Locally Conformally Product Structures

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Part I

Weyl connections on conformal manifolds

Let M be a smooth n-dimensional manifold. A *conformal class* c on M is an equivalence class of Riemannian metrics for the equivalence relation

$$g_1 \sim g_2 \iff \exists f \in C^\infty(M),\ g_1 = e^f g_2.$$

Definition

Weyl connection on (M, c): Torsion-free linear connection D preserving c, i.e. satisfying $Dg = \theta^g \otimes g$ for some (thus all) $g \in c$.

Notation

$\theta^g =$ the Lee form of D with respect to g.

Remark

(i) If $g_1 = e^f g \Longrightarrow \theta^{g_1} = \theta^g + df$.
(ii) $\theta^g = 0 \iff D$ is the Levi-Civita connection of g.

Gauduchon metrics

Theorem (Gauduchon,1975)

For every Weyl connection D on a compact conformal manifold (M, c), there exists a metric g, unique up to a constant, such that $\delta^g \theta^g = 0$.

Proof: elliptic theory, maximum principle.

Important applications in Hermitian (non-Kählerian) geometry, etc.

Definition

A Weyl structure D is called closed (exact) if θ^g is closed (exact) for some $g \in c$ (independent of g) $\iff$ D is locally (globally) the Levi-Civita connection of a metric in c.

Koszul formula: for every $g \in c$ and $X, Y \in TM$,

$$D_X Y = \nabla^g_X Y - \frac{1}{2}\left(\theta^g(X)Y + \theta^g(Y)X - (\theta^g)^{\sharp_g} g(X,Y)\right).$$

Remark

Any 2 of D, g, θ determine the third one (up to a constant for g).

From now on: D is closed, non-exact. Let g be a metric on M. On the universal cover $\tilde{M}$ the lift $\tilde{\theta}^g = d\varphi$ is exact $\implies$ the lift $\tilde{D}$ is the Levi-Civita connection of the metric $h_D = e^{-\varphi}\tilde{g}$. The action of $\pi_1(M)$ on $\tilde{M}$ is by h_D-homotheties (not all isometries).

Conversely, if $(\tilde{M}, h)$ is a simply connected Riemannian manifold on which a discrete group Γ acts by homotheties, ∇^h is the lift to $\tilde{M}$ of a closed Weyl connection D on $M := \tilde{M}/\Gamma$.

Natural correspondence:

Closed Weyl connections on (M, c)

$$\Updownarrow$$

Riemannian manifolds $(\tilde{M}, h)$ with discrete groups of homotheties.

Assume now that M is compact and D has special holonomy ($\iff$ $(\tilde{M}, h_D)$ has special holonomy + co-compact group Γ of homotheties). Berger Holonomy Theorem $\Longrightarrow$

- $(\tilde{M}, h_D)$ is an irreducible symmetric space. Its scalar curvature is then constant and non-zero, so this case is ruled out by the existence of strict homotheties in Γ. ✓
- $\mathrm{Hol}(\tilde{M}, h_D)$ belongs to the Berger list

  $$U(n/2), SU(n/2), Sp(n/4), Sp(n/4).Sp(1), G_2, Spin(7).$$

  The first three items correspond to LCK geometry. In the last two cases, M endowed with the Gauduchon metric of D, is S^1 times a NK 6-manifold or a nearly parallel G_2-manifold. ✓
- $(\tilde{M}, h_D)$ has reducible holonomy. (??)

This last case is less understood, and makes the object of the talk. It motivates the following:

Definition

A locally conformally product (LCP) structure on a compact manifold M consists of a conformal structure c and a closed, non-exact Weyl connection D with reducible holonomy.

Equivalently,

Definition

An LCP structure is a reducible Riemannian metric h on a simply connected manifold $\tilde{M}$ together with a discrete co-compact group of homotheties, not all of them being isometries.

Part II

The theory of LCP manifolds

Examples of LCP manifolds are not easy to construct. In fact:

Theorem (Belgun, -, 2009)

If M is compact and D is closed, non-exact, non-flat and tame (i.e. the lifetime of incomplete geodesics is uniformly bounded on compact subsets of $TM \setminus 0$), then D has irreducible holonomy.

