ON THE ERGODICITY OF THE FRAME FLOW ON EVEN-DIMENSIONAL MANIFOLDS

MIHAJLO CEKIĆ, THIBAULT LEFEUVRE, ANDREI MOROIANU, AND UWE SEMMELMANN

Abstract. It is known that the frame flow on a closed n-dimensional Riemannian manifold with negative sectional curvature is ergodic if n is odd and $n \neq 7$. In this paper we study its ergodicity for $n \geq 4$ even and $n = 7$, and we show that:

(1) if $n \equiv 2 \mod 4$ or $n = 4$, the frame flow is ergodic if the manifold is ~ 0.3-pinched,
(2) if $n \equiv 0 \mod 4$, it is ergodic if the manifold is ~ 0.6-pinched,

except in the three dimensions $n = 7, 8, 134$, where the respective pinching conditions are 0.4962..., 0.6212..., and 0.6716.... In particular, if $n \equiv 2 \mod 4$ or $n = 4$, this almost solves a long-standing conjecture of Brin asserting that 1/4-pinched even-dimensional manifolds have an ergodic frame flow.

Contents

1. Introduction

Let (M^n, g) be a smooth closed (compact, without boundary) oriented Riemannian manifold with negative sectional curvature of dimension $n \geq 3$. Let $SM \to M$ be the unit tangent bundle and let $FM \to M$ be the principal $\mathrm{SO}(n)$-bundle of *oriented orthonormal bases* over M. A point $w \in FM$ over $x \in M$ is the data of an oriented orthonormal basis $(v, \mathbf{e}_2, ..., \mathbf{e}_n)_x$ of T_xM. Equivalently, we will see FM as a principal $\mathrm{SO}(n-1)$-bundle over SM by the projection map $p : FM \to SM$ defined as $p(v, \mathbf{e}_2, ..., \mathbf{e}_n)_x = (x, v)$.

We denote by $(\varphi_t)_{t\in\mathbb{R}}$ the geodesic flow on SM and by $(\Phi_t)_{t\in\mathbb{R}}$ the frame flow on FM, defined in the following way: given $t \in \mathbb{R}, w \in FM$, the point $\Phi_t(w)$ is obtained by flowing (x, v) by the geodesic flow and parallel transport along this geodesic of the remaining vectors $(\mathbf{e}_2, ..., \mathbf{e}_n)$. They satisfy the obvious commutation relation $p \circ \Phi_t = \varphi_t \circ p$, that is the frame flow is an extension of the geodesic flow. When (M, g) has negative sectional curvature (or more generally, when the geodesic flow is *Anosov*, i.e. uniformly hyperbolic), the frame flow is a typical example of a *partially hyperbolic flow*, see [HP06]. Since it preserves a natural smooth measure (the product measure of the Liouville measure on the unit tangent bundle and the Haar measure on the group), one of the main questions from the perspective of dynamical systems is to understand its ergodicity with respect to that measure. In negative curvature, it is known that the frame flow is ergodic when M is odd-dimensional [BG80] and $n \neq 7$, without any further restriction on the metric. However, the situation is more complicated in even dimensions and for $n = 7$.

We will say that the negatively-curved manifold (M, g) has *δ-pinched curvature* for some $\delta \in (0, 1]$ if there exists a constant $K > 0$ such that the sectional curvature κ satisfies the uniform bounds:

$$-K \leq \kappa(u \wedge v) \leq -\delta K, \tag{1.1}$$

for any two-plane $u \wedge v$ in TM. We will say that it has *strictly δ-pinched curvature* if the inequality on the right of (1.1) is strict. Note that in even dimensions, Kähler manifolds cannot have an ergodic frame flow and these are at most 1/4-pinched [Ber60]. Brin thus formulated the natural conjecture (see [Bri82, Conjecture 2.6]):

Conjecture 1.1 (Brin '82)**.** *If (M, g) is strictly $1/4$-pinched, then the frame flow is ergodic.*

More generally, Brin conjectures in the same paper that the frame flow should be ergodic as long as the holonomy group of the manifold is $\mathrm{SO}(n)$, see [Bri82, Conjecture 2.9]. It is also reasonable to expect that the frame flow is ergodic in dimension 7, without any pinching condition. So far, positive answers to the conjectural ergodicity of the frame flow in even dimensions and dimension 7 were obtained for a pinching δ close to 1: strictly greater than 0.8649 in even dimensions different from 8 [BK84], and strictly greater than 0.9805... in dimensions 7 and 8 [BP03][1]. Ergodicity of the frame flow also holds on an open

[1]Note that our convention is different from [BK84, BP03] as our pinching is the square of their pinching.

and dense set of C^3-metrics with negative curvature [Bri82, Section 5]. However, there has been no progress on Conjecture 1.1 in the past twenty years.

In this paper, we prove the following:

Theorem 1.2. *Let (M^n, g) be a closed n-dimensional negatively-curved oriented Riemannian manifold with δ-pinched curvature and $n \geq 3$. Then the frame flow is ergodic if:*

(1) n is odd and $n \neq 7$ [BG80],
(2) n is even or $n = 7$ and $\delta > \delta(n)$, where $\delta(n)$ is given by

$$\begin{array}{ll} 0.2928..., & \text{if } n = 4, \\ 0.2823..., & n = 6, \\ 0.4962..., & n = 7, \\ 0.6212..., & n = 8, \\ 0.6716..., & n = 134, \\ \dfrac{\frac{2}{3}\sqrt{3(n^2-1)}+\frac{1}{2}}{3(n+1)+\frac{2}{3}\sqrt{3(n^2-1)}-\frac{1}{2}}, & \text{if } n \geq 10,\ n \neq 134,\ n \equiv 2 \bmod 4, \\ \dfrac{n+5+\frac{8}{3}\sqrt{(n-1)(n+2)}+\frac{2(n+2)(n+4)}{3(n+1)(n+6)}\left(n+3+\frac{4}{3}\sqrt{3(n^2-1)}\right)}{3(n+1)+\frac{8}{3}\sqrt{(n-1)(n+2)}+\frac{2(n+2)(n+4)}{3(n+1)(n+6)}\left(5n+3+\frac{4}{3}\sqrt{3(n^2-1)}\right)}, & \text{if } n \geq 12,\ n \equiv 0 \bmod 4. \end{array} \tag{1.2}$$

Asymptotically, $\delta(4\ell+2) \to_{\ell\to\infty} 0.2779...$ and $\delta(4\ell) \to_{\ell\to\infty} 0.5572....$ Moreover, the sequence $(\delta(4\ell+2))_{\ell\geq 2}$ is increasing and $\delta(10) = 0.2725...$, while $(\delta(4\ell))_{\ell\geq 3}$ is decreasing and $\delta(12) = 0.5948....$

It can be checked that ergodicity implies mixing for frame flows, see [Lef]. Theorem 1.2 is illustrated by Figure 1.

The strategy of the proof is the following. When the frame flow is not ergodic, one can define a strict subgroup H of $\mathrm{SO}(n-1)$ (defined up to conjugation) called the *transitivity group*, and an H-subbundle of FM on which the flow is ergodic, see [Bri75b, Bri75a], §3.1, and §3.2 for further details. This subgroup gives in particular a reduction of the structure group of the frame bundle over the sphere $\mathbb{S}^{n-1}$. Using topological arguments, one can exclude most subgroups of $\mathrm{SO}(n-1)$ and only a few cases survive, see §3.3. We show by representation theory and the non-Abelian Livšic theory developed in [CL21b], that for each possible subgroup H one can construct a flow-invariant section on the unit tangent bundle SM which takes values either in p-forms (for $p = 1, 2, 3$) or symmetric endomorphisms of the *normal bundle* (the tangent bundle to the spherical fibers), see Theorem 3.8. In turn, the existence of such an invariant section gives rise on the base manifold M to a new object, which we call *normal twisted conformal Killing tensor*: it is a symmetric tensor twisted by some vector bundle, satisfying an algebraic constraint

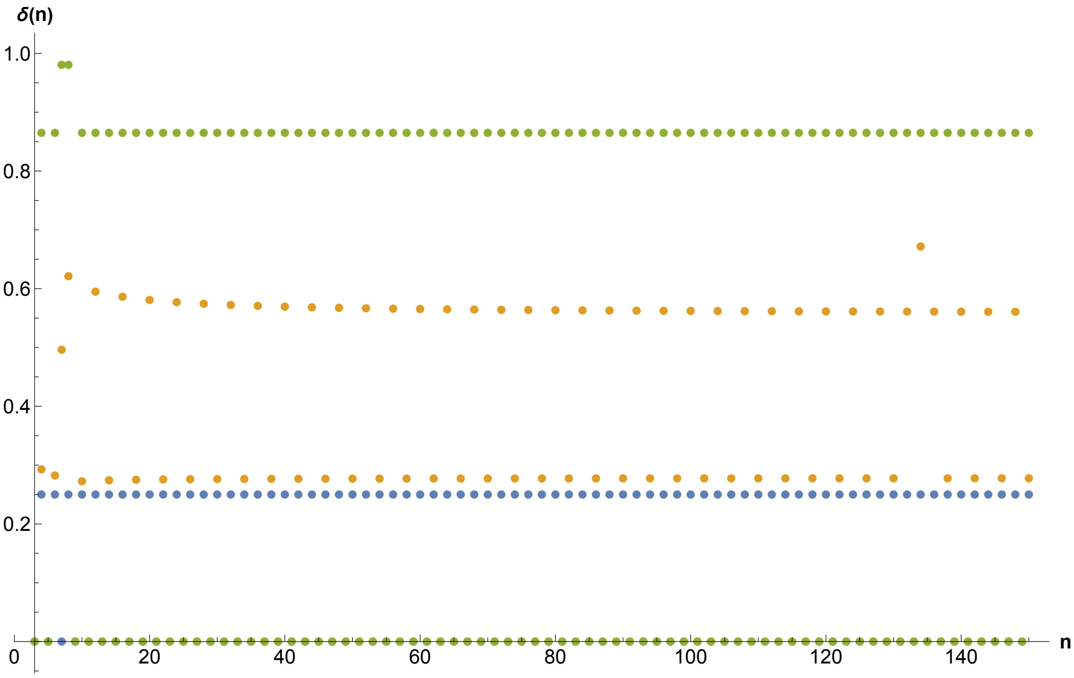

FIGURE 1. In green: the bounds existing in the literature [BG80, BK84, BP03]. In orange: the bounds provided by Theorem 1.2. In blue: the conjectural 1/4 threshold.

(*normal* condition) and a differential equation similar to the conformal Killing equation (see [DS10, HMS16, GPSU16] for further details on the conformal Killing equation). Under some pinching condition, we can then rule out the existence of such a non-trivial object by the use of the twisted Pestov identity, see Theorem 4.1 and §4. In comparison, earlier results on the ergodicity of the frame flow are based on purely topological arguments [BG80] or on geometric arguments on the universal cover of the manifold, see [BK84, BP03].

We believe that the new approach developed in the present paper should eventually lead to a proof of Conjecture 1.1, at least in dimensions 4 and $4\ell+2, \ell > 0$. At this stage, it is not clear whether we use the full strength of the twisted Pestov identity or if some improvements could be achieved in the computations. In particular, numerical experiments could help understand how sharp the inequalities derived from the Pestov identity in §4 are. Moreover, once the normal twisted conformal Killing tensor is obtained in §4, there might also be an alternative approach to the Pestov identity (e.g. another energy identity on the unit tangent bundle) in order to conclude that this tensor is zero. More generally, we believe that the approach of the present paper should allow one to study ergodicity of general isometric extensions (to a compact fiber bundle) of the geodesic flow over a negatively-curved Riemannian manifold, see [Lef] where this is further discussed.

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2. Preliminaries

In this section, we provide technical preliminaries necessary throughout this article.

2.1. **Bounds on the curvature tensor.** Let (M, g) be a smooth Riemannian manifold. We define the curvature tensor $R \in C^\infty(M, \Lambda^2 T^*M \otimes \mathrm{End}(TM))$ as:

$$R(X,Y)Z = \nabla^2 Z(X,Y) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

where ∇ is the Levi-Civita connection, and X, Y, Z are vector fields on M. We view the curvature tensor as a 4-tensor via $R(X,Y,Z,T) := \langle R(X,Y)Z, T\rangle$, where $\langle \bullet, \bullet\rangle = g(\bullet,\bullet)$ is the metric. We denote by $\kappa(a \wedge b) := \frac{R(a,b,b,a)}{|a|^2|b|^2 - \langle a,b\rangle^2}$ the sectional curvature of the plane spanned by a and b. The curvature tensor of a space of constant sectional curvature -1 is given by

$$G(a,b)c = \langle a,c\rangle.b - \langle b,c\rangle.a, \quad G(a,b,c,d) = \langle a,c\rangle.\langle b,d\rangle - \langle b,c\rangle.\langle a,d\rangle, \tag{2.1}$$

for any tangent vectors a, b, c, d. If (M, g) has δ-pinched negative sectional curvature (for some $0 < \delta \leq 1$), that is, the sectional curvature satisfies $-1 \leq \kappa(a \wedge b) \leq -\delta$ for all 2-planes $a \wedge b$, we set:

$$R_0 := R - \tfrac{1+\delta}{2}G. \tag{2.2}$$

Observe that for unit tangent vectors a, b, R_0 "centralizes" the tensor R around zero:

$$|R_0(a,b,b,a)| \leq \tfrac{1-\delta}{2}(1 - \langle a,b\rangle^2) \leq \tfrac{1-\delta}{2}. \tag{2.3}$$

Eventually, using polarisation identities for the curvature, [BK78, Lemma 3.7] shows that:

Lemma 2.1. *For all unit vectors $a, b, c, d \in T_xM$, the following estimate holds:*

$$|R_0(a,b,c,d)| \leq \tfrac{2(1-\delta)}{3}. \tag{2.4}$$

This estimate is sharp for the complex hyperbolic space, with $\delta = \frac{1}{4}$.

For $p \in \{1, ..., n\}$, the connection on $\Lambda^p TM \to M$ is induced by the Levi-Civita connection by asking the Leibniz rule to hold. The induced curvature R^{Λ^p} on the bundle $\Lambda^p TM$ (for $p = 1, ..., n$) is given by:

$$\begin{aligned} R^{\Lambda^p}(a,b)\big(e_1 \wedge ... \wedge e_p\big) =& (R(a,b)e_1) \wedge e_2 \wedge ... \wedge e_p \\ &+ ... + e_1 \wedge ... \wedge e_{p-1} \wedge (R(a,b)e_p) \end{aligned} \tag{2.5}$$

where $x \in M, a, b, e_1, ..., e_p \in T_xM$; the constant curvature map G^{Λ^p} is similarly defined from G. The scalar product on pure states in $\Lambda^p TM$ is given by the determinant, namely

$$\langle \eta_1 \wedge ... \wedge \eta_p, \omega_1 \wedge ... \wedge \omega_p\rangle = \det(\langle \eta_i, \omega_j\rangle)_{1\leq i,j\leq p}.$$

The induced connection on $\Lambda^p TM$ is compatible with this scalar product, that is, it is *orthogonal* (also called metric). As before, the curvature R^{Λ^p} splits as $R^{\Lambda^p} = R_0^{\Lambda^p} + \frac{1+\delta}{2} G^{\Lambda^p}$. Using (2.4), the fact that $R_0(a,b)$ (being skew-symmetric) is diagonalisable over $\mathbb{C}$, inducing a diagonal basis for $R_0^{\Lambda^p}(a,b)$, together with (2.5), we easily see that

$$|\langle R_0^{\Lambda^p}(a,b)\omega, \tau\rangle| \leq \tfrac{2p}{3}(1-\delta)|a||b||\omega||\tau|, \tag{2.6}$$

for every tangent vector a, b and p-forms ω, τ.

In the following, we will identify $\mathrm{Sym}^2 TM \to M$ with the bundle of symmetric endomorphisms of M, whose scalar product is given by

$$\langle A, B\rangle_x := \mathrm{Tr}(A(x)B(x)).$$

The action of the curvature map is extended to $\mathrm{Sym}^2 TM$ by the commutator action, namely for all $x \in M, C \in \mathrm{Sym}^2 T_x M$,

$$R^{\mathrm{Sym}^2}(a,b)C = [R(a,b), C].$$

Similarly, there is a splitting $R^{\mathrm{Sym}^2} = R_0^{\mathrm{Sym}^2} + \frac{1+\delta}{2} G^{\mathrm{Sym}^2}$ and using (2.4), we obtain the estimate:

$$|\langle R_0^{\mathrm{Sym}^2}(a,b)C, D\rangle| \leq \tfrac{4}{3}(1-\delta)|a||b||C||D|, \tag{2.7}$$

for all $x \in M, a, b \in T_x M, C, D \in \mathrm{Sym}^2 T_x M$.

2.2. **Fourier analysis in the fibers.** Further details on this paragraph can be found in [Pat99], [PSU15, Section 2] and [CL21a, Section 5].

2.2.1. *Trivial line bundle.* Let $\pi : SM \to M$ be the projection on the base, where SM is the unit tangent bundle of (M, g). There is a canonical splitting of the tangent bundle of SM as:

$$T(SM) = \mathbb{V} \oplus \mathbb{H} \oplus \mathbb{R}X,$$

where X is the geodesic vector field, $\mathbb{V} := \ker d\pi$ is the vertical space and $\mathbb{H}$ is the horizontal space defined in the following way. Consider the *connection map* $\mathcal{K} : T(SM) \to TM$ defined as follows: let $(x,v) \in SM, w \in T_{(x,v)}(SM)$ and a curve $(-\varepsilon, \varepsilon) \ni t \mapsto z(t) \in SM$ such that $z(0) = (x,v), \dot{z}(0) = w$; write $z(t) = (x(t), v(t))$; then $\mathcal{K}_{(x,v)}(w) := \nabla_{\dot{x}(t)} v(t)|_{t=0}$. Then $\mathbb{H} := \ker \mathcal{K}$ and if we define the *normal bundle* $\mathcal{N} \to SM$ whose fiber at $(x,v) \in SM$ is given by $\mathcal{N}(x,v) := \{v\}^\perp \subset T_x M$, then $d\pi : \mathbb{H} \to \{v\}^\perp, \mathcal{K} : \mathbb{V} \to \{v\}^\perp$ are both identified with isomorphisms $d\pi : \mathbb{H} \to \mathcal{N}, \mathcal{K} : \mathbb{V} \to \mathcal{N}$. In particular, we will think of the normal bundle $\mathcal{N}$ as the tangent bundle to the spheres. We denote by g_{Sas} the Sasaki metric on SM, which is the canonical metric on the unit tangent bundle, defined by:

$$g_{\mathrm{Sas}}(w, w') := g(d\pi(w), d\pi(w')) + g(\mathcal{K}(w), \mathcal{K}(w')).$$

For $x \in M$, the unit sphere

$$S_x M = \left\{ v \in T_x M \mid |v|_x^2 = 1 \right\} \subset SM$$

(endowed with the Sasaki metric) is then isometric to the canonical sphere $(\mathbb{S}^{n-1}, g_{\mathrm{can}})$. Denote by $\Delta_{\mathbb{V}}$ the vertical Laplacian obtained for $f \in C^\infty(SM)$ as $\Delta_{\mathbb{V}} f(x,v) = \Delta_{g_{\mathrm{can}}}(f|_{S_xM})(v)$, where $\Delta_{g_{\mathrm{can}}}$ is the (positive) spherical Laplacian. For $k \geq 0$, we introduce

$$\Omega_k(x) = \ker(\Delta_{\mathbb{V}}(x) - k(n+k-2)),$$

the spherical harmonics of degree k. Observe that $\Omega_k \to M$ is a well-defined vector bundle over M. Given $f \in C^\infty(SM)$, it can be decomposed as $f = \sum_{k\geq 0} \widehat{f_k}$ where $\hat{f}_k \in C^\infty(M, \Omega_k)$ is the projection of f onto spherical harmonics of degree k. We call *Fourier degree* of f, denoted by $\deg(f)$, the maximal integer $k_0 \in \mathbb{Z}_{\geq 0}$ such that $\hat{f}_{k_0} \neq 0$; it takes values in $\{0, 1, ..., +\infty\}$. We will also say that f has *finite Fourier content* if its degree is finite, and that f is *odd* (resp. *even*) if it only contains odd (resp. even) spherical harmonics.

