

CONFORMALLY RELATED KÄHLER METRICS AND THE HOLONOMY OF LCK MANIFOLDS

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ABSTRACT. A locally conformally Kähler (lcK) manifold is a complex manifold (M, J) together with a Hermitian metric g which is conformal to a Kähler metric in the neighbourhood of each point. In this paper we obtain three classification results in locally conformally Kähler geometry. The first one is the classification of conformal classes on compact manifolds containing two non-homothetic Kähler metrics. The second one is the classification of compact Einstein locally conformally Kähler manifolds. The third result is the classification of the possible (restricted) Riemannian holonomy groups of compact locally conformally Kähler manifolds. We show that every locally (but not globally) conformally Kähler compact manifold of dimension $2n$ has holonomy $\mathrm{SO}(2n)$, unless it is Vaisman, in which case it has restricted holonomy $\mathrm{SO}(2n - 1)$. We also show that the restricted holonomy of a proper globally conformally Kähler compact manifold of dimension $2n$ is either $\mathrm{SO}(2n)$, or $\mathrm{SO}(2n - 1)$, or $\mathrm{U}(n)$, and we give the complete description of the possible solutions in the last two cases.

1. INTRODUCTION

It is well-known that on a compact complex manifold, any conformal class admits at most one Kähler metric compatible with the complex structure, up to a positive constant. The situation might change if the complex structure is not fixed. One may thus naturally ask the following question: are there any compact manifolds which admit two non-homothetic metrics in the same conformal class, which are both Kähler (then necessarily with respect to non-conjugate complex structures)? One of the aims of the present paper is to answer this question by describing all such manifolds. This problem can be interpreted in terms of conformally Kähler metrics in real dimension $2n$ with Riemannian holonomy contained in the unitary group $\mathrm{U}(n)$. More generally, we want to classify *locally* conformally Kähler metrics on compact manifolds which are Einstein or have non-generic holonomy.

Recall that a Hermitian manifold (M, g, J) of complex dimension $n \geq 2$ is called locally conformally Kähler (lcK) if around every point in M the metric g can be conformally rescaled to a Kähler metric. If $\Omega := g(J\cdot, \cdot)$ denotes the fundamental 2-form, the above condition is equivalent to the existence of a closed 1-form θ , called the Lee form (which is up to a constant

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equal to the logarithmic differential of the local conformal factors), such that

$$d\Omega = 2\theta \wedge \Omega.$$

If the Lee form θ vanishes, the structure (g, J) is simply Kähler. If the Lee form does not vanish identically, the lcK structure is called *proper*. When θ is exact, there exists a Kähler metric in the conformal class of g , and the manifold is called *globally conformally Kähler* (gcK). If θ is not exact, then (M, g, J) is called *strictly lcK*. A particular class of proper lcK manifolds is the class of Vaisman manifolds, whose Lee form is parallel with respect to the Levi-Civita connexion of the metric. A Vaisman manifold is always strictly lcK since the Lee form, being harmonic, cannot be exact.

In this paper we study three apparently independent – but actually interrelated – classification problems:

P1. The classification of compact conformal manifolds (M^{2n}, c) whose conformal class c contains two non-homothetic Kähler metrics.

P2. The classification of compact proper lcK manifolds (M^{2n}, g, J, θ) with g Einstein.

P3. The classification of compact proper lcK manifolds (M^{2n}, g, J, θ) with reduced (*i.e.* non-generic) holonomy: $\text{Hol}(M, g) \subsetneq \text{SO}(2n)$.

It turns out that P1 and P2 are important steps (but also interesting for their own sake) towards the solution of P3.

We are able to solve each of these problems completely. Their solutions are provided by Theorem 1.1, Theorem 1.2 and Theorem 1.3 below. We now explain briefly these results and describe the methods used to prove them.

The solution of Problem P1 is given by the following:

Theorem 1.1. *Assume that a conformal class on a compact manifold M of real dimension $2n \geq 4$ contains two non-homothetic Kähler metrics g_+ and g_- , that is, there exist complex structures J_+ and J_- and a non-constant function φ such that (g_+, J_+) and $(g_- := e^{-2\varphi}g_+, J_-)$ are Kähler structures. Then J_+ and J_- commute, so that M is ambikähler for $n = 2$. Moreover, for $n \geq 3$, there exists a compact Kähler manifold (N, h, J_N) , a positive real number b , and a function $\ell : (0, b) \rightarrow \mathbb{R}^{>0}$ such that (M, g_+, J_+) and (M, g_-, J_-) are obtained from the construction described in Proposition 4.1.*

The proof, whose details are given in Sections 3 and 4, relies on a commutation result for complex structures compatible with two conformally related non-homothetic Kähler metrics. More precisely, if (g_+, J_+) and $(g_- := e^{-2\varphi}g_+, J_-)$ are Kähler structures on a compact manifold M of real dimension $2n \geq 4$ and φ is non-constant, then (g_+, J_-) is a proper gcK structure on M , whose metric is Kähler with respect to J_+ . In Theorem 3.1 below we show, more generally, that if a Riemannian metric g_+ is lcK with respect to some complex structure J_- and Kähler with respect to another complex structure J_+ , then the initial lcK structure (g_-, J_+) is in fact gcK and *the two complex structures commute*. The argument is based on the fact that for $n \geq 3$ a certain non-negative function, depending on J_+ , J_- , and on the

Lee form of (g_+, J_-) , is the co-differential of a 1-form on M , and thus vanishes by Stokes' theorem.

In dimension $2n = 4$ the commutation of J_+ and J_- was already known. Indeed, they define different orientations since otherwise the associated Kähler metrics would be homothetic, see *e.g.* [24, Proposition 3.2] or Case 1 in the proof of Theorem 3.1 for a short argument. In particular, Problem P1 reduces in dimension 4 to the study of ambikähler structures, according to the terminology introduced by V. Apostolov, D. Calderbank and P. Gauduchon in [2], where the local classification in the toric case is obtained. In full generality, the local classification of ambikähler structures was announced recently by R. Bryant [7].

Once the commutation of J_+ and J_- is established, it turns out that for $n \geq 3$, $J_+ + J_-$ defines a Hamiltonian 2-form of rank 1 (in the sense of [1]) with respect to both Kähler metrics g_+ and g_- . One can then either use the classification of compact manifolds with Hamiltonian forms obtained in [1] (which however is rather involved) or show directly, by a geometric argument given in Proposition 4.2, that the solutions are obtained on the total spaces of some S^2 -bundles over compact Hodge manifolds, by an Ansatz which is reminiscent of Calabi's construction [9], described in Proposition 4.1 below.

As a striking consequence of Theorem 1.1, we obtain that for $n \geq 3$, if (M^{2n}, g, J) is a non-ruled compact Kähler manifold, then every conformal diffeomorphism of $(M, [g])$ is an isometry of g (Corollary 4.4).

Problem P2 has been already solved in complex dimension two, as well as when the scalar curvature is positive. Indeed, C. LeBrun [18] showed, more generally, that if a compact complex surface (M, g) admits an Einstein metric compatible with the complex structure, then (M, g) is either Kähler-Einstein, or homothetic to $\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$ with the Page metric, or to $\mathbb{C}\mathbb{P}^2 \# 2\mathbb{C}\mathbb{P}^2$ with the metric constructed in [10]. Moreover, in the case of positive scalar curvature, Problem P2 reduces, by Myers' theorem, to the study of Einstein gcK metrics, rather than lcK. By changing the point of view, this can be interpreted as the classification of compact conformally-Einstein Kähler manifolds, which has been obtained by A. Derdzinski and G. Maschler, [12].

Our main contribution towards the solution of Problem P2 is Theorem 5.2 below, where we show, using Weitzenböck-type arguments, that every compact Einstein lcK manifold with non-positive scalar curvature has vanishing Lee form.

Altogether, this completes the solution of Problem P2:

Theorem 1.2. *If (g, J, θ) is an Einstein proper lcK structure on a compact manifold M^{2n} , then the Lee form is exact ($\theta = d\varphi$), and the scalar curvature of g is positive. For $n = 2$, (M, g) is homothetic to either $\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$ with the Page metric, or $\mathbb{C}\mathbb{P}^2 \# 2\mathbb{C}\mathbb{P}^2$ with the metric constructed in [10]. For $n \geq 3$, the Kähler manifold $(M, e^{-2\varphi}g, J)$ is one of the conformally-Einstein Kähler manifolds constructed by Bérard-Bergery in [5].*

We now discuss the holonomy problem for compact proper lcK manifolds, that is, Problem P3, whose original motivation stems from [21]. By the Berger-Simons holonomy theorem, an lcK manifold (M^{2n}, g, J) either has reducible restricted holonomy representation, or is locally

symmetric irreducible, or its restricted holonomy group $\text{Hol}_0(M, g)$ is one of the following: $\text{SO}(2n)$, $\text{U}(n)$, $\text{SU}(n)$, $\text{Sp}(\frac{n}{2})$, $\text{Sp}(\frac{n}{2})\text{Sp}(1)$, $\text{Spin}(7)$.

In the reducible case, our crucial result is Theorem 6.2, where we show that a compact proper lcK manifold (M^{2n}, g, J) cannot carry a parallel distribution whose rank d satisfies $2 \leq d \leq 2n - 2$. On the other hand, the cases $d = 1$ and $d = 2n - 1$ were recently classified for $n \geq 3$ by the second named author in [21]. In Theorem 6.6 below we give an alternate proof of this classification, which is not only simpler, but also covers the missing case $n = 2$.

The remaining possible cases given by the Berger-Simons theorem are either Einstein or Kähler (and gcK by Theorem 3.1), and thus fall into the previous classification results. Summarizing, we have the following classification result for the possible (restricted) holonomy groups of compact proper lcK manifolds:

Theorem 1.3. *Let (M^{2n}, g, J, θ) , $n \geq 2$, be a compact proper lcK manifold with non-generic holonomy group $\text{Hol}(M, g) \subsetneq \text{SO}(2n)$. Then the following exclusive possibilities occur:*

1. (M, g, J, θ) is strictly lcK, $\text{Hol}(M, g) \simeq \text{SO}(2n - 1)$ and (M, g, J, θ) is Vaisman (that is, θ is parallel).
2. (M, g, J, θ) is gcK (that is, θ is exact) and either:
 - a) $n \geq 3$, $\text{Hol}_0(M, g) \simeq \text{U}(n)$, and a finite covering of (M, g, J, θ) is obtained by the Calabi Ansatz described in Proposition 4.1, or
 - b) $n = 2$, $\text{Hol}_0(M, g) \simeq \text{U}(2)$ and M is ambikähler, or
 - c) $\text{Hol}_0(M, g) \simeq \text{SO}(2n - 1)$ and a finite covering of (M, g, J, θ) is obtained by the construction described in Theorem 6.6.

2. PRELIMINARIES ON LCK MANIFOLDS

A locally conformally Kähler (lcK) manifold is a connected Hermitian manifold (M, g, J) of real dimension $2n \geq 4$ such that around each point, g is conformal to a metric which is Kähler with respect to J . The covariant derivative of J with respect to the Levi-Civita connection ∇ of g is determined by a closed 1-form θ (called the Lee form) via the formula (see e.g. [21]):

$$(1) \quad \nabla_X J = X \wedge J\theta + JX \wedge \theta, \quad \forall X \in \text{TM}.$$

Recall that if τ is any 1-form on M , $J\tau$ is the 1-form defined by $(J\tau)(X) := -\tau(JX)$ for every $X \in \text{TM}$, and $X \wedge \tau$ denotes the endomorphism of TM defined by $(X \wedge \tau)(Y) := g(X, Y)\tau^\sharp - \tau(Y)X$. We will often identify 1-forms and vector fields via the metric g , which will also be denoted by $\langle \cdot, \cdot \rangle$ when there is no ambiguity.

Let $\Omega := g(J\cdot, \cdot)$ denote the associated 2-form of J . By (1), its exterior derivative and co-differential are given by

$$(2) \quad d\Omega = 2\theta \wedge \Omega,$$

and

$$(3) \quad \delta\Omega = (2 - 2n)J\theta.$$

If $\theta \equiv 0$, the structure (g, J) is simply Kähler. If θ is not identically zero, then the lcK structure (g, J, θ) is called *proper*. If $\theta = d\varphi$ is exact, then $d(e^{-2\varphi}\Omega) = 0$, so the conformally modified structure $(e^{-2\varphi}g, J)$ is Kähler, and the structure (g, J, θ) is called *globally conformally Kähler* (gcK). The lcK structure is called *strictly* lcK if the Lee form θ is not exact and *Vaisman* if θ is parallel with respect to the Levi-Civita connexion of g .

A typical example of strictly lcK manifold, which is actually Vaisman, is $\mathbb{S}^1 \times \mathbb{S}^{2n-1}$, endowed with the complex structure induced by the diffeomorphism

$$(\mathbb{C}^n \setminus \{0\})/\mathbb{Z} \longrightarrow \mathbb{S}^1 \times \mathbb{S}^{2n-1}, \quad [z] \longmapsto \left(e^{2\pi i \ln |z|}, \frac{z}{|z|} \right),$$

where $[z] := \{e^k z \in \mathbb{C}^n \setminus \{0\} \mid k \in \mathbb{Z}\}$. The Lee form of this lcK structure is the length element of \mathbb{S}^1 , which is parallel.

