QUATERNION-KÄHLER MANIFOLDS WITH NON-NEGATIVE QUATERNIONIC SECTIONAL CURVATURE

ANDREI MOROIANU, UWE SEMMELMANN, GREGOR WEINGART

Abstract. Compact Hermitian symmetric spaces are Kähler manifolds with constant scalar curvature and non-negative sectional curvature. A famous result by A. Gray states that, conversely, a compact simply connected Kähler manifold with constant scalar curvature and non-negative sectional curvature is a Hermitian symmetric space. The aim of the present article is to transpose Gray's result to the quaternion-Kähler setting. In order to achieve this, we introduce the quaternionic sectional curvature of quaternion-Kähler manifolds, we show that every Wolf space has non-negative quaternionic sectional curvature, and we prove that, conversely, every quaternion-Kähler manifold with non-negative quaternionic sectional curvature is a Wolf space. The proof makes crucial use of the nearly Kähler twistor spaces of positive quaternion-Kähler manifolds.

1. Introduction

The original inspiration for this article stems from a beautiful result of A. Gray [13] stating that *compact simply connected Kähler manifolds with constant scalar curvature and non-negative sectional curvature are Hermitian symmetric*. Gray's proof was revisited in [19], where it is shown that it basically follows from a Weitzenböck formula applied to the curvature tensor R, together with the fact that the curvature operator $q(R)$ is semi-definite on symmetric tensors whenever the sectional curvature is non-negative, cf. [14].

It is thus natural to ask whether this strategy can be applied to other classes of Riemannian manifolds, like quaternion-Kähler manifolds, which appear in Berger's classification of Riemannian manifolds with special holonomy. In every dimension $4n \geq 8$, the holonomy group of a quaternion-Kähler manifold is contained in $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1) \subset \mathrm{SO}(4n)$. Geometrically, a quaternion-Kähler manifold (M, g) can be characterized by the existence of a parallel rank 3 vector subbundle $\mathcal{E}$ of $\mathrm{End}(\mathrm{T}M)$ locally spanned by three almost Hermitian structures I, J, K satisfying the quaternionic relations $I^2 = J^2 = K^2 = -\mathrm{id}$, and $IJ = K$.

All quaternion-Kähler manifolds are Einstein. The scalar curvature vanishes if and only if the manifold is hyperkähler (i.e. I, J, K can be chosen globally as Kähler structures), a situation which is in general not considered as proper quaternion-Kähler. In the remaining cases, a quaternion-Kähler manifold $(M, g, \mathcal{E})$ is called positive or negative according to the

Date: November 30, 2024.
2010 *Mathematics Subject Classification.* 53B05, 53C25.
Key words and phrases. parallel skew-symmetric torsion, Killing vector fields.

sign of its scalar curvature. While in the negative case it is possible to construct complete non-locally symmetric examples of quaternion-Kähler manifolds, the only examples of complete positive quaternion-Kähler manifolds are the so-called Wolf spaces [26], which are symmetric spaces of compact type associated to each simple compact Lie group G via the choice of a root of G of maximal length.

The famous LeBrun-Salamon conjecture (which was proved only in small dimensions $n = 2$ [23] and $n = 3$ or 4 [5]) states that all positive quaternion-Kähler manifolds are Wolf spaces. A possible weakening of this conjecture would be to add some curvature positivity assumption, and to use Gray's theorem [13]. Indeed, an important feature of positive quaternion-Kähler manifolds is that the sphere bundle of $\mathcal{E}$, called the *twistor space*, can be naturally endowed with a Kähler-Einstein structure [24].

It is thus tempting to try to impose a condition on the sectional curvature of M which is sufficient for the non-negativity of the sectional curvature of the twistor space and conclude by Gray's theorem. However this cannot work, since the twistor spaces of Wolf spaces endowed with the Kähler-Einstein metric are not symmetric, except for the quaternionic projective space $\mathbb{H}P^n$. Such a curvature condition on M would therefore be too strong and would not hold for the other Wolf spaces.

Our strategy in this article is inspired by this naive idea. However, the main new ingredient is that, instead of using the Kähler-Einstein structure of the twistor space, we rather consider its *nearly Kähler structure*, obtained from the former by changing the sign of the complex structure on the vertical distribution, and rescaling the metric by a factor 1/2 in vertical directions. Recall that every nearly Kähler manifold (N, g, J) carries a metric connection $\bar{\nabla}$ with parallel skew-symmetric torsion, called the canonical connection. This connection often is more appropriate for understanding the intrinsic structure of the manifold than the Levi-Civita connection ∇^g, and indeed, it will play a key role in what follows.

Note that a somewhat similar approach was tempted by Chow and Yang in [8]. They introduced a so called quaternionic bisectional curvature and showed that if it is non-negative for a positive quaternion-Kähler manifold, the manifold has to be a Wolf space. In fact, without noticing, they proved that the manifold has to be $\mathbb{H}P^n$. By calling the non-negativity of the quaternionic bisectional curvature "a somewhat weaker assumption" than non-negativity of the sectional curvature, they suggest that the first is implied by the later. Actually, this is not the case, as explained in [19].

Let us now give a quick overview of the organization of the paper. In Section 2 we focus on nearly Kähler manifolds and generalize Gray's result to this setting. In Proposition 2.1 we show that the curvature tensor $\bar{R}$ of the canonical connection is the sum of a curvature tensor of Kähler type and a parallel tensor, which, according to Proposition 2.4, satisfies a Weitzenböck formula similar to the one in the Kähler setting. Assuming the non-negativity of the sectional curvature of $\bar{R}$, it is then possible to show that $\bar{\nabla}\bar{R} = 0$, so the manifold is an Ambrose-Singer homogeneous space (cf. Theorem 2.7).

In Section 3 we study in detail the nearly Kähler structure of the twistor spaces of positive quaternion-Kähler manifolds. Most calculations performed in §3.1 can be found at different places in the literature, but the computations in the references that we could find are not always correct, so we give a self-contained proof of the necessary formulas in Proposition 3.3.

Using these calculations, we reinterpret in §3.2 the non-negativity condition for the sectional curvature of the canonical connection $\bar{\nabla}$ on the nearly Kähler twistor space of a positive quaternion-Kähler manifold, in terms of the Riemannian curvature of the quaternion-Kähler base (cf. Corollary 3.10). It turns out that this is equivalent to the non-negativity of the so-called *quaternionic sectional curvature* of the base.

In Section 4 we show that every Wolf space has non-negative quaternionic sectional curvature (cf. Corollary 4.2) and conversely we obtain our main result (Theorem 4.5) which can be stated as follows: *A compact quaternion-Kähler manifold with non-negative quaternionic sectional curvature is a Wolf space.*

In particular it follows that a compact positive quaternion-Kähler manifold whose sectional curvature κ satisfies for every 2-plane P the inequality

$$\kappa(P) \geq \frac{\mathrm{scal}\cos^2(\theta_P)}{8n(n+2)}$$

(where scal denotes the scalar curvature and θ_P is the angle between P and the quaternionic line passing through a non-zero vector in P), is a Wolf space. Hence, our main result gives further evidence for the following

Conjecture: Any compact quaternion-Kähler manifold with non-negative sectional curvature is a Wolf space.

Recall that by a result of Berger [3], a quaternion-Kähler manifold of (strictly) positive sectional curvature is isometric to $\mathbb{H}P^n$ up to constant rescaling.

Acknowledgments. A.M. was partly supported by the PNRR-III-C9-2023-I8 grant CF 149/31.07.2023 *Conformal Aspects of Geometry and Dynamics*. U.S. was partly supported by the Procope Project No. 57650868 (Germany) / 48959TL (France).

2. Nearly Kähler manifolds of non-negative curvature

A *nearly Kähler manifold* is by definition a Riemannian manifold (N, g) together with an almost complex structure J, orthogonal with respect to g and satisfying the condition $(\nabla^g_X J)X = 0$ for all tangent vectors X. Here ∇^g is the Levi-Civita connection of g. A nearly Kähler manifold is called *strict* if $\nabla^g_X J = 0$ implies that $X = 0$ for all $X \in \mathrm{T}N$. On any nearly Kähler manifold there exists a unique metric connection $\bar{\nabla}$ with skew-symmetric and parallel torsion, preserving the Hermitian structure, i.e. with $\bar{\nabla} J = 0$. It can be written as $\bar{\nabla} = \nabla^g + \tau$ with $\tau := -\frac{1}{2} J \circ \nabla^g J$. Then $\tau(X, Y, Z) = g(\tau_X Y, Z)$ is a $\bar{\nabla}$-parallel 3-form. Note that $\tau_X \in \Lambda^{(2,0)+(0,2)}\,\mathrm{T}N$, i.e. $\tau_X(JY, JZ) = -\tau_X(Y, Z)$ holds for all vectors X, Y, Z. We will call $\bar{\nabla}$ the *canonical connection* of the nearly Kähler manifold (N, g, J).

For the convenience of the reader the following proposition recalls and collects properties of the curvature of the canonical connection on a nearly Kähler manifold. These facts will be important in the rest of our paper.

Proposition 2.1. *Let (N^{2n}, g, J) be a strict nearly Kähler manifold. Then the curvature $\bar{R}$ of the canonical connection $\bar{\nabla}$ is pair symmetric and satisfies the first and second Bianchi identities*

$$\underset{XYZ}{\mathfrak{S}} \big(\bar{R}(X,Y,Z,W) - 4g(\tau_X Y, \tau_Z W)\big) \;=\; 0 \qquad \textit{and} \qquad \underset{XYZ}{\mathfrak{S}} (\bar{\nabla}_X \bar{R})_{Y,Z} \;=\; 0 \;,$$

for all tangent vectors $X, Y, Z, W \in \mathrm{T}$ $(= \mathrm{T}N)$. The Ricci tensor $\overline{\mathrm{Ric}}$ of $\bar{R}$ is symmetric and $\bar{\nabla}$-parallel. Moreover, the curvature $\bar{R}$ takes values in the Lie algebra $\mathfrak{u}(n) \cong \Lambda^{1,1}\,\mathrm{T}$, i.e. $\bar{R} \in \mathrm{Sym}^2\Lambda^{1,1}\,\mathrm{T}$, and $\bar{R}$ can be written as

$$\bar{R} \;=\; R^K + R^0 \;,$$

where R^K is a Kähler curvature tensor and R^0 is a $\bar{\nabla}$-parallel tensor. More precisely, it holds that $R^0_{X,Y} = -(\sigma_{X,Y} + \sigma_{JX,JY})$, where $\sigma := \tau_{e_i} \wedge \tau_{e_i} = \frac{1}{2} d\tau$ (throughout this paper we use the Einstein sum convention).

