TWISTOR AND KILLING FORMS IN RIEMANNIAN GEOMETRY

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Plan of the talk

- Algebraic preliminaries
- Twistor forms on Riemannian manifolds
- Short history
- Main properties of twistor forms
- Examples
- Compact manifolds with non–generic holonomy carrying twistor forms
- Twistor forms on Kähler manifolds
- Open problems

1. Algebraic preliminaries

Let E be a n–dimensional Euclidean space endowed with the scalar product $\langle\cdot,\cdot\rangle$. We identify throughout this talk E and E^*.

$\{e_i\}$ denotes an orthonormal basis of E, (or a local orthonormal frame of the Riemannian manifold in the next sections).

Consider the two natural linear maps

$$\lrcorner : E \otimes \Lambda^k E \to \Lambda^{k-1} E,$$

$$\wedge : E \otimes \Lambda^k E \to \Lambda^{k+1} E.$$

Their meric adjoints (wrt the induced metric on the exterior powers of E) are

$$\lrcorner^*(\tau) = \sum e_i \otimes e_i \wedge \tau \ , \qquad \tau \in \Lambda^{k-1} E,$$

$$\wedge^*(\sigma) = \sum e_i \otimes e_i \lrcorner \sigma \ , \qquad \sigma \in \Lambda^{k+1} E.$$

Since obviously

$$\wedge \circ \lrcorner^* = \lrcorner \circ \wedge^* = 0,$$

one gets the direct sum decomposition

$$E \otimes \Lambda^k E = Im(\lrcorner^*) \oplus Im(\wedge^*) \oplus T^k E$$

where $T^k E$ denotes the orthogonal complement of the direct sum of the first two summands. We denote by π_1, π_2, π_3 the projections on the three summands. The relations

$$\lrcorner \circ \lrcorner^* = (n-k+1) Id_{\Lambda^{k-1}E}$$

$$\wedge \circ \wedge^* = (k+1) Id_{\Lambda^{k+1}E}$$

show that for $\xi \in E \otimes \Lambda^k E$ one has

$$\pi_1 \xi = \frac{1}{k+1} \lrcorner^* \circ \lrcorner \xi,$$

$$\pi_2 \xi = \frac{1}{n-k+1} \wedge^* \circ \wedge \xi,$$

$$\pi_3 \xi = \xi - \frac{1}{k+1} \lrcorner^* \circ \lrcorner \xi - \frac{1}{n-k+1} \wedge^* \circ \wedge \xi.$$

2. Twistor forms on Riemannian manifolds

Let (M^n, g) be a Riemannian manifold. As before, we identify 1–forms and vectors via the metric. Let ∇ denote the covariant derivative of the Levi–Civita connection of M. If u is a k–form, then ∇u is a section of $TM \otimes \Lambda^k M$, where

$$\Lambda^k M := \Lambda^k(T^*M) \simeq \Lambda^k(TM).$$

Using the notations above (for $E = TM$) we define the first order differential operator

$$T : C^\infty(\Lambda^k M) \to C^\infty(TM \otimes \Lambda^k M),$$

$$Tu := \pi_3(\nabla u).$$

Noticing that the exterior differential d and its formal adjoint δ can be writen

$$du = \wedge(\nabla u) \ , \qquad \delta u = - \lrcorner (\nabla u),$$

one gets

$$Tu(X) = \nabla_X u - \frac{1}{k+1} X \lrcorner\, du + \frac{1}{n-k+1} X \wedge \delta u$$

for all $X \in TM$.

Definition 1 *The k–form u is called* twistor form *if $Tu = 0$.*

If, moreover, u is co–closed, then it is called Killing form*.*

Remark: if one takes the wedge or interior product with X in the twistor equation

$$\nabla_X u = \frac{1}{k+1} X \lrcorner\, du - \frac{1}{n-k+1} X \wedge \delta u,$$

put $X = e_i$ and sum over i one gets tautological identities. In case of holonomy reduction, such an approach can be used successfully (see below).

