1})\mathrm{d}h$. Therefore, there exists for each $\gamma \in \pi_1(M)$ a constant $\lambda(\gamma) \in \mathbb{R}$, such that

$$\gamma^*h = \rho(\gamma^{-1})h + \lambda(\gamma),$$

which equivalently reads

$$(\gamma^{-1})^* h = \rho(\gamma) h + \lambda(\gamma^{-1}), \qquad \gamma \in \pi_1(M). \tag{3}$$

We claim that the map $\xi\colon \pi_1(M) \to \mathrm{GL}_2(\mathbb{R})$ defined by

$$\xi(\gamma) := \begin{pmatrix} 1 & \lambda(\gamma^{-1}) \\ 0 & \rho(\gamma) \end{pmatrix} \tag{4}$$

is a group morphism. Indeed, if $\gamma_1, \gamma_2 \in \pi_1(M)$, we have by (3):

$$((\gamma_1\gamma_2)^{-1})^* h = (\gamma_1^{-1})^*(\gamma_2^{-1})^* h = (\gamma_1^{-1})^*(\rho(\gamma_2)h + \lambda(\gamma_2^{-1})) = \rho(\gamma_2)(\rho(\gamma_1)h + \lambda(\gamma_1^{-1})) + \lambda(\gamma_2^{-1}),$$

thus showing that $\rho(\gamma_1\gamma_2) = \rho(\gamma_1)\rho(\gamma_2)$ and $\lambda((\gamma_1\gamma_2)^{-1}) = \rho(\gamma_2)\lambda(\gamma_1^{-1}) + \lambda(\gamma_2^{-1})$. Consequently,

$$\xi(\gamma_1)\xi(\gamma_2) = \begin{pmatrix} 1 & \lambda(\gamma_1^{-1}) \\ 0 & \rho(\gamma_1) \end{pmatrix} \begin{pmatrix} 1 & \lambda(\gamma_2^{-1}) \\ 0 & \rho(\gamma_2) \end{pmatrix} = \begin{pmatrix} 1 & \rho(\gamma_2)\lambda(\gamma_1^{-1}) + \lambda(\gamma_2^{-1}) \\ 0 & \rho(\gamma_1)\rho(\gamma_2) \end{pmatrix} = \xi(\gamma_1\gamma_2).$$

We clearly have that $\det(\xi) = \rho$. It remains to check that ξ is indecomposable. Assuming by contradiction that there exists a one-dimensional subrepresentation $V \subset \mathbb{R}^2$ of ξ with $V \neq \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$, then V is generated by some vector $\begin{pmatrix} c \\ 1 \end{pmatrix} \in \mathbb{R}^2$. By (4), for each $\gamma \in \pi_1(M)$ we have

$$\xi(\gamma) \begin{pmatrix} c \\ 1 \end{pmatrix} = \begin{pmatrix} c + \lambda(\gamma^{-1}) \\ \rho(\gamma) \end{pmatrix}.$$

Thus V is preserved by ξ if and only if $\lambda(\gamma^{-1}) + c = \rho(\gamma)c$ for every $\gamma \in \pi_1(M)$.

Together with (3) we obtain:

$$(\gamma^{-1})^*(h + c) = (\gamma^{-1})^* h + c = \rho(\gamma)h + \lambda(\gamma^{-1}) + c = \rho(\gamma)h + \rho(\gamma)c = \rho(\gamma)(h + c).$$

This shows that $e^{\varphi}(h + c)$ is the pull-back through π of a function on M, *i.e.* there exists $s \in \mathcal{C}^\infty(M)$ such that $h + c = e^{-\varphi}\pi^* s$. However, this yields:

$$e^{-\varphi}\pi^*\alpha = \mathrm{d}h = \mathrm{d}(h + c) = \mathrm{d}(e^{-\varphi}\pi^* s) = e^{-\varphi}\pi^* \mathrm{d}_\theta s,$$

whence $\alpha = \mathrm{d}_\theta s$, contradicting that $[\alpha]_\theta \neq 0$. We thus conclude that ξ is indecomposable.