Proof. The curvature of any metric connection $\bar{\nabla} = \nabla^g + \tau$ with parallel skew-symmetric torsion τ can be decomposed as

$$\bar{R} = R^g - \tau^2 \;, \tag{1}$$

where $\tau^2_{X,Y} := [\tau_X, \tau_Y] - 2\tau_{\tau_X Y}$ (see [9, Eq. (3)]). This formula implies the pair symmetry of $\bar{R}$ and thus also the symmetry of the Ricci tensor $\overline{\mathrm{Ric}}$ of $\bar{R}$. The first and second Bianchi identities follow from [17, Thm. 5.3, Ch. III], (cf. also [9, Cor. 2.3]). The fact that the Riemannian Ricci tensor Ric is $\bar{\nabla}$-parallel was first shown by P.-A. Nagy [20, Cor. 2.1]. By (1), it follows that $\overline{\mathrm{Ric}}$ is $\bar{\nabla}$-parallel as well. Note that in the case of interest for us in this paper, namely when the nearly Kähler structure is the twistor space over a positive quaternion-Kähler manifold, we provide a direct proof for this fact in Proposition 3.12 below.

Since the almost complex structure is $\bar{\nabla}$-parallel, the curvature $\bar{R}$ takes values in $\mathfrak{u}(n)$ and the holonomy of $\bar{\nabla}$ is a subgroup of $\mathrm{U}(n)$. The pair symmetry implies $\bar{R} \in \mathrm{Sym}^2\Lambda^{1,1}\,\mathrm{T}$.

It remains to prove the decomposition of $\bar{R}$. We have $\bar{R} \in \mathrm{Sym}^2(\Lambda^2\,\mathrm{T}) \cong \ker b \oplus \Lambda^4\,\mathrm{T}$, where $\mathrm{b} : \mathrm{Sym}^2(\Lambda^2\,\mathrm{T}) \to \Lambda^4\,\mathrm{T}$ is the Bianchi map, defined by $\mathrm{b}(R) := e_i \wedge e_j \wedge R(e_i \wedge e_j)$. The kernel of b is by definition the space of algebraic Riemannian curvature tensors. Consider the embedding $\imath$ of $\Lambda^4\,\mathrm{T}$ into $\mathrm{Sym}^2(\Lambda^2\,\mathrm{T})$ defined by $\imath(\sigma)(X \wedge Y) := \sigma(X, Y, \cdot, \cdot)$. Since $\mathrm{b} \circ \imath = 12\,\mathrm{Id}_{\Lambda^4 T}$, it follows that $\frac{1}{12}\imath \circ \mathrm{b}$ is the projection onto the subspace of $\mathrm{Sym}^2(\Lambda^2\,\mathrm{T})$ isomorphic to $\Lambda^4\,\mathrm{T}$. An easy computation shows that $\mathrm{b}(\bar{R}) = -\mathrm{b}(\tau^2) = 8\sigma$. Hence, one can write $\bar{R}_{X,Y} =: \tilde{R}_{X,Y} + \frac{1}{12}(\imath \circ \mathrm{b}(\bar{R}))_{X,Y} = \tilde{R}_{X,Y} + \frac{2}{3}\sigma_{X,Y}$, with

$$\sigma_{X,Y} = 2\tau(e_i, X, Y)\tau_{e_i} - 2\tau_{e_i} X \wedge \tau_{e_i} Y =: 2\sigma^-_{X,Y} - 2\sigma^+_{X,Y} \;, \tag{2}$$

where σ^- denotes the first and σ^+ the second summand of σ in the above expression and $\tilde{R}$ is an algebraic Riemannian curvature tensor. Since $\tau_X \in \Lambda^{(2,0)+(0,2)}\,\mathrm{T}$ we see that $\sigma^+_{X,Y} \in \Lambda^{1,1}\,\mathrm{T}$, i.e. $\sigma^+ \in \mathrm{Sym}^2\Lambda^{1,1}\,\mathrm{T}$ and $\sigma^-_{X,Y} \in \Lambda^{(2,0)+(0,2)}\,\mathrm{T}$. In particular, we have $\sigma^+_{X,Y} = \frac{1}{2}(\sigma_{X,Y} + \sigma_{JX,JY})$. Another simple calculation shows $\mathrm{b}(\sigma^-) = 2\sigma$ and $\mathrm{b}(\sigma^+) = -4\sigma$. It follows that $\tilde{\sigma} := 2\sigma^- + \sigma^+$ is in the kernel of the Bianchi map, i.e. $\tilde{\sigma}$ is an algebraic Riemannian curvature tensor, whence $R^K := \tilde{R} + \frac{2}{3}\tilde{\sigma}$ is also an algebraic Riemannian curvature tensor. Moreover, we can write $\sigma = 2\sigma^- - 2\sigma^+ = \tilde{\sigma} - 3\sigma^+$ and

$$\bar{R}_{X,Y} = \tilde{R}_{X,Y} + \frac{2}{3}\sigma_{X,Y} = (\tilde{R}_{X,Y} + \frac{2}{3}\tilde{\sigma}_{X,Y}) - 2\sigma^+_{X,Y} = R^K_{X,Y} - (\sigma_{X,Y} + \sigma_{JX,JY}) \ .$$

Since $\bar{R}$ and σ^+ are in $\mathrm{Sym}^2\Lambda^{1,1}$ the previous calculation shows that R^K is an algebraic Riemannian curvature tensor of Kähler type. From (2) it follows that σ is $\bar{\nabla}$-parallel and thus the same is true for σ^+. Hence R^0 defined by $R^0_{X,Y} := -(\sigma_{X,Y} + \sigma_{JX,JY})$ is a $\bar{\nabla}$-parallel tensor. □

Let $q(\bar{R})$ be the curvature endomorphism of the vector bundle $\Lambda^2\,\mathrm{T} \otimes \Lambda^2\,\mathrm{T}$, defined as

$$q(\bar{R})K \ := \ \frac{1}{2}(e_i \wedge e_j)_* \bar{R}_{e_i,e_j} K \ ,$$

for every section K of $\Lambda^2\,\mathrm{T} \otimes \Lambda^2\,\mathrm{T}$, where $\{e_i\}$ is any local orthonormal basis of T. If $(\omega_\alpha)_{1\le\alpha\le n(n-1)/2}$ is an orthonormal basis of $\Lambda^2\,\mathrm{T}$, then one can write $q(\bar{R})K \ = \ (\omega_\alpha)_* \bar{R}(\omega_\alpha)_* K$.

Remark 2.2. The curvature endomorphism $q(\bar{R})$ is symmetric. This follows from the pair symmetry of $\bar{R}$ which more generally holds for the curvature tensor of all connections with parallel skew-symmetric torsion. In particular, $q(\bar{R})$ is pointwise diagonalizable.

Consider the curvature tensor $\bar{R}$ as an element of $\Omega^2(\Lambda^2\,\mathrm{T})$, i.e. as a 2-form with values in the vector bundle $\Lambda^2\,\mathrm{T}$.

Lemma 2.3. *Let π_{Sym^2} denote the projection from $\Lambda^2\,\mathrm{T} \otimes \Lambda^2\,\mathrm{T}$ to $\mathrm{Sym}^2(\Lambda^2\,\mathrm{T})$. Then*

$$\pi_{\mathrm{Sym}^2}(e_j \wedge e_i \lrcorner \bar{R}_{e_i,e_j}\bar{R}) = \frac{1}{2} q(\bar{R})\bar{R} \ ,$$

where the interior and exterior products on the left hand side are only applied to the first factor of $\Lambda^2\,\mathrm{T} \otimes \Lambda^2\,\mathrm{T}$.

Proof. Let us write $\bar{R} = \bar{R}_{\alpha\beta}\omega_\alpha \otimes \omega_\beta$, with $\bar{R}_{\alpha\beta} = \bar{R}_{\beta\alpha}$ for every $1 \le \alpha, \beta \le n(n-1)/2$. We then get

$$\begin{aligned} q(\bar{R})\bar{R} &= \bar{R}_{\alpha\beta}(\omega_\alpha)_*(\omega_\beta)_*\bar{R} = \bar{R}_{\alpha\beta}\bar{R}_{\gamma\delta}(\omega_\alpha)_*(\omega_\beta)_*(\omega_\gamma \otimes \omega_\delta) \\ &= \bar{R}_{\alpha\beta}\bar{R}_{\gamma\delta}\big((\omega_\alpha)_*(\omega_\beta)_*\omega_\gamma \otimes \omega_\delta + (\omega_\beta)_*\omega_\gamma \otimes (\omega_\alpha)_*\omega_\delta \\ &\qquad + (\omega_\alpha)_*\omega_\gamma \otimes (\omega_\beta)_*\omega_\delta + \omega_\gamma \otimes (\omega_\alpha)_*(\omega_\beta)_*\omega_\delta\big) \\ &= \bar{R}_{\alpha\beta}\bar{R}_{\gamma\delta}\big((\omega_\alpha)_*(\omega_\beta)_*\omega_\gamma \otimes \omega_\delta + 2(\omega_\alpha)_*\omega_\gamma \otimes (\omega_\beta)_*\omega_\delta + \omega_\gamma \otimes (\omega_\alpha)_*(\omega_\beta)_*\omega_\delta\big) \ . \end{aligned}$$

On the other hand, recall that $(e_i \wedge e_j)_*$ acts on $\Lambda^2\,\mathrm{T}$ as $e_j \wedge e_i \lrcorner - e_i \wedge e_j \lrcorner$. Therefore, using that $\bar{R}(\omega_\alpha) = \bar{R}_{\alpha\beta}\omega_\beta$, we can write

$$\begin{aligned} e_j \wedge e_i \lrcorner \bar{R}_{e_i,e_j}\bar{R} &= \frac{1}{2}(e_j \wedge e_i - e_i \wedge e_j \lrcorner)\, \bar{R}_{e_i,e_j}\bar{R} = (\omega_\alpha)_*\bar{R}(\omega_\alpha)\bar{R} \\ &= (\omega_\alpha)_*\big(\bar{R}_{\alpha\beta}\bar{R}_{\gamma\delta}((\omega_\beta)_*\omega_\gamma \otimes \omega_\delta + \omega_\gamma \otimes (\omega_\beta)_*\omega_\delta)\big) \\ &= \bar{R}_{\alpha\beta}\bar{R}_{\gamma\delta}\big((\omega_\alpha)_*(\omega_\beta)_*\omega_\gamma \otimes \omega_\delta + (\omega_\alpha)_*\omega_\gamma \otimes (\omega_\beta)_*\omega_\delta\big) \ , \end{aligned}$$

where we recall that in the first two lines, $(\omega_\alpha)_*$ only acts on the first factor of $\Lambda^2\,\mathrm{T} \otimes \Lambda^2\,\mathrm{T}$.