Remark 2.1. For each lcK manifold (M, g, J, θ) there exists a group homomorphism from $\pi_1(M)$ to $(\mathbb{R}, +)$ which is trivial if and only if the structure is gcK. Indeed, $\pi_1(M)$ acts on the universal covering \widetilde{M} of M , and preserves the induced lcK structure $(\tilde{g}, \tilde{J}, \tilde{\theta})$. Since $\tilde{\theta} = d\varphi$ is exact on \widetilde{M} , for every $\gamma \in \pi_1(M)$ we have $d(\gamma^*\varphi) = \gamma^*(d\varphi) = \gamma^*(\tilde{\theta}) = \tilde{\theta} = d\varphi$, so there exists some real number c_γ such that $\gamma^*\varphi = \varphi + c_\gamma$. The map $\gamma \mapsto c_\gamma$ is clearly a group morphism from $\pi_1(M)$ to $(\mathbb{R}, +)$, which is trivial if and only if θ is exact on M . This shows, in particular, that if $\pi_1(M)$ is finite, then every lcK structure on M is gcK.

For later use, we express, for every lcK structure (g, J, θ) , the action on the Hermitian structure J of the Riemannian curvature tensor of g , defined by

$$R_{X,Y} := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Lemma 2.2. *The following formula holds for every vector fields X, Y on a lcK manifold (M, g, J, θ) :*

$$(4) \quad R_{X,Y}J = \theta(X)Y \wedge J\theta - \theta(Y)X \wedge J\theta - \theta(Y)JX \wedge \theta + \theta(X)JY \wedge \theta \\ - |\theta|^2 Y \wedge JX + |\theta|^2 X \wedge JY + Y \wedge J\nabla_X \theta + JY \wedge \nabla_X \theta - X \wedge J\nabla_Y \theta - JX \wedge \nabla_Y \theta.$$

Proof. Taking X, Y parallel at the point where the computation is done and applying (1), we obtain:

$$R_{X,Y}J = \nabla_X(Y \wedge J\theta + JY \wedge \theta) - \nabla_Y(X \wedge J\theta + JX \wedge \theta) \\ = Y \wedge (\nabla_X J)(\theta) + (\nabla_X J)(Y) \wedge \theta - X \wedge (\nabla_Y J)(\theta) - (\nabla_Y J)(X) \wedge \theta \\ + Y \wedge J\nabla_X \theta + JY \wedge \nabla_X \theta - X \wedge J\nabla_Y \theta - JX \wedge \nabla_Y \theta,$$

which gives (4) after a straightforward calculation using (1) again. \square

Let $\{e_i\}_{i=1, \dots, 2n}$ be a local orthonormal basis of TM . Substituting $Y = e_j$ in (4), taking the interior product with e_j and summing over $j = 1, \dots, 2n$ yields:

$$(5) \quad \sum_{j=1}^{2n} (R_{X, e_j} J)(e_j) = (2n-3) (\theta(X)J\theta - |\theta|^2 JX + J\nabla_X \theta) - \theta(JX)\theta - \nabla_{JX} \theta - JX \delta \theta,$$

since the sum $\sum_{j=1}^{2n} g(J\nabla_{e_j}\theta, e_j)$ vanishes, as $\nabla\theta$ is symmetric.

Corollary 2.3. *If the metric g of a compact lcK manifold (M, g, J, θ) is flat, then $\theta \equiv 0$.*

Proof. If the Riemannian curvature of g vanishes, (5) yields

$$0 = (2n - 3) (\theta(X)J\theta - |\theta|^2 JX + J\nabla_X\theta) - \theta(JX)\theta - \nabla_{JX}\theta - JX\delta\theta.$$

We make the scalar product with JX in this equation for $X = e_j$, where $\{e_j\}_{j=1, \dots, 2n}$ is a local orthonormal basis of TM , and sum over $j = 1, \dots, 2n$ to obtain:

$$\begin{aligned} 0 &= (2n - 3) (|\theta|^2 - 2n|\theta|^2 - \delta\theta) - |\theta|^2 + \delta\theta - 2n\delta\theta \\ &= -(2n - 2)^2|\theta|^2 - 2(2n - 2)\delta\theta. \end{aligned}$$

Since $n \geq 2$, this last equation yields $\delta\theta = (1 - n)|\theta|^2$, which by Stokes' Theorem after integration over M gives $\theta \equiv 0$. \square

The following example shows that the corollary does not hold without the compactness assumption.

Example 2.4. Consider the standard flat Kähler structure (g_0, J_0) on $M := \mathbb{C}^n \setminus \{0\}$. If r denotes the map $x \mapsto r(x) := |x|$, the conformal metric $g := r^{-4}g_0$ on M is gcK with respect to J_0 , with Lee form $\theta = -2d \ln r$. Moreover g is flat, being the pull-back of g_0 through the inversion $x \mapsto x/r^2$.

3. KÄHLER STRUCTURES ON LCK MANIFOLDS

The results of this section constitute an intermediary step towards the solutions of Problem P1 (conformally related Kähler metrics) and P3 (lcK metrics with reduced holonomy). More precisely, we study compact complex manifolds (M, J) admitting a proper lcK metric g which is Kähler with respect to another complex structure I . The striking result here is that I and J must commute. We will see that there are examples of such structures, but they are necessarily gcK. In particular, the Riemannian metric g of a compact strictly lcK manifold (M, g, J, θ) cannot be Kähler with respect to any complex structure on M .

Theorem 3.1. *Let (M, g, J, θ) be a compact proper lcK manifold of complex dimension $n \geq 2$ carrying a complex structure I , such that (M, g, I) is a Kähler manifold. Then I commutes with J and (M, g, J, θ) is globally conformally Kähler.*

Proof. The Riemannian curvature tensor of (M, g) satisfies $R_{X,Y} = R_{IX,IY}$, so in particular we have $R_{X,Y}J = R_{IX,IY}J$, for all vector fields X and Y . Using (4), this identity implies that

$$\begin{aligned} &\langle X, \theta \rangle Y \wedge J\theta - \langle Y, \theta \rangle X \wedge J\theta - \langle Y, \theta \rangle JX \wedge \theta + \langle X, \theta \rangle JY \wedge \theta - |\theta|^2 Y \wedge JX + |\theta|^2 X \wedge JY \\ &\quad + Y \wedge J\nabla_X\theta + JY \wedge \nabla_X\theta - X \wedge J\nabla_Y\theta - JX \wedge \nabla_Y\theta \\ &= \langle IX, \theta \rangle IY \wedge J\theta - \langle IY, \theta \rangle IX \wedge J\theta - \langle IY, \theta \rangle JIX \wedge \theta + \langle IX, \theta \rangle JIY \wedge \theta \\ &- |\theta|^2 IY \wedge JIX + |\theta|^2 IX \wedge JIY + IY \wedge J\nabla_{IX}\theta + JIY \wedge \nabla_{IX}\theta - IX \wedge J\nabla_{IY}\theta - JIX \wedge \nabla_{IY}\theta, \end{aligned}$$

for all vector fields X, Y . Let $\{e_i\}_{i=1, \dots, 2n}$ be a local orthonormal basis of TM , which is parallel at the point where the computation is done. Taking the interior product with X in the above identity and summing over $X = e_i$, we obtain:

$$(6) \quad (4 - 2n)\langle Y, \theta \rangle J\theta + \langle JY, \theta \rangle \theta + (2n - 4)|\theta|^2 JY + (4 - 2n)J\nabla_Y \theta + \nabla_{JY} \theta + \delta\theta JY \\ = \langle IY, \theta \rangle IJ\theta - \langle IJ\theta, \theta \rangle IY - \langle IY, \theta \rangle \text{tr}(JI)\theta + \langle IY, \theta \rangle JI\theta + \langle \theta, IJIY \rangle \theta \\ + |\theta|^2 \text{tr}(IJ)IY - |\theta|^2 IJIY - \sum_{i=1}^{2n} \langle e_i, J\nabla_{Ie_i} \theta \rangle IY + IJ\nabla_{IY} \theta + \nabla_{IJIY} \theta - \text{tr}(JI)\nabla_{IY} \theta + JI\nabla_{IY} \theta.$$

Substituting $Y = e_j$ in (6), taking the scalar product with Je_j and summing over $j = 1, \dots, 2n$ yields:

$$(7) \quad (4n^2 - 10n + 6)|\theta|^2 + (4n - 6)\delta\theta = -2\text{tr}(IJ)\langle I\theta, J\theta \rangle - 2\langle IJ\theta, JI\theta \rangle \\ - (\text{tr}(IJ))^2|\theta|^2 + \text{tr}(IJIJ)|\theta|^2 + 2\text{tr}(IJ) \sum_{i=1}^{2n} \langle IJ\nabla_{e_i} \theta, e_i \rangle - 2 \sum_{i=1}^{2n} \langle IJIJ\nabla_{e_i} \theta, e_i \rangle.$$

On the other hand, a straightforward computation of $\delta(\text{tr}(IJ)JI\theta)$ and $\delta(JIJI\theta)$, using (1) and the fact that I is parallel and $\nabla\theta$ is a symmetric endomorphism, yields the following identities:

$$\text{tr}(IJ) \sum_{i=1}^{2n} \langle e_i, IJ\nabla_{e_i} \theta \rangle = -\delta(\text{tr}(IJ)JI\theta) + (2n - 2)\text{tr}(IJ)\langle I\theta, J\theta \rangle - 2\langle IJ\theta, JI\theta \rangle + 2|\theta|^2, \\ \sum_{i=1}^{2n} \langle e_i, IJIJ\nabla_{e_i} \theta \rangle = -\delta(JIJI\theta) - (2n - 3)\langle IJ\theta, JI\theta \rangle + \text{tr}(IJ)\langle I\theta, J\theta \rangle + |\theta|^2.$$

Substituting these in (7), we obtain

$$(8) \quad (4n^2 - 10n + 4)|\theta|^2 - (4n - 8)\text{tr}(IJ)\langle I\theta, J\theta \rangle - (4n - 12)\langle IJ\theta, JI\theta \rangle \\ - \text{tr}(IJIJ)|\theta|^2 + (\text{tr}(IJ))^2|\theta|^2 = -(4n - 6)\delta\theta - 2\delta(\text{tr}(IJ)JI\theta) + 2\delta(JIJI\theta).$$

In order to exploit this formula we need to distinguish two cases.

Case 1: If $n = 2$, (8) becomes:

$$(9) \quad 4\langle IJ\theta, JI\theta \rangle - \text{tr}(IJIJ)|\theta|^2 + (\text{tr}(IJ))^2|\theta|^2 = -2\delta(\theta + \text{tr}(IJ)JI\theta - JIJI\theta).$$

We claim that I and J define opposite orientations on TM . Assume for a contradiction that they define the same orientation. Recall that complex structures compatible with the orientation on an oriented 4-dimensional Euclidean vector space may be identified with imaginary quaternions of norm 1 acting on \mathbb{H} by left multiplication. For any $q, v \in \mathbb{H}$, we have: $\langle qv, v \rangle = \frac{1}{4}\text{tr}(q)|v|^2$, where $\text{tr}(q)$ denotes the trace of q acting by left multiplication on \mathbb{H} . For every $x \in M$ we can identify $T_x M$ with \mathbb{H} and view I, J as unit quaternions acting by left multiplication. The previous relation gives the following pointwise equality: $4\langle IJIJ\theta, \theta \rangle = \text{tr}(IJIJ)|\theta|^2$. Substituting in (9) and integrating over M , implies that $\text{tr}(IJ) = 0$, so I and J anti-commute.

Equation (9) then further implies that $\delta\theta = 0$. Replacing these two last equalities in (7) yields $|\theta|^2 = 0$. This contradicts the assumption that the lcK structure (g, J, θ) is proper. Hence, I and J define opposite orientations and thus they commute, proving our claim.

Note that, alternatively, the claim also follows from a result by M. Pontecorvo [24, Proposition 3.2].

Now, since (M, g, I) is a compact Kähler manifold, it follows that its first Betti number is even. Y.-T. Siu [25] proved that each compact complex surface with even first Betti number carries a Kähler metric (see also [8] and [16]). On the other hand, I. Vaisman proved that if a complex manifold (M, J) admits a J -compatible Kähler metric, then every lcK metric on (M, J) is gcK [27, Theorem 2.1]. This shows that θ is exact.

Case 2. We assume from now on that $n \geq 3$. The integral over the compact manifold M of the left hand side of (8) is zero, since the right hand side is the co-differential of a 1-form. On the other hand, the following inequalities hold:

$$(10) \quad -(4n - 8)\text{tr}(IJ)\langle I\theta, J\theta \rangle \geq -(4n - 8)|\text{tr}(IJ)||\theta|^2 \geq -((\text{tr}(IJ))^2 + (2n - 4)^2)|\theta|^2,$$

$$(11) \quad -(4n - 12)\langle IJ\theta, JI\theta \rangle \geq -(4n - 12)|\theta|^2,$$

(it is here that the assumption $n \geq 3$ is needed), and

$$(12) \quad -\text{tr}(IJIJ)|\theta|^2 \geq -2n|\theta|^2.$$

Summing up the inequalities (10)–(12) shows that the left hand side of (8) is non-negative. As the right hand side of (8) is a divergence, we deduce that both terms vanish identically, and thus equality holds in (10)–(12).

Let M' denote the set of points where θ is not zero and let M'' denote the interior of $M \setminus M'$. The open set $M' \cup M''$ is clearly dense in M . At each point of M' , the fact that equality holds in (12) shows that $(IJ)^2 = \text{Id}$. Moreover, the endomorphism $(IJ)^2$ is ∇ -parallel along M'' (since $\theta = 0$ along M'' , so (M'', g, J) is Kähler). We deduce that $(IJ)^2$ is ∇ -parallel along $M' \cup M''$, thus along the whole of M by density. Moreover M' is not empty (since by assumption the lcK structure (g, J, θ) is proper). As $(IJ)^2 = \text{Id}$ on M' , we finally get $(IJ)^2 = \text{Id}$ on M , which amounts saying that I and J commute at each point of M .