(2) We denote by M_γ the matrix of $\xi(\gamma)$ with respect to the standard basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, which is of the form $M_\gamma = \begin{pmatrix} 1 & \lambda(\gamma^{-1}) \\ 0 & \rho(\gamma) \end{pmatrix}$. Consider again the function $f \in \mathcal{C}^\infty(\widetilde{M}, \mathbb{R}_+)$ given by Lemma 2.2, and define the function $g\colon \widetilde{M} \to \mathbb{R}^2$ as follows:

$$g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} := \sum_{\gamma \in \pi_1(M)} M_{\gamma^{-1}} \cdot \gamma^* \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

As before, the function g is well-defined and smooth, since the sum is finite in the neighbourhood of any point of $\widetilde{M}$, by property (ii) in Lemma 2.2. Note that the function $g_2 = \sum_{\gamma\in\pi_1(M)} \rho(\gamma^{-1})\gamma^* f$ is positive on $\widetilde{M}$, by property (i) in Lemma 2.2. We compute for any $a \in \pi_1(M)$:

$$\begin{aligned} a^* g &= \sum_{\gamma\in\pi_1(M)} M_{\gamma^{-1}} \cdot a^*\gamma^* \begin{pmatrix} 0 \\ f \end{pmatrix} = \sum_{\gamma\in\pi_1(M)} M_{\gamma^{-1}} \cdot (\gamma a)^* \begin{pmatrix} 0 \\ f \end{pmatrix} \\ &= \sum_{\gamma\in\pi_1(M)} M_{a\gamma^{-1}} \cdot \gamma^* \begin{pmatrix} 0 \\ f \end{pmatrix} = M_a \cdot \sum_{\gamma\in\pi_1(M)} M_{\gamma^{-1}} \cdot \gamma^* \begin{pmatrix} 0 \\ f \end{pmatrix} = M_a \cdot g. \end{aligned}$$

Thus, for any $a \in \pi_1(M)$, we have:

$$\begin{pmatrix} a^* g_1 \\ a^* g_2 \end{pmatrix} = \begin{pmatrix} g_1 + \lambda(a^{-1}) g_2 \\ \rho(a) g_2 \end{pmatrix}.$$

Since $g_2 > 0$ on $\widetilde{M}$ and satisfies $a^* g_2 = \rho(a) g_2$, for all $a \in \pi_1(M)$, we conclude as in the proof of Lemma 2.1, that $\mathrm{d}(\ln g_2)$ is the pull-back of a closed 1-form θ' on M cohomologous to θ. Up to changing the representative, we may assume that $\pi^*\theta = \mathrm{d}(\ln g_2)$.

We define $h\colon \widetilde{M} \to \mathbb{R}$, $h := \frac{g_1}{g_2}$ and compute for every $a \in \pi_1(M)$:

$$a^* h = \frac{a^* g_1}{a^* g_2} = \frac{g_1 + \lambda(a^{-1}) g_2}{\rho(a) g_2} = \rho(a^{-1}) h + \rho(a^{-1}) \lambda(a^{-1}). \tag{5}$$

This shows that $a^*\mathrm{d}h = \rho(a^{-1})\mathrm{d}h$ for all $a \in \pi_1(M)$, so the 1-form $g_2\mathrm{d}h$ is invariant under the action of $\pi_1(M)$. Consequently, there exists $\alpha \in \Omega^1(M)$ with $\pi^*\alpha = g_2\mathrm{d}h$. We now check that α defines a non-trivial twisted cohomology class in $H^1_\theta(M)$. Firstly, α is d_θ closed, because

$$\pi^*(\mathrm{d}_\theta \alpha) = e^{\varphi} \mathrm{d} e^{-\varphi} \pi^*\alpha = g_2 \mathrm{d}\left(\frac{1}{g_2}\pi^*\alpha\right) = g_2 \mathrm{d}(\mathrm{d}h) = 0.$$

We now assume that $[\alpha]_\theta = 0$ in $H^1_\theta(M)$, *i.e.* there exists $s \in \mathcal{C}^\infty(M)$ such that $\alpha = \mathrm{d}_\theta s$. Using (2), this implies

$$g_2 \mathrm{d}h = \pi^*\alpha = \pi^*\mathrm{d}_\theta s = g_2 \mathrm{d}\left(\frac{1}{g_2}\pi^* s\right),$$

hence there exists a constant c such that $h = \frac{1}{g_2}\pi^* s + c$. We claim that the one-dimensional eigenspace spanned by the vector $\begin{pmatrix} c \\ 1 \end{pmatrix} \in \mathbb{R}^2$ is invariant under ξ. Namely, the following equality holds for all $a \in \pi_1(M)$, according to (5) and to the definition of c:

$$\rho(a^{-1})h + \rho(a^{-1})\lambda(a^{-1}) = a^* h = a^*\left(\frac{1}{g_2}\pi^* s + c\right) = \frac{\pi^* s}{\rho(a) g_2} + c = \rho(a^{-1})(h - c) + c,$$

which implies that $c + \lambda(a^{-1}) = \rho(a)c$. Hence, for any $a \in \pi_1(M)$, we have:

$$\xi(a)\begin{pmatrix} c \\ 1 \end{pmatrix} = M_a \begin{pmatrix} c \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda(a^{-1}) \\ 0 & \rho(a) \end{pmatrix}\begin{pmatrix} c \\ 1 \end{pmatrix} = \begin{pmatrix} c + \lambda(a^{-1}) \\ \rho(a) \end{pmatrix} = \rho(a)\begin{pmatrix} c \\ 1 \end{pmatrix}.$$

This contradicts the assumption that ξ is indecomposable, hence we conclude that $[\alpha]_\theta \neq 0$.

□

The indecomposability hypothesis in the above result can be equivalently stated as follows:

Lemma 2.4. *Let $\xi : \Gamma \to \mathrm{GL}_2(\mathbb{R})$ be a two-dimensional representation of a group Γ, which fixes the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2$ and such that $\rho := \det(\xi)$ is non-trivial. Then ξ is decomposable if and only if $[\Gamma, \Gamma] \subset \ker(\xi)$.*

Proof. If ξ is decomposable, then all matrices in $\xi(\Gamma)$ are simultaneously diagonalizable, so they commute, whence $\xi([\Gamma, \Gamma]) = \{\mathrm{I}_2\}$.

Assume, conversely, that $[\Gamma, \Gamma] \subset \ker(\xi)$. By hypothesis, there exists some $\gamma_0 \in \Gamma$ with $\rho(\gamma_0) \neq 1$. Then $\xi(\gamma_0)$ has two distinct eigenvalues, 1 and $\rho(\gamma_0)$, so it has two one-dimensional eigenspaces E_1 and E_2. For every element $\gamma \in \Gamma$, $\xi(\gamma)$ commutes with $\xi(\gamma_0)$, so $\xi(\gamma)$ preserves E_1 and E_2. Thus ξ is decomposable. □

3. Applications

We now derive some consequences of Theorem 2.3.

Corollary 3.1. *Let M be a manifold whose fundamental group $\pi_1(M)$ is nilpotent. Then for any non-trivial cohomology class $[\theta] \in H^1_{\mathrm{dR}}(M)$, we have $H^1_\theta(M) = 0$.*

Proof. Let $[\theta] \in H^1_{\mathrm{dR}}(M)$ with $[\theta] \neq 0$, and let $\rho\colon \pi_1(M) \to (\mathbb{R}^*_+, \times)$ denote the representation associated to $[\theta] \in H^1_{\mathrm{dR}}(M)$, given by Lemma 2.1. Applying Theorem 2.3, we have to show that any representation $\xi\colon \pi_1(M) \to \mathrm{GL}_2(\mathbb{R})$ with $\det \xi = \rho$ and which fixes the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is decomposable. We assume by contradiction that there exists such a representation ξ which is indecomposable.

Since $[\theta] \neq 0$, we have $\rho \neq 1$, so there exists $a \in \pi_1(M)$ such that $\det(\xi(a)) \neq 1$. Then $\xi(a)$ is diagonalizable, so there exists a basis of $\mathbb{R}^2$, such that the matrix of $\xi(a)$ with respect to this basis is given by $M_a = \begin{pmatrix} 1 & 0 \\ 0 & \rho(a) \end{pmatrix}$. Since ξ is assumed to be indecomposable, by Lemma 2.4, there exists $b_0 \in [\pi_1(M), \pi_1(M)]$ with $M_{b_0} = \begin{pmatrix} 1 & \lambda(b_0^{-1}) \\ 0 & \rho(b_0) \end{pmatrix}$ and $\lambda(b_0) \neq 0$. We then obtain for $b_1 := b_0^{-1}a^{-1}b_0a$:

$$\begin{aligned} M_{b_1} &= \begin{pmatrix} 1 & -\frac{\lambda(b_0^{-1})}{\rho(b_0)} \\ 0 & \frac{1}{\rho(b_0)} \end{pmatrix} \begin{pmatrix} 1 & -\frac{\lambda(a^{-1})}{\rho(a)} \\ 0 & \frac{1}{\rho(a)} \end{pmatrix} \begin{pmatrix} 1 & \lambda(b_0^{-1}) \\ 0 & \rho(b_0) \end{pmatrix} \begin{pmatrix} 1 & \lambda(a^{-1}) \\ 0 & \rho(a) \end{pmatrix} \\ &= \begin{pmatrix} 1 & \lambda(b_0^{-1})(\rho(a)-1) \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

which shows that also $\lambda(b_1^{-1}) = \lambda(b_0^{-1})(\rho(a)-1) \neq 0$, because $\rho(a) \neq 1$ and $\lambda(b_0) \neq 0$. If we define for $i \in \mathbb{N}$ inductively $b_{i+1} := b_i^{-1}a_0^{-1}b_ia_0$, then $\lambda(b_i) \neq 0$, for all i, which contradicts the hypothesis that $\pi_1(M)$ is nilpotent.

□

Corollary 3.2. *Let M be a manifold whose commutator subgroup $G := [\pi_1(M), \pi_1(M)]$ is finitely generated. Then the set*

$$\{[\theta] \in H^1_{\mathrm{dR}}(M) \,|\, H^1_\theta(M) \neq 0\}$$

is finite and has at most $\mathrm{rank}(G)^{\mathrm{rank}(\pi_1(M))}$ *elements.*

Proof. Let $\{a_1, \ldots, a_m\}$ be a set of generators of $\pi_1(M)$ and let $\{b_1, \ldots, b_k\}$ be a set of generators of G. Let $[\theta] \in H^1_{\mathrm{dR}}(M)$ with $H^1_\theta(M) \neq 0$. Let $\rho\colon \pi_1(M) \to (\mathbb{R}^*_+, \times)$ denote the representation associated to $[\theta] \in H^1_{\mathrm{dR}}(M)$, given by Lemma 2.1, and let $\xi\colon \pi_1(M) \to \mathrm{GL}_2(\mathbb{R})$ be a representation associated to $[\theta]$, as in Theorem 2.3.

We denote by M_i the matrix of $\xi(b_i)$ with respect to the standard basis of $\mathbb{R}^2$. Since $b_i \in G = [\pi_1(M), \pi_1(M)]$, we have $\rho(b_i) = 1$, so the matrix M_i has the following form: $M_i = \begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix}$, for some $x_i \in \mathbb{R}$. Let us remark that at least one of the numbers x_i does not vanish, since otherwise the restriction of ξ to G would be trivial and then, by Lemma 2.4, ξ would be decomposable.

For any $1 \leq j \leq m$ and $1 \leq i \leq k$, the element $a_j^{-1}b_ia_j$ belongs to G. Therefore, there exist integers $n_{ij\ell}$, for $1 \leq \ell \leq k$, such that $a_j^{-1}b_ia_j = \prod_{\ell=1}^{k} b_\ell^{n_{ij\ell}}$. On the one hand, we compute:

$$\xi(a_j^{-1})\xi(b_i)\xi(a_j) = \begin{pmatrix} 1 & -\frac{\lambda(a_j^{-1})}{\rho(a_j)} \\ 0 & \frac{1}{\rho(a_j)} \end{pmatrix} \begin{pmatrix} 1 & x_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda(a_j^{-1}) \\ 0 & \rho(a_j) \end{pmatrix} = \begin{pmatrix} 1 & x_i\rho(a_j) \\ 0 & 1 \end{pmatrix}.$$

On the other hand, we have:

$$\xi(a_j^{-1})\xi(b_i)\xi(a_j) = \xi(a_j^{-1}b_ia_j) = \xi\left(\prod_{\ell=1}^{k} b_\ell^{n_{ij\ell}}\right) = \prod_{\ell=1}^{k} M_\ell^{n_{ij\ell}} = \begin{pmatrix} 1 & \sum_{\ell=1}^{k} n_{ij\ell}x_\ell \\ 0 & 1 \end{pmatrix}.$$

Hence, for all $1 \leq j \leq m$ and $1 \leq i \leq k$, the following equality holds: $x_i\rho(a_j) = \sum_{\ell=1}^{k} n_{ij\ell}x_\ell$. If for each fixed $j \in \{1, \dots, m\}$, we define the $k \times k$-matrix with integer entries $N_j := (n_{ij\ell})_{i,\ell}$, then the above system of equations for j fixed can be equivalently written as:

$$(N_j - \rho(a_j)\mathrm{I}_k)\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = 0.$$

As previously noticed, at least one of the x_i's is non-zero. Thus $\rho(a_j)$ must be an eigenvalue of N_j, so each $\rho(a_j)$ can take at most k different values. Therefore, when j varies, there are overall at most k^m different possibilities for defining ρ, or, equivalently, for defining a cohomology class $[\theta] \in H^1_{\mathrm{dR}}(M)$ with $H^1_\theta(M) \neq 0$.