Thanks to the symmetry of $\bar{R}$, the last summand belongs to $\mathrm{Sym}^2(\Lambda^2\,\mathrm{T})$, whereas the projection of the first factor onto $\mathrm{Sym}^2(\Lambda^2\,\mathrm{T})$ reads

$$\begin{aligned}\pi_{\mathrm{Sym}^2}(\bar{R}_{\alpha\beta}\bar{R}_{\gamma\delta}(\omega_\alpha)_*(\omega_\beta)_*\omega_\gamma\otimes\omega_\delta) &= \frac{1}{2}\bar{R}_{\alpha\beta}\bar{R}_{\gamma\delta}((\omega_\alpha)_*(\omega_\beta)_*\omega_\gamma\otimes\omega_\delta+\omega_\delta\otimes(\omega_\alpha)_*(\omega_\beta)_*\omega_\gamma)\\ &= \frac{1}{2}\bar{R}_{\alpha\beta}\bar{R}_{\gamma\delta}((\omega_\alpha)_*(\omega_\beta)_*\omega_\gamma\otimes\omega_\delta+\omega_\gamma\otimes(\omega_\alpha)_*(\omega_\beta)_*\omega_\delta)\ .\end{aligned}$$

Comparing the above expressions yields the result. □

Proposition 2.4. *For the curvature tensor of the canonical connection of a nearly Kähler manifold (N, g, J) the following relation holds*

$$\pi_{\mathrm{Sym}^2}(\bar{\nabla}^*\bar{\nabla}\bar{R}) \;=\; -\frac{1}{2}q(\bar{R})\bar{R}\ .$$

Proof. Considering again $\bar{R}$ as an element of $\Omega^2(\Lambda^2\,\mathrm{T})$, it is well known that the second Bianchi identity for $\bar{R}$ is equivalent to $d^{\bar{\nabla}}\bar{R}=0$. Since $\bar{\nabla}\,\overline{\mathrm{Ric}}=0$ we also have $\delta^{\bar{\nabla}}\bar{R}=0$. Indeed, writing $\bar{R}$ as $\bar{R}=\frac{1}{2}\,e_i\wedge e_j\otimes\bar{R}_{e_i,e_j}$ we find (choosing the local orthonormal frame to be $\bar{\nabla}$-parallel at the point where the computation is done):

$$\delta^{\bar{\nabla}}\bar{R} \;=\; -e_i\lrcorner\bar{\nabla}_{e_i}\bar{R} \;=\; -\frac{1}{2}\,e_i\lrcorner(e_j\wedge e_k)\otimes(\bar{\nabla}_{e_i}\bar{R})_{e_j,e_k} \;=\; -(\bar{\nabla}_{e_i}\bar{R})_{e_i,e_k}\ .$$

Using the second Bianchi identity for $\bar{R}$ we obtain

$$\begin{aligned} g((\bar{\nabla}_{e_i}\bar{R})_{e_i,e_k}Y,Z) &= g((\bar{\nabla}_{e_i}\bar{R})_{Y,Z}e_i,X)\\ &= g((\bar{\nabla}_Y\bar{R})_{e_i,Z}e_i,X)\;+\;g((\bar{\nabla}_Z\bar{R})_{Y,e_i}e_i,X)\\ &= Y g(\bar{R}_{e_i,Z}e_i,X)\;+\;Z g(\bar{R}_{Y,e_i}e_i,X)\ .\end{aligned}$$

The last sum vanishes because of $\bar{\nabla}\,\overline{\mathrm{Ric}}=0$. We assume again that the local orthonormal frame is $\bar{\nabla}$-parallel at the point where the computation is done, so in particular $[e_i,e_j]=-\tau_{e_i}e_j+\tau_{e_j}e_i=-2\tau_{e_i}e_j$. Keeping in mind that $d^{\bar{\nabla}}\bar{R}=0$ and $\delta^{\bar{\nabla}}\bar{R}=0$, we then obtain:

$$\begin{aligned} 0=\delta^{\bar{\nabla}}d^{\bar{\nabla}}\bar{R} &= -e_i\lrcorner\bar{\nabla}_{e_i}(e_j\wedge\bar{\nabla}_{e_j}\bar{R})=-e_i\lrcorner(e_j\wedge\bar{\nabla}_{e_i}\bar{\nabla}_{e_j}\bar{R})\\ &= -\bar{\nabla}_{e_i}\bar{\nabla}_{e_i}\bar{R}\;+\;e_j\wedge e_i\lrcorner\bar{\nabla}_{e_i}\bar{\nabla}_{e_j}\bar{R}\\ &= \bar{\nabla}^*\bar{\nabla}\bar{R}\;+\;e_j\wedge e_i\lrcorner\left(\bar{R}_{e_i,e_j}\bar{R}+\bar{\nabla}_{e_j}\bar{\nabla}_{e_i}\bar{R}+\bar{\nabla}_{[e_i,e_j]}\bar{R}\right)\\ &= \bar{\nabla}^*\bar{\nabla}\bar{R}\;-\;d^{\bar{\nabla}}\delta^{\bar{\nabla}}\bar{R}\;+\;e_j\wedge e_i\lrcorner\bar{R}_{e_i,e_j}\bar{R}\;-\;2e_j\wedge e_i\lrcorner\bar{\nabla}_{\tau_{e_i}e_j}\bar{R}\ ,\end{aligned}$$

whence

$$\bar{\nabla}^*\bar{\nabla}R \;=\; -e_j\wedge e_i\lrcorner\bar{R}_{e_i,e_j}\bar{R}\;+2e_j\wedge e_i\lrcorner\bar{\nabla}_{\tau_{e_i}e_j}\bar{R}\ . \tag{3}$$

We claim that the last summand in (3) vanishes. Indeed, the second Bianchi identity yields

$$\begin{aligned} 2\, e_j \wedge e_i \lrcorner \bar\nabla_{\tau_{e_i} e_j} \bar R &= 2 e_j \wedge e_i \lrcorner (\bar\nabla_{e_k} \bar R)\, \tau(e_i, e_j, e_k) \\ &= e_j \wedge e_i \lrcorner (e_a \wedge e_b) \otimes (\bar\nabla_{e_k} \bar R)_{e_a, e_b}\, \tau(e_i, e_j, e_k) \\ &= 2\, e_j \wedge e_b \otimes (\bar\nabla_{e_k} \bar R)_{e_a, e_b}\, \tau(e_a, e_j, e_k) \\ &= 2\, e_j \wedge e_b \otimes \big((\bar\nabla_{e_a} \bar R)_{e_k, e_b} + (\bar\nabla_{e_b} \bar R)_{e_a, e_k}\big)\, \tau(e_a, e_j, e_k) \\ &= e_j \wedge e_b \otimes (\bar\nabla_{e_b} \bar R)_{e_a, e_k}\, \tau(e_a, e_j, e_k)\ . \end{aligned}$$

(The last equality follows by exchanging the indices a and k in the third line). We then replace e_a and e_k by Je_a and Je_k and obtain that the last line is equal to its opposite (as $\bar R_{e_a, e_k} = \bar R_{Je_a, Je_k}$ and $\tau(e_a, e_j, e_k) = -\tau(Je_a, e_j, Je_k)$) so it has to vanish. Consequently, (3) becomes

$$\bar\nabla^* \bar\nabla \bar R = -e_j \wedge e_i \lrcorner \bar R_{e_i, e_j} \bar R\ . \tag{4}$$

The proposition thus follows from Lemma 2.3. □

Remark 2.5. It is easy to check, using the symmetry by pairs of $\bar R$, that $\bar\nabla^* \bar\nabla \bar R$ is actually pair symmetric as well. Equation (4) thus shows that the formula in Lemma 2.3 is actually valid without projecting the left hand term on $\mathrm{Sym}^2(\Lambda^2 \mathrm{T})$.