Moreover, the fact that equality holds in (10) shows that for each point $x \in M'$ there exists $\varepsilon_x \in \{-1, 1\}$ with $I\theta = \varepsilon_x J\theta$ and $\text{tr}(IJ) = \varepsilon_x(2n - 4)$. The function $\text{tr}(IJ)$ is thus locally constant on M' and on M'' (since as before IJ is parallel along M''), so by density, it is constant on M . After replacing I with $-I$ if necessary we may thus assume that $\text{tr}(IJ) = 2n - 4$, and $I\theta = J\theta$ on M (this last relation holds tautologically on $M \setminus M'$).

This shows that the orthogonal involution IJ has two eigenvalues: 1 with multiplicity $2n - 2$ and -1 with multiplicity 2. At each point of M' , since θ and $I\theta$ are eigenvectors of IJ for the eigenvalue -1 , it follows that $IJX = X$, for every X orthogonal on θ and $J\theta$, which can also be expressed by the formula

$$(13) \quad JX = -IX + \frac{2}{|\theta|^2} (\langle X, \theta \rangle I\theta - \langle X, I\theta \rangle \theta), \quad \forall X \in \text{TM}'.$$

We thus have $\Omega^J = -\Omega^I + \frac{2}{|\theta^2|}\theta \wedge I\theta$ on M' . In particular, we have

$$(14) \quad \theta \wedge \Omega^J = -\theta \wedge \Omega^I,$$

at every point of M (as this relation holds tautologically on $M \setminus M'$, where by definition $\theta = 0$). From (2) and (14) we get

$$(15) \quad d\Omega^J = 2\theta \wedge \Omega^J = -2\theta \wedge \Omega^I = -2L^I(\theta),$$

where $L^I : \Lambda^*M \rightarrow \Lambda^*M$, $L^I(\alpha) := \Omega^I \wedge \alpha$ is the Lefschetz operator of the Kähler manifold (M, g, I) .

Using the Hodge decomposition on M , we decompose the closed 1-form θ as $\theta = \theta_H + d\varphi$, where θ_H is the harmonic part of θ and φ is a smooth real-valued function on M . From (15) and the fact that L^I commutes with the exterior differential, we obtain

$$(16) \quad L^I(\theta_H) = -d\left(\frac{1}{2}\Omega^J + L^I\varphi\right).$$

Moreover, since L^I commutes on any Kähler manifold with the Laplace operator (see *e.g.* [20]), the left-hand side of (16) is a harmonic form and the right-hand side is exact. This implies that $L^I\theta_H$ vanishes, so $\theta_H = 0$ since L^I is injective on 1-forms for $n \geq 2$. Thus $\theta = d\varphi$ is exact, so (M, g, J, θ) is globally conformally Kähler. \square

Example 3.2. As in Example 2.4, we consider on $M := \mathbb{C}^n \setminus \{0\}$ the standard flat structure (g_0, J_0) . Let J be a constant complex structure on M , compatible with g_0 and which does not commute with J_0 . Then, $(M, g := r^{-4}g_0, J)$ is gcK and (M, g, I) is Kähler, where I is the pull-back of J_0 through the inversion, but J and I do not commute. This example shows that the compactness assumption in Theorem 3.1 is necessary.

4. CONFORMAL CLASSES WITH NON-HOMOTHETIC KÄHLER METRICS

As an application of Theorem 3.1, we will describe in this section all compact conformal manifolds (M^{2n}, c) with $n \geq 2$, such that the conformal class c contains two non-homothetic Kähler metrics, thus solving Problem P1.

We start by constructing a class of examples, which will be referred to as the *Calabi Ansatz*.

Proposition 4.1. *Let (N, h, J_N, Ω_N) be a Hodge manifold, i.e. a compact Kähler manifold with $[\Omega_N] \in H^2(N, 2\pi\mathbb{Z})$. Let $\pi : S \rightarrow N$ be the principal S^1 -bundle with the connection (given by Chern-Weil theory) whose curvature form is the pull-back to S of $i\Omega_N$. For any positive real number ℓ , let h_ℓ be the unique Riemannian metric on S such that π is a Riemannian submersion with fibers of length $2\pi\ell$. For every $b > 0$ and smooth function $\ell : (0, b) \rightarrow \mathbb{R}^{>0}$, consider the metric $g_\ell := h_{\ell(r)} + dr^2$ on $M' := S \times (0, b)$. Then the metric g_ℓ is globally conformally Kähler with respect to two non-conjugate complex structures J_+, J_- on M' . Moreover, if $\ell^2(r) = r^2(1 + A(r^2))$ near $r = 0$ and $\ell^2(r) = (b - r)^2(1 + B((b - r)^2))$ near $r = b$ for smooth functions A, B defined near 0 with $A(0) = B(0) = 0$, then the metric completion M of (M', g_ℓ) is a smooth manifold diffeomorphic to the total space of an S^2 -bundle over N , and g_ℓ, J_+ , and J_- extend smoothly to M .*

Proof. Let $i\omega \in \Omega^1(S, i\mathbb{R})$ denote the connection form on S satisfying

$$(17) \quad d\omega = \pi^*(\Omega_N).$$

The metric h_ℓ is defined by $h_\ell := \pi^*h + \ell^2\omega \otimes \omega$. Let ξ denote the vector field on S induced by the S^1 -action. By definition ξ verifies $\pi_*\xi = 0$ and $\omega(\xi) = 1$. Let X^* denote the horizontal lift of a vector field X on N (defined by $\omega(X^*) = 0$ and $\pi_*(X^*) = X$). By the equivariance of the connection we have $[\xi, X^*] = 0$ for every vector field X , and from (17) we readily obtain $[X^*, Y^*] = [X, Y]^* - \Omega_N(X, Y)\xi$. The Koszul formula immediately gives the covariant derivative ∇^ℓ of the metric $g_\ell := h_{\ell(r)} + dr^2$ on $M' := S \times (0, b)$:

$$\begin{aligned} \nabla_\xi^\ell \partial_r &= \nabla_{\partial_r}^\ell \xi = \frac{\ell'}{\ell} \xi, \\ \nabla_\xi^\ell \xi &= -\ell \ell' \partial_r, \\ \nabla_{\partial_r}^\ell \partial_r &= \nabla_{X^*}^\ell \partial_r = \nabla_{\partial_r}^\ell X^* = 0, \\ \nabla_{X^*}^\ell \xi &= \nabla_\xi^\ell X^* = \frac{\ell^2}{2} (J_N X)^*, \\ \nabla_{X^*}^\ell Y^* &= (\nabla_X^h Y)^* - \frac{1}{2} \Omega_N(X, Y)\xi, \end{aligned}$$

where ℓ' is the derivative of ℓ .

We now define for $\varepsilon = \pm 1$ the Hermitian structures J_ε on (M', g_ℓ) by

$$J_\varepsilon(X^*) := \varepsilon(J_N X)^*, \quad J_\varepsilon(\xi) := \ell \partial_r, \quad J_\varepsilon(\partial_r) := -\ell^{-1} \xi.$$

A straightforward calculation using the previous formulas yields $\nabla_Z^\ell J_\varepsilon = Z \wedge J_\varepsilon \theta_\varepsilon + J_\varepsilon Z \wedge \theta_\varepsilon$ for every vector field Z on M , where $\theta_\varepsilon := \frac{1}{2} \varepsilon \ell dr$. Thus (g_ℓ, J_ε) are globally conformally Kähler structures on M' with Lee forms

$$\theta_\varepsilon = \varepsilon d\varphi, \quad \text{where} \quad \varphi(r) := \frac{1}{2} \int_0^r \ell(t) dt.$$

The last statement of the proposition follows from a coordinate change (from polar to Euclidean coordinates) in the fibers $S^1 \times (0, b)$ of the Riemannian submersion $M' \rightarrow N$. Indeed, in a neighbourhood of $r = 0$, with Euclidean coordinates $x_1 := r \cos t$ and $x_2 := r \sin t$, we have:

$$\begin{pmatrix} \partial_r \\ \frac{1}{r} \xi \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix},$$

where $\xi = \partial_t$. In these coordinates, we have the following formulas for the complex structures and the metric:

$$\begin{aligned} J_\varepsilon(\partial_{x_1}) &= \frac{\ell}{r} \left(\frac{-A(r^2)}{\ell^2} x_1 x_2 \partial_{x_1} - \left(\frac{A(r^2)}{\ell^2} x_1^2 + 1 \right) \partial_{x_2} \right), \\ J_\varepsilon(\partial_{x_2}) &= \frac{\ell}{r} \left(\frac{A(r^2)}{\ell^2} x_1 x_2 \partial_{x_2} + \left(1 - \frac{A(r^2)}{\ell^2} x_2^2 \right) \partial_{x_1} \right), \\ g &= \pi^*h + \left(1 + \frac{A(r^2)}{r^2} x_2^2 \right) dx_1^2 + \left(1 + \frac{A(r^2)}{r^2} x_1^2 \right) dx_2^2 - 2 \frac{A(r^2)}{r^2} x_1 x_2 dx_1 dx_2. \end{aligned}$$

From the assumption on A , the functions $\frac{\ell}{r}$ and $\frac{A(r^2)}{\ell^2}$ extend smoothly at $r = 0$, therefore, the complex structures J_- , J_+ and the metric g extend smoothly at $r = 0$. The same argument applies to the other extremal point $r = b$. Hence, the metric g_ℓ on M' extends to a smooth metric g_0 on M , and there exist two distinct Kähler structures on M in the conformal class $[g_0]$, whose restrictions to M' are equal to $(g_+ := e^\varphi g_\ell, J_+)$ and $(g_- := e^{-\varphi} g_\ell, J_-)$ respectively. \square

Conversely, the Calabi Ansatz can be characterized geometrically by the following data:

Proposition 4.2. *Let (M, g_0, I) be a compact globally conformally Kähler manifold with non-trivial Lee form $\theta_0 = d\varphi_0$ and denote by ∇^0 the Levi-Civita connection of g_0 . We assume that on M' , the set where θ_0 is not vanishing, its derivative is given by:*

$$(18) \quad \nabla_X^0 \theta_0 = f(\theta_0(X)\theta_0 + I\theta_0(X)I\theta_0), \quad \forall X \in \text{TM}'$$

for some function $f \in C^\infty(M')$. We denote by ξ the metric dual of $I\theta_0$ with respect to g_0 and further assume that there exists a distribution \mathcal{V} on M , such that \mathcal{V}_x is spanned by ξ and $I\xi$, for every $x \in M'$. Then (M, g_0) is obtained from the Calabi Ansatz.

Proof. We first notice that $M' \neq M$. Indeed, θ_0 vanishes at the extrema of the function φ_0 defined on the compact manifold M .

From (18) and (1) we deduce the following formulas on M' :

$$(19) \quad \nabla_X^0(I\xi) = -f(\langle X, I\xi \rangle I\xi + \langle X, \xi \rangle \xi), \quad \forall X \in \text{TM}'$$

$$(20) \quad \nabla_X^0 \xi = (1 + f)(\langle X, \xi \rangle I\xi - \langle X, I\xi \rangle \xi) - |\xi|^2 IX, \quad \forall X \in \text{TM}'$$

which imply that the distribution \mathcal{V} is totally geodesic along M' .

Equation (20) also shows that $\nabla^0 \xi$ is a skew-symmetric endomorphism, hence ξ is a Killing vector field on (M', g_0) . Since ξ is tautologically Killing on the interior of $M \setminus M'$, it is Killing on the whole of M by density. We denote by N one of the connected components of the zero set of ξ , which is thus a compact totally geodesic submanifold of M . Applying (20) at a sequence of points of M' converging to some point of N , we see that $d\xi^b$ has rank at most 2 at each point of N . Moreover ξ is not identically 0, thus showing that N has co-dimension 2, and its normal bundle equals $\mathcal{V}|_N$.

Let Φ_s denote the 1-parameter group of isometries of (M, g_0) induced by ξ and let us fix some $p \in N$. For every $s \in \mathbb{R}$, the differential of Φ_s at p is an isometry of $T_p M$ which fixes $T_p N$, so it is determined by a rotation of angle $k(s)$ in \mathcal{V}_p . From $\Phi_s \circ \Phi_{s'} = \Phi_{s+s'}$ we obtain $k(s) = ks$, for some $k \in \mathbb{R}^*$. For $s_0 = 2\pi/k$, the isometry Φ_{s_0} fixes p and its differential at p is the identity. We obtain that $\Phi_{s_0} = \text{Id}_M$, so ξ has closed orbits. Note that any $p \in N$ is a fixed point of Φ_s , for all $s \in \mathbb{R}$, and that $\Phi_{\frac{s_0}{2}}$ is an orientation preserving isometry whose differential at p squares to the identity, and is the identity on $T_p N = \mathcal{V}_p^\perp$. Hence, $(d\Phi_{\frac{s_0}{2}})_p|_{\mathcal{V}_p}$ is either plus or minus the identity of \mathcal{V}_p . The first possibility would contradict the definition of s_0 , so we have

$$(21) \quad (d\Phi_{\frac{s_0}{2}})_p|_{\mathcal{V}_p} = -\text{Id}_{\mathcal{V}_p}.$$

Let γ be a geodesic of (M, g_0) starting from p , such that $V := \dot{\gamma}(0) \in \mathcal{V}_p$ and $|\dot{\gamma}(0)| = 1$. Since \mathcal{V} is totally geodesic, $\dot{\gamma}(t) \in \mathcal{V}$ for all t . The function $g_0(\xi, \dot{\gamma})$ clearly vanishes at $t = 0$ and its derivative along γ equals $g_0(\nabla_{\dot{\gamma}}^0 \xi, \dot{\gamma}) = 0$, so $g_0(\xi, \dot{\gamma}) \equiv 0$ along γ . We thus have

$$(22) \quad I\xi_{\gamma(t)} = c_{p,V}(t)\dot{\gamma}(t),$$

for some function $c_{p,V} : \mathbb{R} \rightarrow \mathbb{R}$. Clearly $c_{p,V}^2(t) = |\xi_{\gamma(t)}|^2$, so $c_{p,V}$ is smooth at all points t with $\gamma(t) \in M'$. By (19)–(20) we easily check that $[\xi, I\xi] = 0$ on M' , and thus on M by density. Hence, each isometry Φ_s preserves $I\xi$. Moreover, $\Phi_s(\gamma(t))$ is the geodesic starting at p with tangent vector $(\Phi_s)_*(\dot{\gamma}(0))$. This shows that the function $c_p := c_{p,V}$ does not depend on the unit vector V in \mathcal{V}_p defining γ .