□

Corollary 3.3. *If S is a compact Riemann surface of genus $g \geq 2$, then $H^1_\theta(S) \neq 0$ for every closed 1-form θ on S.*

Proof. It is well known that $\pi_1(S)$ has $2g$ generators $\gamma_1, \dots, \gamma_{2g}$ subject to the relation

$$\prod_{j=1}^{g}(\gamma_{2j-1}\gamma_{2j}\gamma_{2j-1}^{-1}\gamma_{2j}^{-1}) = 1. \tag{6}$$

Any representation $\rho : \pi_1(S) \to (\mathbb{R}^*_+, \times)$ is defined by the $2g$ positive real numbers $y_i := \rho(\gamma_i)$. According to Lemma 2.4 and Theorem 2.3, we need to show for every such ρ, there exists a two-dimensional representation $\xi : \pi_1(S) \to \mathrm{GL}_2(\mathbb{R})$ with $\det(\xi) = \rho$, which fixes the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2$ and whose restriction to $[\pi_1(S), \pi_1(S)]$ is non-trivial.

We look for ξ of the form $\xi(\gamma_i) := \begin{pmatrix} 1 & x_i \\ 0 & y_i \end{pmatrix}$. The commutator of two such matrices is

$$\begin{pmatrix} 1 & x_i \\ 0 & y_i \end{pmatrix}\begin{pmatrix} 1 & x_j \\ 0 & y_j \end{pmatrix}\begin{pmatrix} 1 & x_i \\ 0 & y_i \end{pmatrix}^{-1}\begin{pmatrix} 1 & x_j \\ 0 & y_j \end{pmatrix}^{-1} = \begin{pmatrix} 1 & x_i(y_j-1) - x_j(y_i-1) \\ 0 & 1 \end{pmatrix},$$

so by (6), the condition that ξ defines a representation reads

$$\sum_{j=1}^{g} (x_{2j-1}(y_{2j}-1) - x_{2j}(y_{2j-1}-1)) = 0. \tag{7}$$

Moreover, such a representation is non-trivial on $[\pi_1(S), \pi_1(S)]$ provided that

$$\exists\ i, j \in \{1, \dots, 2g\} \ \text{ such that } x_i(y_j - 1) - x_j(y_i - 1) \neq 0. \tag{8}$$

Since $g \geq 2$, for any positive real numbers y_i ($1 \leq i \leq 2g$), one can choose the real numbers x_i such that (7) and (8) are satisfied.

□

4. Examples

Let f_A be the diffeomorphism of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ induced by a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ and let M_A be the mapping torus of f_A. In other words, M_A is the quotient of $\mathbb{T}^2 \times \mathbb{R}$ by the free $\mathbb{Z}$-action generated by the diffeomorphism $(p, t) \mapsto (f_A(p), t+1)$. The fundamental group of M_A is isomorphic to the semidirect product of $\mathbb{Z}$ acting on $\mathbb{Z}^2$: $\pi_1(M_A) \simeq \mathbb{Z}^2 \rtimes_A \mathbb{Z}$.

We pick some non-zero constant $c \in \mathbb{R}$ and denote by θ_c the closed form on M_A whose pull-back to $\mathbb{T}^2 \times \mathbb{R}$ is $c\,\mathrm{d}t$. The associated representation $\rho_c : \pi_1(M_A) \to (\mathbb{R}^*_+, \times)$ maps $\mathbb{Z}^2$ to 1 and the generator of $\mathbb{Z}$ to e^c.