It can be proved that the operator X has the following mapping properties (see [PSU15, Section 3]):

$$X : C^\infty(M, \Omega_k) \to C^\infty(M, \Omega_{k+1}) \oplus C^\infty(M, \Omega_{k-1}). \tag{2.8}$$

This is understood as follows: a section $\hat{f}_k \in C^\infty(M, \Omega_k)$ defines in particular a smooth function in $C^\infty(SM)$; we can differentiate in the X-direction and this only contains spherical harmonics of degree $k-1$ and $k+1$. Taking the projection on higher degree (resp. lower degree), we obtain an operator $X_+ : C^\infty(M, \Omega_k) \to C^\infty(M, \Omega_{k+1})$ of gradient type, i.e. with injective principal symbol (resp. $X_- : C^\infty(M, \Omega_k) \to C^\infty(M, \Omega_{k-1})$ of divergence type) such that $X = X_+ + X_-$ and $X_+^* = -X_-$ (the latter being a mere consequence of the fact that $X^* = -X$, since X preserves the Liouville measure on SM). As X_+ acting on spherical harmonics of degree k has injective principal symbol, its kernel is finite dimensional by elliptic theory if M is compact. We call *conformal Killing tensors* of degree $k \in \mathbb{Z}_{\geq 0}$ the elements in its kernel.

2.2.2. *Twist by a vector bundle.* Let $\mathcal{E} \to M$ be a *real*[2] vector bundle over M equipped with an orthogonal connection $\nabla^{\mathcal{E}}$. Consider the pullback bundle $\pi^*\mathcal{E} \to SM$ equipped with the pullback connection $\pi^*\nabla$ and introduce the first-order differential operator

$$\mathbf{X} := (\pi^*\nabla^{\mathcal{E}})_X : C^\infty(SM, \pi^*\mathcal{E}) \to C^\infty(SM, \pi^*\mathcal{E}).$$

The connection $\pi^*\nabla^{\mathcal{E}}$ also gives rise to differential operators:

$$\nabla^{\mathcal{E}}_{\mathbb{H},\mathbb{V}} : C^\infty(SM, \pi^*\mathcal{E}) \to C^\infty(SM, \pi^*\mathcal{E} \otimes \mathcal{N}),$$

defined in the following way: given $f \in C^\infty(SM, \pi^*\mathcal{E})$, the covariant derivative $\pi^*\nabla^{\mathcal{E}} f \in C^\infty(SM, T^*(SM) \otimes \pi^*\mathcal{E})$ can be identified with an element of $C^\infty(SM, T(SM) \otimes \pi^*\mathcal{E})$ by using the musical isomorphism $T^*(SM) \to T(SM)$ induced by the Sasaki metric. Using the orthogonal projections of $T(SM)$ onto $\mathbb{H}$ and $\mathbb{V}$ one can then define the operators:

$$\nabla^{\mathcal{E}}_{\mathbb{H}} f := d\pi((\pi^*\nabla^{\mathcal{E}} f)_{\mathbb{H}}), \ \nabla^{\mathcal{E}}_{\mathbb{V}} f := \mathcal{K}((\pi^*\nabla^{\mathcal{E}} f)_{\mathbb{V}}),$$

[2] It can also be taken to be complex but we will always consider real bundles throughout this article.

which take values in the bundle $\pi^* \mathcal{E} \otimes \mathcal{N} \to SM$. In local coordinates, these operators have explicit expressions in terms of the connection 1-form and we refer to [GPSU16, Lemma 3.2] for further details.

If $(e_1, ..., e_r)$ is a smooth local orthonormal basis of $\mathcal{E}$ in a neighborhood of a point $x_0 \in M$, then smooth sections $f \in C^\infty(SM, \pi^* \mathcal{E})$ can be written near x_0 as:

$$f(x,v) = \sum_{j=1}^{r} f^{(j)}(x,v) e_j(x) \in \mathcal{E}_x,$$

where $f^{(j)} \in C^\infty(SM)$ is only locally defined. As in §2.2, each $f^{(j)}$ can be in turn decomposed into spherical harmonics. In other words, we can write $f = \sum_{k \geq 0} \widehat{f}_k$, where $\hat{f}_k \in C^\infty(M, \Omega_k \otimes \mathcal{E})$ and pointwise in $x \in M$:

$$\Omega_k \otimes \mathcal{E}(x) := \ker(\Delta_{\mathbb{V}}^{\mathcal{E}}(x) - k(n+k-2)),$$

is the kernel of the vertical Laplacian $\Delta_{\mathbb{V}}^{\mathcal{E}}$ (this Laplacian is independent of the connection $\nabla^{\mathcal{E}}$, it only depends on $\mathcal{E}$ and on g). Elements in this kernel are called the *twisted spherical harmonics of degree* k and they form a well-defined vector bundle $\Omega_k \otimes \mathcal{E} \to M$. As in §2.2, we can define the degree of $f \in C^\infty(SM, \mathcal{E})$ and we say that f has *finite Fourier content* if its expansion in spherical harmonics only contains a finite number of terms.

We call *twisted cohomological equation* an equation of the form $\mathbf{X} f = h$, where $h \in C^\infty(SM, \pi^* \mathcal{E})$ is given. We will be interested more specifically in the case where $h = 0$. Similarly to (2.8), the operator $\mathbf{X}$ maps

$$\mathbf{X} : C^\infty(M, \Omega_k \otimes \mathcal{E}) \to C^\infty(M, \Omega_{k-1} \otimes \mathcal{E}) \oplus C^\infty(M, \Omega_{k+1} \otimes \mathcal{E}) \quad (2.9)$$

and can be decomposed as $\mathbf{X} = \mathbf{X}_+ + \mathbf{X}_-$, where, if $u \in C^\infty(M, \Omega_k \otimes \mathcal{E})$, $\mathbf{X}_+ u \in C^\infty(M, \Omega_{k+1} \otimes \mathcal{E})$ denotes the orthogonal projection on the twisted spherical harmonics of degree $k+1$. The operator $\mathbf{X}_+$ is elliptic and thus has finite-dimensional kernel (when M is compact) which consists of *twisted conformal Killing Tensors* (CKTs) whereas $\mathbf{X}_-$ is of divergence type. Moreover, $\mathbf{X}_+^* = -\mathbf{X}_-$, where the adjoint is computed with respect to the canonical L^2 scalar product on SM induced by the Sasaki metric. We also refer to the original articles of Guillemin-Kazhdan [GK80a, GK80b] for a description of these facts and to [GPSU16] for a more modern exposition. It was shown in [GPSU16, Theorem 4.1] that flow-invariant sections, i.e. smooth sections in $\ker \mathbf{X}$ have *finite Fourier content*.

Let us also mention that if $\mathfrak{o}(\mathcal{E})$ is any vector bundle obtained by a functorial operation $\mathfrak{o}$ from $\mathcal{E}$ (e.g. dual, exterior and symmetric powers, tensor products), there is an induced orthogonal connection $\nabla^{\mathfrak{o}(\mathcal{E})}$ on $\mathfrak{o}(\mathcal{E})$ and thus an induced operator $\mathbf{X}$ acting on $C^\infty(SM, \pi^* \mathfrak{o}(\mathcal{E}))$. In order not to burden the notation, we will keep the notation $\mathbf{X}$ even if it might denote an operator acting on distinct vector bundles. In particular, this will be applied with $\mathfrak{o}(\mathcal{E}) = \Lambda^p \mathcal{E}$ or $\mathfrak{o}(\mathcal{E}) = \mathrm{Sym}^2 \mathcal{E}$, with $\mathcal{E} = TM$.

2.2.3. *Twisted Pestov identity.* The Pestov identity is a classical identity in Riemannian geometry, see [GK80a, CS98, PSU15] and [GPSU16] for the twisted version. If $(\mathcal{E}, \nabla^{\mathcal{E}})$ is a vector bundle with an orthogonal connection, we write $\mathrm{End}_{\mathrm{sk}}(\mathcal{E})$ for skew-symmetric endomorphisms of $\mathcal{E}$ and

$$F_{\nabla} = F_{\nabla^{\mathcal{E}}} = (\nabla^{\mathcal{E}})^{\circ 2} \in C^\infty(M, \Lambda^2 T^*M \otimes \mathrm{End}_{\mathrm{sk}}(\mathcal{E})),$$

for the curvature. Following [GPSU16, Section 3], let $\mathcal{F}^{\mathcal{E}} \in C^\infty(SM, \mathcal{N} \otimes \mathrm{End}_{\mathrm{sk}}(\mathcal{E}))$ be defined by the identity:

$$\langle \mathcal{F}^{\mathcal{E}}(x,v)e, w \otimes e' \rangle := \langle (F_\nabla)_x(v,w)e, e' \rangle, \tag{2.10}$$

where $(x,v) \in SM, e, e' \in \mathcal{E}_x, w \in \mathcal{N}(x,v)$, and the metric on the right-hand side is the tensor product metric on $\mathcal{N}(x,v) \otimes \mathcal{E}_x$. Similarly, we will view the Riemannian curvature tensor as an operator on $\mathcal{N} \otimes \mathcal{E}$, defined by the relation:

$$R(x,v)(w \otimes e) = (R_x(w,v)v) \otimes e, \quad w \in \mathcal{N}(x,v), e \in \mathcal{E}_x\,.$$

Lemma 2.2. *We have, for any orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_n \in T_xM$, and $v \in S_xM$:*

$$\mathcal{F}^{\mathcal{E}}(x,v) = \sum_{i=1}^n \mathbf{e}_i \otimes F_\nabla(v, \mathbf{e}_i). \tag{2.11}$$

Proof. We simply write:

$$\mathcal{F}^{\mathcal{E}}(x,v) = \sum_{i=1}^n \mathbf{e}_i \otimes \mathcal{F}_i(x,v),$$

for some endomorphisms $\mathcal{F}_i(x,v) : \mathcal{E}_x \to \mathcal{E}_x$. Since $\mathcal{F}^{\mathcal{E}}$ is a section of $\mathcal{N} \otimes \mathrm{End}(\mathcal{E})$, for all $B \in \mathrm{End}(\mathcal{E}_x)$ we have:

$$0 = \langle \mathcal{F}^{\mathcal{E}}(x,v), v \otimes B \rangle = \Big\langle \sum_i \langle v, \mathbf{e}_i \rangle . \mathcal{F}_i, B \Big\rangle,$$

that is we have $\sum_i \langle v, \mathbf{e}_i \rangle . \mathcal{F}_i(x,v) = 0$. Thus we compute, by plugging in $w = \mathbf{e}_j - v.\langle v, \mathbf{e}_j \rangle \in \mathcal{N}(x,v)$ in (2.10), for any $e, e' \in \mathcal{E}_x$:

$$\begin{aligned}\langle F_\nabla(v, \mathbf{e}_j)e, e' \rangle &= \langle \mathcal{F}^{\mathcal{E}}(x,v)e, (\mathbf{e}_j - v.\langle v, \mathbf{e}_j \rangle) \otimes e' \rangle = \sum_i \Big\langle \mathbf{e}_i \otimes \mathcal{F}_i e, (\mathbf{e}_j - v.\langle \mathbf{e}_j, v \rangle) \otimes e' \Big\rangle \\ &= \langle \mathcal{F}_j(x,v)e, e' \rangle - \langle \mathbf{e}_j, v \rangle . \Big\langle \underbrace{\sum_i \langle \mathbf{e}_i, v \rangle . \mathcal{F}_i}_{=0} e, e' \Big\rangle,\end{aligned}$$

using that $F_\nabla(v,v) = 0$. Since $e, e' \in \mathcal{E}_x$ are arbitrary, we have $\mathcal{F}_j(x,v) = F_\nabla(v, \mathbf{e}_j)$. ☐

All the norms below are the L^2-norms. In order to avoid repetitions, we suppress the subscript L^2. We call the following identity, the *twisted Pestov identity*. It is slightly different from what [GPSU16] call a twisted identity but the following lemma can be easily recovered from [GPSU16, Proposition 3.5].

Lemma 2.3 (Localized Pestov identity)**.** *Let (M,g) be an n-dimensional Riemannian manifold. The following identity holds: for all $k \in \mathbb{Z}_{\geq 0}$, $u \in C^\infty(M, \Omega_k \otimes \mathcal{E})$,*

$$\frac{(n+k-2)(n+2k-4)}{n+k-3}\|\mathbf{X}_- u\|^2 - \frac{k(n+2k)}{k+1}\|\mathbf{X}_+ u\|^2 + \|Z(u)\|^2 = \langle R\nabla^{\mathcal{E}}_{\mathbb{V}} u, \nabla^{\mathcal{E}}_{\mathbb{V}} u\rangle + \langle \mathcal{F}^{\mathcal{E}} u, \nabla^{\mathcal{E}}_{\mathbb{V}} u\rangle, \tag{2.12}$$

where Z is a first order differential operator which we do not make explicit.

Proof. By [GPSU16, Proposition 3.5], the following equality holds for $u \in C^\infty(M, \Omega_k \otimes \mathcal{E})$:

$$(n+2k-3)\|\mathbf{X}_- u\|^2 + \|\nabla^{\mathcal{E}}_{\mathbb{H}} u\|^2 - \langle R\nabla^{\mathcal{E}}_{\mathbb{V}} u, \nabla^{\mathcal{E}}_{\mathbb{V}} u\rangle - \langle \mathcal{F}^{\mathcal{E}} u, \nabla^{\mathcal{E}}_{\mathbb{V}} u\rangle = (n+2k-1)\|\mathbf{X}_+ u\|^2. \tag{2.13}$$

Moreover, by [GPSU16, Lemma 3.7], we have:

$$\nabla^{\mathcal{E}}_{\mathbb{H}} u = \frac{1}{k+1}\nabla^{\mathcal{E}}_{\mathbb{V}} \mathbf{X}_+ u - \frac{1}{n+k-3}\nabla^{\mathcal{E}}_{\mathbb{V}} \mathbf{X}_- u + Z(u),$$

where $Z(u)$ is some term with vanishing vertical divergence (i.e. $\operatorname{div}^{\mathcal{E}}_{\mathbb{V}} Z(u) = 0$, where $\operatorname{div}^{\mathcal{E}}_{\mathbb{V}}$ is the formal adjoint to the vertical gradient $\nabla^{\mathcal{E}}_{\mathbb{V}}$). As a consequence, taking the L^2-norms, we get:

$$\begin{aligned} \|\nabla^{\mathcal{E}}_{\mathbb{H}} u\|^2 &= \frac{1}{(k+1)^2}\|\nabla^{\mathcal{E}}_{\mathbb{V}} \mathbf{X}_+ u\|^2 + \frac{1}{(n+k-3)^2}\|\nabla^{\mathcal{E}}_{\mathbb{V}} \mathbf{X}_- u\|^2 + \|Z(u)\|^2 \\ &= \frac{n+k-1}{k+1}\|\mathbf{X}_+ u\|^2 + \frac{k-1}{n+k-3}\|\mathbf{X}_- u\|^2 + \|Z(u)\|^2. \end{aligned} \tag{2.14}$$

(Here, we simply use that $\operatorname{div}^{\mathcal{E}}_{\mathbb{V}} \nabla^{\mathcal{E}}_{\mathbb{V}} = \Delta^{\mathcal{E}}_{\mathbb{V}}$.) Plugging (2.14) into (2.13), and after some algebraic simplifications, we obtain the claimed result. □

2.3. **Link with symmetric tensors.** We refer to [HMS16] as well as [GPSU16, Section 3] and [CL21a] for further details on this paragraph. In the following, we will keep identifying TM and T^*M by the metric. Let $\operatorname{Sym}^k TM \to M$ be the vector bundle of symmetric k-tensors over M. If $u = \sum_{i_1,...,i_k=1}^n u_{i_1...i_k}\, \mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_k} \in TM^{\otimes k}$, where $(\mathbf{e}_1, ..., \mathbf{e}_n)$ is a local orthonormal frame, then the orthogonal projection of u onto symmetric tensors $\operatorname{Sym} : TM^{\otimes k} \to \operatorname{Sym}^k TM$ is given by (here S_k denotes the permutation group):

$$\operatorname{Sym}(u) = \frac{1}{k!}\sum_{\sigma \in S_k}\sum_{i_1,...,i_k=1}^n u_{i_1...i_k}\, \mathbf{e}_{i_{\sigma(1)}} \otimes ... \otimes \mathbf{e}_{i_{\sigma(k)}}\,.$$

Let $\mathcal{I} : \operatorname{Sym}^{k+2} TM \to \operatorname{Sym}^k TM$ be the trace operator defined pointwise on M as:

$$\mathcal{I}u := \sum_{i=1}^n u(\mathbf{e}_i, \mathbf{e}_i, \bullet, ..., \bullet).$$

We define $\mathrm{Sym}_0^k TM := \mathrm{Sym}^k TM \cap \ker \mathcal{I}$ to be the space of trace-free tensors. The adjoint of $\mathcal{I}$ with respect to the natural metric[3] on $\mathrm{Sym}^k TM$, which we denote by $\mathcal{J}$, is the symmetric multiplication by the metric g, namely:

$$\mathcal{I}^* u = \mathcal{J} u = \mathrm{Sym}(g \otimes u).$$

The space $\mathrm{Sym}^k TM$ breaks up as the orthogonal sum:

$$\mathrm{Sym}^k TM = \oplus_{i=0}^{\lfloor k/2 \rfloor} \mathcal{J}^i \mathrm{Sym}_0^{k-2i} TM. \tag{2.15}$$

Let

$$\mathcal{P} : \mathrm{Sym}^k TM \to \mathrm{Sym}_0^k TM$$

be the orthogonal projection onto trace-free symmetric tensors, that is, onto the highest degree summand of (2.15).