3. Short history

- Yano (1952) introduces Killing forms

- Tachibana, Kashiwada (1968–1969) introduce and study twistor forms

- Jun, Ayabe, Yamaguchi (1982) study twistor forms on compact Kähler manifolds. They conclude that if $n > 2k \geq 8$, every twistor k–form on a n–dimensional compact Kähler manifold is parallel (?!)

- Since 2001: Semmelmann, M, Belgun et al. study twistor and Killing forms on compact manifolds with reduced holonomy and on symmetric spaces. Several classification results are obtained.

4. Main properties of twistor forms

Geometric interpretation. If $k = 1$, a twistor 1–form is just the dual of a conformal vector field. A Killing 1–form is the dual of a Killing vector field. Remark: twistor k–forms have no geometric interpretation for $k > 1$.

Conformal invariance. If u is a twistor k–form on (M, g) and $\hat{g} := e^{2\lambda} g$ is a conformally equivalent metric, the form $\hat{u} := e^{(k+1)\lambda} u$ is a twistor form on $(M, \hat{g})$. This is a consequence of the conformal invariance of the twistor operator: $\hat{T}(\hat{u}) = \widehat{Tu}$.

Finite dimension. Twistor forms are determined by their 2–jet at a point. More precisely, $(u, du, \delta u, \Delta u)$ is a parallel section of

$$\Lambda^k M \oplus \Lambda^{k+1} M \oplus \Lambda^{k-1} M \oplus \Lambda^k M$$

with respect to some explicit connection on this bundle.

Thus, the space of twistor k–forms has finite dimension $\leq \begin{pmatrix} n+2 \\ k+1 \end{pmatrix}$. This dimensional bound is sharp, equality is obtained on S^n.

Relations to twistor spinors. If (M^n, g) is oriented and spin, endowed with a spin structure, one can consider the (complex) spin bundle ΣM with its canonical Hermitian product $(\cdot,\cdot)$, Clifford product γ and covariant derivative ∇ induced by the Levi-Civita connection. The Dirac operator D is defined as the composition $D := \gamma \circ \nabla$. More explicitly, $D = \sum e_i \cdot \nabla_{e_i}$ in a local ON frame. $TM \otimes \Sigma M$ splits as follows:

$$TM \otimes \Sigma M = Im(\gamma^*) \oplus Ker(\gamma).$$

A spinor ψ is called a *twistor spinor* if the projection of $\nabla\psi$ onto the second summand vanishes. Since $\gamma \circ \gamma^* = -nId_{\Sigma M}$, this translates into

$$\nabla_X \psi + \frac{1}{n} X \cdot D\psi = 0.$$

To every spinor ψ one can associate a k–form ψ_k via the squaring construction:

$$\psi_k := \sum_{i_1 < \ldots < i_k} e_{i_1} \wedge \ldots \wedge e_{i_k} (e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot \psi, \psi).$$

Proposition 2 *(M – Semmelmann, 2003) If ψ is a twistor spinor then ψ_k are twistor k–forms for every k.*

The converse clearly does not hold. The twistor form equation can thus be seen as a weakening of the twistor spinor equation. Similar relations exist between Killing spinors and forms.

5. Examples

- Parallel forms; more generally, if u is a parallel k–form on (M, g), $e^{(k+1)\lambda}u$ is a (non–parallel) twistor form on $(M, e^{2\lambda}g)$.

- The round sphere S^n. Twistor forms are sums of closed and co–closed forms corresponding to the least eigenvalue of the Laplace operator.

- Sasakian manifolds: $d\xi^l,\ \xi \wedge d\xi^l,\ l \geq 0$ are closed (resp. co–closed) twistor forms.

- Weak G_2–manifolds or nearly Kähler manifolds: the distinguished 3–form (resp. the fundamental 2–form) are Killing forms.

- Kähler manifolds: new examples (see below).