Lemma 2.6. *Let (N, g, J) be a nearly Kähler manifold. If the sectional curvature of the curvature $\bar R$ of the canonical connection $\bar\nabla$ is non-negative, then $q(\bar R) \geq 0$ on all symmetric tensors, i.e. all eigenvalues of $q(\bar R)$ are non-negative.*

Proof. Since the curvature $\bar R$ is pair symmetric the proof of the corresponding statement for the Riemannian curvature carries over to the present situation (see the proof of [14, Prop. 6.6.]). □

Theorem 2.7. *A compact, simply connected nearly Kähler manifold (N, g, J) such that the sectional curvature of $\bar R$ is non-negative, is a naturally reductive homogeneous space.*

Proof. We write $\bar R = R^K + R^0$ as in Proposition 2.1. Since R^0 is $\bar\nabla$-parallel, $q(\bar R) R_0 = 0$, so from Proposition 2.4 and Remark 2.5 we get

$$\bar\nabla^* \bar\nabla R^K = \bar\nabla^* \bar\nabla \bar R = -\frac{1}{2} q(\bar R) \bar R = -\frac{1}{2}\, q(\bar R) R^K\ .$$

Taking the scalar product with R^K yields

$$g(R^K, \bar\nabla^* \bar\nabla R^K) = -\frac{1}{2}\, g(R^K, q(\bar R) R^K)\ ,$$

so after integration we find

$$\|\bar\nabla R^K\|^2_{L^2} = -\frac{1}{2}\, \langle q(\bar R) R^K, R^K \rangle_{L^2}\ .$$

On the other hand, as in the proof of [19, Lem. 2.4] we can replace R^K by its holomorphic sectional curvature S, which is a symmetric 4-tensor. Note that the map $R^K \mapsto S$ is injective (see [16, Lem. 2.2]). Then Lemma 2.6 implies that $g(R^K, q(\bar R) R^K) \geq 0$ at every point, whence by the above integral formula $\bar\nabla R^K = 0$ and finally $\bar\nabla \bar R = 0$. It follows that the canonical connection $\bar\nabla$ of the nearly Kähler manifold (M, g, J) has parallel torsion and parallel curvature. Hence M is homogeneous by the Ambrose-Singer Theorem [2]. □

3. The twistor space of quaternion-Kähler manifolds

3.1. **The nearly Kähler structure of the twistor space.** Let $(M^{4n}, g^M, \mathcal{E})$ be a quaternion-Kähler manifold, where $\mathcal{E} \subset \mathrm{End}(\mathrm{T}M)$ is the rank 3 parallel subbundle of $\mathrm{End}(\mathrm{T}M)$ locally spanned by almost complex structures I, J, K compatible with the metric and satisfying the quaternionic relation $IJ = K$. Such a triple $\{I, J, K\}$ will be called a local quaternionic frame, and will sometimes be written as $\{J_\alpha\}$, $\alpha = 1, 2, 3$. We denote with ∇^M the Levi-Civita connection of g^M, with $R^M_{X,Y} := \nabla^M_X \nabla^M_Y - \nabla^M_Y \nabla^M_X - \nabla^M_{[X,Y]}$ its curvature tensor, and with scal the scalar curvature of R^M. The metric g^M is then automatically Einstein as already mentioned in the introduction.

The *twistor space* of M is defined as $\mathcal{Z}M := \{I \in \mathcal{E} \mid I^2 = -\mathrm{id}\}$. It is the total space of a $\mathbb{C}P^1$-fibration $\mathcal{Z}M \to M$. The connection on $\mathcal{E}$ induced by the Levi-Civita of g^M defines the decomposition $\mathrm{T}\mathcal{Z}M = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of $\mathcal{Z}M$ into the horizontal and the vertical tangent spaces. The corresponding decomposition of tangent vectors $\xi \in \mathrm{T}\mathcal{Z}M$ will be written as $\xi = \xi^H + \xi^V$. At a point $I \in \mathcal{Z}M$, the vertical tangent space, i.e. the tangent space to the fibres, can be written as $\mathcal{V}_I = \{A^* \in \mathcal{E} \mid AI + IA = 0\}$. Here and below we will write A^* for an endomorphism $A \in \mathcal{E}$ considered as tangent vector, under the canonical identification of a vector space with its tangent space at a point.

On $\mathcal{Z}M$ we consider the family of metrics $g_c = \pi^* g^M + g_c^{\mathcal{V}}$ $(c > 0)$, with the vertical part $g_c^{\mathcal{V}}$ defined as the restriction $g_c^{\mathcal{E}}|_{\mathcal{V}}$ of the metric $g_c^{\mathcal{E}}(A^*, B^*) = -c^2 \mathrm{tr}(AB)$. For every choice of c, $\pi : \mathcal{Z}M \to M$ is a Riemannian submersion with totally geodesic fibres. Since $|A^*|^2 = 4nc^2$ for all $A \in \mathcal{Z}M$, the twistor fibres $\mathcal{Z}M_p = \pi^{-1}(p)$, are 2-spheres of radius $c\sqrt{4n}$ in the Euclidean vector space $(\mathcal{E}_p, g_c^{\mathcal{E}})$. In particular the fibres $\mathcal{Z}M_p$ have sectional curvature $\frac{1}{4nc^2}$.

For a tangent vector $X \in \mathrm{T}_pM$ we denote with $\tilde X \in \mathrm{T}_I\mathcal{Z}M$ its horizontal lift at $I \in \mathcal{Z}M_p$. The twistor space $\mathcal{Z}M$ carries two natural almost complex structures J^ε. They are defined on the horizontal part $\mathcal{H}_I$ by $J^\varepsilon(\tilde X)_I = \widetilde{IX}$ and on the vertical part $\mathcal{V}_I$ by $J^\varepsilon(A^*)_I = \varepsilon(IA)^*$, with $\varepsilon = \pm 1$.

Lemma 3.1. *For all vertical vector fields A^* and vector fields X, Y on M we have*

(i) $J^\varepsilon[A^*, \tilde X] = [J^\varepsilon A^*, \tilde X]$;

(ii) $[A^*, J^\varepsilon \tilde X]^{\mathcal{H}} = \widetilde{AX}$;

(iii) $[\tilde X, \tilde Y]_I = \widetilde{[X, Y]} - [R^M_{X,Y}, I]^*$ *in a point* $I \in \mathcal{Z}M$.

Proof. All these formulas can be found in [4]: the argument for *(i)* is in §14.72, the formula *(ii)* is exactly equation (14.72). The horizontal part of *(iii)* is clear. The vertical part follows from (9.53a) and (9.53b). □

Lemma 3.2. *Let $\{I, J, K\}$ be a local quaternionic frame. Then the relation*

$$[R^M_{X,Y}, I] = \frac{\mathrm{scal}}{4n(n+2)}\big(- g^M(KX,Y)J^* + g^M(JX,Y)K^*\big)$$

is true for all $X, Y \in \mathrm{T}_pM$. The formula holds for all even permutations of I, J, K.

Proof. The formula appears in [4, Thm. 14.39]. Note that in [4, Lem. 14.40] the wrong factor 2 has to be deleted. □

It is well known that the twistor space of a positive quaternion-Kähler manifold is Kähler, respectively nearly Kähler for certain choices of the parameter ε and c. For the convenience of the reader and since we need the explicit form of the torsion τ, which we could not found in the literature, we give the following proposition together with its proof.

Proposition 3.3. *The twistor space $(\mathcal{Z}M, g_c, J^\varepsilon)$ is Kähler if and only if $\varepsilon = 1$ and $c^2 = \frac{n+2}{\mathrm{scal}}$. It is nearly Kähler if and only if $\varepsilon = -1$ and $c^2 = \frac{n+2}{2\,\mathrm{scal}}$. The torsion τ of the nearly Kähler structure on $\mathcal{Z}M$ is a section of $\Lambda^2\mathcal{H} \otimes \mathcal{V}$. At each point $I \in \mathcal{Z}M$, the torsion is given by*

$$\tau(A^*, \tilde{X}, \tilde{Y})_I = \frac{1}{4} g^M(AIX, Y)\ ,$$

for every $X, Y \in \mathrm{T}_pM$ and $A \in \mathcal{V}_I$.

Proof. In order to simplify the notation, we will denote throughout the proof the metric g_c by g and its Levi-Civita connection by ∇. We need to compute the covariant derivative ∇J^ε. According to the splitting $\mathrm{T}\mathcal{Z}M = \mathcal{H} \oplus \mathcal{V}$, we consider several cases.

First we note that $(\nabla_{A^*} J^\varepsilon)B^* = \nabla_{A^*} J^\varepsilon B^* - J^\varepsilon \nabla_{A^*} B^* = 0$, since the fibres are totally geodesic and J^ε restricted to the fibres is parallel, being the standard complex structure on $S^2 = \mathbb{C}P$.

Next we compute using the Koszul formula

$$\begin{aligned} 2g((\nabla_{\tilde{X}} J^\varepsilon)A^*, B^*) &= 2g(\nabla_{\tilde{X}} J^\varepsilon)A^*, B^*) + 2g(\nabla_{\tilde{X}} A^*, J^\varepsilon B^*) \\ &= \tilde{X} g(J^\varepsilon A^*, B^*) + g([\tilde{X}, J^\varepsilon A^*], B^*) + g([B^*, \tilde{X}], A^*) \\ &\quad + \tilde{X} g(A^*, J^\varepsilon B^*) + g([\tilde{X}, A^*], J^\varepsilon B^*) + g([J^\varepsilon B^*, \tilde{X}], A^*) \end{aligned}$$

Since J^ε is skew-symmetric and using Lemma 3.1 (i) we see that the summands cancel pairwise and we obtain $g((\nabla_{\tilde{X}} J^\varepsilon)A^*, B^*) = 0$.

As a consequence we also have $g((\nabla_{A^*} J^\varepsilon)\tilde{X}, B^*) = -g((\nabla_{A^*} J^\varepsilon)B^*, \tilde{X}) = 0$.

We next claim that $g((\nabla_{\tilde{X}} J^\varepsilon)\tilde{Y}, \tilde{Z}) = 0$ holds for all vector fields X, Y, Z on M. Indeed, applying again the Koszul formula we find

$$\begin{aligned} 2g((\nabla_{\tilde{X}} J^\varepsilon)\tilde{Y}, \tilde{Z}) &= \tilde{X} g(J^\varepsilon \tilde{Y}, \tilde{Z}) + (J^\varepsilon \tilde{Y}) g(\tilde{Z}, \tilde{X}) - \tilde{Z} g(\tilde{X}, J^\varepsilon \tilde{Y}) \\ &\quad + \ g([\tilde{X}, J^\varepsilon \tilde{Y}], \tilde{Z}) - g([J^\varepsilon \tilde{Y}, \tilde{Z}], \tilde{X}) + g([\tilde{Z}, \tilde{X}], J^\varepsilon \tilde{Y})\ . \end{aligned}$$

we can assume X, Y, Z and $I \in \mathcal{Z}M_p$ to be parallel at p. Then it is clear the the second and the last summand in the sum above vanish at p. Moreover we compute at p:

$$\tilde{X} g(J^\varepsilon \tilde{Y}, \tilde{Z}) = \frac{d}{dt}\Big|_{t=0} g(I_t Y, Z) = 0 \ ,$$

where I_t is the parallel transport of I along the flow of X. Hence the scalar product is constant. It follows that the first and third summands in the sum above vanish as well. If φ_t denotes the flow of $\tilde{X}$, then $\varphi_t(I) = I_t$ and $(J^\varepsilon \tilde{Y})_{I_t} = \widetilde{I_t Y}$. For the remaining two summands we compute $[\tilde{X}, J^\varepsilon \tilde{Y}]^{\mathcal{H}} = [\tilde{X}, \widetilde{AY}]^{\mathcal{H}}$, where A is an endomorphism on M with $A_p = I$ and A is parallel along the integral curve of X through p. Then the commutator vanishes since we have $[X, AY] = 0$ at p.