Example (Matveev, Nikolayevsky, 2015)

Let $A \in SL_2(\mathbb{Z})$ be symmetric, with $tr(A) > 2$, and let e^1, e^2 be the dual of an orthonormal basis of $\mathbb{R}^2$ of eigenvectors of A for the eigenvalues λ and λ^{-1}. Define $\tilde{M} := \mathbb{R}^2 \times \mathbb{R}^*_+$, with metric $h := (e^1)^2 + t^4(e^2)^2 + dt^2$ and let $\Gamma \simeq \mathbb{Z}^2 \ltimes \mathbb{Z}$ be generated by the integer translations in $\mathbb{R}^2$ and by the map $(x, t) \mapsto (Ax, \lambda t)$.

Then $M := \tilde{M}/\Gamma$ has an LCP structure. Other examples: OT-manifolds (Oeljeklaus-Toma).

The universal cover $(\tilde{M}, h)$ of an LCP manifold is never complete $\Longrightarrow$ the global de Rham Theorem does not hold. Nevertheless:

Theorem (Kourganoff, 2019)

The universal cover $(\tilde{M}, h)$ of a compact LCP manifold (M, c, D) is globally isometric to a Riemannian product $\mathbb{R}^q \times (N, g_N)$, where $\mathbb{R}^q$ $(q \geq 1)$ is the flat Euclidean space, and (N, g_N) is an incomplete Riemannian manifold with irreducible holonomy.

This is a striking result for several reasons.

The foliation tangent to flat factor $\mathbb{R}^q$ is preserved by $\pi_1(M)$ so induces a foliation $\mathcal{F}$ on M.

Definition (Flamencourt, 2022)

Let (M, c, D) be an LCP manifold. A Riemannian metric $g \in c$ is adapted if θ^g vanishes on the flat foliation $\mathcal{F}$

Remark (Flamencourt, 2022)

The Riemannian product of an adapted LCP manifold with any compact Riemannian manifold is again adapted LCP.

Proof.

If the metric g is adapted, its lift $\tilde{g}$ to $\tilde{M} = \mathbb{R}^q \times N$ can be written $\tilde{g} = e^{\varphi} h_D$ with $\varphi \in C^\infty(N)$ (since $\widetilde{\theta^g} = \theta^{\tilde{g}} = d\varphi$). For every compact (M', g'), $\pi_1(M \times M')$ acts by homotheties on the metric

$$e^{-\varphi}(\tilde{g} + \tilde{g}') = h_D + e^{-\varphi}\tilde{g}' = g_{\mathbb{R}^q} + (g_N + e^{-\varphi}\tilde{g}')$$

which thus has reducible holonomy. □

Existence results:

Theorem (Flamencourt, 2022)

Every compact LCP manifold has adapted metrics.

The proof uses averaging and smoothing arguments.

Theorem (-, Pilca, 2023)

The Gauduchon metric on every compact LCP manifold is adapted.

The proof uses averaging arguments together with deep results of Kourganoff concerning the structure of the fundamental groups of LCP manifolds.

Part III

LCP solvmanifolds

Definition

A Riemannian Lie group is a Lie group G endowed with a left invariant Riemannian metric g. The restriction of g to the Lie algebra $\mathfrak{g}$ of G is a scalar product, also denoted by g.

A left invariant 1-form on G is equivalent to an element $\theta \in \mathfrak{g}^*$. The 1-form θ is closed if and only if

$$\theta|_{\mathfrak{g}'} = 0,$$

where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. The Levi-Civita connection of g induces a linear map $x \mapsto \nabla^g_x$, from $\mathfrak{g}$ to $\mathfrak{so}(\mathfrak{g})$, defined by the Koszul formula

$$g(\nabla^g_x y, z) = \frac{1}{2}\left(g([x,y],z) - g([x,z],y) - g([y,z],x)\right)$$

for all $x, y, z \in \mathfrak{g}$.