Lemma 4.1. *$H^1_{\theta_c}(M_A) \neq 0$ if and only if e^c is an eigenvalue of A.*

Proof. If $H^1_{\theta_c}(M_A) \neq 0$, Theorem 2.3 shows that there exists an indecomposable representation $\xi : \pi_1(M_A) \to \mathrm{GL}_2(\mathbb{R})$ which fixes the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2$ and such that $\det(\xi) = \rho_c$. This means that for every $v \in \mathbb{Z}^2$ there exists $\lambda(v) \in \mathbb{R}$ such that $\xi(v) = \begin{pmatrix} 1 & \lambda(v) \\ 0 & 1 \end{pmatrix}$ and if a denotes the generator of the subgroup $\mathbb{Z} \subset \pi_1(M_A)$, there exists $x \in \mathbb{R}$ such that $\xi(a) = \begin{pmatrix} 1 & x \\ 0 & e^c \end{pmatrix}$.

The map λ is clearly a group morphism from $\mathbb{Z}^2$ to $(\mathbb{R}, +)$, so

$$\lambda(v_1, v_2) = \lambda_1 v_1 + \lambda_2 v_2, \qquad \forall v = (v_1, v_2) \in \mathbb{Z}^2. \tag{9}$$

Moreover, by Lemma 2.4, λ is not identically zero since $[\pi_1(M_A), \pi_1(M_A)] = \mathbb{Z}^2$.

Since $ava^{-1} = Av$, we get

$$\begin{pmatrix} 1 & \lambda(Av) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & e^c \end{pmatrix} \begin{pmatrix} 1 & \lambda(v) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & e^c \end{pmatrix}^{-1} = \begin{pmatrix} 1 & e^{-c}\lambda(v) \\ 0 & 1 \end{pmatrix},$$

whence

$$\lambda(Av) = e^{-c}\lambda(v), \qquad \forall v \in \mathbb{Z}^2. \tag{10}$$

By (9), this is equivalent to

$${}^tA \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = e^{-c} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}. \tag{11}$$

Thus e^{-c} is an eigenvalue of ${}^tA$, and since the spectra of A and ${}^tA$ are the same and $\det(A) = 1$, it follows that e^c is an eigenvalue of A.

Conversely, if e^c is an eigenvalue of A, then there exists $(\lambda_1, \lambda_2) \in \mathbb{R}^2 \setminus \{0\}$ such that (11) holds. Then (10) also holds for λ defined by (9).

We can then define a representation $\xi : \pi_1(M_A) \simeq \mathbb{Z}^2 \rtimes_A \mathbb{Z} \to \mathrm{GL}_2(\mathbb{R})$ by $\xi(v) := \begin{pmatrix} 1 & \lambda(v) \\ 0 & 1 \end{pmatrix}$, for $v \in \mathbb{Z}^2$ and $\xi(k) := \begin{pmatrix} 1 & 0 \\ 0 & e^{ck} \end{pmatrix}$, for $k \in \mathbb{Z}$. By Lemma 2.4, this representation is indecomposable, so by Theorem 2.3, we conclude that $H^1_{\theta_c}(M_A) \neq 0$.

□

Example 4.2. Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, inducing a diffeomorphism f_A of $\mathbb{T}^2$ and let M_A denote the mapping torus of f_A as before. Since $\frac{3+\sqrt{5}}{2}$ is an eigenvalue of A, Lemma 4.1 shows that for $c := \ln \frac{3+\sqrt{5}}{2}$, the first twisted cohomology group associated to the nowhere vanishing 1-form $\theta_c := c\,\mathrm{d}t$ on M is non-zero: $H^1_{\theta_c}(M_A) \neq 0$.

By [2, Theorem 4.5], the twisted cohomology associated to a closed 1-form which is parallel with respect to some Riemannian metric, vanishes. The above example thus shows the existence of compact manifolds carrying nowhere vanishing closed 1-forms which are not parallel with respect to any Riemannian metric.

Our last example concerns the twisted cohomology on Vaisman manifolds. Recall that a Vaisman manifold is a locally conformally Kähler manifold with parallel Lee form [8]. The space of harmonic 1-forms on a compact Vaisman manifold (M, g, J) with Lee form θ decomposes as follows:

$$\mathcal{H}^1(M, g) = \mathrm{span}\{\theta\} \oplus \mathcal{H}^1_0(M, g), \tag{12}$$

where $\mathcal{H}^1_0(M, g)$ is J-invariant and consists of harmonic 1-forms pointwise orthogonal to θ and $J\theta$ (see for instance [3, Lemma 5.2]). That means that every harmonic 1-form on M can be written as $\beta = t\theta + \alpha$, with $t \in \mathbb{R}$ and $\alpha \in \mathcal{H}^1_0(M, g)$.