We can consider symmetric tensors in $\mathrm{Sym}^k TM$ as homogeneous polynomials of degree k on TM, or, by restricting to the unit sphere, as spherical harmonics of degree $\leq k$. In fact, for any $x \in M$, $k \in \mathbb{Z}_{\geq 0}$, we introduce the *pullback operator* defined pointwise at x by:

$$(\pi_k^* u)(x, v) := u_x(v^{\otimes k}), \qquad \pi_k^* : \mathrm{Sym}^k TM(x) \otimes \mathcal{E}(x) \xrightarrow{\sim} \bigoplus_{i=0}^{\lfloor k/2 \rfloor} \Omega_{k-2i}(x) \otimes \mathcal{E}(x), \tag{2.16}$$

and this operator is a graded isomorphism

$$\pi_k^* : \mathrm{Sym}_0^k TM(x) \otimes \mathcal{E}(x) \xrightarrow{\sim} \Omega_k(x) \otimes \mathcal{E}(x).$$

Given $f \in C^\infty(SM, \pi^* \mathcal{E})$, we have $\deg(f) < \infty$ if and only if there exists $k_1, k_2 \in \mathbb{Z}_{\geq 0}$, $f_{\mathrm{even}} \in C^\infty(M, \mathrm{Sym}^{2k_1} TM \otimes \mathcal{E})$, $f_{\mathrm{odd}} \in C^\infty(M, \mathrm{Sym}^{2k_2+1} TM \otimes \mathcal{E})$ such that

$$f = \pi_{2k_1}^* f_{\mathrm{even}} + \pi_{2k_2+1}^* f_{\mathrm{odd}}.$$

In other words, sections on SM with finite Fourier content can always be seen as sections defined on the base. We now relate the operators $\mathbf{X}$ and $\mathbf{X}_+$ with the usual *symmetrized covariant derivative* D and $\mathrm{D}_{\mathcal{E}}$. We introduce:

$$\mathrm{D} : C^\infty(M, \mathrm{Sym}^k TM) \to C^\infty(M, \mathrm{Sym}^{k+1} TM), \ \mathrm{D} := \mathrm{Sym} \circ \nabla,$$

where ∇ is the Levi-Civita connection induced by g. Similarly, in the twisted case, we can consider:

$$\mathrm{D}_{\mathcal{E}} : C^\infty(M, \mathrm{Sym}^k TM \otimes \mathcal{E}) \to C^\infty(M, \mathrm{Sym}^{k+1} TM \otimes \mathcal{E}), \ \mathrm{D}_{\mathcal{E}} := \mathrm{Sym} \circ \nabla^{\mathcal{E}}.$$

[3]The scalar product on $TM^{\otimes k}$ is given by:

$$g_{TM^{\otimes k}}(v_1 \otimes ... \otimes v_k, w_1 \otimes ... \otimes w_k) := \prod_{j=1}^k g(v_j, w_j),$$

where $v_i, w_i \in TM$ and this induces a scalar product on $\mathrm{Sym}^k TM$ by restriction of the metric.

For the sake of simplicity, we will drop the subscript $\mathcal{E}$ and write D, even in the twisted case. We have the following relations, see [GPSU16, Section 3.6]:

Lemma 2.4. *The following relations hold:*

(1) For $f \in C^\infty(M, \mathrm{Sym}^k TM \otimes \mathcal{E})$, $\pi_{k+1}^* \mathrm{D}\, f = \mathbf{X} \pi_k^* f$;
(2) For $f \in C^\infty(M, \mathrm{Sym}_0^k TM \otimes \mathcal{E})$, $\pi_{k+1}^* \mathcal{P}\, \mathrm{D}\, f = \mathbf{X}_+ \pi_k^* f$.

Elements in $\ker \mathrm{D}$ are called *twisted Killing Tensors* while elements in $\ker \mathcal{P}\,\mathrm{D}$ are called *twisted Conformal Killing Tensors*. In the non-twisted case, when the metric has negative sectional curvatures, it is known that there are no conformal Killing tensors for $k \geq 1$, see [DS10, HMS16]. Investigating the (non-)existence of twisted Conformal Killing Tensors satisfying certain algebraic properties will play an important role in this article and this is due to the following observation: if $f \in C^\infty(SM, \pi^* \mathcal{E}) \cap \ker \mathbf{X}$, then f has finite degree by [GPSU16, Theorem 4.1]. Hence, we can decompose f into Fourier modes $f = \hat{f}_0 + ... + \hat{f}_k$ with $k \geq 0$, $\hat{f}_k \neq 0$, and we have $\mathbf{X}_+ \hat{f}_k = 0$ by the mapping properties (2.9) of $\mathbf{X}$. By the previous Lemma 2.4, this implies the existence of a non-trivial trace-free twisted conformal Killing tensor of degree $k \geq 0$.

3. Dynamics of the frame flow and topology of the frame bundle

3.1. **Extensions of Anosov flows to principal bundles.** Frame flows on negatively-curved manifolds are typical examples of partially hyperbolic systems arising as the extension on a principal bundle of an Anosov flow. This was originally studied by Brin [Bri75b, Bri75a] (see also the survey [Bri82]) who treated the general case of a principal bundle over a manifold with transitive Anosov flow. In this paragraph, we consider the following setting: we let $\mathcal{M}$ be a smooth closed manifold equipped with a volume-preserving Anosov flow $(\varphi_t)_{t\in\mathbb{R}}$ and we let $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ be its generator, $\pi : P \to \mathcal{M}$ is a principal G-bundle over $\mathcal{M}$ and $(\Phi_t)_{t\in\mathbb{R}}$ is an extension of $(\varphi_t)_{t\in\mathbb{R}}$ in the sense that it satisfies the relations: for all $t \in \mathbb{R}, g \in G$,

$$\pi \circ \Phi_t = \varphi_t \circ \pi, \;\; R_g \circ \Phi_t = \Phi_t \circ R_g, \tag{3.1}$$

where $R_g : P \to P$ denotes the right-action in the fibers.

In order to describe the flow $(\Phi_t)_{t\in\mathbb{R}}$, we adopt a slightly different point of view than the original approach of Brin [Bri75b, Bri75a] and use the point of view of [CL21b]. Proofs of the following facts can be found in [Lef]. We fix an arbitrary periodic point $x_\star \in \mathcal{M}$ for the flow $(\varphi_t)_{t\in\mathbb{R}}$. We let $\mathcal{H}$ be the set of *homoclinic orbits* to $x_\star$, namely the set of all orbits for the flow $(\varphi_t)_{t\in\mathbb{R}}$ converging in the past and in the future to the periodic orbit $\gamma_\star$ of $x_\star$. We then introduce *Parry's free monoid* $\mathbf{G}$ as the following formal free monoid:

$$\mathbf{G} := \{\gamma_1 \cdot ... \cdot \gamma_k \mid k \in \mathbb{N}, \forall i \in \{1, ..., k\}, \gamma_i \in \mathcal{H}\}.$$

We denote by $W^{s,u}_{\mathcal{M}}$ the strong stable/unstable foliation of the flow on $\mathcal{M}$. Given $x \in \mathcal{M}$ and $x' \in W^s_{\mathcal{M}}(x)$, one can define for x close to x' a holonomy map $\mathrm{Hol}^s_{x\to x'} : P_x \to P_{x'}$ in the following way:

$$\mathrm{Hol}^s_{x\to x'} w := \lim_{t\to+\infty} \Phi_{-t} \circ C_{\varphi_t(x)\to\varphi_t(x')} \circ \Phi_t(w), \tag{3.2}$$

where $C_{x_1\to x_2} : P_{x_1} \to P_{x_2}$ is defined, for pairs of points that are close enough, as the parallel transport (with respect to an arbitrary connection on P) along the unique short geodesic (with respect to an arbitrary metric on $\mathcal{M}$) joining x_1 to x_2. Convergence of (3.2) is ensured by the Ambrose-Singer formula together with the fact that the distance between $\varphi_t(x)$ and $\varphi_t(x')$ converges exponentially fast to 0, see [CL21b, Section 3.2.2] for instance. Alternatively, one can define the holonomy map $w' := \mathrm{Hol}^s_{x\to x'} w$ as the unique point in the intersection

$$W^s_P(w) \cap P_{x'} = \{w'\},$$

where $W^{s,u}_P$ denotes the strong stable/unstable foliation in the principal bundle. Similarly, we define $\mathrm{Hol}^u_{x\to x'}$ for $x' \in W^u_{\mathcal{M}}(x)$ by taking the limit as $t \to -\infty$. Eventually, for x' on the same flowline as x, there is also a natural holonomy map $\mathrm{Hol}^c_{x\to x'}$ given by the flow $(\Phi_t)_{t\in\mathbb{R}}$ itself. Let $P_\star := P_{x_\star}$ be the fiber over $x_\star$. After an arbitrary choice of point $w_\star \in P_\star$, the fiber $P_\star$ gets identified with the group by the map $G \to P_\star, g \mapsto R_g w_\star$.

This formalism allows to define a representation of the monoid $\mathbf{G}$. We call *Parry's representation* the representation $\rho : \mathbf{G} \to G$ of the free monoid obtained by the following process. For $w \in G \simeq P_\star$, we set $\rho(\gamma_\star)w = \Phi_{T_\star}(x_\star)w$, and for $\gamma \in \mathcal{H}$ with $\gamma \neq \gamma_\star$, we pick two arbitrary points $x_1 \in W^u_{\mathcal{M}}(x_\star) \cap \gamma, x_2 \in W^s_{\mathcal{M}}(x_\star) \cap \gamma$ (where $x_{1,2}$ are chosen close to $x_\star$) and define

$$\rho(\gamma)w := \mathrm{Hol}^s_{x_2\to x_\star} \circ \mathrm{Hol}^c_{x_1\to x_2} \circ \mathrm{Hol}^u_{x_\star\to x_1} w \in G. \tag{3.3}$$

Note that $\rho(\gamma)$ is an isometry of the group and it commutes with the right action by (3.1) so it is a left action and can therefore be identified with an element of the group G itself, namely $\rho(\gamma) \in G$.

In [CL21b] Parry's representation was defined by choosing a $k_n \to \infty$ such that we have $\Phi_{T_\star}(x_\star)^{k_n} \to \mathrm{Id}_{x_\star}$, and specialising to $t = k_n T_\star$ in (3.2); in particular, the two definitions agree. The following notion plays a central role and was identified by Brin [Bri75b, Bri75a]:

Definition 3.1. The *transitivity group* of the flow $(\Phi_t)_{t\in\mathbb{R}}$ is $H := \overline{\rho(\mathbf{G})}$.

Observe that H is a closed subgroup of a compact Lie group, it is thus a Lie group [Hel01, Theorem 2.3]. Moreover, H is independent of the choice of points x_1, x_2 in (3.3). However, the transitivity group H does depend on the choice of point $w_\star$ in $P_\star$ and changing $w_\star$ by $w'_\star$, one obtains another group H' which is conjugate to H. In other words, the transitivity group is only well-defined up to conjugacy. We observe that Brin does not use exactly the same definition of transitivity group but the two notions coincide.

Proposition 3.2. *There exists an H-principal bundle $Q \to \mathcal{M}$ such that $w_\star \in Q$, $Q \subset P$ is a flow-invariant subbundle, and the restriction of $(\Phi_t)_{t\in\mathbb{R}}$ to Q is ergodic. In particular, if $H = G$, then the flow $(\Phi_t)_{t\in\mathbb{R}}$ is ergodic.*

For a proof, we refer to [Lef] (see also the papers of Brin [Bri75b, Bri75a]). From a topological perspective, the reduction to a subgroup $H \leqslant G$ is a strong constraint called a *reduction of the structure group* of the principal bundle (see §3.3).

3.2. **Transitivity group of the frame flow.** We now specify the previous discussion to the case where $\mathcal{M} = SM$, X is the geodesic vector field and $(\varphi_t)_{t\in\mathbb{R}}$ is the geodesic flow, $(\Phi_t)_{t\in\mathbb{R}}$ is the frame flow on the principal $\mathrm{SO}(n-1)$-bundle $FM \to SM$. A point $w \in FM_{(x,v)}$ over $(x, v) \in SM$ is seen from now on as an isometry $w : \mathbb{R}^{n-1} \to v^\perp$. For $g \in \mathrm{SO}(n-1)$, the right-action R_g in the fibers is given by composition, namely $R_g w = w \circ g$.

We fix an arbitrary periodic point $(x_\star, v_\star)$ and set $\mathcal{N}_\star := \mathcal{N}(x_\star, v_\star)$. By Proposition 3.2, the closure H of the representation of Parry's free monoid $\rho : \mathbf{G} \to \mathrm{SO}(\mathcal{N}_\star) \simeq \mathrm{SO}(n-1)$ gives a flow-invariant reduction of the structure group of the frame bundle. Note that the identification $\mathbb{R}^{n-1} \simeq \mathcal{N}_\star$ is made by choosing an arbitrary isometry $w_\star : \mathbb{R}^{n-1} \to \mathcal{N}_\star$, and this corresponds to choosing a point in the frame bundle over $(x_\star, v_\star)$. Changing $w_\star$ by another isometry $w'_\star$, we would obtain another conjugate group H'. We start by the following observation, mostly due to Brin [Bri75b]:

Lemma 3.3. *Let (M, g) be a negatively-curved Riemannian manifold of dimension ≥ 3. Let $H \leqslant \mathrm{SO}(n-1)$ be the transitivity group of the frame flow and $H_0 \leqslant H$ be its identity component. Then, there is a finite Riemannian cover $(\widehat{M}, \widehat{g}) \to (M, g)$ such that the frame flow has transitivity group equal to H_0.*

The conclusion also holds by replacing the negatively-curved assumption by Anosov.

Proof. Let $w_\star \in FM$ and $Q(w_\star)$ be its ergodic component given by Proposition 3.2. Since the restriction of the flow to $Q(w_\star)$ is transitive, the bundle $Q(w_\star) \to SM$ is a connected H-principal bundle and thus $Q(w_\star)/H_0 \to SM$ is a finite cover of SM with deck transformation group $G := H/H_0$. Now, by the long exact sequence in homotopy [Hat02, Theorem 4.41], we have:

$$... \longrightarrow \underbrace{\pi_1(\mathbb{S}^{n-1})}_{=0} \longrightarrow \pi_1(SM) \longrightarrow \pi_1(M) \longrightarrow \underbrace{\pi_0(\mathbb{S}^{n-1})}_{=0} \longrightarrow ...,$$

that is $\pi_1(SM) \simeq \pi_1(M)$. Thus there is a subgroup $\Gamma \leqslant \pi_1(M)$ such that $\widehat{M} := \widetilde{M}/\Gamma$ is a Riemannian cover of M (equipped with the pullback metric) with deck transformation group G and a diffeomorphism $F : S\widehat{M} \to Q(w_\star)/H_0$; here $\widetilde{M}$ denotes the universal cover of M. Moreover, the geodesic flow on $S\widehat{M}$ and the frame flow on $Q(w_\star)/H_0$ are G-equivariant and both project to the geodesic flow on SM so they are conjugate by F. Eventually, it suffices to observe that the frame flow on $S\widehat{M}$ has transitivity group equal to H_0. □

By construction, the group H acts on $\mathcal{N}_\star$ and thus on any vector space obtained functorially from $\mathcal{N}_\star$. In other words, if Vec denotes the category of finite-dimensional Euclidean (or Hermitian) vector spaces and

$$\mathfrak{o} : \mathrm{Vec} \to \mathrm{Vec}$$

is an operation of (finite-dimensional) vector spaces (e.g. exterior power, symmetric or tensor power, dual), we obtain an induced representation

$$\rho_{\mathfrak{o}} : H \to \mathrm{SO}(\mathfrak{o}(\mathcal{N}_\star)),$$

which simply corresponds to parallel transport along homoclinic orbits of vectors of $\mathfrak{o}(\mathcal{N}_\star)$. We define:

$$\mathfrak{f}_{\mathfrak{o}} := \{v \in \mathfrak{o}(\mathcal{N}_\star) \mid \forall h \in H, \rho_{\mathfrak{o}}(h)v = v\},$$

i.e. these are all the elements of $\mathfrak{o}(\mathcal{N}_\star)$ which are invariant by the induced action of H.

Proposition 3.4. *Let $\mathfrak{o}$: Vec → Vec be an operation of vector spaces. Then, the evaluation map:*

$$\Phi : C^\infty(SM, \mathfrak{o}(\mathcal{N})) \cap \ker \mathbf{X} \to \mathfrak{f}_{\mathfrak{o}}, \ \ \Phi(f) := f(x_\star, v_\star),$$

is an isomorphism.

In other words, if the induced representation on $\mathfrak{o}(\mathcal{N}_\star)$ fixes a vector, then there exists a unique flow-invariant section of $\mathfrak{o}(\mathcal{N}) \to SM$ whose value at $(x_\star, v_\star)$ is given by that vector. We only sketch the proof, which can be found in [CL21b, Lemma 3.6]. The presence of the operation $\mathfrak{o}$ is purely functorial, so it is harmless to assume that $\mathfrak{o} = \mathbb{1}$ (identity) and to work with $\mathcal{N}_\star$.

Idea of proof. The fact that the map Φ is well-defined is almost tautological. The injectivity of Φ is immediate. Indeed, if $\mathbf{X}f = 0$, we have $X|f|^2 = 2\langle \mathbf{X}f, f\rangle = 0$ that is $|f|^2$ is constant by ergodicity of the geodesic flow. Hence if $\Phi(f) = f(x_\star, v_\star) = 0$, we get $f = 0$. Let us now show surjectivity. We consider a vector $e_\star \in \mathcal{N}_\star$ such that for any homoclinic orbit $\gamma \in \mathbf{G}$, we have $\rho(\gamma)e_\star = e_\star$. We denote by $(\Phi_t^{(2)})_{t\in\mathbb{R}}$ the flow on 2-frames (i.e. the parallel transport along the geodesic flow of a vector that is orthogonal to the geodesic). Given $\gamma \in \mathbf{G}$, we can then define the invariant section f on γ in the following way: consider an arbitrary point $(x_0, v_0) \in \gamma \cap W^u_{SM}(x_\star, v_\star)$ and define for $(x, v) \in \gamma$, $f(x, v) := \Phi_t^{(2)} \circ \mathrm{Hol}^u_{(x_\star,v_\star)\to(x_0,v_0)} e_\star$, where $t \in \mathbb{R}$ is such that $\varphi_t(x_0, v_0) = (x, v)$. Using that $e_\star$ is preserved by the holonomy along $\gamma_\star$, one can check that this definition is independent of the choice of (x_0, v_0). Moreover, f is obviously flow-invariant on γ by construction.