The trace form $H^{\mathfrak{g}} \in \mathfrak{g}^*$ is defined by $H^{\mathfrak{g}}(x) := \mathrm{tr}(\mathrm{ad}_x),\ \forall\, x \in \mathfrak{g}$. For every $x, y \in \mathfrak{g}$ we have

$$H^{\mathfrak{g}}([x,y]) = \mathrm{tr}(\mathrm{ad}_{[x,y]}) \ = \mathrm{tr}[\mathrm{ad}_x, \mathrm{ad}_y] \ = 0.$$

so $H^{\mathfrak{g}}$ is closed. Definition: $\mathfrak{g}$ is unimodular if $H^{\mathfrak{g}} = 0$. If G has a lattice then $\mathfrak{g}$ is unimodular (Milnor, 1976).

Let c denote the conformal structure on G defined by some left invariant metric g. Every Weyl connection can be expressed as $D^\theta := \nabla^g - \frac{1}{2}\tilde{\theta}$, where $\tilde{\theta}$ is the $(2,1)$ tensor defined by some 1-form $\theta \in \Omega^1(G)$ by the formula:

$$\tilde{\theta}_X Y := \theta(X)Y + \theta(X)Y - g(X,Y)\theta^{\sharp_g}.$$

If θ is left invariant, the Weyl connection is called left invariant.

Let $(\mathfrak{g}, g)$ be a metric Lie algebra, $\theta \in \mathfrak{g}^*$ a closed 1-form and $D^\theta = \nabla^g - \frac{1}{2}\tilde{\theta}$ the closed Weyl connection defined above. We denote by R^θ the curvature tensor of D^θ, that is

$$R^\theta_{x,y} = [D^\theta_x, D^\theta_y] - D^\theta_{[x,y]} \qquad \forall\, x, y \in \mathfrak{g}.$$

A subspace $\mathfrak{u} \subset \mathfrak{g}$ is D^θ-parallel if $D^\theta_x z \in \mathfrak{u}$ for all $x \in \mathfrak{g},\ z \in \mathfrak{u}$, and D^θ-flat if it is D^θ-parallel and $R^\theta_{x,y} z = 0$ for all $x, y \in \mathfrak{g},\ z \in \mathfrak{u}$.

Definition

A locally conformally product (LCP) Lie algebra is a quadruple $(\mathfrak{g}, g, \theta, \mathfrak{u})$ where $(\mathfrak{g}, g)$ is a metric Lie algebra, θ is a non-zero closed 1-form on $\mathfrak{g}^*$ and $\mathfrak{u}$ is a D^θ-flat subspace of $(\mathfrak{g}, g)$. The LCP structure is called *adapted* if $\theta|_{\mathfrak{u}} = 0$, *degenerate* if $\mathfrak{u} = 0$, and *conformally flat* if $\mathfrak{u} = \mathfrak{g}$.

D^θ does not change if the metric g is replaced by λg for some $\lambda > 0 \implies$ the LCP Lie algebras are preserved by constant rescalings of the metric.

Remark

Conformally flat Lie algebras have been classified by Maier.

Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$ and assume that $\Gamma \subset G$ is a lattice.

Consider the quotient $M := \Gamma \backslash G$. Every element in the tensor algebra of $\mathfrak{g}$ defines a left invariant tensor on G, which is in particular Γ-invariant, so projects to a tensor on M.

From now on, let $\theta \in \mathfrak{g}^*$ and $g \in \mathrm{Sym}^2(\mathfrak{g}^*)$ be a closed 1-form and a scalar product. We will denote by the same letters the corresponding 1-form and Riemannian metric on G and on M.

Proposition

The pair $(g, \theta) \in \mathrm{Sym}^2(T^*M) \times \Omega^1(M)$ on M induced by a scalar product $g \in \mathrm{Sym}^2(\mathfrak{g}^*)$ and a closed non-zero 1-form $\mathfrak{g}^*$ is an LCP structure if and only if there exists a vector subspace $\mathfrak{u} \subset \mathfrak{g}$ such that $(\mathfrak{g}, g, \theta, \mathfrak{u})$ is an LCP Lie algebra.