6. Classification program

Let (M^n, g) be a compact, simply connected, oriented Riemannian manifold with holonomy $\neq \mathrm{SO}_n$. By the Berger–Simons Holonomy Theorem, one of the 3 following cases occurs:

- M is a symmetric space of compact type.

- M is a Riemannian product $M = M_1 \times M_2$.

- M has reduced holonomy.

A. Symmetric spaces. The existence problem for twistor forms is not yet completely solved. For Killing forms one has the following result:

Theorem 3 *(Belgun – M – Semmelmann, 2004) A symmetric space of compact type carries a non–parallel Killing form if and only if it has a Riemannian factor isometric to a round sphere.*

B. Riemannian products. Twistor forms are completely understood in this case:

Theorem 4 *(M – Semmelmann, 2004) A twistor form on a Riemannian product is a sum of parallel forms, Killing forms on one of the factors, and their Hodge duals.*

C. Reduced holonomy. We distinguish three sub–cases:

(i) <u>Kähler geometries</u> (holonomy group U_m, SU_m or Sp_l). Killing forms are parallel and twistor forms are related to Hamiltonian forms (see below).

(ii) Quaternion–Kähler geometry (holonomy group $\mathrm{Sp}_1 \cdot \mathrm{Sp}_l$, $l > 1$).

Theorem 5 *(M – Semmelmann, 2004) Every Killing k–form ($k > 1$) on a quaternion–Kähler manifold is parallel.*

The similar question for twistor forms is still open.

(iii) Joyce geometries (holonomy group G_2 or Spin_7).

Theorem 6 *(Semmelmann, 2002) Every Killing k–form on a Joyce manifold is parallel. There are no twistor k–forms on G_2–manifolds for $k = 1, 2, 5, 6$.*

7. An example: twistor forms on Kähler manifolds

Let (M^{2m}, g, J) be a Kähler manifold with Kähler form denoted by Ω.

Definition 7 *(Apostolov – Calderbank – Gauduchon) A* 2*–form* $\omega \in \Lambda^{1,1}M$ *is called* Hamiltonian *if*

$$\nabla_X \omega = X \wedge J\mu + \mu \wedge JX, \qquad \forall X \in TM,$$

for some 1*–form* μ *(which necessarily satisfies* $\mu = \frac{1}{2} d\langle \omega, \Omega \rangle$*).*

Main feature: if A denotes the endomorphism associated to ω, the coefficients of the characteristic polynomial χ_A are Hamiltonians of *commuting* Killing vector fields on M (toric geometry). In a sequence of recent papers, A–C–G obtain the classification of compact Kähler manifolds with Hamiltonian forms.

For the study of twistor forms one uses the Kählerian operators

$$d^c := \sum Je_i \wedge \nabla_{e_i} \ , \ \ \delta^c := -\sum Je_i \lrcorner \nabla_{e_i},$$

$$L := \Omega\wedge = \frac{1}{2} e_i \wedge Je_i \wedge \ , \ \Lambda := L^* = \frac{1}{2}\sum Je_i \lrcorner \, e_i \lrcorner \, ,$$

$$J := \sum Je_i \wedge e_i \lrcorner$$

and the relations between them:

$$d^c = -[\delta, L] = -[d, J] \ , \ \ \delta^c = [d, \Lambda] = -[\delta, L],$$

$$d = [\delta^c, L] = [d^c, J] \ , \ \ \delta = -[d^c, \Lambda] = [\delta^c, L],$$

$$\Delta = d\delta + \delta d = d^c\delta^c + \delta^c d^c \ , \ [\Lambda, L] = (m-k) Id_{\Lambda^k},$$

as well as the vanishing of the following commutators resp. anti–commutators

$$0 = [d, L] = [d^c, L] = [\delta, \Lambda] = [\delta^c, \Lambda] = [\Lambda, J] = [J, L],$$

$$0 = \delta d^c + d^c\delta = dd^c + d^c d = \delta\delta^c + \delta^c\delta = d\delta^c + \delta^c d.$$

(21 relations)

Theorem 8 *(M – Semmelmann, 2002) Let u be a twistor k–form on a compact Kähler manifold (M^{2m}, g, J) and suppose that $k \neq m$. Then k is even, $k = 2p$, and there exists a Hamiltonian 2–form ψ with*

$$u = L^{p-1}\psi - \frac{1}{2p}L^p\langle\psi, \Omega\rangle$$

up to parallel forms.