The union of all homoclinic orbits turns out to be dense in SM, see [CL21b, Lemma 3.11]. Following the same proof as in [CL21b, Lemma 3.21] relying on the local product structure of the geodesic flow, one can then show that f is Lipschitz-continuous on the set of all homoclinic orbits (the key point in this proof is that $e_\star$ satisfies $h \cdot e_\star = e_\star$ for all $h \in H$; this implies that $f(x, v)$ is also equal to $\Phi_{t'}^{(2)} \circ \mathrm{Hol}^s_{(x_\star,v_\star)\to(x_1,v_1)} e_\star$, where $(x_1, v_1) \in \gamma \cap W^s_{SM}(x_\star, v_\star)$

and $t' \in \mathbb{R}$ is such that $(x,v) = \varphi_{t'}(x_1,v_1))$. Since homoclinic orbits are dense in SM, this shows that f extends as a Lipschitz-continuous section $f \in C^{\mathrm{Lip}}(SM,\mathcal{N})$ that is flow-invariant, i.e. such that $\mathbf{X}f = 0$. Since the *threshold*[4] of the operator $\mathbf{X}$ is 0, we can apply [GL20, Theorem 1.4] to obtain that f is actually smooth. □

To conclude, let us make the following important remark:

Remark 3.5. The *type* of an H-invariant object is always preserved. For instance, if $n = 8$, $H = \mathrm{G}_2$, $\mathfrak{o} = \Lambda^3$, then H fixes an invariant vector $f_\star \in \Lambda^3\mathfrak{o}(\mathcal{N}_\star)$, the 3-form defining the G_2-structure. Consider the section $f \in C^\infty(SM, \Lambda^3\mathcal{N})$ such that $\mathbf{X}f = 0$, $f(x_\star, v_\star) = f_\star$, provided by Proposition 3.4. Then $f(x,v)$ is a 3-form with G_2-stabilizer on $\mathcal{N}(x,v)$ for all $(x,v) \in SM$. This can be easily seen by the following argument. First of all, in order to define $f_\star$, one has first chosen an implicit arbitrary isometry $w_\star : \mathbb{R}^7 \to \mathcal{N}(x_\star, v_\star)$ and then $f_\star := w_\star(\xi_0)$, where ξ_0 is the 3-form G_2-structure on $\mathbb{R}^7$. The isometry $w_\star$ is also a point in the frame bundle over SM. By Proposition 3.2, there exists a (unique) principal G_2-bundle $Q \to SM$ which is flow-invariant and such that $w_\star \in Q$. Let $(x,v) \in SM$ and $w_{(x,v)} \in Q_{(x,v)}$ seen as an isometry $w_{(x,v)} : \mathbb{R}^7 \to \mathcal{N}(x,v)$. Then we get an induced isometry $w_{(x,v)} : \Lambda^3\mathbb{R}^7 \to \Lambda^3\mathcal{N}(x,v)$ and we claim that $f(x,v) = w_{(x,v)}(\xi_0)$. Indeed, this is clearly independent of the choice of point $w_{(x,v)} \in Q_{(x,v)}$ since any other point $w'_{(x,v)}$ is obtained as $w'_{(x,v)} = w_{(x,v)} \circ h$, where $h \in \mathrm{G}_2$ and $w'_{(x,v)}(\xi_0) = w_{(x,v)}(h\xi_0) = w_{(x,v)}(\xi_0)$. Moreover, this is flow-invariant since Q is flow-invariant and $w_{(x,v)}(\xi_0)$ agrees with f at $(x_\star, v_\star)$ so by Proposition 3.4 they are equal.

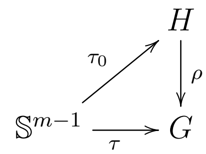
3.3. **Topological reduction of the structure group.** We refer to [DK01] for the background in algebraic topology. On the sphere $\mathbb{S}^m$ (for $m \geq 2$), the topology of principal bundles is easier than on general CW-complexes since it does not require the use of classifying spaces. A principal G-bundle $P \to \mathbb{S}^m$ is determined by the homotopy class of its *classifying map* $\tau : \mathbb{S}^{m-1} \to G$. We will use the following standard terminology of algebraic topology:

Definition 3.6. Let $P \to \mathbb{S}^m$ be a principal G-bundle over $\mathbb{S}^m$, where G is a compact Lie group. We say that it admits a reduction of its structure group to (H,ρ)[5], where H is a compact Lie group and $\rho : H \to G$ is a homomorphism, if the classifying map $\tau : \mathbb{S}^{m-1} \to G$ can be factored (up to homotopy) through the map ρ, that is there exists a classifying map

[4]The threshold in this case is the exponential growth of the norm of the propagator $e^{t\mathbf{X}}$ acting on $L^\infty(SM,\mathcal{N})$. The connection being unitary, this operator is unitary, that is $\|e^{tX}\|_{L^\infty \to L^\infty} \leq 1$.

[5]Note that in the literature, the letter for the group reduction is G and the usual terminology is that of a G-*structure*. In order to avoid a notational clash with the letter used for the transitivity group, we stick to the letter H.

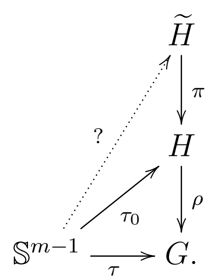
τ_0 such that:



Another accepted (but not exactly equivalent) definition is to restrict only to subgroups $H \leqslant G$ and then $\rho = \iota$ where $\iota : H \to G$ is the embedding of H into G. If we had added the requirement that ρ is faithful in Definition 3.6 these would be equivalent, but it is sometimes useful to allow for non-faithful homomorphisms ρ. We will need the following:

Lemma 3.7. *Let $\tau : \mathbb{S}^{m-1} \to G$ be a principal G-bundle over $\mathbb{S}^m$, identified with its classifying map. If (H, ρ) is a reduction of the structure group, H is a compact semisimple*[6] *Lie group, and $\pi : \widetilde{H} \to H$ is a cover of H, then $(\widetilde{H}, \rho \circ \pi)$ is a reduction of the structure group.*

Proof. We are looking for a lift of the map τ_0 defined as:



Since $\pi_1(\mathbb{S}^{m-1}) = 0$, the existence of such a lift is immediate. □

Recall that a section $f \in C^\infty(SM, \pi^* \mathcal{E})$ is said to be *odd* or *even* if its Fourier degrees are either all odd or even and invariant (or flow-invariant) if it satisfies $\mathbf{X}f = 0$. If (M, g) has negative sectional curvature (or, more generally, ergodic geodesic flow) any flow-invariant object has constant norm on SM since $X|f|^2 = 2\langle \mathbf{X}f, f\rangle = 0$ and the geodesic flow is ergodic. A typical flow-invariant odd object on the unit tangent bundle is the *tautological section* $s \in C^\infty(SM, \pi^*TM)$ given by $s(x, v) := v$. It can be easily checked that it satisfies $\mathbf{X}s = 0$. In order not to burden the notation, we will simply denote it by v. The following statement shows that the non-ergodicity of the frame flow implies the existence of new invariant objects on the unit tangent bundle. Recall that the Radon-Hurwitz numbers $\rho(n)$ are defined as follows: writing $n = (2a + 1)2^b$, $b = c + 4d$ with $0 \leq c \leq 3$, we have $\rho(n) := 2^c + 8d$. The number $\rho(n) - 1$ corresponds to the maximal number of linearly independent vector fields on the sphere $\mathbb{S}^{n-1}$, see [Ada62]. Note that $\rho(8) - 1 = 7$, and for $\ell \in \mathbb{Z}_{\geq 0}$, we have $\rho(4\ell + 2) - 1 = 1, \rho(8\ell + 4) - 1 = 3$.

[6] Recall that a compact connected Lie group H is said to be semisimple if it possesses no Abelian connected normal subgroup other than $\{1\}$.

Theorem 3.8. *Let (M^n, g) be a smooth closed negatively-curved Riemannian manifold of even dimension or dimension 7. If the frame flow is not ergodic, then there exists a finite Riemannian cover $(\widehat{M}, \widehat{g})$ such that the following holds. If $n \neq 7, 8, 134$, then:*

- *If $n = 4$ or $n \equiv 2 \bmod 4$, there exists an invariant unit vector field $f \in C^\infty(S\widehat{M}, \mathcal{N})$ of odd degree,*
- *If $n \equiv 0 \bmod 4$, there exists an invariant orthogonal projector $f \in C^\infty(S\widehat{M}, \mathrm{Sym}^2\mathcal{N})$ of even degree such that $\mathrm{rank}(f) \leq \min\big(\rho(n) - 1, (n-2)/2\big)$.*

In the three exceptional cases, we have:

- *If $n = 7$, there exists an invariant almost complex-structure $f \in C^\infty(S\widehat{M}, \Lambda^2\mathcal{N})$ of odd degree,*
- *If $n = 8$, there exists an invariant G_2-structure $f \in C^\infty(S\widehat{M}, \Lambda^3\mathcal{N})$ of odd degree or there exists an invariant orthogonal projector $f \in C^\infty(S\widehat{M}, \mathrm{Sym}^2\mathcal{N})$ of even degree such that $\mathrm{rank}(f) \leq 3$,*
- *If $n = 134$, there exists an invariant Lie bracket[7] $f \in C^\infty(S\widehat{M}, \Lambda^3\mathcal{N})$ or there exists an invariant unit vector field $f \in C^\infty(S\widehat{M}, \mathcal{N})$ of odd degree.*

In all cases, the invariant elements have finite Fourier degree [GPSU16].

In the proof, it will be convenient to use the contraction operator by the tautological section $\imath_v$ applied to an object f taking values in Λ^pTM or Sym^2TM. For $f \in C^\infty(SM, \pi^*\Lambda^pTM)$, $\imath_v f$ is the usual contraction of forms. For $f \in \mathrm{Sym}^2TM$, viewed as a symmetric endomorphism, $\imath_v f$ is just the vector $f(v)$. In particular, all objects obtained from Theorem 3.8 take values in $\mathcal{N}$ so they satisfy $\imath_v f = 0$.

Proof. The proof is divided in two steps. First of all, we show the existence of these invariant structures and then we show that their Fourier degree has to be odd or even.

Step 1: existence of invariant structures. Let (M^n, g) be a closed Riemannian manifold with negative sectional curvature and further assume that the frame flow is not ergodic. By Proposition 3.2, the transitivity group H is a strict proper subgroup of $\mathrm{SO}(n-1)$. If H is not connected, we can apply Lemma 3.3 to produce a finite cover $(\widehat{M}, \widehat{g})$ whose transitivity group of the frame flow is the identity component H_0 of H. Thus, without loss of generality, we can directly assume that H is connected. This implies in particular that (H, ι) is a reduction of the structure group of the frame bundle $F\mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ over the sphere (which is a principal $\mathrm{SO}(n-1)$-bundle), where $\iota : H \to \mathrm{SO}(n-1)$ is the embedding[8]. We let $H \leqslant \widehat{H} \lneqq \mathrm{SO}(n-1)$ be a maximal subgroup containing H. Note that if $\widehat{H}$ fixes an

[7]Invariant 3-form ω such that the bracket defined by $\langle [a, b], c\rangle := \omega(a, b, c)$ satisfies the Jacobi identity.

[8]Note that, here, we see H as a subgroup of $\mathrm{SO}(n-1)$, that is, equivalently, we ask the representation $\iota : H \to \mathrm{SO}(n-1)$ to be faithful.

element in some exterior or symmetric power of $\mathbb{R}^{n-1}$, then so does H. We claim that the following holds. If $n \neq 7, 8, 134$, then:

- If n is odd, there is no reduction of the structure group of $\mathrm{SO}(n-1)$,
- If $n \equiv 2 \mod 4$ or $n = 4$, H fixes a vector of $\mathbb{R}^{n-1}$,
- If $n \equiv 0 \mod 4$, H acts reducibly on $\mathbb{R}^{n-1} = V \oplus V^{\perp}$. Moreover, assuming without loss of generality that $\dim V \leq \dim V^{\perp}$, we have $\dim V \leq \rho(n) - 1$.

The following exceptional cases hold:

- If $n = 7$, then $\widehat{H} = \mathrm{U}(3)$ and H fixes an almost-complex structure structure in $\mathbb{R}^6$,
- If $n = 8$, then either: $\widehat{H} = \mathrm{G}_2$ and H fixes a non-zero 3-form in $\mathbb{R}^7$ (with G_2 stabilizer), *or* H acts reducibly on $\mathbb{R}^7 = V \oplus V^{\perp}$ with $\dim V \leq 3$;
- If $n = 134$, then either: $\widehat{H} = \mathrm{E}_7/\mathbb{Z}_2$ and H fixes a non-zero 3-form on $\mathbb{R}^{133}$ (a Lie bracket), *or* H fixes a non-zero vector in $\mathbb{R}^{133}$.

We believe that the $\mathrm{E}_7/\mathbb{Z}_2$-structure on $\mathbb{S}^{133}$ actually never occurs and could be ruled out by purely topological arguments but we are unable to prove this.

The claim mainly follows from [Ada62, Leo71, ČC06] and requires only a few additional arguments which are given below. Taking this claim for granted, the proof of Theorem 3.8 is then a straightforward consequence of Proposition 3.4 which produces flow-invariant objects from vectors fixed by the representation, and parity properties shown in Step 2. When H acts reducibly, the object considered is the orthogonal projector onto one of the summands (the one with the lowest dimension).

We now prove the claim. For n odd, this follows from [Leo71, Theorem 1] and we assume from now on that n is even. We further assume that $n \geq 10$, the low-dimensional cases are dealt afterwards. We look at the following dichotomy:

A. The action of $\widehat{H}$ on $\mathbb{R}^{n-1}$ is reducible. Then H acts also reducibly, that is, $\mathbb{R}^{n-1} = V \oplus V^{\perp}$, where each summand is H-invariant. Up to changing the roles of V and $V^{\perp}$, we can assume that $\dim V \leq \dim V^{\perp}$. By [Leo71, Theorem 2.A], there must exist at least $\dim V$ vector fields on $\mathbb{S}^{n-1}$, that is $\dim V \leq \rho(n) - 1$. Since $\rho(4\ell + 2) - 1 = 1$, we see that $\widehat{H}$ fixes a non-zero vector in $\mathbb{R}^{n-1}$.

B. The action of $\widehat{H}$ on $\mathbb{R}^{n-1}$ is irreducible. Then $\widehat{H}$ has to be simple[9] by [Leo71, Theorem 3]. We say that $\widehat{H}$ has type $\mathfrak{x}$, where $\mathfrak{x}$ is a Lie algebra, if the Lie algebra $\widehat{\mathfrak{h}}$ of $\widehat{H}$ is isomorphic to $\mathfrak{x}$.

[9]Recall that a group G is said to be simple if it is connected, non-Abelian, and every closed connected normal subgroup is either the identity or the whole group.

B.1. Since an irreducible representation of $\mathfrak{so}(2)$ has dimension at most 2, $\mathfrak{so}(4)$ is not simple, and $\mathfrak{sp}(1) = \mathfrak{so}(3) = \mathfrak{su}(2)$, $\widehat{H}$ has type $\mathfrak{so}(k)$ with $k \geq 5$ or $\mathfrak{sp}(k), \mathfrak{su}(k)$ with $k \geq 2$.

B.1.1. Let us first deal with the last two cases $\mathfrak{sp}(k)$ or $\mathfrak{su}(k)$. This implies that $\widehat{H}$ is a quotient of $\mathrm{Sp}(k)$ or $\mathrm{SU}(k)$ by a subgroup of its center. By Lemma 3.7, we thus know that the reduction of the structure group $\iota : \widehat{H} \to \mathrm{SO}(n-1)$ lifts to a reduction $\widetilde{\iota} : \mathrm{Sp}(k), \mathrm{SU}(k) \to \mathrm{SO}(n-1)$. If $n \geq 10$, the existence of such an irreducible reduction $\widetilde{\iota}$ is impossible by [ČC06, Theorem 2.1 (B), (C)] so $\widehat{H}$ cannot be of type $\mathfrak{sp}(k)$ or $\mathfrak{su}(k)$.

B.1.2. We now deal with the case where $\widehat{H}$ has type $\mathfrak{so}(k)$. There are a few possibilities for $\widehat{H}$, namely it can be $\mathrm{Spin}(k), \mathrm{SO}(k)$, or additionally $\mathrm{PSO}(k)$ when k is even, or additionally the semi-spin group $\mathrm{SemiSpin}(k)$ if $k \equiv 0 \mod 4$.

B.1.2.1. If $\widehat{H}$ is $\mathrm{PSO}(k)$, then there is also a reduction to $\mathrm{SO}(k)$ by Lemma 3.7, in which case [ČC06, Theorem 2.1 (A)] shows that $\mathrm{SO}(k) \hookrightarrow \mathrm{SO}(n-1)$ is the standard diagonal embedding so it cannot be irreducible.

B.1.2.2. If $\widehat{H} = \mathrm{SO}(k)$, the conclusion follows from the previous point.

B.1.2.3. We now deal with $\mathrm{Spin}(k)$ and $\mathrm{SemiSpin}(k)$ (when $k \equiv 0 \mod 4$). In the former case, it suffices to show that a group morphism $\iota : \mathrm{Spin}(k) \to \mathrm{SO}(n-1)$ cannot be faithful. Observe that $\iota(-1) = \pm 1$ since -1 is in the center of $\mathrm{Spin}(k)$ and $\iota(-1)^2 = 1$ (we use that $\widehat{H}$ is irreducible). If $\iota(-1) = 1$, then we obtain that ι is not faithful and thus the representation factorizes through $\mathrm{SO}(k)$. Hence $\iota(-1) = -1$ but since $\mathbf{e}_1 \cdot \mathbf{e}_2 \in \mathrm{Spin}(k)$ squares to -1 (where $(\mathbf{e}_1, ..., \mathbf{e}_k)$ is an orthonormal basis of $\mathbb{R}^k$), we get that $\iota(\mathbf{e}_1 \cdot \mathbf{e}_2)$ is a complex structure on $\mathbb{R}^{n-1}$ and this is a contradiction since $n-1$ is odd. The same argument works for $\mathrm{SemiSpin}(k)$.

B.2. $\widehat{H}$ has exceptional type, that is $\widehat{\mathfrak{h}}$ is isomorphic to one of the five exceptional simple Lie algebras and $\widehat{H}$ is isomorphic to (a finite quotient of) one of the exceptional Lie groups. So it suffices to look at real irreducible representations of exceptional Lie groups on odd dimensional vector spaces. Moreover, by [ČC06, Proposition 3.1], writing $n - 1 = \dim \widehat{H} + k + 1$ for some integer k, there must exist at least k vector fields on the sphere $\mathbb{S}^{n-1}$, so the Radon-Hurwitz number satisfies

$$\rho(n) \geq n - 1 - \dim H. \tag{3.4}$$

For $n \geq 10$ we have $\rho(n) \leq \frac{n}{2}$, unless $n = 16$. Since no exceptional Lie group has an irreducible representation in dimension 15, it follows that $n-1$ has to be the dimension of an irreducible representation of an exceptional Lie group H, with

$$9 \leq n-1 \leq 2\dim H + 1. \tag{3.5}$$

The following possibilities may occur:

B.2.1. $\widehat{\mathfrak{h}} = \mathfrak{g}_2$, $\dim H = 14$. The only odd-dimensional irreducible representation of $\mathfrak{g}_2$ satisfying (3.5) has dimension $n-1 = 27$. However, $\rho(28) = 4$, so it does not satisfy (3.4).

B.2.2. $\widehat{\mathfrak{h}} = \mathfrak{f}_4$, $\dim H = 52$. There is no odd-dimensional irreducible representation of $\mathfrak{f}_4$ satisfying (3.5).

B.2.3. $\widehat{\mathfrak{h}} = \mathfrak{e}_6$, $\dim H = 78$. Again, there is no odd-dimensional irreducible representation of $\mathfrak{e}_6$ satisfying (3.5).

B.2.4. $\widehat{\mathfrak{h}} = \mathfrak{e}_7$, $\dim H = 133$. The only odd-dimensional irreducible representation of $\mathfrak{e}_7$ satisfying (3.5) is its adjoint representation. Note that the adjoint representation of E_7 on $\mathbb{R}^{133}$ is not faithful since E_7 has non-trivial center, so $\widehat{H} = \mathrm{E}_7/\mathbb{Z}_2$. In this case, $\widehat{H}$ and thus H fixes the Lie bracket on $\mathbb{R}^{133} = \mathfrak{e}_7$ which we can view as a vector in $\Lambda^3\mathbb{R}^{133}$.

B.2.5. $\widehat{\mathfrak{h}} = \mathfrak{e}_8$, $\dim H = 248$. There is no odd-dimensional irreducible representation of $\mathfrak{e}_8$ satisfying (3.5).