Main objective: study LCP structures on compact Riemannian manifolds obtained as quotients of Riemannian Lie groups by lattices.

Proposition

Let $(\mathfrak{g}, g)$ be a metric Lie algebra, $\theta \in \mathfrak{g}^*$ a non-zero closed 1-form, and $\mathfrak{u} \subset \mathfrak{g}$ a vector subspace. Then $(\mathfrak{g}, g, \theta, \mathfrak{u})$ is LCP if and only if the following conditions hold:

1. *$\mathfrak{u}$ and $\mathfrak{u}^\perp$ are Lie subalgebras;*
2. *for every $u \in \mathfrak{u}$ and $x \in \mathfrak{u}^\perp$,*

$$g([u,x],x) = \theta(u)|x|^2 \qquad \text{and} \qquad g([x,u],u) = \theta(x)|u|^2$$

   (in particular θ is uniquely determined by $H^{\mathfrak{u}}$ and $H^{\mathfrak{u}^\perp}$);
3. *the map $\nabla^\theta : \mathfrak{g} \to \mathfrak{so}(\mathfrak{u}) \oplus \mathbb{R}\mathrm{id}_{\mathfrak{u}} \subset \mathfrak{gl}(\mathfrak{u})$, defined by $x \mapsto \nabla^\theta_x|_{\mathfrak{u}}$, is a Lie algebra representation.*

Definition

An LCP Lie algebra $(\mathfrak{g}, g, \theta, \mathfrak{u})$ is called adapted if $\theta|_{\mathfrak{u}} = 0$.

The extension construction of Flamencourt has the following Lie algebraic counterpart:

Proposition

If $(\mathfrak{g}, g, \theta, \mathfrak{u})$ *is an adapted LCP Lie algebra and* $(\mathfrak{k}, k)$ *is an arbitrary metric Lie algebra, then* $(\tilde{\mathfrak{g}} := \mathfrak{g} \oplus \mathfrak{k},\ \tilde{g} := g + k,\ \tilde{\theta} := \theta,\ \tilde{\mathfrak{u}} := \mathfrak{u})$ *is again an adapted LCP Lie algebra.*

Motivated by a result below, we give here a general construction Ansatz for LCP Lie algebras.

Proposition

Let $(\mathfrak{h}, h)$ be a non-unimodular metric Lie algebra with trace form $H^{\mathfrak{h}} \in \mathfrak{h}^*$ and let $q \geq 1$ be an integer. Let $\beta : \mathfrak{h} \to \mathfrak{so}(\mathbb{R}^q)$ be a Lie algebra representation such that $\beta|_{\mathfrak{h}'} = 0$ (e.g. $\beta = 0$).
Then $\alpha : \mathfrak{h} \to \mathfrak{gl}(\mathbb{R}^q)$ defined by $\alpha(x) := -\frac{1}{q} H^{\mathfrak{h}}(x)\mathrm{id}_{\mathbb{R}^q} + \beta(x)$ for $x \in \mathfrak{h}$ is a Lie algebra representation, and $(\mathfrak{g}, g, \theta, \mathfrak{u})$ is a non-degenerate unimodular LCP Lie algebra for
$\mathfrak{g} := \mathfrak{h} \ltimes_\alpha \mathbb{R}^q, \quad g := h + \langle\cdot,\cdot\rangle_{\mathbb{R}^q}, \quad \theta := -\frac{1}{q} H^{\mathfrak{h}}, \mathfrak{u} := \mathbb{R}^q.$
Moreover, $\mathfrak{g}$ is solvable if and only if $\mathfrak{h}$ is solvable.

Remark

In particular, any non-unimodular metric Lie algebra $(\mathfrak{h}, h)$ is a codimension 1 subalgebra of an LCP Lie algebra. Indeed, it suffices to take $q = 1$ and $\beta = 0$ in the proposition above.