We claim that in fact, for all $p, q \in N$, $c_p(t) = c_q(t)$, for all t . In other words, the norm of $\xi_{\gamma(t)}$ only depends on t and not on the initial data of γ starting in N . For a fixed $t \in \mathbb{R}$, we consider the map $F : SN \rightarrow M$, $F(V) := \exp(tV)$, where SN denotes the unit normal bundle of N . By the Gauss' Lemma, we know that $dF_V(T_V SN) \subset (\dot{\gamma}_{p,V}(t))^\perp$, where $\gamma_{p,V}$ denotes the geodesic starting at p with unit speed vector V . Since ξ is Killing, the function $g_0(\dot{\gamma}_{p,V}, \xi)$ is constant along $\gamma_{p,V}$ and thus identically zero, because ξ vanishes on N . As $\dot{\gamma}_{p,V} \in \mathcal{V}$, it follows that $\dot{\gamma}_{p,V}$ is proportional to $I\xi$, which is the metric dual of $-\theta_0$. On the other hand, (18) immediately gives $d|\theta_0|^2 = 2f|\theta_0|^2\theta_0$. Therefore, $d|\theta_0|^2$ vanishes on $dF_V(T_V SN)$, showing that the norm of $\xi_{\gamma(t)}$ does not depend on the starting point either. Hence, we further denote the function $c_p = c_{p,V}$ simply by $c : \mathbb{R} \rightarrow \mathbb{R}$.

Differentiating the relation $\gamma_{p,V}(t) = \gamma_{p,-V}(-t)$ which holds for all geodesics and for all t , yields $\dot{\gamma}_{p,V}(t) = -\dot{\gamma}_{p,-V}(-t)$. Therefore, from (22) we conclude that $c(-t) = -c(t)$, for all t . Moreover, $c(t)$ is non-vanishing for $|t| \neq 0$ and sufficiently small. By replacing I with $-I$ if necessary, we thus can assume that c is negative on some interval $(0, \varepsilon)$ and positive on $(-\varepsilon, 0)$. Since $(\varphi_0(\gamma(t)))' = \theta_0(\dot{\gamma}(t)) = -c(t)$, we conclude that N is a connected component of the level set of a local minimum of φ_0 .

By compactness of N , the exponential map defined on the normal bundle of N is surjective, so its image contains points where φ_0 attains its absolute maximum. At such a point, the vector field ξ vanishes, so (22) shows that $t_0 := \inf\{t > 0 \mid c(t) = 0\}$ is well-defined and positive. Let N' be a connected component of the inverse image through φ_0 of $\varphi_0(\exp_p(t_0V))$, for some $p \in N$ and some unit vector V in \mathcal{V}_p . The above argument, applied to N' instead of N , shows that N' is a connected component of the level set of a local maximum of φ_0 . It also shows that $\exp_q(t_0W) \in N'$ for any $q \in N$ and any unit vector $W \in \mathcal{V}_q$. From (21) it follows that $\dot{\gamma}_{p,-V}(t_0) = -\dot{\gamma}_{p,V}(t_0)$, for any $p \in N$ and any unit vector $V \in \mathcal{V}_p$. In other words, if a geodesic starting at a point p of N with unit speed vector $V \in \mathcal{V}_p$ arrives after time t_0 in a point $p' \in N'$ with speed vector $V' \in \mathcal{V}_{p'}$, then the geodesic starting at p with speed vector $-V$ arrives after time t_0 in p' with speed vector $-V'$, showing that these two geodesics close up to one geodesic. Hence, M equals the image through the exponential map of the compact subset of the normal bundle of N consisting of vectors of norm $\leq t_0$, thus showing that $M \setminus M' = N \cup N'$.

Consequently, the function φ_0 attains its minimum on N and its maximum on N' and has no other critical point. Let S be some level set corresponding to a regular value of φ_0 .

Consider the unit vector field $\zeta := \frac{I\xi}{|I\xi|}$ on M' (see Figure 1 for a visualization of the vector fields ξ and ζ and of the level sets of φ_0).

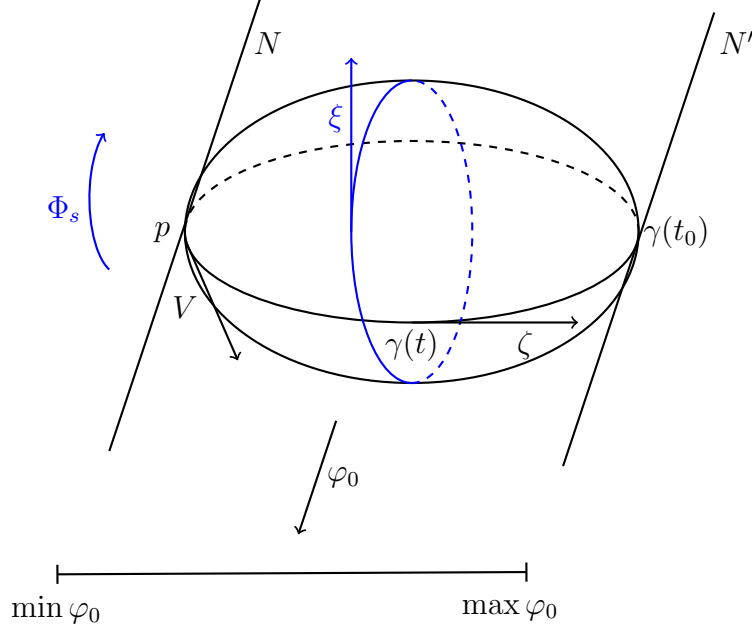


FIGURE 1

From (19) and (20) we readily compute on M' :

$$(23) \quad \nabla_X^0 \zeta = -\frac{f}{|\xi|} \langle X, \xi \rangle \xi, \quad \forall X \in TM',$$

and

$$(24) \quad \nabla_\zeta^0 I\zeta = 0.$$

In particular, we have $\nabla_\zeta^0 \zeta = 0$, so if Ψ denotes the (local) flow of ζ , the curve $t \mapsto \Psi_t(x)$ is a geodesic for every $x \in M'$, that is, $\Psi_t(x) = \exp_x(t\zeta)$. Note that by (23), we have $d\zeta^b = 0$ so the Cartan formula implies $\mathcal{L}_\zeta \zeta^b = d(\zeta \lrcorner \zeta^b) + \zeta \lrcorner d\zeta^b = 0$, which can also be written as

$$(25) \quad (\mathcal{L}_\zeta g_0)(\zeta, X) = 0, \quad \forall X \in TM'.$$

We claim that for fixed t , $\varphi_0(\Psi_t(x))$ does not depend on $x \in S$. To see this, let $X \in T_x S$. By definition $d\varphi_0(X) = 0$, whence $g_0(X, \zeta) = 0$. We need to show that $d\varphi_0((\Psi_t)_*(X)) = 0$. This is equivalent to $0 = g_0(\zeta, (\Psi_t)_*(X)) = (\Psi_t^* g_0)(\zeta, X)$, which clearly holds at $t = 0$. Moreover, from (25) we see that the derivative of the function $(\Psi_t^* g_0)(\zeta, X)$ vanishes:

$$\frac{d}{dt}((\Psi_t^* g_0)(\zeta, X)) = (\Psi_t^* \mathcal{L}_\zeta g_0)(\zeta, X) = (\mathcal{L}_\zeta g_0)(\zeta, (\Psi_t)_*(X)) = 0.$$

This shows that for every $x \in S$, $\exp_x(t\zeta)$ belongs to the same level set of φ_0 . Moreover, $\varphi_0(\exp_x(t\zeta))$ is decreasing in t since its derivative equals $d\varphi_0(\zeta) = \theta_0(\zeta) = -|\xi|$. Take the smallest $t_1 > 0$ such that $\pi(x) := \exp_x(t_1\zeta) \in N$ for every $x \in S$.

Claim. The map π is a Riemannian submersion from $(S, g_0|_S)$ to $(N, g_0|_N)$ with totally geodesic 1-dimensional fibers tangent to ξ .

Proof of the Claim. First, the Killing vector field ξ commutes with ζ , so $(\Psi_t)_*\xi = \xi$ for all $t < t_1$. Making t tend to t_1 implies $\pi_*(\xi_x) = \xi_{\pi(x)} = 0$ for every $x \in S$, since $\pi(x) \in N$. Thus ξ is tangent to the fibers of π . From (20) we get $\nabla_{I\zeta}I\zeta = f|\xi|\zeta$, so $I\zeta$ is a geodesic vector field on S . Since $I\zeta$ is proportional to ξ , it is also tangent to the fibres of π .

Take now any tangent vector $X \in T_xS$ orthogonal to $I\zeta$ and denote by $X_t := (\Psi_t)_*(X)$, which makes sense for all $t < t_1$. By construction we have $\pi_*(X) = \lim_{t \rightarrow t_1} X_t$. Since $0 = \langle \zeta, X_t \rangle = \nabla_{\zeta}^0 X_t - \nabla_{X_t}^0 \zeta$, we get by (23) and (24):

$$\zeta(\langle X_t, I\zeta \rangle) = \langle \nabla_{\zeta}^0 X_t, I\zeta \rangle + \langle X_t, \nabla_{\zeta}^0 I\zeta \rangle = \langle \nabla_{X_t}^0 \zeta, I\zeta \rangle = -f|\xi|\langle X_t, I\zeta \rangle.$$

The function $\langle X_t, I\zeta \rangle$ vanishes at $t = 0$ and satisfies a first order linear ODE along the geodesic $\gamma(t) := \exp_x(t\zeta)$, so it vanishes identically. Thus, X_t is orthogonal to $I\zeta$ for all $t < t_1$. Moreover, the vector field X_t along γ has constant norm:

$$(26) \quad \zeta(|X_t|^2) = 2\langle \nabla_{\zeta}^0 X_t, X_t \rangle = 2\langle \nabla_{X_t}^0 \zeta, X_t \rangle \stackrel{(23)}{=} -\frac{2f}{|\xi|}\langle X_t, \xi \rangle^2 = -2f|\xi|\langle X_t, I\zeta \rangle^2 = 0.$$

This shows that $|\pi_*(X)|^2 = |X|^2$, thus proving the claim.

Let us now consider the smallest $t_2 > 0$ such that $\pi(x) := \exp_x(-t_2\zeta) \in N'$ for every $x \in S$ and let $b := t_1 + t_2$. The flow of the geodesic vector field ζ defines a diffeomorphism between $(0, b) \times S$ and M' , which maps (r, x) onto $\exp_x((r-t_2)\zeta)$. With respect to this diffeomorphism, the vector field ζ is identified to ∂_r , the metric reads $g_0 = dr^2 + k_r$, where k_r is a family of Riemannian metrics on S , and the function $|\theta_0|$ only depends on r , say $|\theta_0| = \alpha(r)$. It follows that $\theta_0 = \alpha dr$ and since $d\varphi_0 = \theta_0$, we see that $\varphi_0 = \varphi_0(r)$ and $\varphi_0' = \alpha$.

The previous claim actually shows that for every $r \in (0, b)$, $k_r = \pi^*(h) + \tau_r \otimes \tau_r$, where $\tau_r := I\zeta^b$ and $h := g_0|_N$. From (23) and (24) we readily obtain

$$\dot{\tau}_r = \mathcal{L}_{\zeta}(I\zeta^b) = -f\alpha I\zeta^b = -f\alpha\tau_r.$$

This shows that $\tau_r = \ell(r)\omega$ with $\ell(r) := e^{-\int_0^r f(t)\alpha(t)dt}$, where ω denotes the connection 1-form on the S^1 -bundle $S \rightarrow N$ induced by the Riemannian submersion π . Finally, the metric on M' reads $g_0 = dr^2 + \pi^*(h) + \ell^2\omega \otimes \omega$, showing that g_0 has the form of the metric described in Proposition 4.1. \square

4.1. Proof of Theorem 1.1. We can now finish the classification of compact manifolds carrying two conformally related non-homothetic Kähler metrics. Assume that (g_+, J_+) and (g_-, J_-) are Kähler structures on a compact manifold M of real dimension $2n \geq 4$ with $g_+ = e^{2\varphi}g_-$ for some non-constant function φ . Note that J_+ is not conjugate to J_- . Indeed, if J_+ were equal to $\pm J_-$, then $\Omega_+ = \pm e^{2\varphi}\Omega_-$, so $0 = d\Omega_+ = \pm 2e^{2\varphi}d\varphi \wedge \Omega_-$ would imply $d\varphi = 0$, so φ would be constant.

We introduce the following notation, in order to use the results from Section 3:

$$g := g_+, \quad I := J_+, \quad J := J_-, \quad \Omega^I := \Omega_+ = g(J_+ \cdot, \cdot), \quad \Omega^J := g(J \cdot, \cdot) = e^{2\varphi} \Omega_-.$$

Then (M, g, I) is Kähler, and (M, g, J) is lcK (in fact globally conformally Kähler), with Lee form $\theta := d\varphi$. This last statement follows from (2), since $d\Omega^J = 2e^{2\varphi} d\varphi \wedge \Omega_- = 2d\varphi \wedge \Omega^J$.