Eventually, it remains to deal with the low-dimensional cases $n = 4, 6, 8$. For $n = 4$, the only strict proper connected subgroup of $\mathrm{SO}(3)$ is $\mathrm{SO}(2)$ and this fixes a vector in $\mathbb{R}^3$. For $n = 6$ or 8, if the subgroup $\widehat{H}$ of $\mathrm{SO}(5)$ or $\mathrm{SO}(7)$ acts reducibly on $\mathbb{R}^5$ or $\mathbb{R}^7$ respectively, then it falls into case **A.** Hence we may assume $\widehat{H}$ acts irreducibly. Using that the groups $\mathrm{SO}(5)$, $\mathrm{SO}(7)$ have respective ranks 2 and 3, it is straightforward to check that among their subgroups acting irreducibly on $\mathbb{R}^5$ or $\mathbb{R}^7$, the only possibilities are $\widehat{H} = \mathrm{SO}(3)$ for $n = 6$, and $H = \mathrm{SO}(3)$ or G_2 for $n = 8$ (just in this case we argue on H rather than on $\widehat{H}$). The case $n = 6$ and $\widehat{H} = \mathrm{SO}(3)$ is ruled out by [ABBF11], while in the latter one, note that the irreducible $\mathrm{SO}(3)$ representation of dimension 7 is the restriction of the irreducible 7-dimensional G_2 representation, and so the group fixes a invariant vector in $\Lambda^3\mathbb{R}^7$, the G_2-structure.

Step 2: topological reduction of the Fourier degree. We now show that the degree must be odd or even in certain cases. We start with the following observation: if $f \in C^\infty(SM, \Lambda^p\mathcal{N})$ or $f \in C^\infty(SM, \mathrm{Sym}^2\mathcal{N})$ is such that $\mathbf{X}f = 0$, then writing $f = f_{\mathrm{odd}} + f_{\mathrm{even}}$ where each term has respectively odd or even Fourier degree, we have $0 = \imath_v f = \imath_v f_{\mathrm{odd}} = \imath_v f_{\mathrm{even}}$ and

$0 = \mathbf{X}f = \mathbf{X}f_{\mathrm{odd}} = \mathbf{X}f_{\mathrm{even}}$. (The proof is immediate as both operators $\mathbf{X}$ and $\imath_v$ shift the Fourier degrees by ± 1.)

Lemma 3.9. *If the dimension n of M is even, then every section $f \in C^\infty(SM, \mathcal{N})$ which satisfies $\mathbf{X}f = 0$ has odd degree.*

Proof. Write $f = f_{\mathrm{even}} + f_{\mathrm{odd}}$, the decomposition of f into even and odd Fourier degrees. Both terms have constant norm on SM and satisfy $\imath_v f_{\mathrm{even}} = \imath_v f_{\mathrm{odd}} = 0$. Assume for a contradiction that $f_{\mathrm{even}} \neq 0$. Up to rescaling, we can assume its norm is constant equal to 1. Fixing an arbitrary point $x_0 \in M$ and identifying $S_{x_0}M \simeq \mathbb{S}^{n-1}$, the section f_{even} defines a non-vanishing vector field $Z(v) := f(x_0, v)$ such that $Z \in C^\infty(\mathbb{S}^{n-1}, T\mathbb{S}^{n-1})$ satisfying $Z(v) = Z(-v)$ by evenness. We can see Z as a smooth map $Z : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. Now, Z is clearly homotopic to the identity by

$$Z_t(v) := \cos\left(\tfrac{\pi t}{2}\right) v + \sin\left(\tfrac{\pi t}{2}\right) Z(v),$$

hence its topological degree is 1. On the other hand, any regular value of Z has an even number of preimages since $Z(v) = Z(-v)$, so the topological degree of Z is even. This is a contradiction. □

Lemma 3.10. *Assume that n is even and there exists a section $f \in C^\infty(SM, \mathrm{Sym}^2\mathcal{N})$ not proportional to the tautological section $\mathbb{1}$, such that $\mathbf{X}f = 0$. Then, there exists a flow-invariant orthogonal projector $f' \in C^\infty(SM, \mathrm{Sym}^2\mathcal{N})$ of even degree such that $1 \leq \mathrm{rank}(f') \leq \min\big(\rho(n) - 1, (n-2)/2\big)$.*

Proof. We write $f = f_{\mathrm{even}} + f_{\mathrm{odd}}$, where both terms are flow-invariant and take values in $\mathrm{Sym}^2\mathcal{N}$. Each of them can be diagonalized and has constant (real) eigenvalues. Following [CL21a, Section 5], one can show that $f_{\mathrm{even}} = \sum_i \lambda_i \Pi_{\lambda_i}$, where λ_i are the constant eigenvalues and Π_{λ_i} are the orthogonal projectors onto the corresponding eigenspaces. It can easily be checked that these projectors are all even. For the odd part of f, it can be shown that if λ is an eigenvalue of f_{odd}, then so is $-\lambda$ and $\Pi_\lambda(-v) = \Pi_{-\lambda}(v)$. Hence $f_{\mathrm{odd}} = \sum_i \lambda_i \Pi_{\lambda_i} - \lambda_i \Pi_{-\lambda_i}$ and $\Pi_{\lambda_i} + \Pi_{-\lambda_i}$ is an even orthogonal projection for each i.

In any case, we obtain a non-zero orthogonal projector f' with even Fourier degree which is different from 0 and $\mathbb{1}$. Let F be the image of f'. Up to changing f' by $\mathbb{1} - f'$ (and F by $F^\perp$), we can assume that $1 \leq r := \mathrm{rank}(f') \leq (n-2)/2$. Observe that F is in particular a subbundle of the tangent bundle to the sphere and this also forces $r \leq \rho(n) - 1$, see [Leo71]. □

In practice, the following lemma is only applied for $n = 7$.

Lemma 3.11. *If n is odd and $f \in C^\infty(SM, \Lambda^2\mathcal{N})$ satisfies $\mathbf{X}f = 0$, then f has odd degree.*

Proof. As before, we can write $f = f_{\mathrm{odd}} + f_{\mathrm{even}}$ with $\imath_v f_{\mathrm{odd}} = \imath_v f_{\mathrm{even}} = 0$, that is, $f_{\mathrm{even}} \in C^\infty(SM, \Lambda^2\mathcal{N})$ is even and has constant norm. Assume that f_{even} is non-vanishing and is non-invertible viewed as skew-symmetric endomorphism. Its kernel then defines a strict flow-invariant subbundle $\mathcal{F} \subset \mathcal{N}$. In particular, at a point $x_0 \in M$, we obtain a subbundle $\mathcal{F}(x_0) \to S_{x_0}M \simeq \mathbb{S}^{n-1}$, which is impossible by [Ste51, Theorem 27.18] since $n-1$ is even. If f_{even} is invertible, writing $n = 2m+1$, this implies that $f_{\mathrm{even}}^{\wedge m} \in C^\infty(SM, \Lambda^{2m}\mathcal{N})$ is flow-invariant and nowhere vanishing. At x_0, we then obtain a section $f_{\mathrm{even}}^{\wedge m}(x_0, \cdot) \in C^\infty(\mathbb{S}^{n-1}, \Lambda^{n-1}T\mathbb{S}^{n-1})$. Writing $\omega_{\mathbb{S}^{n-1}}$ for the volume form on $\mathbb{S}^{n-1}$, we have $f_{\mathrm{even}}^{\wedge m}(x_0, \cdot) = h\, \omega_{\mathbb{S}^{n-1}}$ for some smooth real-valued and nowhere vanishing function $h \in C^\infty(\mathbb{S}^{n-1})$. By assumption, f_{even} is even and so is $f_{\mathrm{even}}^{\wedge m}(x_0, \cdot)$. Hence:

$$f_{\mathrm{even}}^{\wedge m}(x_0, -v) = f_{\mathrm{even}}^{\wedge m}(x_0, v) = h(v)\omega_{\mathbb{S}^{n-1}}(v) = h(-v)\omega_{\mathbb{S}^{n-1}}(-v). \tag{3.6}$$

Now, if $(\mathbf{e}_1, ..., \mathbf{e}_n)$ for an orthonormal basis of $T_{x_0}M$, writing $\omega_{\mathbb{R}^n} := \mathbf{e}_1 \wedge ... \wedge \mathbf{e}_n$, we have $\omega_{\mathbb{S}^{n-1}}(v) = \iota_v \omega$, that is $\omega_{\mathbb{S}^{n-1}}(-v) = -\omega_{\mathbb{S}^{n-1}}(v)$. This contradicts (3.6) as h is non-vanishing. Thus $f_{\mathrm{even}} = 0$. □

Eventually, we treat the case of flow-invariant SO(3)- or G_2-structures in dimension 8.

Lemma 3.12. *Assume $n = 8$ and H acts irreducibly on $\mathbb{R}^7$. Then $H = \mathrm{SO}(3)$ or G_2 and H fixes an invariant vector $f_\star$ in $\Lambda^3\mathbb{R}^7$, the G_2-structure. Let $f \in C^\infty(SM, \Lambda^3\mathcal{N})$ be the G_2-structure such that $\mathbf{X}f = 0$ and $f(x_\star, v_\star) = f_\star$. Then f has odd degree.*

Proof. In both cases $H = \mathrm{SO}(3)$ or G_2, it can be checked that the unique H-invariant elements in $\Lambda^3\mathbb{R}^7$ are in the span of the G_2-structure on $\mathbb{R}^7$. Let $f = f_{\mathrm{even}} + f_{\mathrm{odd}}$ be the flow-invariant G_2-structure obtained by Proposition 3.4. Observe that both f_{even} and f_{odd} are flow-invariant, thus fixed by H so by uniqueness, they must be proportional which implies that one of them is zero. We argue by contradiction and assume that $f_{\mathrm{odd}} = 0, f_{\mathrm{even}} \neq 0$. For every $v \in \mathbb{S}^7 \simeq S_{x_0}M$, $f(x_0, v) \in \Lambda^3 v^\perp$ is a G_2-structure on $v^\perp$, so for every $w \in v^\perp$ we have $(\iota_w f(x_0, v)) \wedge (\iota_w f(x_0, v)) \wedge f(x_0, v) = 6|w|^2\omega_{\mathbb{S}^7}(v) = 6\iota_v\omega_{\mathbb{R}^7}$ (see the proof of [Bry87, Theorem 1]). Replacing v by $-v$ yields

$$\begin{aligned}6|w|^2\iota_{-v}\omega_{\mathbb{R}^7} &= (\iota_w f(x_0, -v)) \wedge (\iota_w f(x_0, -v)) \wedge f(x_0, -v)\\ &= (\iota_w f(x_0, v)) \wedge (\iota_w f(x_0, v)) \wedge f(x_0, v) = 6|w|^2\iota_v\omega_{\mathbb{R}^7},\end{aligned}$$

which is a contradiction. □

We finally prove the following result.

Lemma 3.13. *Assume that $n = 134$ and H is a subgroup of $\mathrm{E}_7/\mathbb{Z}_2$. Let $f \in C^\infty(SM, \Lambda^3\mathcal{N})$ be the flow-invariant Lie bracket obtained by Proposition 3.4, i.e. a section which is equivalent at every point (x, v) to the canonical 3-form of the Lie algebra $\mathfrak{e}_7$, and such that $\mathbf{X}f = 0$. Then f has degree at least 2.*

Proof. Assume for the sake of a contradiction that the degree of f is ≤ 1. Then $f = f_0 + f_1$ and $\iota_v f_0 = \iota_v f_1 = 0$. Since f_0 can be identified with an element in $C^\infty(M, \Lambda^3 TM)$, the condition $\iota_v f_0 = 0$ implies $f_0 = 0$. Hence $f = f_1 \in C^\infty(M, TM \otimes \Lambda^3 TM)$. The condition $\imath_v f = 0$ reads $f(v, v, \bullet, \bullet) = 0$, that is f is a 4-form on M which we denote by $\phi \in C^\infty(M, \Lambda^4 TM)$.

The hypothesis on f shows that for every unit vector v, there exists an isometry between $v^\perp$ and $\mathfrak{e}_7$, such that the 3-form $\iota_v \phi$ is the pull-back of the canonical 3-form of $\mathfrak{e}_7$. We will identify them by a slight abuse of notation. For every $u \in v^\perp$ one can thus interpret the 2-form $\phi(v, u)$ as corresponding to the endomorphism ad_u acting on $\mathfrak{e}_7$. By the irreducibility of the adjoint representation of E_7 we have $|\mathrm{ad}_u|^2 = c|u|^2$ for some positive constant c which only depends on the structure of E_7 (indeed, $\mathfrak{e}_7 \ni u \mapsto |\mathrm{ad}_u|^2$ is an E_7-invariant quadratic form so by Schur's lemma, it is proportional to the metric).

We can write this as

$$|\phi(u, v)|^2 = c|u \wedge v|^2, \tag{3.7}$$

for every $u, v \in TM$. We can also see ϕ as a symmetric endomorphism $\phi : \Lambda^2 TM \to \Lambda^2 TM$ (the symmetry comes from the relation $\phi(u, v, w, z) = \langle \phi(u \wedge v), w \wedge z\rangle = \phi(w, z, u, v) = \langle \phi(w, z), u \wedge v\rangle$) and (3.7) says that ϕ is an isometry on decomposable elements.

By polarization in u we obtain $\langle \phi(u, v), \phi(w, v)\rangle = c\langle u \wedge v, w \wedge v\rangle$, and by polarization in v we get:

$$\langle \phi(u, z), \phi(w, v)\rangle + \langle \phi(u, v), \phi(w, z)\rangle = c\left(\langle u \wedge z, w \wedge v\rangle + \langle u \wedge v, w \wedge z\rangle\right). \tag{3.8}$$

We now fix a unit vector $a \in TM$ and consider $u, v, w, z \in a^\perp$. The Jacobi identity on $a^\perp \simeq \mathfrak{e}_7$ implies that the following cyclic sum vanishes:

$$\mathfrak{S}_{u,v,w}\langle \mathrm{ad}_v w, \mathrm{ad}_u z\rangle = 0 = \mathfrak{S}_{u,v,w}\langle \phi(a, v, w), \phi(a, u, z)\rangle = \mathfrak{S}_{u,v,w}\langle \phi(w, v, a), \phi(u, z, a)\rangle.$$

This identity also holds for $u, v, w, z \in TM$ since $\phi(a, a, \bullet, \bullet) = 0$ so it holds for all $a, u, v, w, z \in TM$. Taking the trace in a over an orthonormal basis $(\mathbf{e}_1, ..., \mathbf{e}_n)$ of TM, we get:

$$0 = \mathfrak{S}_{u,v,w}\langle \phi(w, v), \phi(u, z)\rangle,$$

that is

$$\langle \phi(u, z), \phi(w, v)\rangle + \langle \phi(w, z), \phi(v, u)\rangle + \langle \phi(v, z), \phi(u, w)\rangle = 0. \tag{3.9}$$

Similarly, we have

$$\langle u \wedge z, w \wedge v\rangle + \langle w \wedge z, v \wedge u\rangle + \langle v \wedge z, u \wedge w\rangle = 0. \tag{3.10}$$

As a consequence, setting $F := \phi^2 - c\mathbb{1}_{\Lambda^2}$, we get that F is a symmetric endomorphism and using (3.8), (3.9) and (3.10) it satisfies the relations:

$$\begin{aligned}
&\langle F(u \wedge z), w \wedge v\rangle + \langle F(u \wedge v), w \wedge z\rangle = 0,\\
&\langle F(u \wedge z), w \wedge v\rangle + \langle F(w \wedge z), v \wedge u\rangle + \langle F(v \wedge z), u \wedge w\rangle = 0.
\end{aligned}$$

It is straightforward to check that these relations imply that $F = 0$. Thus, setting $\phi' := \frac{1}{\sqrt{c}}\phi$, we get that ϕ' is symmetric and $\phi'^2 = \mathbb{1}_{\Lambda^2}$, that is ϕ' is an orthogonal symmetry. Thus the trace of ϕ' is equal to the difference of the dimensions of its 1 and -1 eigenspaces, so it is an odd integer since $\Lambda^2(TM)$ has odd dimension $\frac{134\times 133}{2} = 67 \times 133$. However, this is a contradiction since the trace of ϕ' is

$$\mathrm{Tr}(\phi') = \sum_{i<j} \langle \phi'(\mathbf{e}_i \wedge \mathbf{e}_j), \mathbf{e}_i \wedge \mathbf{e}_j \rangle = \frac{1}{\sqrt{c}} \sum_{i<j} \phi(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j) = 0.$$

□

We observe that the above result actually holds more generally by replacing $\mathrm{E}_7/\mathbb{Z}_2$ with any simple compact Lie group of dimension $n-1 = 4k+1$.

The proof of Theorem 3.8 is now complete.

□

Remark 3.14. As a concluding remark, we observe that the ergodicity of the frame flow on negatively-curved manifolds of odd dimension $n \neq 7$ proved by Brin-Gromov [BG80] is actually an immediate consequence of [Leo71, Theorem 1.A] which shows that there is no reduction of the structure group of $F\mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$, unless $n = 7$ and $H = \mathrm{U}(3)$ or $\mathrm{SU}(3)$.

4. Non-existence of flow-invariant structures under pinching conditions

Theorem 3.8 shows that non-ergodicity of the frame flow implies the existence of flow-invariant structures on SM. We now show that this is impossible under some pinching conditions.

Theorem 4.1. *Let (M^n, g) be a closed Riemannian manifold with δ-pinched negative sectional curvature. Then:*

- *There exists $\delta_{\Lambda^1}(n)$ given in (4.1) such that if $\delta > \delta_{\Lambda^1}(n)$, then there are no non-trivial odd flow-invariant sections $f \in C^\infty(SM, \mathcal{N})$,*
- *There exists $\delta_{\mathrm{Sym}^2}(n)$ given in (4.2) such that if $\delta > \delta_{\mathrm{Sym}^2}(n)$, then there are no non-trivial even flow-invariant orthogonal projectors $f \in C^\infty(SM, \mathrm{Sym}^2\mathcal{N})$ of rank $r \leq \min\big(\rho(n) - 1, (n-1)/2\big)$.*

Moreover, the following exceptional cases hold:

- *If $n = 7$ and $\delta > \delta_{\mathrm{U}(3)}(7) := 0.4962...$, then there are no non-trivial odd flow-invariant almost-complex structures $f \in C^\infty(SM, \Lambda^2\mathcal{N})$,*
- *If $n = 8$ and $\delta > \delta_{\mathrm{G}_2}(8) := 0.6212...$, then there are no non-trivial odd flow-invariant G_2-structures $f \in C^\infty(SM, \Lambda^3\mathcal{N})$,*
- *If $n = 134$ and $\delta > \delta_{\mathrm{E}_7}(134) := 0.6716...$, then there are no non-trivial flow-invariant Lie brackets $f \in C^\infty(SM, \Lambda^3\mathcal{N})$ of degree ≥ 2.*

The expressions for the thresholds are given by:

$$\delta_{\Lambda^1}(n) = \begin{cases} \frac{\frac{2}{3}\sqrt{3(n^2-1)}+\frac{1}{2}(n+3)}{3(n+1)+\frac{2}{3}\sqrt{3(n^2-1)}-\frac{1}{2}+\frac{1}{2}\frac{(n+2)(5n+2)}{n+4}}, & \text{if } n \leq 8, \\ \frac{\frac{2}{3}\sqrt{3(n^2-1)}+\frac{1}{2}}{3(n+1)+\frac{2}{3}\sqrt{3(n^2-1)}-\frac{1}{2}}, & \text{if } n \geq 10, \end{cases} \tag{4.1}$$

and:

$$\delta_{\mathrm{Sym}^2}(n) = \frac{n+5+\frac{8}{3}\sqrt{(n-1)(n+2)}+\frac{2(n+2)(n+4)}{3(n+1)(n+6)}\left(n+3+\frac{4}{3}\sqrt{3(n^2-1)}\right)}{3(n+1)+\frac{8}{3}\sqrt{(n-1)(n+2)}+\frac{2(n+2)(n+4)}{3(n+1)(n+6)}\left(5n+3+\frac{4}{3}\sqrt{3(n^2-1)}\right)}. \tag{4.2}$$

The first values for δ_{Λ^1} are $\delta_{\Lambda^1}(4) = 0.2928...$, $\delta_{\Lambda^1}(6) = 0.2823...$ and appear in Theorem 1.2. The combination of Theorem 3.8 and Theorem 4.1 immediately proves Theorem 1.2. The remaining part of the paper is devoted to the proof of Theorem 4.1.