The parameter ε and c will be fixed by the following calculations. For every $I \in \mathcal{Z}M$ and $A^* \in \mathcal{V}_I$ we have

$$g((\nabla_{\tilde{X}} J^\varepsilon)A^*, \tilde{Y})_I = g(\nabla_{\tilde{X}}(J^\varepsilon A^*), \tilde{Y})_I + g(\nabla_{\tilde{X}} A^*, J^\varepsilon \tilde{Y})_I = -g(\nabla_{\tilde{X}} \tilde{Y}, J^\varepsilon A^*)_I + g(\nabla_{\tilde{X}} A^*, \widetilde{IY})_I \ .$$

We will calculate the two summands separately. If we write $A = aJ + bK$, then $(IA)^* = aK^* - bJ^*$, so using the Koszul formula, Lemma 3.1 (3) and Lemma 3.2, together with $|K^*|^2 = |J^*|^2 = 4nc^2$, we obtain

$$\begin{aligned} -g(\nabla_{\tilde{X}} \tilde{Y}, J^\varepsilon A^*)_I &= \frac{1}{2} g([\tilde{Y}, \tilde{X}], \varepsilon(IA)^*)_I \\ &= \frac{\mathrm{scal}}{8n(n+2)} g\big(-g^M(KX, Y)J^* + g^M(JX, Y)K^*, \varepsilon(IA)^*\big)_I \\ &= \frac{\mathrm{scal}\, \varepsilon\, c^2}{2(n+2)} g^M(AX, Y) \ . \end{aligned}$$

Similarly we compute (by extending I to a local section of $\mathcal{E}$):

$$\begin{aligned} g(\nabla_{\tilde{X}} A^*, \widetilde{IY})_I &= \frac{1}{2} g([\widetilde{IY}, \tilde{X}], A^*)_I \\ &= \frac{\mathrm{scal}}{8n(n+2)} g\big(-g^M(KX, IY)J^* + g^M(JX, IY)K^*, A^*\big)_I \\ &= -\frac{\mathrm{scal}\, c^2}{2(n+2)} g^M(AX, Y) \ . \end{aligned}$$

Combining these two calculations we get: $g((\nabla_{\tilde{X}} J^\varepsilon)A^*, \tilde{Y})_I = \frac{\mathrm{scal}\, c^2}{2(n+2)}(\varepsilon - 1) g^M(AX, Y)$. For later use we note that the formula for the second summand also proves the equation

$$g([\widetilde{IY}, \tilde{X}], A^*)_I = -\frac{\mathrm{scal}\, c^2}{n+2} g^M(AX, Y) \ . \tag{5}$$

Finally we have to compute the expression

$$g((\nabla_{A^*} J^\varepsilon)\tilde{X}, \tilde{Y})_I = g(\nabla_{A^*}(J^\varepsilon \tilde{X}), \tilde{Y})_I + g(\nabla_{A^*} \tilde{X}, \widetilde{IY})_I \ .$$

Since $[\tilde{X}, A^*]^H = 0$, the second summand follows from the calculation in the previous case:

$$g(\nabla_{A^*}\tilde{X}, \widetilde{IY})_I = g(\nabla_{\tilde{X}}A^*, \widetilde{IY})_I = -\frac{\operatorname{scal} c^2}{2(n+2)} g^M(AX, Y) \ .$$

For the first summand we again use the Koszul formula to obtain

$$g(\nabla_{A^*}J^\varepsilon\tilde{X}, \tilde{Y})_I = \frac{1}{2}\big(A^* g(J^\varepsilon\tilde{X}, \tilde{Y})_I + g([A^*, J^\varepsilon\tilde{X}], \tilde{Y})_I - g([J^\varepsilon\tilde{X}, \tilde{Y}], A^*)_I\big) \ . \tag{6}$$

We fix a curve $A_t \in \mathcal{Z}M$ with $A_0 = I$ and $\dot{A}_0 = A$. Then the first summand in (6) is equal to $\frac{1}{2}\frac{d}{dt}|_{t=0} g(\widetilde{A_t X}, \tilde{Y}) = \frac{1}{2}g^M(AX, Y)$. Due to Lemma 3.1, the second summand in (6) is equal to $\frac{1}{2}g(\widetilde{AX}, \tilde{Y}) = \frac{1}{2}g^M(AX, Y)$. At last, by tensoriality in X, Y and by (5), the third summand in (6) can be computed as $-\frac{1}{2}g([\widetilde{IX}, \tilde{Y}], A^*)_I = -\frac{1}{2}\frac{\operatorname{scal} c^2}{n+2} g^M(AX, Y)$. Altogether we obtain

$$g((\nabla_{A^*}J^\varepsilon)\tilde{X}, \tilde{Y})_I = \left(1 - \frac{\operatorname{scal} c^2}{n+2}\right) g^M(AX, Y) \ . \tag{7}$$

Summarizing we see that (g, J^ε) is Kähler, i.e. J^ε parallel, if and only if $\operatorname{scal} c^2 = n+2$ and $\varepsilon = 1$. The twistor space (g, J^ε) is nearly Kähler, i.e. $(\nabla_{\tilde{X}}J^\varepsilon)A^* + (\nabla_{A^*}J^\varepsilon)\tilde{X} = 0$, if and only if $\varepsilon = -1$ and $2\operatorname{scal} c^2 = n+2$.

We still have to compute the torsion 3-form τ of the nearly Kähler structure on $\mathcal{Z}M$. From the calculation of ∇J^ε given above it is clear that $\tau \in \Lambda^2\mathcal{H} \otimes \mathcal{V}$. The explicit form of τ then follows from (7) with $2\operatorname{scal} c^2 = n+2$ and $\varepsilon = -1$. We find

$$\tau(A^*, \tilde{X}, \tilde{Y})_I \;=\; \frac{1}{2}\, g((\nabla_{A^*}J^\varepsilon)\tilde{X}, J^\varepsilon\tilde{Y})_I \;=\; \frac{1}{4}\, g^M(AX, IY) \;=\; \frac{1}{4}\, g^M(AIX, Y) \ .$$

Here we still use that A, representing the tangent vector A^* in $\mathcal{V}_I$, anti-commutes with I. □

Remark 3.4. Our values of ε and c in the Kähler respectively nearly Kähler case confirm the results in [1]. However, the definition of the constant c in [1] has to be modified since it is not consistent with the rest of the calculations.

From now on we will only consider the nearly Kähler structure on the twistor space $\mathcal{Z}M$, i.e. we fix $\varepsilon = -1$ and $c^2 = \frac{n+2}{2\operatorname{scal}}$. For this choice we will write $J = J^\varepsilon$, $g = g_c$, and denote as before by $\bar{\nabla} = \nabla^g + \tau$ the canonical connection of the nearly Kähler manifold $(\mathcal{Z}M, g, J)$.

Lemma 3.5. *For any vector fields X, Y on M and any vertical vector V in $\mathcal{V}$ we have*

(i) $\bar{\nabla}_{\tilde{X}}\tilde{Y} = \widetilde{\nabla^M_X Y}$*;*

(ii) $[\tilde{X}, \tilde{Y}]^{\mathcal{V}} = -2\tau_{\tilde{X}}\tilde{Y}$*;*

(iii) $g(\bar{\nabla}_V\tilde{X}, \tilde{Y}) = 2\tau(V, \tilde{X}, \tilde{Y}) = 2g(\tau_{\tilde{X}}\tilde{Y}, V)$.

In particular, the torsion τ coincides (up to a sign) with the O'Neill tensor A.

Proof. All three statements are consequences of the fact that the connection $\bar\nabla$ preserves the splitting $\mathrm{T}\mathcal{Z}M = \mathcal{H}\oplus\mathcal{V}$ in horizontal and vertical vectors. For (i) we compute

$$\bar\nabla_{\tilde X}\tilde Y \;=\; (\bar\nabla_{\tilde X}\tilde Y)^{\mathcal{H}} \;=\; (\nabla_{\tilde X}\tilde Y + \tau_{\tilde X}\tilde Y)^{\mathcal{H}} \;=\; (\nabla_{\tilde X}\tilde Y)^{\mathcal{H}} \;=\; \widetilde{\nabla^M_X Y}\;,$$

where we also use the property that the torsion vanishes on three horizontal vectors. Similarly we derive for (ii):

$$[\tilde X,\tilde Y]^{\mathcal{V}} \;=\; (\nabla_{\tilde X}\tilde Y - \nabla_{\tilde Y}\tilde X)^{\mathcal{V}} \;=\; (\bar\nabla_{\tilde X}\tilde Y - \tau_{\tilde X}\tilde Y - \bar\nabla_{\tilde Y}\tilde X + \tau_{\tilde Y}\tilde X)^{\mathcal{V}} \;=\; -2\tau_{\tilde X}\tilde Y\;.$$

To prove (iii) it is enough to compute the scalar product with a horizontal lift $\tilde Z$. The statement then follows from

$$g(\bar\nabla_{\tilde X}\tilde Y,\tilde Z) \;=\; g(\nabla_{\tilde X}\tilde Y,\tilde Z) \;=\; g^M(\nabla^M_X Y, Z) \;=\; g(\widetilde{\nabla^M_X Y},\tilde Z)\;.$$

For the first equality we used the fact that the torsion vanishes on three horizontal vectors. □

3.2. **The curvature of the nearly Kähler twistor space.** In this section we will describe the relation between the curvature $\bar R$ of the canonical nearly Kähler connection of the twistor space $\mathcal{Z}M$ and the curvature R^M of the Levi-Civita connection on the quaternion-Kähler manifold M. Since the decomposition $\mathrm{T}\mathcal{Z}M = \mathcal{V}\oplus\mathcal{H}$ is preserved by $\bar\nabla$, it follows that $\bar R$ can be considered as map $\bar R : \Lambda^2\,\mathrm{T}\mathcal{Z}M \to \Lambda^2\mathcal{V}\oplus\Lambda^2\mathcal{H}$. Hence, the curvature $\bar R$ is determined by the following three lemmas.