The first part of Theorem 3.1 shows that I and J commute, which proves the statement of Theorem 1.1 for $n = 2$.

Assume from now on that $n \geq 3$. The proof of the Theorem 3.1 shows that after replacing I by $-I$ if necessary, one has $I\theta = J\theta$ and $\text{tr}(IJ) = 2n - 4$.

Let us consider the 2-form $\sigma := \frac{1}{2}\Omega^I + \frac{1}{2}\Omega^J$, corresponding to the endomorphism $I + J$ of TM via the metric g . We denote again by M' the open set where θ is non-vanishing. By (13), on M' we have

$$(27) \quad \sigma = \frac{1}{|\theta|^2} \theta \wedge I\theta.$$

Since I is ∇ -parallel (where ∇ is the Levi-Civita connection of g), we obtain by (1) that $\nabla_X \sigma = \frac{1}{2}(X \wedge J\theta + JX \wedge \theta)$. Substituting $\Omega^J = 2\sigma - \Omega^I$, and using the fact that $\sigma(\theta) = I\theta$ we obtain the following formula for the covariant derivative of σ :

$$\begin{aligned} \nabla_X \sigma &= \frac{1}{2} \nabla_X \Omega^J = \frac{1}{2} (X \wedge J\theta + JX \wedge \theta) \\ &= \frac{1}{2} (X \wedge (2\sigma - I)\theta + (2\sigma - I)X \wedge \theta) \\ &= \frac{1}{2} (X \wedge I\theta - IX \wedge \theta) + \sigma(X) \wedge \theta. \end{aligned}$$

Since (27) gives $\theta \wedge \sigma = 0$, we get $0 = X \lrcorner (\theta \wedge \sigma) = \langle X, \theta \rangle \sigma - \theta \wedge \sigma(X)$ for every $X \in TM$. The previous computation thus yields

$$(28) \quad \nabla_X \sigma = \frac{1}{2} (X \wedge I\theta - IX \wedge \theta) - \langle X, \theta \rangle \sigma, \quad \forall X \in TM.$$

We consider now the 2-form $\tilde{\sigma} := e^\varphi \sigma$. By (28), its covariant differential reads:

$$\nabla_X \tilde{\sigma} = \frac{e^\varphi}{2} (X \wedge I\theta - IX \wedge \theta), \quad \forall X \in TM.$$

Equivalently, this equation can be written as

$$(29) \quad \nabla_X \tilde{\sigma} = \frac{1}{2} (d(\text{tr } \tilde{\sigma}) \wedge IX - d^c(\text{tr } \tilde{\sigma}) \wedge X), \quad \forall X \in TM,$$

where $\text{tr } \tilde{\sigma} := \langle \tilde{\sigma}, \Omega^I \rangle = e^\varphi$ is the trace with respect to the Kähler form Ω^I and d^c denotes the twisted exterior differential defined by $d^c \alpha := \sum_i I e_i \wedge \nabla_{e_i} \alpha$, for any form α .

A real $(1, 1)$ -form on a Kähler manifold (M, g, I, Ω^I) satisfying (29) is called a *Hamiltonian 2-form* (see [1]). Compact Kähler manifolds carrying such forms are completely described in [3, Theorem 5]. In the case where the Hamiltonian form has rank 2, these are exactly the manifolds obtained from the Calabi Ansatz described in Proposition 4.1.

However, the statement and the proof of [3, Theorem 5] are rather involved, and it is not completely clear that the construction described in that theorem is equivalent to the Calabi Ansatz. We will thus provide here a more direct proof.

All we need is to show that the globally conformally Kähler structure on M determined by $g_0 := e^\varphi g_- = e^{-\varphi} g_+$ and $I := J_+$ satisfies the hypotheses of Proposition 4.2. We start with the following:

Lemma 4.3. *On the open set M' where θ is not vanishing, the covariant derivative of θ with respect to g is given by*

$$(30) \quad \nabla_X \theta = \frac{1}{2} |\theta|^2 X - \frac{1}{2} \left(\frac{\delta \theta}{|\theta|^2} + n + 1 \right) \langle X, \theta \rangle \theta - \frac{1}{2} \left(\frac{\delta \theta}{|\theta|^2} + n - 1 \right) \langle X, I\theta \rangle I\theta.$$

Proof. Using the fact that I and J commute, $I\theta = J\theta$ and $\text{tr}(IJ) = 2n - 4$, (7) simplifies to

$$(31) \quad \sum_{i=1}^{2n} \langle IJ\nabla_{e_i} \theta, e_i \rangle = 2(n-1)|\theta|^2 + \delta\theta.$$

Substituting this into (6), we obtain

$$(32) \quad 2(2-n)\langle Y, \theta \rangle J\theta - 2n\langle Y, J\theta \rangle \theta + (2n-5)|\theta|^2 JY + |\theta|^2 IY + \delta\theta(JY + IY) \\ + 2(2-n)J\nabla_Y \theta + 2\nabla_{JY} \theta + 2(n-2)\nabla_{IY} \theta - 2IJ\nabla_{IY} \theta = 0.$$

Differentiating (27) on M' yields

$$(33) \quad \nabla_X \sigma = -\frac{2\langle \nabla_X \theta, \theta \rangle}{|\theta|^4} \theta \wedge I\theta + \frac{1}{|\theta|^2} (\nabla_X \theta \wedge I\theta + \theta \wedge I\nabla_X \theta).$$

Comparing with (28), we obtain

$$(34) \quad \frac{1}{2}(X \wedge I\theta - IX \wedge \theta) - \langle X, \theta \rangle \sigma = -\frac{2\langle \nabla_X \theta, \theta \rangle}{|\theta|^4} \theta \wedge I\theta + \frac{1}{|\theta|^2} (\nabla_X \theta \wedge I\theta + \theta \wedge I\nabla_X \theta).$$

Taking the interior product with $I\theta$ in the last equality, we get

$$(35) \quad \frac{1}{2}\langle X, I\theta \rangle I\theta - \frac{1}{2}|\theta|^2 X + \frac{1}{2}\langle X, \theta \rangle \theta = \frac{\langle \nabla_X \theta, \theta \rangle}{|\theta|^2} \theta + \frac{1}{|\theta|^2} \langle \nabla_X \theta, I\theta \rangle I\theta - \nabla_X \theta.$$

We deduce that the following equality holds:

$$(36) \quad \nabla_X \theta = \frac{1}{2} |\theta|^2 X + \alpha(X)\theta + \beta(X)I\theta,$$

where α and β are the following 1-forms:

$$(37) \quad \alpha = \frac{1}{2} \left(\frac{d(|\theta|^2)}{|\theta|^2} - \theta \right), \quad \beta = \frac{1}{|\theta|^2} \nabla_{I\theta} \theta - \frac{1}{2} I\theta.$$

Since θ is closed, (36) yields $0 = \alpha \wedge \theta + \beta \wedge I\theta$. Therefore, there exist $a, b, c \in C^\infty(M')$, such that

$$\alpha = a\theta + bI\theta \quad \text{and} \quad \beta = b\theta + cI\theta.$$

Moreover, (37) shows that α is closed, so $da \wedge \theta + db \wedge I\theta + bd(I\theta) = 0$. On the other hand, by (36), we have $d(I\theta) = |\theta|^2\Omega^I + \alpha \wedge I\theta - \beta \wedge \theta = |\theta|^2\Omega^I + (a+c)\theta \wedge I\theta$. Hence,

$$da \wedge \theta + db \wedge I\theta + b|\theta|^2\Omega^I + b(a+c)\theta \wedge I\theta = 0.$$

Applying the last equality to X and IX , for a non-zero vector field X orthogonal to θ and $I\theta$ yields $b = 0$. By (36) again we have

$$(38) \quad -\delta\theta = \sum_{i=1}^{2n} \langle e_i, \nabla_{e_i}\theta \rangle = (n+a+c)|\theta|^2.$$

Substituting Y by θ in (32) and using (36), we obtain

$$(39) \quad (\delta\theta + (1 + (2-n)a + nc)|\theta|^2) I\theta = 0.$$

From (38) and (39), it follows that

$$a = -\frac{1}{2} \left(\frac{\delta\theta}{|\theta|^2} + n + 1 \right) \quad \text{and} \quad c = -\frac{1}{2} \left(\frac{\delta\theta}{|\theta|^2} + n - 1 \right).$$

This proves the lemma. \square

We write (30) as

$$(40) \quad \nabla_X\theta = \frac{1}{2}g(\theta, \theta)X^\flat - \frac{1}{2}(f+2)\theta(X)\theta - \frac{1}{2}fI\theta(X)I\theta,$$

where $f := \left(\frac{\delta\theta}{|\theta|^2} + n - 1 \right)$. Note that we no longer identify vectors and 1-forms in this relation, since we will now perform a conformal change of the metric.

Namely, we consider the ‘‘average metric’’ $g_0 := e^\varphi g_- = e^{-\varphi} g_+$ and denote by ∇^0 its Levi-Civita covariant derivative, by θ_0 the Lee form of $I := J_+$ with respect to g_0 and by $\Omega_0 := g_0(I \cdot, \cdot)$. Since $d\Omega_0 = d(e^{-\varphi}\Omega_+) = -e^{-\varphi}d\varphi \wedge \Omega_+ = -d\varphi \wedge \Omega_0$, we get $\theta_0 = -\frac{1}{2}d\varphi = -\frac{1}{2}\theta$.

From (40) we immediately get

$$(41) \quad \nabla_X\theta_0 = -g(\theta_0, \theta_0)X^\flat + (f+2)\theta_0(X)\theta_0 + fI\theta_0(X)I\theta_0.$$

The classical formula relating the covariant derivatives of g and $g_0 = e^{-\varphi}g$ on 1-forms reads

$$\nabla_X^0\eta = \nabla_X\eta + g(\theta_0, \eta)X^\flat - \eta(X)\theta_0 - \theta_0(X)\eta, \quad \forall X \in TM, \forall \eta \in \Omega^1(M),$$

where \flat is the index lowering with respect to g . For $\eta = \theta_0$, (41) becomes exactly (18).

From the proof of Theorem 3.1 it is clear that the distribution $\mathcal{V} := \ker(I - J)$ is spanned along M' by ξ and $I\xi$, where ξ denotes the vector field on M corresponding to $I\theta_0$ via the metric g_0 . This shows that the hypotheses of Proposition 4.2 are verified, thus concluding the proof of Theorem 1.1.

Corollary 4.4. *If $n \geq 3$ and (M^{2n}, g, J) is a non-ruled compact Kähler manifold, then every conformal diffeomorphism of $(M, [g])$ is an isometry of g .*

Proof. Let $f : M \rightarrow M$ be a conformal diffeomorphism of $(M, [g])$. Since compact Riemannian manifolds do not admit strict homotheties, f is either an isometry of g , or $f^*g = e^{-2\varphi}g$ for some smooth non-constant function φ . In the latter case, (g, J) and (f^*g, f^*J) are conformally related non-homothetic Kähler structures on M , so by Theorem 1.1, (M, g, J) is obtained from the Calabi Ansatz of Proposition 4.1, thus contradicting the hypothesis that it is non-ruled. \square

The novelty of this result comes from the fact that it gives information about every single conformal transformation. Indeed, at the infinitesimal level, it is a classical fact that conformal Killing vector fields on compact Kähler manifolds are automatically Killing. On the other hand, this does not imply that every conformal transformation of a compact Kähler manifold is isometric, and this is actually not true on some ruled manifolds: the transformation defined by the antipodal map of the fibers in the Calabi Ansatz is conformal but not isometric, since it maps the Kähler metric g_+ to a scalar multiple of $g_- = e^{-2\varphi}g_+$.

5. COMPACT EINSTEIN LCK MANIFOLDS

The purpose of this section is to classify compact Einstein proper lcK manifolds (Problem P2). Note that this problem is open only in dimension $n \geq 3$, since C. LeBrun has recently shown in [18, Theorem A] that an Einstein and Hermitian metric on a compact surface is either Kähler-Einstein, or homothetic to the Page metric [23] on $\mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2$, or to the metric constructed by X. Chen, C. LeBrun and B. Weber in [10] on $\mathbb{C}\mathbb{P}^2 \# 2\mathbb{C}\mathbb{P}^2$.

Assume that (M^{2n}, g, J, θ) is a compact proper lcK manifold and g is Einstein, with positive scalar curvature. By Myers' Theorem and Remark 2.1, (M, g, J) is gcK, so there exists a function φ such that $(M, e^{-2\varphi}g, J)$ is Kähler (and of course conformally Einstein). The classification of conformally Einstein compact Kähler manifolds in complex dimension $n \geq 3$ has been obtained by A. Derdzinski and G. Maschler in a series of three papers [12, 13, 14]. They showed that the only examples are given by the construction of L. Bérard-Bergery, [5].

According to the above, Problem P2 is still open only when the scalar curvature is non-positive and $n \geq 3$. Using the Bochner formula and a compactness argument, we will show in Theorem 5.2 below that in this case the Lee form of the lcK structure must vanish, hence the manifold is already Kähler. Note that our proof works for $n = 2$ as well. We start with the following:

Lemma 5.1. *On an lcK manifold (M, g, J, θ) with g Einstein, the symmetric 2-tensor S defined by*

$$(42) \quad S := \nabla\theta + \theta \otimes \theta,$$

is of type $(1, 1)$ with respect to J , i.e. it satisfies $S(J \cdot, J \cdot) = S(\cdot, \cdot)$.