4.1. **Normal twisted conformal Killing tensors.** In the following, we let $\mathcal{E} = \Lambda^p TM$ or $\mathcal{E} = \mathrm{Sym}^2 TM$. We have proved in Theorem 3.8 that non-ergodicity of the frame flow gives rise to a flow-invariant smooth section $f \in C^\infty(SM, \pi^* \mathcal{E})$ such that $\mathbf{X}f = 0$ and $\imath_v f = 0$. By [GPSU16], such an invariant section must have finite Fourier content, i.e. it can be written as $f = f_0 + f_1 + \ldots + f_k$, where $f_i \in C^\infty(M, \Omega_i \otimes \mathcal{E})$ and $f_k \neq 0$. We now define $u := f_k$. The two conditions $\mathbf{X}f = 0$ and $\imath_v f = 0$ then translate into the fact that $\mathbf{X}_+ u = 0$ and $\imath_v u \in C^\infty(M, \Omega_{k-1} \otimes \mathcal{E})$, i.e. $\imath_v u$ is two degrees less than expected. Alternatively, we can see u as the pullback to SM of a twisted conformal Killing tensor on the base, i.e. there exists $K \in C^\infty(M, \mathrm{Sym}^k_0 TM \otimes \mathcal{E})$ such that $u = \pi_k^* K$, $\mathcal{P}DK = 0$ (conformal Killing condition) and satisfying also the algebraic condition $\imath_v \pi_k^* K$ is of degree $k-1$. In the case where $\mathcal{E} = \Lambda^p TM$, we also observe that $\imath_v \imath_v u = 0$ since u is a form. In the case where $\mathcal{E} = \mathrm{Sym}^2 TM$ and f is an even orthogonal projector, we have:

Lemma 4.2. *Let $f \in C^\infty(SM, \mathrm{Sym}^2 \mathcal{N})$ be an orthogonal projector of rank r with (finite) even degree k and let $u := f_k$. Then $\imath_v u$ is of degree $k-1$ and $\imath_v \imath_v u$ is of degree $k-2$. Moreover, if $k = 2$, then we have $f = \frac{r}{n}\mathbb{1}_{\mathcal{N}} + f_2$, where $f_2 \in C^\infty(M, \Omega_2 \otimes \mathrm{Sym}^2 TM)$.*

Proof. We write $f = \pi_k^* K$ for some tensor $K \in C^\infty(M, \mathrm{Sym}^k TM \otimes \mathrm{Sym}^2 TM)$. The condition $\imath_v f = 0$ is the same as $K(v, ..., v, v, \bullet) = 0$. Differentiating (on SM) in v and taking $w = \partial_v$, using the symmetries of the tensor, we get $kK(w, v, ..., v, v, \bullet) + K(v, ..., v, w, \bullet) = 0$. Applying to v and w, this gives the relations

$$kK(w, v, ..., v, v) + K(v, ..., v, w) = 0, \;\; kK(w, v, ..., v, v, w) + K(v, ..., v, w, w) = 0. \tag{4.3}$$

Differentiating once again the first of these relations, we get

$$3kK(w, v, ..., v, w) + k(k-1)K(w, w, v, ..., v, v) + K(v, ..., v, w, w) = 0. \tag{4.4}$$

Combining the second relation of (4.3) and (4.4), we get

$$K(v,...,v,w,w) = \frac{k(k-1)}{2} K(w,w,v,...,v). \tag{4.5}$$

By (2.15), we can write $K = \sum_{i=0}^{k/2} \mathcal{J}^{k-2i} K_{2i}$, where $K_{2i} \in \mathrm{Sym}_0^{2i} TM \otimes \mathrm{Sym}^2 TM$ and we have $f = \pi_k^* K = \sum_{i=0}^{k/2} f_{2i}$ with $f_{2i} = \pi_{2i}^* K_{2i}$. Taking the trace in the w-variable in (4.5), we then obtain:

$$\begin{aligned}
\sum_{i=1}^n K(v,...,v,\mathbf{e}_i,\mathbf{e}_i) = \mathrm{Tr}(f(v)) = r \\
&= \frac{k(k-1)}{2} \sum_{i=1}^n K(\mathbf{e}_i,\mathbf{e}_i,v,...,v) \\
&= \frac{k(k-1)}{2} (\mathcal{I}K)(v,...,v) \\
&= \frac{k(k-1)}{2} \sum_{i=0}^{k/2-1} c_i \mathcal{J}^{k-2-2i} K_{2i}(v,...,v) = \frac{k(k-1)}{2} \sum_{i=0}^{k/2-1} c_i \imath_v \imath_v f_{2i}(v)
\end{aligned}$$

where $c_i > 0$ are some positive constants. The term of highest degree in the last sum is $\imath_v \imath_v f_{k-2}(v)$ which, in principle, could be a sum of spherical harmonics of degrees $k-4$, $k-2$ and k. But since the total sum is equal to r, the highest degree vanishes so each term in the sum has degree $\leq k-2$. Using the relation $\imath_v \imath_v f = 0 = \imath_v \imath_v u + \sum_{i=0}^{k/2-1} \imath_v \imath_v f_{2i}$, we then deduce that $\imath_v \imath_v u$ has degree $k-2$.

In the particular case where $k=2$, taking unit v and w, (4.5) gives that

$$\imath_w \imath_w (f(v)) = \imath_v \imath_v (f_0 + f_2(w)),$$

and taking the trace in w, we obtain $\mathrm{Tr}(f(v)) = r = n\langle f_0 v, v\rangle$ for all v, that is $f_0 = \frac{r}{n} \mathbb{1}_{TM}$. □

It is now worth introducing the following terminology:

Definition 4.3. We say that $K \in C^\infty(M, \mathrm{Sym}_0^k TM \otimes \mathcal{E})$ is a *normal twisted conformal Killing tensor*, if it satisfies $\mathcal{P}DK = 0$ (twisted conformal Killing condition), $\imath_v \pi_k^* K$ is of degree $k-1$ and $\imath_v \imath_v \pi_k^* K$ is of degree $k-2$ (normal condition).

For $\mathcal{E} = \Lambda^p TM$, the condition on $\imath_v \imath_v \pi_k^* K$ is automatically satisfied since it is always 0, while for $\mathcal{E} = \mathrm{Sym}^2 TM$, in our situation it is guaranteed by Lemma 4.2. The proof of Theorem 4.1 will be a consequence of Proposition 4.4 below. First of all, we introduce the constants:

$$\begin{aligned}
B^{\Lambda^p}_{n,k,\delta} &:= \delta k(n+k-2) - \frac{(1+\delta)p}{2} - \frac{2p}{3}(1-\delta)\left[k(n+k-2)(n-1)\right]^{1/2}, \\
B^{\mathrm{Sym}^2}_{n,k,\delta} &:= B^{\Lambda^2}_{n,k,\delta},
\end{aligned} \tag{4.6}$$

for $k, p \geq 0$, and

$$\begin{aligned} C^{\Lambda^p}_{n,k,\delta} &:= \frac{k(n+k-2)(n+2k-4)}{(n+k-3)(k-1)(n+2k-2)} B^{\Lambda^{p-1}}_{n,k-1,\delta} - \frac{(n+2k-4)(1+\delta)}{2}, \\ C^{\mathrm{Sym}^2}_{n,k,\delta} &:= \frac{k(n+k-2)(n+2k-4)}{(n+k-3)(k-1)(n+2k-2)} B^{\Lambda^1}_{n,k-1,\delta} - (n+2k-4)(1+\delta), \end{aligned} \tag{4.7}$$

for $k \geq 2$, $p \geq 1$. For $k = 1$, we define $C^{\Lambda^p}_{n,1,\delta} = -(n-2)\frac{1+\delta}{2}$ and $C^{\mathrm{Sym}^2}_{n,1,\delta} = -(n-2)(1+\delta)$. We eventually define

$$D_{n,k,\delta} := \frac{(n+2k-6)k(n+k-2)(n+2k-4)}{(n+k-3)(k-1)(n+2k-2)} \frac{(1+\delta)}{2},$$

with the convention that this is 0 for $k = 1$.

Proposition 4.4. *Let $K \in C^\infty(M, \mathrm{Sym}^k_0 TM \otimes \mathcal{E})$ be a normal twisted conformal Killing tensor and further assume that $k \geq 2$ if $\mathcal{E} = \Lambda^p TM$ and $k \geq 3$ if $\mathcal{E} = \mathrm{Sym}^2 TM$. If $u := \pi_k^* K \in C^\infty(M, \Omega_k \otimes \mathcal{E})$ denotes the corresponding section of $\pi^* \mathcal{E}$ over SM, then we have:*

$$B^{\mathcal{E}}_{n,k,\delta} \|u\|^2 + C^{\mathcal{E}}_{n,k,\delta} \|\imath_v u\|^2 - D_{n,k,\delta} \|\imath_v \imath_v u\|^2 \leq 0. \tag{4.8}$$

Proof. The normal twisted conformal Killing condition reads:

$$\mathbf{X}_+ u = 0, \ \ \imath_v u \text{ has degree } k-1, \ \ \imath_v \imath_v u \text{ has degree } k-2. \tag{4.9}$$

We consider $u \in C^\infty(M, \Omega_k \otimes \mathcal{E})$ satisfying (4.9). The $\mathbf{X}_+ u$ term in the localized Pestov identity (2.12) vanishes and we get:

$$\frac{(n+k-2)(n+2k-4)}{n+k-3} \|\mathbf{X}_- u\|^2 + \|Z(u)\|^2 = \langle R \nabla^{\mathcal{E}}_{\mathbb{V}} u, \nabla^{\mathcal{E}}_{\mathbb{V}} u \rangle + \langle \mathcal{F}^{\mathcal{E}} u, \nabla^{\mathcal{E}}_{\mathbb{V}} u \rangle. \tag{4.10}$$

We will bound the terms on the left-hand side of (4.10) from below while we will bound the terms on the right-hand side from above. The term $Z(u)$ seems difficult to control and we simply use $\|Z(u)\|^2 \geq 0$. The first term on the right-hand side of (4.10) involves the curvature tensor $R^{\mathcal{E}}$ can be decomposed according to §2.1 as $R^{\mathcal{E}} = R^{\mathcal{E}}_0 + \frac{1+\delta}{2} G^{\mathcal{E}}$. Using (2.11), we write correspondingly for any orthonormal basis $(\mathbf{e}_i)_{i=1}^n \subset T_x M$

$$\mathcal{F}^{\mathcal{E}} = \mathcal{F}^{\mathcal{E}}_0 + \frac{1+\delta}{2} . \mathcal{G}^{\mathcal{E}}, \quad \mathcal{F}^{\mathcal{E}}_0(x,v) = \sum_{i=1}^n \mathbf{e}_i \otimes R^{\mathcal{E}}_0(v, \mathbf{e}_i), \quad \mathcal{G}^{\mathcal{E}}(x,v) = \sum_{i=1}^n \mathbf{e}_i \otimes G^{\mathcal{E}}(v, \mathbf{e}_i). \tag{4.11}$$

We first deal with the term $\mathcal{F}^{\mathcal{E}}_0$:

Lemma 4.5. *Given $u \in C^\infty(M, \Omega_k \otimes \mathcal{E})$, one has:*

$$|\langle \mathcal{F}^{\mathcal{E}}_0 u, \nabla^{\mathcal{E}}_{\mathbb{V}} u \rangle| \leq \frac{2p}{3} (1-\delta) \left[k(n+k-2)(n-1)\right]^{1/2} \|u\|^2,$$

with the convention that $p = 2$ if $\mathcal{E} = \mathrm{Sym}^2 TM$.

Proof. We fix $x \in M$. Below, all the scalar products below are the ones on $L^2(S_xM)$, $(\mathbf{e}_i)_{1\leq i\leq n}$ is an orthonormal basis of T_xM and $(\mathbf{e}_\alpha)_\alpha$ is an orthonormal basis of $\mathcal{E}$. We have, writing $\nabla_{\mathbb{V}} u_\alpha = \sum_i \langle \nabla_{\mathbb{V}} u_\alpha, \mathbf{e}_i\rangle\, \mathbf{e}_i$ that:

$$\begin{aligned}\langle \mathcal{F}_0^{\mathcal{E}} u, \nabla_{\mathbb{V}}^{\mathcal{E}} u\rangle_{L^2(S_xM)} &= \sum_\alpha \int_{S_xM} \langle R_0^{\mathcal{E}}(v, \nabla_{\mathbb{V}} u_\alpha)u, \mathbf{e}_\alpha\rangle dv \\ &= \sum_i \int_{S_xM} \langle R_0^{\mathcal{E}}(v, \mathbf{e}_i - v_i.v)u, \sum_\alpha \langle \nabla_{\mathbb{V}} u_\alpha, \mathbf{e}_i\rangle\, \mathbf{e}_\alpha\rangle dv.\end{aligned}$$

By (2.6) and (2.7), we have:

$$|\langle R_0^{\mathcal{E}}(X,Y)\omega, \tau\rangle| \leq \tfrac{2p}{3}(1-\delta)|X||Y||\omega||\tau|,$$

for every tangent vector X, Y and p-forms or symmetric 2-tensors ω, τ (where $p = 2$ in the latter case). Hence, by Cauchy-Schwarz:

$$\begin{aligned}&|\langle \mathcal{F}_0^{\mathcal{E}} u, \nabla_{\mathbb{V}}^{\mathcal{E}} u\rangle_{L^2(S_xM)}| \\ &\leq \frac{2p}{3}(1-\delta)\sum_i \int_{S_xM} |\,\mathbf{e}_i - v_i.v||u| \left|\sum_\alpha \langle \nabla_{\mathbb{V}} u_\alpha, e_i\rangle\, \mathbf{e}_\alpha\right| dv \\ &\leq \frac{2p}{3}(1-\delta)\left(\sum_i \int_{S_xM} |\,\mathbf{e}_i - v_i.v|^2|u|^2 dv\right)^{1/2} \left(\sum_i \int_{S_xM} \left|\sum_\alpha \langle \nabla_{\mathbb{V}} u_\alpha, \mathbf{e}_i\rangle\, \mathbf{e}_\alpha\right|^2 dv\right)^{1/2} \\ &= \frac{2p}{3}(1-\delta)\left(\int_{S_xM} (n-1)|u|^2 dv\right)^{1/2} \left(\sum_i \int_{S_xM} \sum_\alpha \langle \nabla_{\mathbb{V}} u_\alpha, \mathbf{e}_i\rangle^2 dv\right)^{1/2} \\ &= \frac{2p}{3}(1-\delta)(n-1)^{1/2}\|u\| \left(\sum_i \int_{S_xM} \sum_\alpha \langle \nabla_{\mathbb{V}} u_\alpha, \mathbf{e}_i\rangle^2 dv\right)^{1/2}.\end{aligned} \tag{4.12}$$

Observe that we may compute fibre-wise in T_xM, where ∂_i denotes differentiation in $\mathbf{e}_i$:

$$\langle \nabla_{\mathbb{V}} u_\alpha, \mathbf{e}_i\rangle = \langle \nabla_{\mathbb{V}}^{\mathrm{tot}} u_\alpha - v.\langle \nabla_{\mathbb{V}}^{\mathrm{tot}} u_\alpha, v\rangle, \mathbf{e}_i\rangle = \partial_i u_\alpha - v_i.k u_\alpha, \tag{4.13}$$

where $\nabla_{\mathbb{V}}^{\mathrm{tot}}(\bullet) = \nabla_{\mathbb{V}}(\bullet) + v.\langle \bullet, v\rangle$ is the total gradient, and the last equality follows from Euler's relation on homogeneous functions. Using the same relations, we obtain:

$$\begin{aligned}\sum_{\alpha,i} \langle \nabla_{\mathbb{V}} u_\alpha, \mathbf{e}_i\rangle^2 &= \sum_{\alpha,i} |\partial_i u_\alpha|^2 - \sum_\alpha 2k u_\alpha \sum_i v_i \partial_i u_\alpha + \sum_\alpha k^2 \sum_i v_i^2 u_\alpha^2 = \sum_{\alpha,i} |\partial_i u_\alpha|^2 - k^2|u|^2 \\ &= \sum_\alpha |\nabla_{\mathbb{V}}^{\mathrm{tot}} u_\alpha|^2 - k^2|u|^2 = \sum_\alpha |\nabla_{\mathbb{V}} u_\alpha|^2 = |\nabla_{\mathbb{V}}^{\mathcal{E}} u|^2.\end{aligned}$$

Integrating on the sphere, we get

$$\int_{S_xM} \sum_{\alpha,i} \langle \nabla_{\mathbb{V}} u_\alpha, \mathbf{e}_i \rangle^2 = \|\nabla_{\mathbb{V}}^{\mathcal{E}} u\|^2_{L^2(S_xM)} = \langle \Delta_{\mathbb{V}}^{\mathcal{E}} u, u \rangle = k(n+k-2)\|u\|^2.$$

This completes the proof. □

We now deal with the term involving $\mathcal{G}^{\mathcal{E}}$ in (4.11):

Lemma 4.6. *Let $u \in C^\infty(M, \Omega_k \otimes \mathcal{E})$ such that $\imath_v u$ is of degree $k-1$. Then we have:*

$$\langle \mathcal{G}^{\Lambda^p} u, \nabla_{\mathbb{V}}^{\Lambda^p} u \rangle = (n+2k-4)\|\imath_v u\|^2 + p\|u\|^2,$$

and

$$\langle \mathcal{G}^{\mathrm{Sym}^2} u, \nabla_{\mathbb{V}}^{\mathrm{Sym}^2} u \rangle = 2(n+2k-4)\|\imath_v u\|^2 + 2\|u\|^2.$$

Proof. We treat the two cases separately.