By [4, Lemma 3.3], every harmonic form $\beta = t\theta + \alpha$ with $t > 0$ is the Lee form of a Vaisman metric on M. In particular, for every non-vanishing t, there exists a metric on M whith respect to which β is parallel. By [2, Theorem 4.5], the twisted cohomology $H^*_{t\theta+\alpha}(M)$

vanishes for all $t \neq 0$ and $\alpha \in \mathcal{H}^1_0(M,g)$. It remains to understand the case where $t=0$, *i.e.* the twisted cohomology associated to forms $\alpha \in \mathcal{H}^1_0(M,g)$.

It turns out that there exist Vaisman manifolds (M,g) with $\mathcal{H}^1_0(M,g) \neq 0$, for which $H^*_\alpha(M)$ is non-zero for every $\alpha \in \mathcal{H}^1_0(M,g) \setminus \{0\}$.

Example 4.3. Let S be a compact oriented Riemann surface and let $\pi : N \to S$ be the principal S^1-bundle whose first Chern class is the positive generator $e \in H^2(S,\mathbb{Z})$. For every Riemannian metric g_S on S, the 3-dimensional manifold N carries a Riemannian metric g_N making π a Riemannian submersion, and which is Sasakian. Consequently, the Riemannian product $(M,g) := S^1 \times (N, g_N)$ is Vaisman. Its Lee form is just the length element of S^1, denoted by $\theta = \mathrm{d}t$.

The Gysin exact sequence associated to the fibration $\pi : N \to S$ reads

$$0 \mapsto H^1_{\mathrm{dR}}(S) \xrightarrow{\pi^*} H^1_{\mathrm{dR}}(N) \xrightarrow{\pi_*} H^0_{\mathrm{dR}}(S) \xrightarrow{c_1(N)\wedge} H^2_{\mathrm{dR}}(S) \longrightarrow \cdots .$$

By the choice of $c_1(N) = e$, the last arrow is an isomorphism, thus showing that $\pi^* : H^1_{\mathrm{dR}}(S) \to H^1_{\mathrm{dR}}(N)$ is an isomorphism too. Since $\pi : (N, g_N) \to (S, g_S)$ is a Riemannian submersion, we thus have $\pi^*(\mathcal{H}^1(S,g_S)) = \mathcal{H}^1(N,g_N)$.

Moreover, if $p_2 : M = S^1 \times N \to N$ denotes the projection on the second factor, we clearly have $\mathcal{H}^1(M,g) = \mathrm{span}\{\theta\} \oplus p_2^*(\mathcal{H}^1(N,g_N))$.

Denoting by $p := \pi \circ p_2$, the decomposition (12) becomes

$$\mathcal{H}^1(M,g) = \mathrm{span}\{\theta\} \oplus p^*(\mathcal{H}^1(S,g_S)). \tag{13}$$

Let α be a non-zero harmonic form in $\mathcal{H}^1(S,g_S)$ and let $\rho\colon \pi_1(S) \to (\mathbb{R}^*_+, \times)$ be the character of $\pi_1(S)$ associated to α, given by Lemma 2.1. Clearly, the character of $\pi_1(M)$ associated to $p^*\alpha$ is $\tilde\rho := \rho \circ p_*$, where $p_* : \pi_1(M) \to \pi_1(S)$ is the induced morphism of the fundamental groups. Note that, since the fibers of $p : M \to S$ are connected, the exact homotopy sequence shows that p_* is surjective.

By the proof of Corollary 3.3, there exists a two-dimensional representation $\xi : \pi_1(S) \to \mathrm{GL}_2(\mathbb{R})$ with $\det(\xi) = \rho$, which fixes the vector $\begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{R}^2$ and whose restriction to the commutator $[\pi_1(S), \pi_1(S)]$ is non-trivial.

Composing ξ with p_* yieds a two-dimensional representation $\tilde\xi := \xi \circ p_* : \pi_1(M) \to \mathrm{GL}_2(\mathbb{R})$ with $\det(\tilde\xi) = \tilde\rho$, which fixes the vector $\begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{R}^2$ and whose restriction to $[\pi_1(M), \pi_1(M)]$ is non-trivial (since p_* is surjective). By Theorem 2.3, the first twisted cohomology group $H^1_{p^*\alpha}(M)$ is non-vanishing.

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