We obtain a criterion for the existence of non-degenerate LCP structures on a given unimodular Lie algebra:

Corollary

Let $\mathfrak{g}$ be an unimodular Lie algebra carrying a non-unimodular subalgebra $\mathfrak{h}$ and an abelian ideal $\mathfrak{u}$ of dimension $q \geq 1$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{u}$. Assume that $\mathrm{ad}_x|_{\mathfrak{u}} = 0$ for every $x \in \mathfrak{h}'$ and that there exists a scalar product $g_{\mathfrak{u}}$ on $\mathfrak{u}$ such that $\mathrm{ad}_x|_{\mathfrak{u}} - \theta(x)\mathrm{id}_{\mathfrak{u}} \in \mathfrak{so}(\mathfrak{u})$ for every $x \in \mathfrak{h}$. Then for every scalar product $g_{\mathfrak{h}}$ on $\mathfrak{h}$, $(\mathfrak{g}, g_{\mathfrak{h}} + g_{\mathfrak{u}}, \theta, \mathfrak{u})$ is a non-degenerate LCP structure on $\mathfrak{g}$, where θ is the closed 1-form $-\frac{1}{q}H^{\flat}$ extended to $\mathfrak{g}$ by 0 on $\mathfrak{u}$.

Example

Consider the Lie algebra $\mathfrak{g} := \mathfrak{g}_{5.35}^{-2,0}$ with Lie brackets

$$[e_1, e_2] = [e_5, e_3] = e_3,\ [e_5, e_2] = -[e_1, e_3] = e_2,\ [e_5, e_4] = -2e_4,$$

and metric g in which the basis (e_i) is ON. Then $\mathfrak{g} = \mathfrak{h} \ltimes_\alpha \mathfrak{u}$, with $\mathfrak{h} := \langle e_1, e_4, e_5 \rangle$, $\mathfrak{u} := \langle e_2, e_3 \rangle$, and $\alpha : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{u})$ defined by $\alpha(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\alpha(e_4) = 0$, $\alpha(e_5) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

By the above proposition, $(\mathfrak{g}, g, \theta, \mathfrak{u})$ is an LCP Lie algebra for $\theta := -\frac{1}{2} H^{\mathfrak{h}} = e^5$. Indeed, $\mathrm{ad}_x|_{\mathfrak{u}} = 0$ for every $x \in \mathfrak{h}'$ and $\mathrm{ad}_x|_{\mathfrak{u}} - \theta(x)\mathrm{id}_{\mathfrak{u}} \in \mathfrak{so}(\mathfrak{u})$ for every $x \in \mathfrak{h}$.

Theorem

Let $(g, \theta, \mathfrak{u})$ be an LCP structure on a unimodular solvable Lie algebra $\mathfrak{g}$. Then the following holds:

1. $\mathfrak{u}$ *is an abelian ideal of* $\mathfrak{g}$ *contained in the centre* $\mathfrak{z}(\mathfrak{g}')$ *of* $\mathfrak{g}'$*;*
2. $\nabla^{\theta}_{x} y = \mathrm{ad}_x y$ *for every* $x \in \mathfrak{g}$ *and* $y \in \mathfrak{u}$*.*

Corollary

There is a one-to-one correspondence between non-degenerate LCP structures on unimodular solvable Lie algebras $(\mathfrak{g}, g, \theta, \mathfrak{u})$ and triples $(\mathfrak{h}, h, \beta)$, where $(\mathfrak{h}, h)$ is a solvable non-unimodular metric Lie algebra, and $\beta : \mathfrak{h} \to \mathfrak{so}(q)$ is a Lie algebra representation, for some $q \in \mathbb{N}^*$, which vanishes on $\mathfrak{h}'$.

Using this result, one can obtain the classification of low-dimensional solvable unimodular LCP Lie algebras.

What about lattices?