Case $\mathcal{E} = \Lambda^p TM$: We use the same conventions as in the proof of Lemma 4.5. Firstly, note that $\imath_v u$ being of degree $k-1$ is equivalent to:

$$\imath_v u = \sum_\alpha u_\alpha . \imath_v \mathbf{e}_\alpha = \sum_{\alpha,i} u_\alpha v_i . \imath_{\mathbf{e}_i} \mathbf{e}_\alpha = (n+2(k-1))^{-1} \partial_i u_\alpha |v|^2 \imath_{\mathbf{e}_i} \mathbf{e}_\alpha . \tag{4.14}$$

By (2.11), we may write $\mathcal{G}^{\Lambda^p} = \sum_i \mathbf{e}_i \otimes G^{\Lambda^p}(v, \mathbf{e}_i)$, so that on S_xM:

$$\begin{aligned}
\langle \mathcal{G}^{\Lambda^p} u, \nabla_{\mathbb{V}}^{\Lambda^p} u \rangle_{L^2} &= \sum_{\alpha,\beta} \int_{S_xM} u_\alpha . \langle G^{\Lambda^p}(v, \nabla_{\mathbb{V}} u_\beta)\, \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle \\
&= \sum_{\alpha,\beta,i,j} \int_{S_xM} u_\alpha . \langle \ldots \wedge \langle G(v, \nabla_{\mathbb{V}} u_\beta)\, \mathbf{e}_{\alpha_i}, \mathbf{e}_j \rangle . \mathbf{e}_j \wedge \ldots , \mathbf{e}_\beta \rangle \\
&= \sum_{\alpha,\beta,i,j} \int_{S_xM} u_\alpha . G(v, \nabla_{\mathbb{V}} u_\beta, \mathbf{e}_{\alpha_i}, \mathbf{e}_j) . \langle \mathbf{e}_j \wedge \imath_{\mathbf{e}_{\alpha_i}} \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle \\
&= \sum_{\alpha,\beta,i,j} \int_{S_xM} u_\alpha . \Big(\langle v, \mathbf{e}_i \rangle . \langle \nabla_{\mathbb{V}} u_\beta, \mathbf{e}_j \rangle - \langle v, \mathbf{e}_j \rangle . \langle \nabla_{\mathbb{V}} u_\beta, \mathbf{e}_i \rangle \Big) . \langle \imath_{\mathbf{e}_i} \mathbf{e}_\alpha, \imath_{\mathbf{e}_j} \mathbf{e}_\beta \rangle,
\end{aligned}$$

where we used that the wedge product is adjoint to contraction. Denote by A and B the first and second terms in the last expression, respectively. We compute:

$$\begin{aligned}
A &= \sum_{\alpha,\beta,i,j} \int_{S_xM} u_\alpha . \langle v, \mathbf{e}_i \rangle . \langle \nabla_{\mathbb{V}} u_\beta, \mathbf{e}_j \rangle . \langle \imath_{\mathbf{e}_i} \mathbf{e}_\alpha, \imath_{\mathbf{e}_j} \mathbf{e}_\beta \rangle = \sum_{\beta,i} \int_{S_xM} (\partial_j u_\beta - k . u_\beta v_j) . \langle \imath_v u, \imath_{\mathbf{e}_j} \mathbf{e}_\beta \rangle \\
&= (n+k-2) \int_{S_xM} |\imath_v u|^2,
\end{aligned}$$

where we used (4.13) in the first line and (4.14) in the second one. Next, for B we have:

$$\begin{aligned}
B &= \sum_{\alpha,\beta,i,j} \int_{S_xM} u_\alpha.\langle v, \mathbf{e}_j\rangle.(\partial_i u_\beta - k.u_\beta v_i).\langle \imath_{\mathbf{e}_i}\mathbf{e}_\alpha, \imath_{\mathbf{e}_j}\mathbf{e}_\beta\rangle \\
&= -k\int_{S_xM} |\imath_v u|^2 + \sum_{\alpha,\beta,i,j}\int_{S_xM} u_\alpha.(\partial_i(u_\beta v_j) - u_\beta.\delta_{ij})\langle \imath_{\mathbf{e}_i}\mathbf{e}_\alpha, \imath_{\mathbf{e}_j}\mathbf{e}_\beta\rangle \\
&= -k\int_{S_xM} |\imath_v u|^2 - p\int_{S_xM}|u|^2 + (n+2(k-1))^{-1}\sum_{\alpha,\beta,i,j}\int_{S_xM} u_\alpha.\partial_i(\langle \imath_{\mathbf{e}_i}\mathbf{e}_\alpha, \partial_j u_\beta.\imath_{\mathbf{e}_j}\mathbf{e}_\beta\rangle) \\
&= -k\int_{S_xM} |\imath_v u|^2 - p\int_{S_xM}|u|^2 + (n+2(k-1))^{-1}\sum_{\alpha,\beta,i,j}\int_{S_xM} u_\alpha \\
&\times(2v_i.\partial_j u_\beta + \partial_i\partial_j u_\beta).\langle \imath_{\mathbf{e}_i}\mathbf{e}_\alpha, \imath_{\mathbf{e}_j}\mathbf{e}_\beta\rangle = -(k-2)\int_{S_xM}|\imath_v u|^2 - p\int_{S_xM}|u|^2,
\end{aligned}$$

where we used (4.13) in the first line, (4.14) in the third and final lines, as well as the fact that u_α's are of degree k while $\partial_i\partial_j u_\alpha$ is of strictly lesser degree. We also used the following identity on p-forms: $\sum_i \mathbf{e}_i \wedge \imath_{\mathbf{e}_i} = p.\mathrm{Id}$. This completes the proof when $\mathcal{E} = \Lambda^p TM$.

Case $\mathcal{E} = \mathrm{Sym}^2 TM$*:* Write $u = \sum_{i,j} u_{ij}\mathbf{e}_{ij}$ where $\mathbf{e}_{ij} = \mathbf{e}_i^* \otimes \mathbf{e}_j$ and $u_{ij} = u_{ji}$ by symmetry of u. We begin by observing that $u(v)$ is of degree $k-1$ translates into:

$$u(v) = \sum_{i,j} u_{ij} v_i \mathbf{e}_j = (n+2(k-1))^{-1}\sum_{i,j}\partial_i u_{ij}\mathbf{e}_j\,. \tag{4.15}$$

Therefore

$$\begin{aligned}
\langle \mathcal{G}^{\mathrm{Sym}^2}u, \nabla_{\mathbb{V}}^{\mathrm{Sym}^2}u\rangle &= \sum_{i,\ell,m}\int \langle \mathbf{e}_i, \nabla_{\mathbb{V}}u_{\ell m}\rangle.\langle G^{\mathrm{Sym}^2}(v,\mathbf{e}_i)u, \mathbf{e}_{\ell m}\rangle = \sum_{i,j,\ell,m}\int u_{ij}\langle [G(v,\nabla_{\mathbb{V}}u_{\ell m}), \mathbf{e}_{ij}], \mathbf{e}_{\ell m}\rangle \\
&= \sum_{i,j,\ell,m}\int u_{ij}\big\langle \mathbf{e}_i^*\otimes G(v,\nabla_{\mathbb{V}}u_{\ell m})\,\mathbf{e}_j + (G(v,\nabla_{\mathbb{V}}u_{\ell m})\,\mathbf{e}_i)^*\otimes\mathbf{e}_j, \mathbf{e}_{\ell m}\big\rangle \\
&= \sum_{i,j,\ell,m}\int u_{ij}\big(\delta_{i\ell}\langle G(v,\nabla_{\mathbb{V}}u_{\ell m})\,\mathbf{e}_j, \mathbf{e}_m\rangle + \delta_{jm}\langle G(v,\nabla_{\mathbb{V}}u_{\ell m})\,\mathbf{e}_i, \mathbf{e}_\ell\rangle\big) \\
&= 2\sum_{i,j,\ell}\int u_{ij}\big\langle G(v,\nabla_{\mathbb{V}}u_{i\ell})\,\mathbf{e}_j, \mathbf{e}_\ell\big\rangle = 2\sum_{i,j,\ell}\int u_{ij}\Big(\langle v,\mathbf{e}_j\rangle.\langle\nabla_{\mathbb{V}}u_{i\ell},\mathbf{e}_\ell\rangle - \langle v,\mathbf{e}_\ell\rangle.\langle\nabla_{\mathbb{V}}u_{i\ell},\mathbf{e}_j\rangle\Big).
\end{aligned}$$

Note that we used the symmetry $u_{ij} = u_{ji}$ in the last line. Denoting by A the first and by B the second term, we get using (4.15):

$$A = \sum_{i,j,\ell}\int u_{ij}.v_j.(\partial_\ell u_{i\ell} - k u_{i\ell}v_\ell) = -k\|u(v)\|^2 + (n+2k-2)\|u(v)\|^2 = (n+k-2)\|\imath_v u\|^2.$$

For the term B, we have:

$$\begin{aligned}B &= \sum_{i,j,\ell} \int u_{ij}.v_\ell.(\partial_j u_{i\ell} - k.v_j.u_{i\ell}) = -k\|\imath_v u\|^2 + \sum_{i,j,\ell} \int u_{ij}(\partial_j(v_\ell u_{i\ell}) - \delta_{j\ell} u_{i\ell}) \\ &= -k\|\imath_v u\|^2 - \|u\|^2 + (n+2(k-1))^{-1} \sum_{i,j,\ell} \int u_{ij} \partial_j (|v|^2 \partial_\ell u_{i\ell}) \\ &= -k\|\imath_v u\|^2 - \|u\|^2 + 2(n+2(k-1))^{-1} \sum_{i,j,\ell} \int u_{ij} v_j \partial_\ell u_{i\ell} = -(k-2)\|\imath_v u\|^2 - \|u\|^2.\end{aligned}$$

Here we used (4.15) in the second and last lines, and the fact that $\partial_j \partial_\ell u_{i\ell}$ is of degree at most $k-2$, while u_{ij} is of degree k. This completes the proof. □

Lemma 4.7. *Given $u \in C^\infty(M, \Omega_k \otimes \mathcal{E})$*

$$\langle R\nabla^{\mathcal{E}}_{\mathbb{V}} u, \nabla^{\mathcal{E}}_{\mathbb{V}} u\rangle \leq -\delta k(n+k-2)\|u\|^2.$$

Proof. This is immediate using the upper bound on the sectional curvature:

$$\langle R\nabla^{\mathcal{E}}_{\mathbb{V}} u, \nabla^{\mathcal{E}}_{\mathbb{V}} u\rangle \leq -\delta\|\nabla_{\mathbb{V}} u\|^2 = -\delta\langle \Delta_{\mathbb{V}} u, u\rangle = -\delta k(n+k-2)\|u\|^2.$$

□

Overall, the right-hand side of (4.10) can be bounded using Lemmas 4.5, 4.6, 4.7. We derive from the three previous lemmas the following lower bound:

Lemma 4.8. *Let $u \in C^\infty(M, \Omega_k \otimes \Lambda^p TM)$ such that $\imath_v u$ is of degree $k-1$, and $k \geq 1$. Then:*

$$\|\mathbf{X}_+ u\|^2 \geq \frac{k+1}{k(n+2k)} \left(B^{\Lambda^p}_{n,k,\delta} \|u\|^2 - \frac{1+\delta}{2}(n+2k-4)\|\imath_v u\|^2 \right),$$

where $B^{\Lambda^p}_{n,k,\delta}$ is defined in (4.6).

Proof. We apply the localized Pestov identity (2.12). The terms $\|\mathbf{X}_- u\|^2$ and $\|Z(u)\|^2$ are simply bounded from below by ≥ 0. Using Lemmas 4.5, 4.6, 4.7, we obtain:

$$\begin{aligned}\frac{k(n+2k)}{k+1}\|\mathbf{X}_+ u\|^2 &\geq -\langle R\nabla^{\Lambda^p}_{\mathbb{V}} u, \nabla^{\Lambda^p}_{\mathbb{V}} u\rangle - \langle \mathcal{F}^{\Lambda^p} u, \nabla^{\mathcal{E}}_{\mathbb{V}} u\rangle \\ &\geq \left(\delta k(n+k-2) - \frac{(1+\delta)p}{2} - \frac{2p}{3}(1-\delta)\left[k(n+k-2)(n-1)\right]^{1/2} \right) \|u\|^2 \\ &\qquad - \frac{1+\delta}{2}(n+2k-4)\|\imath_v u\|^2 \\ &= B^{\Lambda^p}_{n,k,\delta}\|u\|^2 - \frac{1+\delta}{2}(n+2k-4)\|\imath_v u\|^2.\end{aligned}$$

□

We now have:

Lemma 4.9. *Under the assumptions* (4.9)*, and if* $k \geq 2$*, we have:*

$$\|\mathbf{X}_- u\|^2 \geq \frac{k}{(k-1)(n+2k-2)} \left(B^{\Lambda^{p-1}}_{n,k-1,\delta} \|\imath_v u\|^2 - \frac{1+\delta}{2}(n+2k-6)\|\imath_v \imath_v u\|^2 \right),$$

with the convention that $p = 2$ *if* $\mathcal{E} = \mathrm{Sym}^2 TM$.

Proof. Observe that $\mathbf{X}\imath_v u = \imath_v \mathbf{X} u = \imath_v \mathbf{X}_- u$, and thus applying Lemma 4.8 with $\imath_v u$, we get:

$$\begin{aligned}
\|\mathbf{X}_- u\|^2 &\geq \|\imath_v \mathbf{X}_- u\|^2 \\
&= \|\mathbf{X}\imath_v u\|^2 = \|\mathbf{X}_- \imath_v u\|^2 + \|\mathbf{X}_+ \imath_v u\|^2 \\
&\geq \frac{k}{(k-1)(n+2k-2)} \left(B^{\Lambda^{p-1}}_{n,k-1,\delta} \|\imath_v u\|^2 - \frac{1+\delta}{2}(n+2k-6)\|\imath_v \imath_v u\|^2 \right).
\end{aligned}$$

□

We can now complete the proof of Proposition 4.4. Inserting the bounds of Lemmas 4.5, 4.6, 4.7, 4.9 in (4.10), we obtain:

$$\begin{aligned}
&\frac{(n+k-2)(n+2k-4)}{n+k-3} \frac{k}{(k-1)(n+2k-2)} \left(B^{\Lambda^{p-1}}_{n,k-1,\delta} \|\imath_v u\|^2 - \frac{1+\delta}{2}(n+2k-6)\|\imath_v \imath_v u\|^2 \right) \\
&\leq \text{RHS of } (4.10) \\
&\leq \frac{1+\delta}{2} \left(\varepsilon(\mathcal{E})(n+2k-4)\|\imath_v u\|^2 + p\|u\|^2 \right) + \frac{2p}{3}(1-\delta) \left[k(n+k-2)(n-1) \right]^{1/2} \|u\|^2 \\
&\qquad - \delta k(n+k-2)\|u\|^2,
\end{aligned} \tag{4.16}$$

where $\varepsilon(\mathcal{E}) = 2$ if $\mathcal{E} = \mathrm{Sym}^2 TM$ and $\varepsilon(\mathcal{E}) = 1$ if $\mathcal{E} = \Lambda^p TM$. This inequality can now be rearranged as

$$B^{\mathcal{E}}_{n,k,\delta}\|u\|^2 + C^{\mathcal{E}}_{n,k,\delta}\|\imath_v u\|^2 - D_{n,k,\delta}\|\imath_v \imath_v u\|^2 \leq 0,$$

□

Remark 4.10. It can be easily checked that the Cauchy-Schwarz estimate in (4.12) is *sharp for* $k = 1$, $p = 1$. We believe that for degrees $k > 1$ this estimate is not sharp. Let us assume $p = 1$ and $k = 3$ in what follows. Let $C(n) > 0$ be the optimal constant such that $F(u) := \sum_{i=1}^n \int_{S_x M} |u - u_i\, \mathbf{e}_i\, |.|\nabla_{\mathbb{V}} u_i|.\omega_i(v) \leq C(n)\|u\|^2$ holds for $u = \sum_{i=1}^n u_i\, \mathbf{e}_i$, where u_i are spherical harmonics of degree $k = 3$ on $\mathbb{S}^{n-1}$; here $\omega_i(v) \in [\frac{1}{2}, 1]$ are certain weights that are obtained by collecting leftover terms in the proof of Lemma 2.1 (these are not necessary in the article). By our estimate (4.12) we get $C(n) \leq \sqrt{3(n^2-1)}$. The space of spherical harmonics of degree $k = 3$ for $n = 4, 6, 8$ has dimension $16, 50, 112$, respectively. Results of a computer program evaluating $F(u)$ at random spherical harmonics in these cases are given in the following table. Here $\delta_{\Lambda^1,\mathrm{new}}(n)$ denotes the new corresponding value of $\delta_{\Lambda^1}(n)$ in Theorem 4.1. Results also indicate that the quotient $\frac{C(n)}{\sqrt{3(n^2-1)}}$ converges to 1 as $n \to \infty$, i.e. that the Cauchy-Schwarz bound is asymptotically optimal, and that this particular estimate does not suffice to prove Conjecture 1.1 directly (however an optimal estimate would improve Theorem 1.2).

$n =$	4	6	8
$C(n) \sim$	5.294	8.614	12.193
$\frac{C(n)}{\sqrt{3(n^2-1)}} \sim$	0.789	0.8407	0.886
$\delta_{\Lambda^1,\mathrm{new}}(n) \sim$	0.267	0.262	0.261
# random points	500	50	5

4.2. **Non-existence of invariant structures.** We can now deduce Theorem 4.1 from the previous paragraph.

Proof of Theorem 4.1. We treat separately the Sym^2TM and the Λ^pTM cases.

Case $\mathcal{E} = \Lambda^pTM$: We consider an element $f \in C^\infty(SM, \Lambda^p\mathcal{N})$ such that $\mathbf{X}f = 0$. By [GPSU16], it has finite degree, so we can write $f = f_0 + ... + f_k$, where $f_i \in C^\infty(M, \Omega_i \otimes \Lambda^pTM)$ and $f_k \neq 0$. We set $u := f_k$. Then u is a normal conformal Killing tensor in the sense of Definition 4.3 and it thus satisfies the conclusions of Proposition 4.4. Moreover, $\imath_v\imath_v u = 0$. We then argue as follows: if $B^{\Lambda^p}_{n,k,\delta} > 0$ and $B^{\Lambda^p}_{n,k,\delta} + C^{\Lambda^p}_{n,k,\delta} > 0$, then $u = 0$. Indeed, assume that this condition is satisfied. Then, if $C^{\Lambda^p}_{n,k,\delta} \geq 0$, we obtain $B^{\Lambda^p}_{n,k,\delta}\|u\|^2 \leq 0$, hence $u = 0$. If $C^{\Lambda^p}_{n,k,\delta} \leq 0$, then we simply use the bound $\|\imath_v u\|^2 \leq \|u\|^2$ which gives $(B^{\Lambda^p}_{n,k,\delta} + C^{\Lambda^p}_{n,k,\delta})\|u\|^2 \leq 0$ and thus $u = 0$. We shall now see that this condition translates into a pinching condition on δ.