Lemma 3.6. *Let $\{I,J,K\}$ be a local quaternionic frame defined in the neighborhood of some $p\in M$. Then at $I_p\in\mathcal{Z}M$ we have*

$$\bar R(\tilde X,\tilde Y,\tilde Z,\tilde V) \;=\; R^M(X,Y,Z,V) - \frac{\mathrm{scal}}{8n(n+2)}\big(g^M(JX,Y)g(JV,Z) + g^M(KX,Y)g(KV,Z)\big)$$

for all $X,Y,Z,V\in\mathrm{T}_pM$.

Proof. Using $[\tilde X,\tilde Y]^{\mathcal{H}} = \widetilde{[X,Y]}$ and the third formula of Lemma 3.5 we obtain

$$\bar R(\tilde X,\tilde Y)\tilde Z \;=\; \bar\nabla_{\tilde X}\bar\nabla_{\tilde Y}\tilde Z - \bar\nabla_{\tilde Y}\bar\nabla_{\tilde X}\tilde Z - \bar\nabla_{[\tilde X,\tilde Y]}\tilde Z \;=\; \widetilde{R^M(X,Y)Z} - \bar\nabla_{[\tilde X,\tilde Y]^{\mathcal{V}}}\tilde Z\;.$$

Taking the scalar product with $\tilde V$ in the last summand and applying the first and second equation of Lemma 3.5 gives

$$\begin{aligned}
-g(\bar\nabla_{[\tilde X,\tilde Y]^{\mathcal{V}}}\tilde Z,\tilde V) &= 2g(\bar\nabla_{\tau_{\tilde X}\tilde Y}\tilde Z,\tilde V) \;=\; 4g(\tau_{\tilde Z}\tilde V,\tau_{\tilde X}\tilde Y)\\
&= 4\sum_{i,j=1}^{2}\frac{1}{|A_i^*|^2}\,g(\tau_{\tilde Z}\tilde V,A_i^*)\,g(\tau_{\tilde X}\tilde Y,A_j^*)\\
&= \frac{\mathrm{scal}}{8n(n+2)}\big(g^M(KZ,V)g^M(KX,Y) + g^M(JZ,V)g^M(JX,Y)\big)\;.
\end{aligned}$$

Here $\{A_i^*\}$ is an orthogonal basis of $\mathcal{V}_I$, which we can take to be $\{J^*,K^*\}$, where $\{I,J,K\}$ is a local quaternionic frame with $|J^*|^2 = |K^*|^2 = 4nc^2 = \frac{2n(n+2)}{\mathrm{scal}}$. The last displayed equation follows from the explicit form of the torsion given in Proposition 3.3. □

Consider $V_1, V_2 \in \mathcal{V}_I$ two arbitrary vertical tangent vectors. Then, writing $V_1 = a_1 J^* + b_1 K^*$ and $V_2 = a_2 J^* + b_2 K^*$, the determinant of the family V_1, V_2 with respect to every oriented orthonormal basis of $\mathcal{V}_I$ is given by

$$\det(V_1, V_2) = (a_1 b_2 - a_2 b_1) \frac{2n(n+2)}{\mathrm{scal}} \ . \tag{8}$$

Lemma 3.7. *For vertical tangent vectors $V_1, V_2 \in \mathcal{V}_I$ the following holds*

$$\bar{R}(V_1, V_2, V_2, V_1) = \frac{\mathrm{scal}}{2n(n+2)} \det(V_1, V_2)^2 = \frac{\mathrm{scal}}{2n(n+2)} |V_1 \wedge V_2|^2 \ .$$

Proof. Since $\tau_{V_1} V_2 = 0$ for vertical vectors V_1, V_2 we have $\bar{R}(V_1, V_2, V_2, V_1) = R(V_1, V_2, V_2, V_1)$ (here R denotes the Riemannian curvature tensor of g on $\mathcal{Z}M$). Thus the statement of the lemma follows from the fact that the fibres of $\mathcal{Z}M$ are totally geodesic and have sectional curvature $\frac{1}{4nc^2} = \frac{\mathrm{scal}}{2n(n+2)}$ in the nearly Kähler case. □

Lemma 3.8. *For vertical tangent vectors $V_1, V_2 \in \mathcal{V}_I$ and tangent vectors $X_1, X_2 \in \mathrm{T}_{\pi(I)} M$ we have*

$$\bar{R}(\tilde{X}_1, \tilde{X}_2, V_2, V_1) = -\frac{\mathrm{scal}}{4n(n+2)} g^M(IX_2, X_1) \det(V_1, V_2) \ .$$

Proof. We apply the first Bianchi identity for connections with parallel and skew-symmetric torsion (see [9, Cor. 2.3]). Then two of the curvature terms vanish since $\bar{\nabla}$ preserves the splitting $\mathrm{T}\mathcal{Z}M = \mathcal{H} \oplus \mathcal{V}$ and in the cyclic sum over the torsion terms one summand vanishes as well because the torsion is zero when applied to two vertical vectors. It follows

$$\begin{aligned} \bar{R}(\tilde{X}_1, \tilde{X}_2, V_2, V_1) &= 4\big(g(\tau_{\tilde{X}_2} V_2, \tau_{\tilde{X}_1} V_1) \ + \ g(\tau_{V_2} \tilde{X}_1, \tau_{\tilde{X}_2} V_1)\big) \\ &= 4\big(- g(\tau_{V_1} \tau_{V_2} \tilde{X}_2, \tilde{X}_1) \ + \ g(\tilde{X}_1, \tau_{V_2} \tau_{V_1} \tilde{X}_2)\big) \\ &= -4\, g([\tau_{V_1}, \tau_{V_2}] \tilde{X}_2, \tilde{X}_1) \ . \end{aligned}$$

It remains to compute the commutator $[\tau_{V_1}, \tau_{V_2}]$. From the explicit form of the torsion, given in Proposition 3.3, we see that $\tau_{J^*} = \frac{1}{4}\tilde{K}$ and $\tau_{K^*} = -\frac{1}{4}\tilde{J}$, where $\tilde{J}$ and $\tilde{K}$ denote the natural lifts of J and K to $\mathcal{H}_I$. Hence, writing again $V_1 = a_1 J^* + b_1 K^*$ and $V_2 = a_2 J^* + b_2 K^*$, we obtain

$$[\tau_{V_1}, \tau_{V_2}] = \frac{1}{8} a_1 b_2 \tilde{I} - \frac{1}{8} b_1 a_2 \tilde{I} = \frac{1}{8}(a_1 b_2 - b_1 a_2)\tilde{I} = \frac{1}{8} \det(V_1, V_2) \frac{\mathrm{scal}}{2n(n+2)} \tilde{I} \ .$$

The statement now follows from the fact that $g(\tilde{I}\tilde{X}_2, \tilde{X}_1) = g^M(IX_2, X_1)$. □

Combining the curvature expressions of the preceding three lemmas we obtain a formula for the sectional curvature of $\bar{R}$. Note that any two tangent vectors in $\mathrm{T}_I \mathcal{Z}M$ can be written as $\tilde{X}_1 + V_1, \tilde{X}_2 + V_2$ for $V_1, V_2 \in \mathcal{V}_I$ and $X_1, X_2 \in \mathrm{T}_{\pi(I)} M$.

Proposition 3.9. *Let V_1, V_2 be vertical vectors at some $I \in \mathcal{Z}M$, and let $X_1,\ X_2$ be tangent vectors to M at the point $\pi(I)$. Then*

$$\begin{aligned}&\bar{R}(\tilde{X}_1 + V_1, \tilde{X}_2 + V_2, \tilde{X}_2 + V_2, \tilde{X}_1 + V_1)\\ &\quad = R^M(X_1, X_2, X_2, X_1) - \frac{\mathrm{scal}}{8n(n+2)} \sum_{\alpha=1}^{3} g^M(J_\alpha X_1, X_2)^2\\ &\qquad\qquad + \frac{\mathrm{scal}}{8n(n+2)} \left(g^M(IX_2, X_1) - 2\det(V_1, V_2)\right)^2 .\end{aligned}$$

Proof. Using the fact that $\bar{\nabla}$ preserves the splitting $\mathrm{T}\mathcal{Z}M = \mathcal{H} \oplus \mathcal{V}$, and the pair symmetry of $\bar{R}$ we obtain

$$\begin{aligned}\bar{R}(\tilde{X}_1 + V_1, \tilde{X}_2 + V_2, \tilde{X}_2 + V_2, \tilde{X}_1 + V_1) \ &= \ \bar{R}(\tilde{X}_1, \tilde{X}_2, \tilde{X}_2, \tilde{X}_1) \ + \ \bar{R}(V_1, V_2, V_2, V_1)\\ &\qquad + 2\bar{R}(\tilde{X}_1, \tilde{X}_2, V_2, V_1) \ .\end{aligned}$$

If we denote as before the local quaternionic frame $\{I, J, K\}$ by $\{J_\alpha\}$, $\alpha = 1, 2, 3$, then substituting the curvature expressions of Lemmas 3.6, 3.7 and 3.8 and using the tautological formula

$$g^M(JX, Y)^2 + g^M(KX, Y)^2 = \sum_{\alpha=1}^{3} g^M(J_\alpha X, Y)^2 - g^M(IX, Y)^2$$

yields the result. □

As a direct consequence of this curvature formula we obtain:

Corollary 3.10. *Let $(M, g^M, \mathcal{E})$ be a positive quaternion-Kähler manifold and let $\bar{R}$ be the curvature of the canonical connection $\bar{\nabla}$ of the nearly Kähler structure on the twistor space $\mathcal{Z}M$. Then the sectional curvature of $\bar{R}$ is non-negative if and only if*

$$R^M(X, Y, Y, X) \geq \frac{\mathrm{scal}}{8n(n+2)} \sum_{\alpha=1}^{3} g(J_\alpha X, Y)^2 \tag{9}$$

holds for all vectors $X, Y \in \mathrm{T}M$, where $\{J_\alpha\}$ is a local quaternionic frame and scal *is the scalar curvature of g^M.*

Proof. From Proposition 3.9 it follows that the assumption in (9) implies that the sectional curvature of $\bar{R}$ is non-negative.