<u>Step 1</u>. (difficult) Ju is parallel (i.e. $u \in \Lambda^{p,p}+$ parallel form).

<u>Step 2</u>. du and δu are eigenforms of ΛL with explicit eigenvalues.

<u>Step 3</u>. The LePage decomposition

$$\omega = \omega_0 + L\omega_1 + L^2\omega_2 + \dots$$

implies

$$du = L^p v \ , \ \delta u = L^{p-1}w, \ v, w \in TM.$$

Step 4. Using the twistor equation one gets

$$u = L^{p-1}\omega + L^p f, \ \omega \in \Lambda^{1,1}M, \ f \in C^\infty M.$$

Step 5. For a right choice of ω and f,

$$u = L^{p-1}\psi - \frac{1}{2p}\Omega^p\langle\psi, \Omega\rangle + \text{parallel form}.$$

Remark. A similar approach can be used to study twistor forms on QK manifolds. If J_α ($\alpha = 1, 2, 3$) denotes a local ON frame of almost complex structures, one can define (besides d and δ) 6 first order natural diferential operators

$$d^+ := \sum_{i,\alpha} L_\alpha J_\alpha(e_i) \wedge \nabla_{e_i},$$

$$d^- := \sum_{i,\alpha} \Lambda_\alpha J_\alpha(e_i) \wedge \nabla_{e_i},$$

$$d^c := \sum_{i,\alpha} J_\alpha J_\alpha(e_i) \wedge \nabla_{e_i},$$

$$\delta^+ := -\sum_{i,\alpha} L_\alpha J_\alpha(e_i) \lrcorner \nabla_{e_i},$$

$$\delta^- := -\sum_{i,\alpha} \Lambda_\alpha J_\alpha(e_i) \lrcorner \nabla_{e_i},$$

$$\delta^c := -\sum_{i,\alpha} J_\alpha J_\alpha(e_i) \lrcorner \nabla_{e_i}.$$

and 6 linear operators

$$L := \sum_\alpha L_\alpha \circ L_\alpha,\ L^- := \sum_\alpha L_\alpha \circ J_\alpha,\ J := \sum_\alpha J_\alpha \circ J_\alpha,$$

$$\Lambda := \sum_\alpha \Lambda_\alpha \circ \Lambda_\alpha,\ \Lambda^+ := \sum_\alpha \Lambda_\alpha \circ J_\alpha,\ C := \sum_\alpha L_\alpha \circ \Lambda_\alpha.$$

This gives rise to 91 commutation relations, e.g

$$\begin{array}{llll}
[d,\Lambda] & = 2\delta^- & [\delta, L] & = -2d^+ \\
[d, L^-] & = -d^+ & [\delta, L^-] & = -\delta^+ - d^c - 3d \\
[d, \Lambda^+] & = -d^- + \delta^c + 3\delta & [\delta, \Lambda^+] & = \delta^- \\
[d, J] & = -2d^c - 3d & [\delta, J] & = -2\delta^c - 3\delta \\
[d, C] & = \delta^+ & [\delta, C] & = -d^- + 3
\end{array}$$

...

8. Open problems

In view of the previous results, the existence of Killing forms on simply connected compact manifolds M with non–generic holonomy is completely understood: there exists a non–parallel Killing k–form ($k > 1$) on M iff M has a factor isometric to a Riemannian sphere S^p, $p \geq 2$.

The similar problem for twistor k–forms is still open

- on symmetric spaces
- on quaternionic–Kähler manifolds
- on Spin_7–manifolds
- on G_2–manifolds (for $k = 3, 4$)
- on Kähler manifolds (for $k = dim_{\mathbb{C}} M$).