Proof. Since the statement is local, we may assume without loss of generality that the Lee form is exact, $\theta = d\varphi$, which means that $g_K := e^{-2\varphi}g$ is Kähler with respect to J . We denote the Einstein constant of g by λ .

The formula relating the Ricci tensors of conformally equivalent metrics [6, Theorem 1.159] reads:

$$\text{Ric}^K = \text{Ric}^g + 2(n-1)(\nabla^g d\varphi + d\varphi \otimes d\varphi) - (\Delta^g \varphi + 2(n-1)g(d\varphi, d\varphi))g.$$

Since g^K is Kähler, $\text{Ric}^K(J\cdot, J\cdot) = \text{Ric}^K(\cdot, \cdot)$. Using this fact, together with $g(J\cdot, J\cdot) = g(\cdot, \cdot)$ and $\text{Ric}^g = \lambda g$ in the above formula, we infer:

$$(43) \quad (\nabla^g d\varphi + d\varphi \otimes d\varphi)(J\cdot, J\cdot) = (\nabla^g d\varphi + d\varphi \otimes d\varphi)(\cdot, \cdot),$$

which is equivalent to $SJ = JS$ (when identifying S with a symmetric endomorphism via the metric g). \square

The main result of this section is the following:

Theorem 5.2. *If (M^{2n}, g, J, θ) is a compact lcK manifold and g is Einstein with non-positive scalar curvature, then $\theta \equiv 0$, so (M, g, J) is a Kähler-Einstein manifold.*

Proof. Let $\{e_i\}_{i=1, \dots, 2n}$ be a local orthonormal basis which is parallel at the point where the computation is done. We denote by $\lambda \leq 0$ the Einstein constant of the metric g , so $\text{Ric} = \lambda g$. The strategy of the proof is to apply the Bochner formula to the 1-forms θ and $J\theta$ in order to obtain a formula relating the Einstein constant, the co-differential of the Lee form and its square norm, which leads to a contradiction (if θ is not identically zero) at a point where $\delta\theta + |\theta|^2$ attains its maximum.

Let S denote as above the endomorphism $S = \nabla\theta + \theta \otimes \theta$. In particular, we have

$$(44) \quad S\theta = \nabla_\theta \theta + |\theta|^2 \theta = \frac{1}{2}d|\theta|^2 + |\theta|^2 \theta$$

and the trace of S is computed as follows

$$(45) \quad \text{tr}(S) = |\theta|^2 - \delta\theta.$$

In the sequel, we use Lemma 5.1, ensuring that S commutes with J . We start by computing the covariant derivative of $J\theta$:

$$(46) \quad \begin{aligned} \nabla_X J\theta &= (\nabla_X J)(\theta) + J(\nabla_X \theta) \stackrel{(1)}{=} (X \wedge J\theta + JX \wedge \theta)(\theta) + J(SX - \theta(X)\theta) \\ &= JSX - J\theta(X)\theta - |\theta|^2 JX. \end{aligned}$$

The exterior differential of $J\theta$ is then given by the following formula:

$$(47) \quad dJ\theta = \sum_{i=1}^{2n} e_i \wedge \nabla_{e_i} J\theta = 2JS + \theta \wedge J\theta - 2|\theta|^2 \Omega.$$

We further compute the Lie bracket between θ and $J\theta$ (viewed as vector fields):

$$(48) \quad [\theta, J\theta] = \nabla_\theta J\theta - \nabla_{J\theta} \theta \stackrel{(42), (46)}{=} JS\theta - |\theta|^2 J\theta - SJ\theta = -|\theta|^2 J\theta.$$

By (3), we have $\delta J\theta = 0$. Using the following identities:

$$(49) \quad \delta(\theta \wedge J\theta) = (\delta\theta)J\theta - \delta(J\theta)\theta - [\theta, J\theta] \stackrel{(48)}{=} (\delta\theta + |\theta|^2)J\theta,$$

$$(50) \quad \delta(|\theta|^2\Omega) = -J(d|\theta|^2) + |\theta|^2\delta\Omega \stackrel{(3)}{=} -J(d|\theta|^2) + (2 - 2n)|\theta|^2J\theta,$$

we compute the Laplacian of $J\theta$:

$$(51) \quad \begin{aligned} \Delta J\theta &= \delta dJ\theta \stackrel{(47)}{=} \delta(2JS + \theta \wedge J\theta - 2|\theta|^2\Omega) \\ &\stackrel{(49),(50)}{=} 2\delta(JS) + (\delta\theta + |\theta|^2)J\theta + 2J(d|\theta|^2) + 2(2n - 2)|\theta|^2J\theta \\ &= 2\delta(JS) + \delta\theta J\theta + 2J(d|\theta|^2) + (4n - 3)|\theta|^2J\theta. \end{aligned}$$

We next compute the rough Laplacian of $J\theta$:

$$(52) \quad \begin{aligned} \nabla^*\nabla J\theta &= -\sum_{i=1}^{2n} \nabla_{e_i} \nabla_{e_i} J\theta \stackrel{(46)}{=} -\sum_{i=1}^{2n} \nabla_{e_i} (JS e_i - J\theta(e_i)\theta - |\theta|^2 J e_i) \\ &= \delta(JS) + \nabla_{J\theta}\theta + Jd(|\theta|^2) + |\theta|^2 \sum_{i=1}^{2n} (\nabla_{e_i} J)(e_i) \\ &\stackrel{(3)}{=} \delta(JS) + SJ\theta + Jd(|\theta|^2) + (2n - 2)|\theta|^2J\theta \\ &\stackrel{(44)}{=} \delta(JS) + |\theta|^2J\theta + \frac{3}{2}Jd(|\theta|^2) + (2n - 2)|\theta|^2J\theta. \end{aligned}$$

Using the Bochner formula $\Delta J\theta = \nabla^*\nabla J\theta + \text{Ric}(J\theta)$ together with (51) and (52) yields:

$$\delta(JS) = -(\delta\theta)J\theta - \frac{1}{2}J(d|\theta|^2) - 2(n - 1)|\theta|^2J\theta + \lambda J\theta,$$

which, after applying J on both sides, reads:

$$(53) \quad J\delta(JS) = (\delta\theta)\theta + \frac{1}{2}d|\theta|^2 + 2(n - 1)|\theta|^2\theta - \lambda\theta.$$

The rough Laplacian of θ is computed as follows:

$$(54) \quad \nabla^*\nabla\theta = -\sum_{i=1}^{2n} \nabla_{e_i} \nabla_{e_i} \theta = -\sum_{i=1}^{2n} \nabla_{e_i} (S e_i - \theta(e_i)\theta) = \delta S - (\delta\theta)\theta + \frac{1}{2}d|\theta|^2.$$

The Bochner formula $\Delta\theta = \nabla^*\nabla\theta + \text{Ric}(\theta)$, together with (54) yields

$$(55) \quad \delta S = (\delta\theta)\theta - \frac{1}{2}d|\theta|^2 - \lambda\theta + d\delta\theta.$$

On the other hand, we have:

$$\begin{aligned} \delta(JS) &= -\sum_{i=1}^{2n} (\nabla_{e_i} JS)(e_i) = -\sum_{i=1}^{2n} (\nabla_{e_i} J)(S e_i) - \sum_{i=1}^{2n} J(\nabla_{e_i} S)(e_i) \\ &\stackrel{(1)}{=} -\sum_{i=1}^{2n} (e_i \wedge J\theta + J e_i \wedge \theta)(S e_i) + J(\delta S) \\ &= -\text{tr}(S)J\theta + 2JS\theta + J(\delta S) \stackrel{(44),(45)}{=} (\delta\theta)J\theta + J(d|\theta|^2) + |\theta|^2J\theta + J(\delta S). \end{aligned}$$

Applying J to this equality yields

$$(56) \quad J\delta(JS) + \delta S = -(\delta\theta)\theta - d|\theta|^2 - |\theta|^2\theta.$$

Summing up (53) and (55), and comparing with (56), we obtain:

$$(57) \quad 3(\delta\theta)\theta - 2\lambda\theta + d\delta\theta + d|\theta|^2 + (2n-1)|\theta|^2\theta = 0.$$

After introducing the function $f := \delta\theta + |\theta|^2$, (57) reads:

$$(58) \quad df = (2\lambda - 3f + (4 - 2n)|\theta|^2)\theta.$$

We argue by contradiction and assume that θ is not identically zero. In this case, the integral of f over M is positive. As M is compact, there exists $p_0 \in M$ at which f attains its maximum, $f(p_0) > 0$. In particular, we have $(df)_{p_0} = 0$ and $(\Delta f)(p_0) \geq 0$. Applying (58) at the point p_0 yields that $\theta_{p_0} = 0$, because $2\lambda - 3f(p_0) + (4 - 2n)|\theta_{p_0}|^2 < 0$. From the definition of f , it follows that $\delta\theta(p_0) > 0$.

On the other hand, taking the co-differential of (58), we obtain:

$$\Delta f = (2\lambda - 3f + (4 - 2n)|\theta|^2)\delta\theta + 3\theta(f) + (2n - 4)\theta(|\theta|^2).$$

Evaluating at p_0 leads to a contradiction, since the left-hand side is non-negative and the right-hand side is negative, as $\theta_{p_0} = 0$ and $(2\lambda - 3f(p_0))\delta\theta(p_0) < 0$. Thus, $\theta \equiv 0$. \square

Theorem 5.2 and the results mentioned at the beginning of this section complete the proof of Theorem 1.2.

6. THE HOLONOMY PROBLEM FOR COMPACT LCK MANIFOLDS

In this last section, we will classify compact proper lcK manifolds (M, g, J, θ) of complex dimension $n \geq 2$ with non-generic holonomy group: $\text{Hol}_0(M, g) \subsetneq \text{SO}(2n)$. We will use the previous main results (Theorems 1.1 and 1.2), as well as an irreducibility result (Theorem 6.2 below) stating that a parallel distribution on a compact proper lcK manifold can only have dimension (or co-dimension) equal to 1.

By the Berger-Simons holonomy theorem, the following exclusive possibilities may occur:

- The restricted holonomy group $\text{Hol}_0(M, g)$ is reducible;
- $\text{Hol}_0(M, g)$ is irreducible and (M, g) is locally symmetric;
- M is not locally symmetric, and $\text{Hol}_0(M, g)$ belongs to the following list: $\text{U}(n)$, $\text{SU}(n)$, $\text{Sp}(n/2)$, $\text{Sp}(n/2)\text{Sp}(1)$, $\text{Spin}(7)$ (for $n = 4$).

6.1. The reducible case. We start by recalling the following general fact (for a proof see for instance the first part of the proof of [4, Theorem 4.1]):

Lemma 6.1. *If (M, g) is a compact Riemannian manifold with $\text{Hol}_0(M, g)$ reducible, then there exists a finite covering \bar{M} of M , such that $\text{Hol}(\bar{M}, \bar{g})$ is reducible, where \bar{g} denotes the pull-back of g to \bar{M} .*

Let (M, g, J, θ) be a compact proper lcK manifold of complex dimension $n \geq 2$ with $\text{Hol}_0(M, g)$ reducible. Lemma 6.1 shows that by replacing M with some (compact) finite covering \overline{M} , and by pulling back the lcK structure to \overline{M} , one may assume that the tangent bundle can be decomposed as $\text{TM} = D_1 \oplus D_2$, where D_1 and D_2 are two parallel orthogonal oriented distributions of rank n_1 , respectively n_2 , with $2n = n_1 + n_2$. By taking a further double covering if necessary, we may assume that the distributions are oriented. The main result of this section is the following:

Theorem 6.2. *Let (M, g, J, θ) be a compact lcK manifold of complex dimension $n \geq 2$. If there exist two orthogonal parallel oriented distributions D_1 and D_2 , of respective ranks $n_1 \geq 2$ and $n_2 \geq 2$, such that $\text{TM} = D_1 \oplus D_2$, then $\theta \equiv 0$.*

Proof. Since the arguments for $n = 2$ and $n \geq 3$ are of different nature, we treat the two cases separately. Consider first the case of complex dimension $n = 2$. Then both distributions D_1 and D_2 have rank 2, and their volume forms Ω_1 and Ω_2 define two Kähler structures on M compatible with g by the formula $g(I_{\pm} \cdot, \cdot) = \Omega_1 \pm \Omega_2$. Using the case $n = 2$ in Theorem 3.1 above, we deduce that J commutes with I_+ and with I_- . In particular, J preserves the ± 1 eigenspaces of $I_+ I_-$, which are exactly the distributions D_1 and D_2 . Since J is also orthogonal, its restriction to D_1 and D_2 coincides up to sign with the restriction of I_+ to D_1 and D_2 . Thus $J = \pm I_+$ or $J = \pm I_-$. In each case, the structure (g, J) is Kähler, so $\theta \equiv 0$.

We consider now the case $n \geq 3$. Let $\theta = \theta_1 + \theta_2$ be the corresponding splitting of the Lee form. We fix a local orthonormal basis $\{e_i\}_{i=1, \dots, 2n}$, which is parallel at the point where the computation is done and denote by e_i^a the projection of e_i onto D_a , for $a \in \{1, 2\}$.

The exterior differential and Ω split with respect to the decomposition of the tangent bundle as follows: $d = d_1 + d_2$ and $\Omega = \Omega_{11} + 2\Omega_{12} + \Omega_{22}$, where for $a, b \in \{1, 2\}$ we define:

$$d_a := \sum_{i=1}^{2n} e_i^a \wedge \nabla e_i^a, \quad \Omega_{ab} := \frac{1}{2} \sum_{i=1}^{2n} e_i^a \wedge (J e_i)^b = \frac{1}{2} \sum_{i=1}^{2n} e_i^a \wedge (J e_i^a)^b.$$

The last equality follows for instance by considering a local orthonormal basis of TM , whose first n_1 vectors are tangent to D_1 .