Conversely, if the sectional curvature of $\bar{R}$ is non-negative, then for any $I \in \mathcal{Z}M$ and tangent vectors $X, Y \in \mathrm{T}_{\pi(I)}M$, we can choose vertical vectors $V_1, V_2 \in \mathcal{V}_I$ with $\det(V_1, V_2) = \frac{1}{2} g^M(IX_2, X_1)$. For such a choice of V_1, V_2, the last summand in the curvature formula of Proposition 3.9 vanishes and the left hand side is non-negative, thus proving (9). □

The curvature formulas for $\bar{R}$ on the twistor space also have another important application.

Corollary 3.11. *Let M be a compact quaternion-Kähler manifold with positive scalar curvature. Then the nearly Kähler twistor space $\mathcal{Z}M$ satisfies $\bar\nabla\bar R = 0$ if and only if M is a Wolf space.*

Proof. We need to show that $\bar\nabla\bar R = 0$ holds if and only if $\nabla^M R^M = 0$ holds. Notice that if $\bar\nabla\bar R = 0$, then the twistor space is an Ambrose-Singer space, since the torsion is already $\bar\nabla$-parallel. Hence it is homogeneous by the Ambrose-Singer Theorem [2].

First, we remark that the composition of $\bar R : \Lambda^2 \mathrm{T}\mathcal{Z}M \to \Lambda^2\mathcal{V} \oplus \Lambda^2\mathcal{H}$ with the projection onto the first summand is $\bar\nabla$-parallel. This follows from a general result on the curvature of connections with parallel skew-symmetric torsion (see [9, Prop. 3.13]). Thus we only need to consider the covariant derivative of the purely horizontal part.

From the first step in the proof of Lemma 3.6 we obtain the formula

$$\bar R(\tilde X, \tilde Y, \tilde Z, \tilde V) \;=\; R^M(X,Y,Z,V)\circ\pi \;+\; 4g(\tau_{\tilde Z}\tilde V, \tau_{\tilde X}\tilde Y)\;. \tag{10}$$

Taking the derivative into the direction of a further horizontal vector $\tilde U$ and recalling that the torsion τ is $\bar\nabla$-parallel together with Lemma 3.5, (3), immediately gives

$$(\bar\nabla_{\tilde U}\bar R)(\tilde X, \tilde Y, \tilde Z, \tilde V) \;=\; (\nabla_U R^M)(X,Y,Z,V)\circ\pi\;.$$

This already proves one direction of the statement, i.e. that $\bar\nabla\bar R = 0$ implies $\nabla^M R^M = 0$ and thus that the quaternion-Kähler base M is symmetric.

For the other direction we still have to consider the derivative into vertical directions. But then the derivative of the first summand on the right side of (10) vanishes since it is constant along the fibres and the derivative of the second summand vanishes again since the torsion is parallel. □

Finally we note that the explicit curvature formulas above can be used to compute the Ricci curvature of $\bar R$, thus verifying directly that $\overline{\mathrm{Ric}}$ is $\bar\nabla$-parallel. We obtain

Proposition 3.12. *Let X be a horizontal tangent vector of $\mathcal{Z}M$ and let V be a vertical tangent vector. Then the $\bar\nabla$-Ricci curvature $\overline{\mathrm{Ric}}$ is completely described by $\overline{\mathrm{Ric}}(X,V) = 0$ and*

$$\overline{\mathrm{Ric}}(X,X) \;=\; \frac{(n+1)\,\mathrm{scal}}{4n(n+2)}\,|X|_g^2 \qquad \textit{and} \qquad \overline{\mathrm{Ric}}(V,V) \;=\; \frac{\mathrm{scal}}{2n(n+2)}\,|V|_g^2\;.$$

In particular $\overline{\mathrm{Ric}}$ is $\bar\nabla$-parallel.

Proof. For arbitrary tangent vectors $A, B \in \mathrm{T}\mathcal{Z}M$ the $\bar\nabla$-Ricci curvature $\overline{\mathrm{Ric}}$ is defined as $\overline{\mathrm{Ric}}(A,B) = \sum_i \bar R(e_i, A, B, e_i)$, where $\{e_i\}, i = 1, \ldots, 4n+2$, is a local orthonormal frame of $\mathcal{Z}M$, which can be assumed to be adapted to the decomposition $\mathrm{T}\mathcal{Z}M = \mathcal{V}\oplus\mathcal{H}$. Now, let X be a horizontal and V be a vertical tangent vector. Then, since $\bar\nabla$ preserves this splitting, we immediately have $\overline{\mathrm{Ric}}(X,V) = 0$. Recall that the Ricci tensor of a connection with parallel skew-symmetric torsion is symmetric. Hence, it remains to compute $\overline{\mathrm{Ric}}(V,V)$

and $\overline{\mathrm{Ric}}(X,X)$. First, we obtain from Lemma 3.7

$$\overline{\mathrm{Ric}}(V,V) \;=\; \sum_{i=1}^{4n+2} \bar{R}(e_i,V,V,e_i) \;=\; \sum_{i=1}^{2} \frac{\mathrm{scal}}{2n(n+2)} |V_i \wedge V|_g^2 \;=\; \frac{\mathrm{scal}}{2n(n+2)} |V|_g^2 \;,$$

where $\{V_i\}, i = 1,2$, can be taken to be an orthonormal basis of the vertical tangent space $\mathcal{V}_I$. Similarly, but this time using a local orthonormal basis $\{X_i\}, i = 1, \dots, 4n$, of the horizontal tangent space $\mathcal{H}_I$, the curvature formula of Lemma 3.6 leads to

$$\begin{aligned}
\overline{\mathrm{Ric}}(X,X) \;&=\; \sum_{i=1}^{4n+2} \bar{R}(e_i,X,X,e_i) \;=\; \sum_{i=1}^{4n} \bar{R}(X_i,X,X,X_i) \\
&=\; \mathrm{Ric}^M(\pi_*(X),\pi_*(X)) \\
&\quad - \sum_{i=1}^{4n} \frac{\mathrm{scal}}{8n(n+2)} \big(g^M(J\pi_*(X_i),\pi_*(X))^2 + g^M(K\pi_*(X_i),\pi_*(X))^2 \big) \\
&=\; \frac{\mathrm{scal}}{4n} g^M(\pi_*(X),\pi_*(X)) - \frac{\mathrm{scal}}{4n(n+2)} g^M(\pi_*(X),\pi_*(X)) \\
&=\; \frac{(n+1)\,\mathrm{scal}}{4n(n+2)} g^M(\pi_*(X),\pi_*(X)) \;.
\end{aligned}$$

Since $\pi : \mathcal{Z}M \to M$ is a Riemannian submersion we have $g^M(\pi_*(X),\pi_*(X)) = g(X,X)$, thus finishing the proof. $\square$

4. Sectional curvature of quaternion-Kähler manifolds

Let $(M,g,\mathcal{E})$ be a quaternion-Kähler manifold. Then any tangent vector $X \in \mathrm{T}_pM$ spans a well-defined 4-dimensional subspace $L(X) := \mathrm{span}\{X, IX, JX, KX\} \subset \mathrm{T}_pM$, where $\{I,J,K\}$ is a local quaternionic frame. The space $L(X)$ is called the *quaternionic line* spanned by X. We also introduce the 3-dimensional subspace $Q(X) := \mathrm{span}\{IX, JX, KX\} \subset \mathrm{T}_pM$, which is in some sense the imaginary part of $L(X)$. Note that the right hand side of the inequality in (9) vanishes if Y is orthogonal to $Q(X)$. If $Y \in Q(X)$, we will call $\mathrm{span}\{X,Y\}$ a *quaternionic plane*, and if Y is orthogonal to $Q(X)$ (but not collinear with X), then $\mathrm{span}\{X,Y\}$ is called a *totally real plane.*

Let X and Y be non-collinear tangent vectors and let $\mathrm{pr}_{Q(X)}$ denote the orthogonal projection of Y onto $Q(X)$. Then the inequality (9) can be reformulated as

$$\kappa(X,Y) \;\geq\; \frac{\mathrm{scal}}{8n(n+2)} \frac{|X|_g^2\,|\mathrm{pr}_{Q(X)}(Y)|_g^2}{|X \wedge Y|_g^2} \;, \tag{11}$$

where $\kappa(X,Y)$ is the sectional curvature of the plane spanned by X and Y. Since $\mathrm{pr}_{Q(X)}(Y) = Y$ for $Y \in Q(X)$, we see that when (11) holds, the sectional curvature restricted to quaternionic planes is bounded from below by $\frac{\mathrm{scal}}{8n(n+2)}$.

The above considerations motivate the following:

Definition 4.1. Let $(M, g, \mathcal{E})$ be a quaternion-Kähler manifold of dimension $4n \geq 8$. For every $p \in M$ and non-collinear tangent vectors $X, Y \in \mathrm{T}_p M$, the *quaternionic sectional curvature* $\kappa_{\mathcal{H}}(X, Y)$ defined by

$$\kappa_{\mathcal{H}}(X,Y) := \kappa(X,Y) - \frac{\mathrm{scal}}{8n(n+2)} \frac{|X|_g^2 \, |\mathrm{pr}_{Q(X)}(Y)|_g^2}{|X \wedge Y|_g^2} \, , \tag{12}$$

where $Q(X)$ is the imaginary part of the quaternionic line generated by X and $\kappa(X,Y)$ is the sectional curvature of the plane spanned by X, Y.