In many cases the solvable LCP are almost abelian (i.e. they have a co-dimension 1 abelian ideal). Then the following result of Bock applies:

Theorem

A unimodular almost abelian Lie group $G = \mathbb{R} \ltimes_\rho \mathbb{R}^{n-1}$ *with Lie algebra* $\mathfrak{g} = \mathbb{R}b \ltimes \mathfrak{k}$ *admits a lattice if and only if there is a basis* $\mathcal{B}$ *of* $\mathfrak{k} := \mathbb{R}^{n-1}$ *and* $t_0 \neq 0$ *such that the matrix of* $\rho(t_0) = e^{t_0 \mathrm{ad}_b} \in \mathrm{Aut}(\mathfrak{k})$ *in that basis is in* $\mathrm{SL}(n-1, \mathbb{Z})$.

In this case a lattice can be given by $\Gamma = t_0\mathbb{Z} \ltimes_\rho \Gamma_0$, where $\Gamma_0 \simeq \mathbb{Z}^{n-1}$ is the lattice of $\mathbb{R}^{n-1}$ spanned by $\mathcal{B}$.

Using the above results, we (Andrada, del Barco, -) were able to classify all solvable LCP Lie algebras of dimension ≤ 5 whose corresponding simply connected Lie group admits lattices.

Lie algebra	Brackets	$\dim \mathfrak{u}$	Lattices
$\mathfrak{e}(1,1)$	$[e_1, e_2] = e_2,\ [e_1, e_3] = -e_3$	1	yes

Table: 3-dimensional LCP Lie algebra

Lie algebra	Brackets	$\dim \mathfrak{u}$	Lattices
$\mathfrak{e}(1,1) \oplus \mathbb{R}$	$[e_1, e_2] = e_2,\ [e_1, e_3] = -e_3$	1	yes
$\mathfrak{g}_{4.2}^{-2}$	$[e_1, e_2] = e_2,\ [e_1, e_3] = e_2 + e_3,$ $[e_1, e_4] = -2e_4$	1	no
$\mathfrak{g}_{4.5}^{p,-p-1}$	$[e_1, e_2] = e_2,\ [e_1, e_3] = pe_3,$ $[e_1, e_4] = -(p+1)e_4,\ -\frac{1}{2} \leq p < 0$	1 or 2 for $p = -\frac{1}{2}$	for some $p \neq -\frac{1}{2}$
$\mathfrak{g}_{4.6}^{-2p,p}$	$[e_1, e_2] = -2pe_2,\ [e_1, e_3] = pe_3 - e_4,$ $[e_1, e_4] = e_3 + pe_4,\ p > 0$	1 or 2	for some p