Lemma 6.3. *With the above notation, for any vector fields $X_1 \in D_1$ and $X_2 \in D_2$, the following relations hold:*

$$(59) \quad \nabla_{X_1} \theta_2 = -\theta_1(X_1) \theta_2, \quad \nabla_{X_2} \theta_1 = -\theta_2(X_2) \theta_1.$$

Proof. Note that $d\theta = 0$ implies $d_a \theta_b + d_b \theta_a = 0$, for all $a, b \in \{1, 2\}$. For $c \in \{1, 2\}$ we compute:

$$\begin{aligned} d_c \Omega_{ab} &\stackrel{(1)}{=} \frac{1}{2} \sum_{i,j=1}^{2n} e_j^c \wedge e_i^a \wedge (\langle e_j^c, e_i \rangle J\theta - \langle J\theta, e_i \rangle e_j^c + \langle J e_j^c, e_i \rangle \theta - \langle \theta, e_i \rangle J e_j^c)^b \\ &= \frac{1}{2} \sum_{i=1}^{2n} (e_i^c \wedge e_i^a \wedge J\theta_b - (J e_i)^c \wedge e_i^a \wedge \theta_b - e_i^c \wedge \theta_a \wedge (J e_i^c)^b) \\ &= \Omega_{ac} \wedge \theta_b + \Omega_{cb} \wedge \theta_a, \end{aligned}$$

so for all $a, b, c \in \{1, 2\}$ we have

$$(60) \quad d_c \Omega_{ab} = \Omega_{ac} \wedge \theta_b + \Omega_{cb} \wedge \theta_a.$$

Using the fact that $d_c^2 = 0$ for $c \in \{1, 2\}$, we obtain

$$(61) \quad 0 = d_c^2 \Omega_{ab} = \Omega_{ac} \wedge (d_c \theta_b + \theta_c \wedge \theta_b) + \Omega_{cb} \wedge (d_c \theta_a + \theta_c \wedge \theta_a).$$

For $c = a \neq b$ in (61), we get $\Omega_{cc} \wedge (d_c \theta_b + \theta_c \wedge \theta_b) = 0$ and for $a = b$: $\Omega_{cb} \wedge (d_c \theta_b + \theta_c \wedge \theta_b) = 0$. Summing up, we obtain that $\Omega \wedge (d_c \theta_b + \theta_c \wedge \theta_b) = 0$, which by the injectivity of $\Omega \wedge \cdot$ on manifolds of complex dimension greater than 2, implies that $d_c \theta_b = -\theta_c \wedge \theta_b$. Applying this identity for $b \neq c$ to $X_c \in D_c$ and $X_b \in D_b$ yields (59). \square

The symmetries of the Riemannian curvature tensor imply that $R_{X_1, X_2} = 0$, and thus $R_{X_1, X_2} J = [R_{X_1, X_2}, J] = 0$, for every $X_1 \in D_1$ and $X_2 \in D_2$.

Using (4) for $X := X_1$ and $Y := X_2$ and applying Lemma 6.3, we obtain:

$$(62) \quad 0 = \langle X_1, \theta_1 \rangle X_2 \wedge J\theta_1 - \langle X_2, \theta_2 \rangle X_1 \wedge J\theta_2 - \langle X_2, \theta_2 \rangle JX_1 \wedge \theta_2 + \langle X_1, \theta_1 \rangle JX_2 \wedge \theta_1 \\ - |\theta|^2 X_2 \wedge JX_1 + |\theta|^2 X_1 \wedge JX_2 + X_2 \wedge J\nabla_{X_1} \theta_1 + JX_2 \wedge \nabla_{X_1} \theta_1 - X_1 \wedge J\nabla_{X_2} \theta_2 - JX_1 \wedge \nabla_{X_2} \theta_2,$$

for every $X_1 \in D_1$ and $X_2 \in D_2$.

Lemma 6.4. *The following formula holds:*

$$(63) \quad \nabla_{X_1} \theta_1 = -\langle X_1, \theta_1 \rangle \theta_1 + \frac{1}{n_1} (|\theta_1|^2 - \delta \theta_1) X_1, \quad \forall X_1 \in D_1.$$

Proof. Let \mathcal{U} denote the open set $\mathcal{U} := \{x \in M \mid (JD_2)_x \not\subset (D_1)_x\}$. By continuity, it is enough to prove the result on the open sets $M \setminus \overline{\mathcal{U}}$ and \mathcal{U} .

Let \mathcal{O} be some open subset of $M \setminus \overline{\mathcal{U}}$, i.e. at every point x of \mathcal{O} the inclusion $(JD_2)_x \subset (D_1)_x$ holds. On \mathcal{O} , let X be some vector field and Y_2, Z_2 vector fields tangent to D_2 . By assumption, we have $JY_2 \in D_1$, hence $\nabla_X JY_2 \in D_1$ and $\nabla_X Y_2 \in D_2$, thus $J\nabla_X Y_2 \in D_1$. Applying (1), we obtain

$$0 = \langle \nabla_X JY_2, Z_2 \rangle - \langle J\nabla_X Y_2, Z_2 \rangle = \langle (\nabla_X J)Y_2, Z_2 \rangle \\ = \langle X, Y_2 \rangle (J\theta)(Z_2) - \langle X, Z_2 \rangle (J\theta)(Y_2) - \langle X, JY_2 \rangle \theta(Z_2) + \langle X, JZ_2 \rangle \theta(Y_2).$$

Since $n_2 \geq 2$, for any $Y_2 \in D_2$ there exists a non-zero $Z_2 \in D_2$ orthogonal to Y_2 . Taking $X = JZ_2 \in D_1$ in the above formula yields $\theta(Y_2) = 0$. This shows that $\theta_2 = 0$, so $\theta = \theta_1$. Taking $X = Z_2 \in D_2$ in the above formula yields $\theta_1(JY_2) = 0$, for all $Y_2 \in D_2$. Substituting into (62), we obtain for all $X_1 \in D_1$ and $Y_2 \in D_2$:

$$(64) \quad \langle X_1, \theta_1 \rangle Y_2 \wedge J\theta_1 + \langle X_1, \theta_1 \rangle JY_2 \wedge \theta_1 - |\theta_1|^2 Y_2 \wedge JX_1 + |\theta_1|^2 X_1 \wedge JY_2 \\ + Y_2 \wedge J\nabla_{X_1} \theta_1 + JY_2 \wedge \nabla_{X_1} \theta_1 = 0.$$

Let us now consider the decomposition $D_1 = JD_2 \oplus D'_1$, where D'_1 denotes the orthogonal complement of JD_2 in D_1 . Note that D'_1 is J -invariant, since it is also the orthogonal complement in TM of the J -invariant distribution $D_2 \oplus JD_2$. Let $X_1 = JV_2 + V_1$ and $\nabla_{X_1} \theta_1 = JW_2 + W_1$ be the decomposition of X_1 , respectively of $\nabla_{X_1} \theta_1$, with respect to this

splitting, *i.e.* $V_2, W_2 \in D_2$ and $V_1, W_1 \in D'_1$. As shown above, θ_1 vanishes on JD_2 , meaning that $\theta_1 \in D'_1$.

Taking the trace with respect to Y_2 in (64) yields

$$(65) \quad n_2 \langle X_1, \theta_1 \rangle J\theta_1 + |\theta_1|^2 [(n_2 - 1)V_2 - n_2 J V_1] + n_2 J W_1 - (n_2 - 1)W_2 = 0,$$

which further implies, by projecting onto D_2 and D'_1 , that $W_1 = -\langle X_1, \theta_1 \rangle \theta_1 + |\theta_1|^2 V_1$ and $W_2 = |\theta_1|^2 V_2$. Hence, $\nabla_{X_1} \theta_1 = |\theta_1|^2 X_1 - \langle X_1, \theta_1 \rangle \theta_1$, which in particular implies $\delta\theta_1 = (1 - n_1)|\theta_1|^2$, proving (63) on $M \setminus \overline{\mathcal{U}}$.

We further show that the formula (63) holds on \mathcal{U} . At every point x of \mathcal{U} there exist vectors $X_2, Y_2 \in (D_2)_x$ such that $X_2 \perp Y_2$ and $\langle Y_2, JX_2 \rangle \neq 0$. Indeed, by definition there exists $Y_2 \in (D_2)_x$ such that $JY_2 \notin D_1$, and we can take X_2 to be the D_2 -projection of JY_2 .

For any vector $X_1 \in (D_1)_x$ we take the scalar product with $X_1 \wedge Y_2$ in (62) and obtain:

$$(66) \quad \langle JX_2, Y_2 \rangle (\langle \nabla_{X_1} \theta_1, X_1 \rangle + |\langle X_1, \theta_1 \rangle|^2) = \\ - |X_1|^2 (\langle X_2, \theta_2 \rangle \langle J\theta_2, Y_2 \rangle - |\theta|^2 \langle JX_2, Y_2 \rangle + \langle J\nabla_{X_2} \theta_2, Y_2 \rangle).$$

We thus get $\langle \nabla_{X_1} \theta_1, X_1 \rangle + |\langle X_1, \theta_1(x) \rangle|^2 = f_1(x)|X_1|^2$, for every $X_1 \in (D_1)_x$, where the real number $f_1(x)$ does not depend on X_1 . By polarization, we obtain:

$$(67) \quad \nabla_{X_1} \theta_1 = -\langle X_1, \theta_1(x) \rangle \theta_1(x) + f_1(x)X_1, \quad \forall X_1 \in (D_1)_x.$$

Taking the trace with respect to X_1 in this formula and using (59) we obtain $(\delta\theta_1)_x = |\theta_1(x)|^2 - n_1 f_1(x)$, whence:

$$(68) \quad f_1(x) = \frac{1}{n_1} (|\theta_1|^2 - \delta\theta_1)(x), \quad \forall x \in \mathcal{U}.$$

From (67) and (68) we obtain (63) on \mathcal{U} . This proves the lemma. \square

A similar argument yields

$$(69) \quad \nabla_{X_2} \theta_2 = -\langle X_2, \theta_2 \rangle \theta_2 + \frac{1}{n_2} (|\theta_2|^2 - \delta\theta_2) X_2, \quad \forall X_2 \in D_2.$$

Substituting (63) and (69) into (62), we obtain

$$\left(\frac{1}{n_1} (|\theta_1|^2 - \delta\theta_1) + \frac{1}{n_2} (|\theta_2|^2 - \delta\theta_2) - |\theta|^2 \right) (X_2 \wedge JX_1 - X_1 \wedge JX_2) = 0, \quad \forall X_1 \in D_1, X_2 \in D_2.$$

Note that for every $X_1 \in D_1$, $X_2 \in D_2$ the two-forms $X_2 \wedge JX_1$ and $X_1 \wedge JX_2$ are mutually orthogonal. So, choosing X_1 non-collinear to JX_2 (which is possible as $n_1 \geq 2$), the 2-form appearing in the previous formula is non-zero. Hence, we necessarily have

$$\frac{1}{n_1} (|\theta_1|^2 - \delta\theta_1) + \frac{1}{n_2} (|\theta_2|^2 - \delta\theta_2) - |\theta|^2 = 0.$$

Integrating this relation over M , we get

$$\int_M |\theta|^2 d\mu_g = \frac{1}{n_1} \int_M |\theta_1|^2 d\mu_g + \frac{1}{n_2} \int_M |\theta_2|^2 d\mu_g.$$

Since $|\theta|^2 = |\theta_1|^2 + |\theta_2|^2$, we obtain $\left(1 - \frac{1}{n_1}\right) \int_M |\theta_1|^2 d\mu_g + \left(1 - \frac{1}{n_2}\right) \int_M |\theta_2|^2 d\mu_g = 0$. As $n_1, n_2 \geq 2$, it follows that $\theta \equiv 0$. This concludes the proof of the theorem. \square

Remark 6.5. For every $n \geq 2$, the tangent bundle $T(\mathbb{C}^n \setminus \{0\})$ endowed with the flat metric g defined in Example 2.4 can be written as an orthogonal direct sum of two parallel distributions of ranks at least 2 in infinitely many ways, but the gcK structure (g, J_0) on $\mathbb{C}^n \setminus \{0\}$ has non-vanishing Lee form $\theta = -2 \ln r$. The compactness assumption in Theorem 6.2 is thus necessary.

It remains to consider the case when one of the two oriented parallel distributions has rank 1, and is thus spanned by a globally defined parallel unit vector field. This case was studied by the second named author in [21, Theorem 3.5] for $n \geq 3$. We will give here a simpler proof of his result, which also extends it to the missing case $n = 2$.