Note that the quaternionic sectional curvature $\kappa_{\mathcal{H}}$ coincides with the Riemannian sectional curvature κ on totally real planes, whereas on quaternionic planes, $\kappa_{\mathcal{H}}$ is equal to $\kappa - \frac{s}{8n(n+2)}$. More generally, for every plane P and every basis X, Y of P, the quantity $\frac{|X|_g^2 \, |\mathrm{pr}_{Q(X)}(Y)|_g^2}{|X \wedge Y|_g^2}$ is equal to $\cos^2(\theta_P)$, where θ_P is the angle between P and the quaternionic line spanned by any of its non-zero vectors, whence

$$\kappa_{\mathcal{H}}(P) = \kappa(P) - \frac{\mathrm{scal}}{8n(n+2)} \cos^2(\theta_P) \, , \tag{13}$$

for every 2-plane P.

As a consequence of Theorem 3.10, we have the following:

Corollary 4.2. *The quaternionic sectional curvature of compact Wolf spaces is non-negative. In particular, the sectional curvature on quaternionic planes is bounded from below by* $\frac{\mathrm{scal}}{8n(n+2)}$.

Proof. The twistor space $\mathcal{Z}M$ of a compact Wolf space M is a simply connected homogeneous strict nearly Kähler manifold and thus it is a compact naturally reductive 3-symmetric space $\mathcal{Z}M = G/K$ equipped with its canonical complex structure and the adapted reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ (see [6], [21]). It is known hat the canonical connection of the canonical almost Hermitian structure of a Riemannian 3-symmetric space coincides with its canonical homogeneous connection (see [11, Prop. 4.1]). Moreover, the metric g on $\mathcal{Z}M$ is then defined by the restriction to $\mathfrak{m}$ of a bi-invariant product on $\mathfrak{g}$ (see [12, p. 360] or [11, Sec. 6]). Hence we can compute the sectional curvature of the connection $\bar{\nabla}$ on the twistor space of a Wolf space by using [18, Thm. 2.6, Ch. X]. In the base point o and for tangent vectors $X, Y \in \mathfrak{m} \cong \mathrm{T}_o G/K$ we find

$$g(\bar{R}(X,Y)Y,X) = -g([[X,Y]_{\mathfrak{k}}, Y], X) = g([X,Y]_{\mathfrak{k}}, [X,Y]_{\mathfrak{k}}) \geq 0 \, .$$

It follows that the assumption in Theorem 3.10 is satisfied and we can conclude estimate (9), which can be reformulated as (11). This finishes the proof. □

Remark 4.3. Q.-S. Chi proved in [7, Thm. 1] that the maximum of the sectional curvature on a positive quaternion-Kähler manifold is obtained on quaternionic planes. Moreover, S. Helgason proved in [15, Thm. 1.1] that on each compact irreducible Riemannian symmetric space the maximum of the sectional curvature is given by the length of the highest restricted root. This root coincides with the so-called Wolf root on all Wolf spaces except the quaternion projective space. The length of the Wolf root was computed in [25, (4.1)]. It follows that the maximum of the sectional curvature on compact Wolf spaces different from $\mathbb{H}P^n$ is $\frac{\mathrm{scal}}{2n(n+2)}$. The length of all highest restricted roots can also be found in [22, Table 1, p. 175].

Remark 4.4. The quaternionic projective space $\mathbb{H}P^n$ is a rank one symmetric space. In particular it has positive sectional curvature. All other compact Wolf spaces are symmetric spaces of higher rank. Hence the sectional curvature is non-negative and has zero as minimal value. The minimum of the sectional curvature on $\mathbb{H}P^n$ is $\frac{\mathrm{scal}}{16n(n+2)}$. The maximum is $\frac{\mathrm{scal}}{4n(n+2)}$, which is the same value for all quaternionic planes. Indeed, the sectional curvature of $\mathbb{H}P^n$ on a plane spanned by an orthonormal basis $\{X, Y\}$ is given by

$$\kappa(X,Y) \;=\; \frac{\mathrm{scal}}{16n(n+2)}\Big(1+3\sum_{\alpha=1}^{3} g^{\mathbb{H}P^n}(J_\alpha X, Y)^2\Big)\;.$$

Thus the minimal value of the sectional curvature $\kappa(X, Y)$ is attained on totally real planes, whereas the maximal value is realized on quaternionic planes.

Finally we can state the main result of this article, which can also be seen as a converse to Corollary 4.2.

Theorem 4.5. *Let $(M^{4n}, g, \mathcal{E})$ be a compact positive quaternion-Kähler manifold. If the quaternionic sectional curvature $\kappa_{\mathcal{H}}$ is non-negative then M is symmetric, i.e. M is isometric to a Wolf space.*

Proof. From Corollary 3.10 it follows that $\kappa_{\mathcal{H}} \geq 0$ implies that the curvature of the canonical connection on the nearly Kähler twistor space is non-negative. Hence by Theorem 2.7 the twistor space $\mathcal{Z}M$ is homogeneous and it follows from Corollary 3.11 that the quaternion-Kähler manifold M is symmetric, i.e. is a Wolf space. □

References

[1] B. Alexandrov, G. Grantcharov, S. Ivanov: *Curvature properties of twistor spaces of quaternionic Kähler manifolds*, J. Geom. **62** (1998), no. 1-2, 1–12.
[2] W. Ambrose, I.M. Singer: *On homogeneous Riemannian manifolds*, Duke Math. J. **25** (1958), 647–669.
[3] M. Berger: *Trois remarques sur les variétés riemanniennes à courbure positive*, C. R. Acad. Sci. Paris Sér. A–B 263 (1966), A76–A78.
[4] A. L. Besse: *Einstein manifolds*, Classics in Mathematics, Springer, 1987.
[5] J. Buczynski, J.A. Wiśniewski: *Algebraic torus actions on contact manifolds. With an appendix by A. Weber*, J. Differ. Geom. **121** (2022), no. 2, 227–289.
[6] J.-B. Butruille: *Classification des variétés approximativement kähleriennes homogénes*, Ann. Global Anal. Geom. **27** (2005), no. 3, 201–225.
[7] Q.-S. Chi: *Quaternionic Kaehler manifolds and a curvature characterization of two-point homogeneous spaces*, Illinois J. Math. **35** (1991), no. 3, 408–418.
[8] B. Chow, D. Yang: *Rigidity of nonnegatively curved compact quaternionic-Kähler manifolds*, J. Differ. Geom. **29** (1991), no. 2, 361–372.
[9] R. Cleyton, A. Moroianu, U. Semmelmann: *Metric connections with parallel skew-symmetric torsion*, Adv. Math. **378** (2021), Paper No. 107519, 50 pp.

[10] Th. Friedrich, S. Ivanov: *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. **6** (2002), no. 2, 303–335.
[11] J.C. González Dávila, F. Martin Cabrera: *Homogeneous nearly Kähler manifolds*, Ann. Global Anal. Geom. **42** (2012), no. 2, 147–170.
[12] A. Gray: *Riemannian manifolds with geodesic symmetries of order* 3, J. Differential Geometry **7** (1972), 343–369.
[13] A. Gray: *Compact Kähler manifolds with nonnegative sectional curvature*, Inventiones Mathematicae **41** (1977), no. 1, 33–43.
[14] K. Heil, A. Moroianu, U. Semmelmann: *Killing and conformal Killing tensors*, J. Geom. Phys. **106** (2016), 383–400.
[15] S. Helgason: *Totally geodesic spheres in compact symmetric spaces*, Math. Ann. **165** (1966), 309–317.
[16] T. Jentsch, G. Weingart: *Riemannian and Kählerian normal coordinates*, Asian J. Math. **24** (2020), no. 3, 369–416.
[17] S. Kobayashi, K. Nomizu: *Foundations of differential geometry, Vol I*, John Wiley & Sons, Inc., New York-London, 1963.
[18] S. Kobayashi, K. Nomizu: *Foundations of differential geometry, Vol II*, John Wiley & Sons, Inc., New York-London, 1969.
[19] O. Macia, U. Semmelmann, G. Weingart: *On quaternionic bisectional curvature*, Math. Ann., https://doi.org/10.1007/s00208-024-03028-y
[20] P.-A. Nagy: *On nearly-Kähler geometry*, Ann. Global Anal. Geom. **22** (2002), no. 2, 167–178.
[21] P.-A. Nagy: *Nearly Kähler geometry and Riemannian foliations*, Asian J. Math. **6** (2002), no. 3, 481–504.
[22] Y. Ohnita: *On stability of minimal submanifolds in compact symmetric spaces*, Compositio Math. **64** (1987), no. 2, 157–189.
[23] Y.S. Poon, S.M. Salamon: *Quaternionic Kähler 8-manifolds with positive scalar curvature*, J. Differ. Geom. **33** (1991), no. 2, 363–378.
[24] S.M. Salamon: *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), no. 1, 143–171.
[25] U. Semmelmann, G. Weingart: *An upper bound for a Hilbert polynomial on quaternionic Kähler manifolds*, J. Geom. Anal. **14** (2004), no. 1, 151–170.
[26] J. A. Wolf: *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, Journal of Mathematical Mechanics **14** (1965), 1033–1047.

Andrei Moroianu, Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, 91405 Orsay, France, and Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 21 Calea Grivitei, 010702 Bucharest, Romania

Email address: andrei.moroianu@math.cnrs.fr

Uwe Semmelmann, Institut für Geometrie und Topologie, Fachbereich Mathematik, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

Email address: uwe.semmelmann@mathematik.uni-stuttgart.de

Gregor Weingart, Instituto de Matemáticas, Universidad Nacional Autónoma de México, Avenida Universidad s/n, Colonia Lomas de Chamilpa, 62210 Cuernavaca, Morelos, Mexico

Email address: gw@matcuer.unam.mx