We introduce the notation $r_{n,p,k} := \frac{2p}{3}\sqrt{\frac{n-1}{k(n+k-2)}}$ and $s_{n,p,k} := (n+2k-2)r_{n,p,k}$. Then one can write

$$\begin{aligned} B^{\Lambda^p}_{n,k,\delta} &= \delta\left(k(n+k-2) - \frac{p}{2} + k(n+k-2)r_{n,p,k}\right) - \left(\frac{p}{2} + k(n+k-2)r_{n,p,k}\right) \\ &= k(n+k-2)\left(\delta\left(1 - \frac{p}{2k(n+k-2)} + r_{n,p,k}\right) - \left(\frac{p}{2k(n+k-2)} + r_{n,p,k}\right)\right), \end{aligned}$$

and

$$\begin{aligned} C^{\Lambda^p}_{n,k,\delta} &= \frac{k(n+k-2)(n+2k-4)}{(k-1)(n+k-3)(n+2k-2)} B^{\Lambda^{p-1}}_{n,k-1,\delta} - \frac{(n+2k-4)(1+\delta)}{2} \\ &= \frac{k(n+k-2)(n+2k-4)}{n+2k-2}\left[\delta\left(1 - \frac{(p-1)}{2(k-1)(n+k-3)} + r_{n,p-1,k-1} - \frac{(n+2k-2)}{2k(n+k-2)}\right)\right. \\ &\quad \left. - \left(\frac{(p-1)}{2(n+k-3)(k-1)} + r_{n,p-1,k-1} + \frac{(n+2k-2)}{2k(n+k-2)}\right)\right]. \end{aligned}$$

It follows that $B^{\Lambda^p}_{n,k,\delta} > 0$ if and only if

$$\delta > \delta_1 := \frac{\frac{p}{2k(n+k-2)} + r_{n,p,k}}{1 - \frac{p}{2k(n+k-2)} + r_{n,p,k}} = \frac{\frac{p(n+2k-2)}{2k(n+k-2)} + s_{n,p,k}}{n+2k-2-\frac{p(n+2k-2)}{2k(n+k-2)} + s_{n,p,k}} \tag{4.17}$$

and $B^{\Lambda^p}_{n,k,\delta} + C^{\Lambda^p}_{n,k,\delta} > 0$ if and only if

$$\begin{aligned}\delta > \delta_2 &:= \frac{\frac{(p-1)}{2(n+k-3)(k-1)} + r_{n,p-1,k-1} + \frac{(n+2k-2)}{2k(n+k-2)} + \frac{n+2k-2}{n+2k-4}\left(\frac{p}{2k(n+k-2)} + r_{n,p,k}\right)}{1-\frac{(p-1)}{2(n+k-3)(k-1)} + r_{n,p-1,k-1} - \frac{(n+2k-2)}{2k(n+k-2)} + \frac{n+2k-2}{n+2k-4}\left(1-\frac{p}{2k(n+k-2)} + r_{n,p,k}\right)} \\ &= \frac{\frac{(p-1)(n+2k-4)}{2(n+k-3)(k-1)} + s_{n,p-1,k-1} + \frac{(n+2k-2)(n+2k-4)}{2k(n+k-2)} + \frac{p(n+2k-2)}{2k(n+k-2)} + s_{n,p,k}}{n+2k-4-\frac{(p-1)(n+2k-4)}{2(n+k-3)(k-1)} + s_{n,p-1,k-1} - \frac{(n+2k-2)(n+2k-4)}{2k(n+k-2)} + n+2k-2-\frac{p(n+2k-2)}{2k(n+k-2)} + s_{n,p,k}}.\end{aligned}$$

In other words, we get:

$$\left(B^{\Lambda^p}_{n,k,\delta} > 0 \text{ and } B^{\Lambda^p}_{n,k,\delta} + C^{\Lambda^p}_{n,k,\delta} > 0\right) \Leftrightarrow \delta > \max(\delta_1, \delta_2).$$

We further claim that the following holds:

Lemma 4.11. *The functions* $(k,p) \mapsto \delta_1(n,k,p), \delta_2(n,k,p)$ *are increasing in* p *and decreasing in* k*, for* $k \geq 2$*,* $p \geq 1$*.*

Proof. Note that

$$s_{n,p,k} = \frac{2p\sqrt{n-1}}{3}\sqrt{\frac{(n+2k-2)^2}{k(n+k-2)}} = \frac{2p\sqrt{n-1}}{3}\sqrt{4+\frac{(n-2)^2}{k(n+k-2)}}$$

is decreasing in k and increasing in p, and by (4.17) one can write

$$\frac{1}{\delta_1} - 1 = \frac{n+2k-2-\frac{p(n+2k-2)}{k(n+k-2)}}{\frac{p(n+2k-2)}{2k(n+k-2)} + s_{n,p,k}} = \frac{n+2k-2-\frac{p}{n+k-2}-\frac{p}{k}}{\frac{1}{2}\left(\frac{p}{n+k-2}+\frac{p}{k}\right) + s_{n,p,k}},$$

thus showing that δ_1 is decreasing in k, increasing in p.

We claim that δ_2 is also decreasing in k. For that, we introduce

$$F_{n,k,p} := \frac{(p-1)(n+2k-4)}{2(n+k-3)(k-1)} + \frac{(n+2k-2)(n+2k-4)}{2k(n+k-2)} + \frac{p(n+2k-2)}{2k(n+k-2)}.$$

From the above expression for δ_2, we get

$$\frac{1}{\delta_2} - 1 = \frac{2n+4k-6-2F_{n,k,p}}{s_{n,p-1,k-1} + s_{n,p,k} + F_{n,k,p}}$$

so our claim would follow if $F_{n,k,p}$ is decreasing in k and increasing in p. The statement for p is immediate. As to the statement for k, this indeed holds for $p \geq 2$ since one can write the above expression as

$$2F_{n,k,p} = \frac{p-1}{k-1} + \frac{p-1}{n+k-3} + 4 + \frac{(n-2)^2}{k(n+k-2)} + \frac{p-2}{k} + \frac{p-2}{n+k-2},$$

which is sum of decreasing functions. For $p = 1$ this argument fails, but in this case one can write

$$\frac{1}{\delta_2} - 1 = \frac{2n + 4k - 6 - \frac{(n+2k-2)(n+2k-3)}{k(n+k-2)}}{\frac{(n+2k-2)(n+2k-3)}{2k(n+k-2)} + s_{n,1,k}} = \frac{2 - \frac{1}{k} - \frac{1}{n+k-2}}{\frac{1}{2k} + \frac{1}{2(n+k-2)} + \frac{s_{n,1,k}}{n+2k-3}},$$

and the numerator is increasing in k, whereas the denominator is decreasing in k. □

We then set

$$\begin{aligned}
\delta_{\Lambda^1}(n) &:= \max\big(\delta_1(n, k = 3, p = 1), \delta_2(n, k = 3, p = 1)\big),\\
\delta_{\mathrm{U}(3)}(7) &:= \max\big(\delta_1(n = 7, k = 3, p = 2), \delta_2(n = 7, k = 3, p = 2)\big) = 0.4962...,\\
\delta_{\mathrm{G}_2}(8) &:= \max\big(\delta_1(n = 8, k = 3, p = 3), \delta_2(n = 8, k = 3, p = 3)\big) = 0.6212...,\\
\delta_{\mathrm{E}_7}(134) &:= \max\big(\delta_1(n = 134, k = 2, p = 3), \delta_2(n = 134, k = 2, p = 3)\big) = 0.6716...
\end{aligned}$$

We deal separately with different cases. The non-zero invariant section f has constant norm and, up to rescaling, we can always assume that it is equal to 1.

(1) We first consider the case where $p = 1$ and $f \in C^\infty(SM, \mathcal{N})$ is flow-invariant and odd. Assuming that the manifold (M^n, g) is δ-pinched and $\delta > \delta_{\Lambda^1}(n)$, we then obtain that $f_k = 0$ if $k \geq 3$ since Lemma 4.11 ensures that $\delta > \delta_{\Lambda^1}(n) > \max\big(\delta_1(n, k, p = 1), \delta_2(n, k, p = 1)\big)$ for all $k \geq 3$. The invariant section $f \in C^\infty(SM, \mathcal{N})$ is thus of degree ≤ 2, hence of pure degree 1 since it is odd. As a consequence, $f = \pi_1^* f_1$ for some $f_1 \in C^\infty(M, TM \otimes TM)$. This implies the existence of $J \in C^\infty(M, \mathrm{End}(TM))$ such that $(f_1)_x(v) = J(x)v$ for all $x \in M, v \in T_xM$. Moreover, the properties $f_1(v) \in v^\perp$ and $|f_1(v)|^2 = 1$ implies that J is skew-symmetric and orthogonal, hence an almost-complex structure. The flow-invariance property $\mathbf{X}f = 0$ then translates into $(\nabla_v J)v = 0$, that is J endows (M^n, g) with a nearly-Kähler structure [Gra76, Nag02]. By [Nag02, Proposition 2.1] this implies that the universal cover $(\widetilde{M}, \widetilde{g})$ splits as a product of a Kähler manifold and a strictly nearly-Kähler manifold (i.e. such that $\nabla_v J \neq 0$ whenever $v \neq 0$). Since $(\widetilde{M}, \widetilde{g})$ has negative sectional curvature, it cannot be a Riemannian product, so one of the factors is trivial. Hence: either (M, g, J) is strictly nearly-Kähler, in which case g has positive scalar curvature by [Nag02, Theorem 1.1] and this is impossible, or it is Kähler, in which case the pinching satisfies $\delta \leq 1/4$ by [Ber60][10] and this is also a contradiction.

(2) We now deal with the exceptional case where $n = 7$ and $f \in C^\infty(SM, \Lambda^2\mathcal{N})$ is an odd flow-invariant almost-complex structure. If $\delta > \delta_{\mathrm{U}(3)}(7)$, we conclude that f has pure degree 1 since it is odd. As before, we obtain that $f = \pi_1^* f_1$

[10] Berger [Ber60] covers the positive sectional curvature case but the proof is verbatim the same in the negatively curved case.

for some $f_1 \in C^\infty(M, \Lambda^3 TM)$ with the property that for all v, $f_1(v, \bullet, \bullet)$ is an almost-complex structure on $v^\perp$. This implies that f_1 is a G_2-structure on M. The flow-invariance condition $\mathbf{X}f = 0$ then translates into $(\nabla_v f_1)(v, \bullet, \bullet) = 0$, that is f_1 is nearly-parallel (see [FG82, Theorem 5.2, second line] or [AS12, Proposition 2.4, case (6)] for definitions). By [FKMS97, Proposition 3.10], nearly-parallel G_2-manifolds are Einstein with non-negative scalar curvature and this contradicts the negative sectional curvature of g.

(3) Consider now the case $n = 8$ and $f \in C^\infty(SM, \Lambda^3 \mathcal{N})$ is an odd flow-invariant G_2-structure on SM. If $\delta > \delta_{\mathrm{G}_2}(8)$, the same argument as before shows that $f = \pi_1^* f_1$, for some $f_1 \in C^\infty(M, \Lambda^4 TM)$ with the property that for every v, $f_1(v, \bullet, \bullet, \bullet) = 0$ is a G_2-structure on $v^\perp$, that is f_1 is a $\mathrm{Spin}(7)$-structure. Moreover, the flow-invariance condition $\mathbf{X}f = 0$ translates into $(\nabla_v f_1)(v, \bullet, \bullet, \bullet)$, that is f_1 is nearly-parallel. By [Fer86, Lemma 4.4], nearly-parallel $\mathrm{Spin}(7)$-structures are necessarily parallel and this implies that g is Ricci-flat [Bon66], which is a contradiction.

(4) Eventually, we deal with the case where $n = 134$ and $f \in C^\infty(SM, \Lambda^3 \mathcal{N})$ is a flow-invariant Lie bracket of degree $k \geq 2$. Taking $u := f_k \neq 0$, we get that u is a normal twisted conformal Killing tensor of degree ≥ 2. If $\delta > \delta_{\mathrm{E}_7}(134)$, then we get as before that $u = 0$ which is a contradiction.

Case $\mathcal{E} = \mathrm{Sym}^2 TM$*:* We treat separately the case $k = 2$ and $k \geq 4$. First of all, let us assume $f \in C^\infty(SM, \mathrm{Sym}^2 \mathcal{N})$ is a flow-invariant even projector with (even) degree $k \geq 4$. Let $u := f_k \neq 0$. Then u is a normal twisted conformal Killing tensor satisfying the conclusions of Proposition 4.4. We use the bound $\|\imath_v \imath_v u\|^2 \leq \|\imath_v u\|^2$, which gives, setting $\overline{C}^{\mathrm{Sym}^2}_{n,k,\delta} := C^{\mathrm{Sym}^2}_{n,k,\delta} - D_{n,k,\delta}$, that:

$$B^{\mathrm{Sym}^2}_{n,k,\delta} \|u\|^2 + \overline{C}^{\mathrm{Sym}^2}_{n,k,\delta} \|\imath_v u\|^2 \leq 0.$$

As before, if $B^{\mathrm{Sym}^2}_{n,k,\delta} > 0$ and $B^{\mathrm{Sym}^2}_{n,k,\delta} + \overline{C}^{\mathrm{Sym}^2}_{n,k,\delta} > 0$, then we can conclude that $u = 0$. Now, we have:

$$\left(B^{\mathrm{Sym}^2}_{n,k,\delta} > 0 \text{ and } B^{\mathrm{Sym}^2}_{n,k,\delta} + \overline{C}^{\mathrm{Sym}^2}_{n,k,\delta} > 0\right) \Leftrightarrow \delta > \max(\delta_1, \delta_2'),$$

where δ_1 is the same as before (4.17) (with $p = 2$) and

$$\begin{aligned}\delta_2' &:= \frac{\frac{1}{2(k-1)(n+k-3)} + r_{n,1,k-1} + \frac{(n+2k-2)}{k(n+k-2)} + \frac{n+2k-2}{n+2k-4}\left(\frac{1}{k(n+k-2)} + r_{n,2,k}\right) + \frac{n+2k-6}{2(k-1)(n+k-3)}}{1 - \frac{1}{2(k-1)(n+k-3)} + r_{n,1,k-1} - \frac{(n+2k-2)}{k(n+k-2)} + \frac{n+2k-2}{n+2k-4}\left(1 - \frac{1}{k(n+k-2)} + r_{n,2,k}\right) - \frac{n+2k-6}{2(k-1)(n+k-3)}} \\ &= \frac{\frac{(n+2k-4)(n+2k-5)}{2(k-1)(n+k-3)} + s_{n,1,k-1} + \frac{(n+2k-2)(n+2k-3)}{k(n+k-2)} + s_{n,2,k}}{2n + 4k - 6 - \frac{(n+2k-4)(n+2k-5)}{2(k-1)(n+k-3)} + s_{n,1,k-1} - \frac{(n+2k-2)(n+2k-3)}{k(n+k-2)} + s_{n,2,k}},\end{aligned}$$

is defined for $k \geq 4$. We claim that the following holds:

Lemma 4.12. *For $n \geq 7$, the function $k \mapsto \delta_2'(n,k)$ is decreasing in k.*

Proof. We have

$$\frac{1}{\delta_2'} - 1 = \frac{2n+4k-6 - \frac{(n+2k-4)(n+2k-5)}{(k-1)(n+k-3)} - \frac{2(n+2k-2)(n+2k-3)}{k(n+k-2)}}{\frac{(n+2k-4)(n+2k-5)}{2(k-1)(n+k-3)} + s_{n,1,k-1} + \frac{(n+2k-2)(n+2k-3)}{k(n+k-2)} + s_{n,2,k}}$$
$$= \frac{2 - \frac{(n+2k-4)(n+2k-5)}{(k-1)(n+k-3)(n+2k-3)} - \frac{2}{k} - \frac{2}{n+k-2}}{\frac{(n+2k-4)(n+2k-5)}{2(k-1)(n+k-3)(n+2k-3)} + \frac{s_{n,1,k-1}}{n+2k-3} + \frac{(n+2k-2)}{k(n+k-2)} + \frac{s_{n,2,k}}{n+2k-3}}.$$

The numerator is increasing in k and the denominator is decreasing in k since for $n \geq 7$ the expression

$$\frac{(n+2k-4)(n+2k-5)}{(k-1)(n+k-3)(n+2k-3)} = \frac{1}{2(k-1)} + \frac{n-5}{2(k-1)(n+2k-3)} + \frac{3}{2(n+k-3)} + \frac{n-7}{2(n+k-3)(n+2k-3)}$$

is decreasing in k. ☐

As a consequence, we deduce that for $\delta > \max\big(\delta_1(n, k=4), \delta_2'(n, k=4)\big)$, the flow-invariant even orthogonal projection f has degree ≤ 2, so it now remains to study the case of degree 2.

In this case, by Lemma 4.2, we know that $f = \frac{r}{n}\mathbb{1}_{TM} + f_2$, with $u = f_2$. Since $\mathbf{X}\mathbb{1}_{TM} = 0$, we get that $\mathbf{X}_\pm u = 0$. Following the proof of Proposition 4.4 (the $\mathbf{X}_- u$ term disappears from the computation) we obtain similarly as in (4.16) that:

$$0 \leq \frac{1+\delta}{2}\left(2(n+2\cdot 2-4)\|\imath_v u\|^2 + 2\|u\|^2\right) + \frac{2\cdot 2}{3}(1-\delta)\left[2(n+2-2)(n-1)\right]^{1/2}\|u\|^2 - \delta\cdot 2(n+2-2)\|u\|^2. \tag{4.18}$$

Observe that $\imath_v f = 0 = \frac{r}{n}v + \imath_v u$. Moreover, at a given point $x_0 \in M$, we have

$$\|f\|^2_{L^2(S_{x_0}M)} = \int_{S_{x_0}M} \mathrm{Tr}(f^2)dv = r\,\mathrm{vol}(\mathbb{S}^{n-1})$$
$$= \|f_0\|^2_{L^2(S_{x_0}M)} + \|u\|^2_{L^2(S_{x_0}M)} = \frac{r^2}{n}\mathrm{vol}(\mathbb{S}^{n-1}) + \|u\|^2_{L^2(S_{x_0}M)},$$

and we obtain

$$\|u\|^2_{L^2(S_{x_0}M)} = r\left(1-\tfrac{r}{n}\right)\mathrm{vol}(\mathbb{S}^{n-1}),$$
$$\|\imath_v u\|^2_{L^2(S_{x_0}M)} = \tfrac{r^2}{n^2}\mathrm{vol}(\mathbb{S}^{n-1}) = \tfrac{r}{n(n-r)}\|u\|^2_{L^2(S_{x_0}M)}.$$

Plugging the previous equality in (4.18), we then obtain:

$$\left(2\delta n - \frac{4}{3}(1-\delta)[2n(n-1)]^{1/2} - (1+\delta)\frac{n}{n-r}\right)\|u\|^2 \leq 0.$$

The term in the brackets is positive if and only if

$$\delta > \delta_2'(n, k=2, r) := \frac{\frac{4}{3}[2n(n-1)]^{1/2} + \frac{n}{n-r}}{2n + \frac{4}{3}[2n(n-1)]^{1/2} - \frac{n}{n-r}}.$$

This term is clearly increasing in r and $r \leq \min\left(\rho(n)-1, \frac{n-2}{2}\right)$. As a consequence, we deduce that for

$$\delta > \max\left(\delta_2'\left(n, k=2, r=\min\left(\rho(n)-1, \frac{n-2}{2}\right)\right), \delta_2'(n,k=4)\right) = \delta_2'(n,k=4),$$

the flow-invariant even orthogonal projector f has degree 0, that is $f = \pi_0^* f_0$ for some orthogonal projector $f_0 \in C^\infty(M, \mathrm{Sym}^2 TM)$. The flow-invariance condition $\mathbf{X} f = 0$ then translates into $\nabla f_0 = 0$, that is f_0 is parallel. This implies that the holonomy is reducible and by the de Rham decomposition Theorem, the universal cover $(\widetilde{M}, \widetilde{g})$ of (M,g) is a Riemannian product, which contradicts the negative curvature assumption. □

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Current address: Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

Laboratoire de Mathématiques d'Orsay, CNRS, Université Paris-Saclay, 91405 Orsay, France

E-mail address: mihajlo.cekic@math.uzh.ch

Université de Paris and Sorbonne Université, CNRS, IMJ-PRG, F-75006 Paris, France.

E-mail address: tlefeuvre@imj-prg.fr

Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, 91405, Orsay, France

E-mail address: andrei.moroianu@math.cnrs.fr

Institut für Geometrie und Topologie, Fachbereich Mathematik, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

E-mail address: uwe.semmelmann@mathematik.uni-stuttgart.de