Theorem 6.6 (cf. [21, Theorem 3.5]). *Let (M, g, J, θ) be a compact proper lcK manifold of complex dimension $n \geq 2$ admitting a non-trivial parallel vector field V . Then, the following exclusive possibilities occur:*

- (i) *The Lee form θ is a non-zero constant multiple of V^\flat , so M is a Vaisman manifold.*
- (ii) *The Lee form θ is exact, so (M, g, Ω, θ) is gcK, and there exists a complete simply connected Kähler manifold (N, g_N, Ω_N) of real dimension $2n - 2$, a smooth non-constant real function $c : \mathbb{R} \rightarrow \mathbb{R}$ and a discrete co-compact group Γ acting freely and totally discontinuously on $\mathbb{R}^2 \times N$, preserving the metric $ds^2 + dt^2 + e^{2c(t)}g_N$, the Hermitian 2-form $ds \wedge dt + e^{2c(t)}\Omega_N$ and the vector fields ∂_s and ∂_t , such that M is diffeomorphic to $\Gamma \backslash (\mathbb{R}^2 \times N)$, and the structure (g, Ω, θ) corresponds to $(ds^2 + dt^2 + e^{2c(t)}g_N, ds \wedge dt + e^{2c(t)}\Omega_N, dc)$ through this diffeomorphism.*

Proof. Let V be a parallel vector field of unit length on M . We identify as usual 1-forms with vectors using the metric g and decompose the Lee form as $\theta = aV + bJV + \theta_0$, where $a := \langle \theta, V \rangle$, $b := \langle \theta, JV \rangle$ and θ_0 is orthogonal onto V and JV . We compute:

$$(70) \quad \delta\theta = -V(a) - JV(b) + b\delta JV + \delta\theta_0.$$

On the other hand, we have:

$$\begin{aligned} \delta JV &= - \sum_{i=1}^{2n} \langle (\nabla_{e_i} J)V, e_i \rangle \stackrel{(1)}{=} \sum_{i=1}^{2n} -\langle (e_i \wedge J\theta + Je_i \wedge \theta)(V), e_i \rangle \\ &= (2 - 2n)\langle \theta, JV \rangle = (2 - 2n)b, \end{aligned}$$

which together with (70) yields

$$(71) \quad \delta\theta = -V(a) - JV(b) + (2 - 2n)b^2 + \delta\theta_0.$$

Replacing X by V in (5) and using that V is parallel, we obtain:

$$(2n - 3)(aJ\theta - |\theta|^2 JV + J\nabla_V \theta) - b\theta - \nabla_{JV} \theta - JV\delta\theta = 0.$$

Taking the scalar product with JV yields

$$(72) \quad (2n - 3)(a^2 - |\theta|^2 + \langle \nabla_V \theta, V \rangle) - b^2 - \langle \nabla_{JV} \theta, JV \rangle - \delta\theta = 0.$$

Further, we compute

$$\langle \nabla_V \theta, V \rangle = V(\langle \theta, V \rangle) = V(a),$$

$\langle \nabla_{JV} \theta, JV \rangle = JV(b) - \langle \theta, (\nabla_{JV} J)V \rangle \stackrel{(1)}{=} JV(b) - \langle \theta, bJV + aV - \theta \rangle = JV(b) + |\theta_0|^2$,
which together with (71) and (72) imply that

$$(2n - 2)(V(a) - |\theta_0|^2) = \delta\theta_0.$$

Integrating over M , we obtain $\int_M |\theta_0|^2 d\mu_g = 0$, because $\int_M V(a) d\mu_g = \int_M a \delta V d\mu_g = 0$, as V is parallel. Hence, $\theta_0 = 0$, showing that $\theta = aV + bJV$.

Claim. The function a is constant and $ab = 0$.

Proof of the Claim. Equation (1) yields

$$(73) \quad \nabla_X JV = \langle X, V \rangle(-bV + aJV) + bX - \langle X, JV \rangle(aV + bJV) - aJX,$$

which allows us to compute the exterior differential of JV , as follows:

$$(74) \quad dJV = 2a(V \wedge JV - \Omega).$$

From the fact that θ is closed and V is parallel, we obtain

$$0 = d\theta = da \wedge V + db \wedge JV + b dJV = da \wedge V + db \wedge JV + 2ab(V \wedge JV - \Omega),$$

which implies that $ab = 0$, for instance, by taking the scalar product with $X \wedge JX$ for some vector field X orthogonal to V and JV . In particular, we have

$$(75) \quad da \wedge V + db \wedge JV = 0.$$

Differentiating again (74) yields

$$0 = da \wedge (V \wedge JV - \Omega) + a(-V \wedge dJV - d\Omega) = da \wedge (V \wedge JV - \Omega) - 2abJV \wedge \Omega = da \wedge (V \wedge JV - \Omega),$$

which together with (75) shows that $da = 0$, thus proving the claim.

If a is non-zero, the second part of the claim shows that $b \equiv 0$, so $\theta = aV$ is parallel and (M, g, J, θ) is Vaisman.

If $a = 0$, Equation (73) becomes:

$$\nabla_X JV = b(X - \langle X, V \rangle V - \langle X, JV \rangle JV).$$

We conclude that in this case the metric structure on M is given as in (ii) by applying Lemma 3.3 and Lemma 3.4 in [21]. \square

Corollary 6.7. *Let (M, g, J, θ) be a compact proper lcK manifold of complex dimension $n \geq 2$. If (M, g) has reducible holonomy, then its restricted holonomy group $\text{Hol}_0(M, g)$ is conjugate to $\text{SO}(2n - 1)$.*

Proof. By Lemma 6.1, Theorem 6.2 and Theorem 6.6, we need to distinguish two cases:

Case 1. If (M, g, J, θ) is Vaisman. Then the Lee form θ is parallel (and non-vanishing), so (M, g) is locally isometric to $\mathbb{R} \times S$ for some Riemannian manifold (S, g_S) . It is well known that S is a Sasakian manifold, but since we want to avoid introducing this class of manifolds, we will derive the necessary formulas directly.

As θ is parallel, we can rescale the metric of M such that $|\theta| = 1$. Equation (4) applied to vector fields X, Y tangent to S (i.e. orthogonal to θ) then yields:

$$R_{X,Y}J = X \wedge JY - Y \wedge JX, \quad \forall X, Y \in \ker(\theta).$$

In particular, applying this formula to θ (seen as vector field) and using the fact that $R_{X,Y}\theta = 0$ gives

$$R_{X,Y}(J\theta) = (R_{X,Y}J)(\theta) = \langle Y, J\theta \rangle X - \langle X, J\theta \rangle Y, \quad \forall X, Y \in \ker(\theta).$$

The metric dual ξ of $J\theta$ is parallel in the direction of θ , so it is actually a vector field on S , and the previous relation reads

$$(76) \quad R_{X,Y}^S \xi = g_S(Y, \xi)X - g_S(X, \xi)Y, \quad \forall X, Y \in \text{TS},$$

where R^S is the Riemannian curvature tensor of (S, g_S) .

Assume, for a contradiction, that $\text{Hol}_0(M, g)$ is strictly contained in $\text{SO}(2n-1)$. Then the same holds for $\text{Hol}_0(S, g_S)$, so by the Berger-Simons theorem, we have three possibilities:

- (S, g_S) has reducible holonomy; this would contradict Theorem 6.2 since then (M, g) would have a holonomy reduction with both factors of dimension at least 2.
- $\text{Hol}_0(S, g_S)$ belongs to the Berger list; the unique group in this list corresponding to an odd-dimensional manifold is G_2 (for $2n-1 = 7$). However, a manifold with holonomy G_2 is Ricci-flat, whereas $\text{Ric}^S(\xi) = (2n-2)\xi$ by taking a trace in (76). This case is thus impossible too.
- (S, g_S) is locally symmetric. Then R^S is parallel, so by taking a further covariant derivative in (76) we get

$$(77) \quad R_{X,Y}^S(\nabla_Z \xi) = g_S(Y, \nabla_Z \xi)X - g_S(X, \nabla_Z \xi)Y, \quad \forall X, Y, Z \in \text{TS}.$$

On the other hand, from (1) we see that $\nabla_Z \xi = -JZ$ when Z is orthogonal to θ and $J\theta$, so the set $\{\nabla_Z \xi \mid Z \in \text{TS}\}$ is equal to the orthogonal of ξ in TS . From (76) and (77) we thus obtain that S has constant sectional curvature 1, i.e. it is locally isometric to the round sphere, and has maximal holonomy group $\text{Hol}_0(S, g_S) = \text{SO}(2n-1)$, which contradicts our assumption.

Case 2. The universal covering of (M, g) is isometric to a Riemannian product $\mathbb{R} \times S$, where $S = \mathbb{R} \times N$ has a warped product metric $g_S = dt^2 + e^{2c(t)}g_N$ with periodic, but non-constant, warping function c . Denoting for convenience $f(t) := e^{c(t)}$, one of the O'Neill formulas for the curvature of warped products (cf. [22, p. 210]) reads:

$$(78) \quad R_{X, \partial_t}^S \partial_t = -\frac{\ddot{f}}{f}X, \quad \forall X \in C^\infty(\text{TN}).$$

Assume now that $\text{Hol}_0(M, g) = \text{Hol}(S, g_S)$ is strictly contained in $\text{SO}(2n-1)$. Like before, Theorem 6.2 shows that (S, g_S) has irreducible holonomy.

Next, if $\text{Hol}(S, g_S)$ belongs to the Berger list, then S is a G_2 -manifold since it has odd dimension, and therefore is Ricci-flat. On the other hand, taking the trace in (78) immediately

shows that

$$(79) \quad \text{Ric}^S(\partial_t, \partial_t) = (1 - 2n) \frac{\ddot{f}}{f}.$$

Thus $\text{Ric}^S = 0$ implies $\ddot{f} = 0$, which is impossible since f is a non-constant periodic function.

It remains to treat the case where (S, g_S) is an irreducible symmetric space. In particular S is Einstein with Einstein constant λ and from (79) we get $\ddot{f} = \frac{\lambda}{1-2n}f$. As f is non-constant and periodic, we necessarily have $\lambda > 0$ and

$$(80) \quad f(t) = \sin(\mu t + \nu)$$

for some real constants μ and ν with $\mu^2 = \frac{\lambda}{2n-1}$. This is a contradiction, since the periodic function $f = e^c$ does not vanish at any point of \mathbb{R} . This shows that $\text{Hol}_0(M, g)$ is conjugate to $\text{SO}(2n-1)$, and thus finishes the proof. \square

Summarizing, if $\text{Hol}_0(M, g)$ is reducible, Theorem 6.2 shows that $\text{Hol}_0(M, g)$ is (up to conjugation) a subgroup of $\text{SO}(2n-1)$ acting irreducibly on \mathbb{R}^{2n-1} and Theorem 6.6 implies that (M^{2n}, g, J, θ) satisfies either case 1. or case 2.c) in Theorem 1.3. Moreover, Corollary 6.7 shows that the restricted holonomy group $\text{Hol}_0(M, g)$ is conjugate to $\text{SO}(2n-1)$ in both cases.

6.2. The irreducible locally symmetric case. In this section we show the following result:

Proposition 6.8. *Every compact irreducible locally symmetric lcK manifold (M^{2n}, g, J, θ) has vanishing Lee form.*

Proof. An irreducible locally symmetric space is Einstein. If the scalar curvature of M is non-positive, the result follows directly from Theorem 5.2.

Assume now that M has positive scalar curvature. By Myers' Theorem and Remark 2.1, (M, g, J) is gcK, so $\theta = d\varphi$ for some function φ , and $g_K := e^{-2\varphi}g$ is a Kähler metric. Let X be a Killing vector field of g . Then X is a conformal Killing vector field of the metric g_K . By a result of Lichnerowicz [19] and Tashiro [26], every conformal Killing vector field with respect to a Kähler metric on a compact manifold is Killing. This shows that X is a Killing vector field for both conformal metrics g and g_K , hence X preserves the conformal factor, *i.e.* $X(\varphi) = 0$. As (M, g) is homogeneous and $X(\varphi) = 0$ for each Killing vector field X of g , it follows that the function φ is constant. Thus $\theta = d\varphi = 0$. \square

In conclusion, there exist no compact irreducible locally symmetric proper lcK manifolds.

6.3. Compact irreducible lcK manifolds with special holonomy. We finally consider compact lcK manifolds (M, g, J, θ) of complex dimension $n \geq 2$, whose restricted holonomy group $\text{Hol}_0(M, g)$ is in the Berger list.

If $\text{Hol}_0(M, g) = \text{U}(n)$, the universal covering $(\widetilde{M}, \widetilde{g})$ has holonomy $\text{Hol}(\widetilde{M}, \widetilde{g}) = \text{U}(n)$, so \widetilde{g} is Kähler with respect to some complex structure \widetilde{I} . Every deck transformation γ of \widetilde{M} is an isometry of \widetilde{g} , so $\gamma^*\widetilde{I}$ is parallel with respect to the Levi-Civita connection of

\tilde{g} . As $\text{Hol}(\widetilde{M}, \tilde{g}) = \text{U}(n)$, we necessarily have $\gamma^* \tilde{I} = \pm \tilde{I}$ for every $\gamma \in \pi_1(M) \subset \text{Iso}(\widetilde{M})$. The group of \tilde{I} -holomorphic deck transformations is thus a subgroup of index at most 2 of $\pi_1(M)$, showing that after replacing M with some double covering if necessary, there exists an integrable complex structure I , such that (M, g, I) is a Kähler manifold. By Theorem 3.1, I and J commute and (M, g, J, θ) is gcK, hence the conformal class of g contains two non-homothetic Kähler metrics. We conclude then by Theorem 1.1 that (M, g, J, θ) falls in one of the cases 2.a) or 2.b) in Theorem 1.3.

If $\text{Hol}_0(M, g)$ is one of $\text{SU}(n)$, $\text{Sp}(n/2)$, or $\text{Spin}(7)$ (for $n = 4$), the metric g is Ricci-flat and $\theta \equiv 0$ by Theorem 5.2.

If $\text{Hol}_0(M, g) = \text{Sp}(n/2)\text{Sp}(1)$, the metric g is quaternion-Kähler, hence Einstein with either positive or negative scalar curvature. In the negative case one has $\theta \equiv 0$ by Theorem 5.2. On the other hand, P. Gauduchon, A. Moroianu and U. Semmelmann, have shown in [15], that the only compact quaternion-Kähler manifolds of positive scalar curvature which carry an almost complex structure are the complex Grassmanians of 2-planes, which are symmetric, thus again $\theta \equiv 0$ by Proposition 6.8.

This completes the classification of compact proper lcK manifolds (M^{2n}, g, J, θ) with non-generic holonomy stated in Theorem 1.3.

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