Table: 4-dimensional LCP Lie algebra

Lie algebra	Brackets	dim $\mathfrak{u}$	Lattices
$\mathfrak{e}(1,1)\oplus\mathbb{R}^2$	$[e_1,e_2]=e_2,\ [e_1,e_3]=-e_3$	1	yes
$\mathfrak{g}_{4.2}^{-2}\oplus\mathbb{R}$	$[e_1,e_2]=e_2,\ [e_1,e_3]=e_2+e_3,$ $[e_1,e_4]=-2e_4$	1	no
$\mathfrak{g}_{4.5}^{p,-p-1}\oplus\mathbb{R}$	$[e_1,e_2]=e_2,\ [e_1,e_3]=pe_3,$ $[e_1,e_4]=-(p+1)e_4,\ -\frac{1}{2}\leq p<0$	1 or 2 for $p=-\frac{1}{2}$	for some $p\neq-\frac{1}{2}$
$\mathfrak{g}_{4.6}^{-2p,p}\oplus\mathbb{R}$	$[e_1,e_2]=-2pe_2,\ [e_1,e_3]=pe_3-e_4,$ $[e_1,e_4]=e_3+pe_4,\ p>0$	1 or 2	for some p
$\mathfrak{g}_{5.7}^{p,q,r}$	$[e_5,e_1]=pe_1,[e_5,e_2]=qe_2,[e_5,e_3]=re_3,$ $[e_5,e_4]=-e_4$ $pqr\neq0,\ p+q+r=1$ $-1\leq p\leq q\leq r\leq 1$	1 or 2 if $p\neq r$ and $q\in\{p,r\}$ or 3 if $p=q=r$	for some $p\neq q\neq r$ or $q=r=1$
$\mathfrak{g}_{5.8}^{-1}$	$[e_5,e_1]=e_1,\ [e_5,e_3]=e_2,\ [e_5,e_4]=-e_4$	1	yes
$\mathfrak{g}_{5.9}^{p,-2-p}$	$[e_5,e_1]=pe_1,\ [e_5,e_2]=e_2,\ [e_5,e_3]=e_2+e_3,$ $[e_5,e_4]=(-2-p)e_4,\quad p\geq-1$	1 or 2 if $p=-1$	no
$\mathfrak{g}_{5.11}^{-3}$	$[e_5,e_1]=e_1,\ [e_5,e_2]=e_1+e_2,,$ $[e_5,e_3]=e_2+e_3,\ [e_5,e_4]=-3e_4$	1	no
$\mathfrak{g}_{5.13}^{-1-2q,q,r}$	$[e_5,e_1]=e_1,\ [e_5,e_2]=qe_2-re_3,$ $[e_5,e_3]=re_2+qe_3,\ [e_5,e_4]=(-1-2q)e_4$ $r>0,\ q\in[-1,0],\ q\neq-\frac{1}{2}$	1 or 2 or 3 if $q=-\frac{1}{3}$	for some $q\neq-\frac{1}{3},r$
$\mathfrak{g}_{5.16}^{-1,q}$	$[e_5,e_1]=e_1,\ [e_5,e_2]=e_1+e_2,$ $[e_5,e_3]=-e_3-qe_4,\ [e_5,e_4]=qe_3-e_4,\ q>0$	2	no
$\mathfrak{g}_{5.17}^{p,-p,r}$	$[e_5,e_1]=pe_1-e_2,\ [e_5,e_2]=e_1+pe_2,$ $[e_5,e_3]=-pe_3-re_4,\ [e_5,e_4]=re_3-pe_4$ $p\geq0,\ r>0$	2 if $p\neq0$	for some p,r

Lie algebra	Brackets	dim $\mathfrak{u}$	Lattices
$\mathfrak{g}_{5.17}^{p,-p,r}$	$[e_5, e_1] = pe_1 - e_2$, $[e_5, e_2] = e_1 + pe_2$, $[e_5, e_3] = -pe_3 - re_4$, $[e_5, e_4] = re_3 - pe_4$ $p \geq 0$, $r > 0$	2 if $p \neq 0$	for some p, r
$\mathfrak{g}_{5.19}^{p,-2p-2}$	$[e_1, e_2] = e_3$, $[e_5, e_1] = e_1$, $[e_5, e_2] = pe_2$, $[e_5, e_3] = (p+1)e_3$, $[e_5, e_4] = -2(p+1)e_4$, $p \neq -1$	1	no
$\mathfrak{g}_{5.23}^{-4}$	$[e_1, e_2] = e_3$, $[e_5, e_1] = e_1$, $[e_5, e_2] = e_1 + e_2$, $[e_5, e_3] = 2e_3$, $[e_5, e_4] = -4e_4$	1	no
$\mathfrak{g}_{5.25}^{p,4p}$	$[e_1, e_2] = e_3$, $[e_5, e_1] = pe_1 + e_2$, $[e_5, e_2] = -e_1 + pe_2$, $[e_5, e_3] = 2pe_3$, $[e_5, e_4] = -4pe_4$, $p > 0$	1	no
$\mathfrak{g}_{5.33}^{-1,-1}$	$[e_1, e_2] = e_2$, $[e_1, e_3] = -e_3$, $[e_5, e_3] = e_3$, $[e_5, e_4] = -e_4$	1	yes
$\mathfrak{g}_{5.35}^{-2,0}$	$[e_1, e_2] = e_3$, $[e_1, e_3] = -e_2$, $[e_5, e_2] = e_2$, $[e_5, e_3] = e_3$, $[e_5, e_4] = -2e_4$	1 or 2	yes

Table: 5-dimensional LCP Lie algebras