Open problems in LCK geometry

Andrei Moroianu, CNRS – Univ. Paris-Saclay

Conformal structures in geometry
– On the occasion of Liviu Ornea’s 60th birthday –

Zoom, July 16, 2020
The many facets of Liviu Ornea
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Professor

Andrei Moroianu, CNRS – Univ. Paris-Saclay
Open problems in LCK geometry
The many facets of Liviu Ornea

Theater critic
The many facets of Liviu Ornea

Columnist at *Observatorul Cultural*
The many facets of Liviu Ornea

Actor (*Aferim*, 2015)
The many facets of Liviu Ornea

Founder of LCK geometry in Romania
LCK structures:
Definition and first properties
Kähler structures

**Definition**

A Kähler metric on a complex manifold \((M, J)\) is a Riemannian metric \(g\) such that

- \(g\) is Hermitian: \(g(J\cdot , \cdot ) = -g(\cdot , J\cdot)\)
- the associated 2-form \(\omega := g(J\cdot , \cdot)\) is closed: \(d\omega = 0\).

**Remark**

If \(M\) is compact \(\implies\) topological obstructions:

- odd degree Betti numbers are even: \(b_{2k+1} \in 2\mathbb{Z}\).
- even degree Betti numbers are non-zero: \(b_{2k} > 0\), \(\forall k \leq \dim_{\mathbb{C}} M\).
Corollary

No Kähler metrics on simple complex manifolds, e.g. $S^1 \times S^{2n-1}$ for $n \geq 2$.

Definition (Vaisman, 80’s)

An LCK metric on $(M, J)$ is a Hermitian metric which is conformal to a Kähler metric around each point. Equivalently, the fundamental 2-form satisfies $d\omega = \theta \wedge \omega$ for some closed 1-form $\theta$ (the Lee form).

Remark

The Lee form $\theta = 0 \iff \omega$ is Kähler.
Remark

Conformal invariance: \((\omega, \theta) \text{ LCK} \iff (e^f \omega, \theta + df) \text{ LCK}\).

Corollary

\(\theta\) exact \(\implies g\) is globally conformally Kähler (GCK). The converse holds if the complex dimension is at least 2.

Theorem (Vaisman)

If \((M, J)\) satisfies the \(\partial\bar{\partial}\)-Lemma (in particular if it carries a Kähler metric), then any LCK metric on \((M, J)\) is GCK.
Examples
### Examples of LCK manifolds

**Example (Compact complex manifolds admitting LCK metrics)**

- **Hopf manifolds:** $\mathbb{Z}$-quotients of $\mathbb{C}^n \setminus \{0\}$, diffeomorphic to $S^1 \times S^{2n-1}$ (Vaisman)

- **Most complex surfaces** (Gauduchon-Ornea, LeBrun, Belgun...)

- **Some OT manifolds** (Oeljeklaus-Toma): quotients of $\mathbb{C}^t \times \mathbb{H}^s$ by co-compact lattices sitting in the ring of algebraic integers $\mathcal{O}_K$ of a number field $K$ with $s$ real embeddings and $2t$ complex embeddings. OT manifolds are LCK for $t = 1$.

- **LCK metrics with potential:** If $(\tilde{M}, J)$ has a positive PSH function $\varphi$ which is automorphic wrt the action of a discrete co-compact group $\Gamma$ of holomorphisms of $\tilde{M}$ (i.e. $\gamma^* \varphi = c_\gamma \varphi$, $\forall \gamma \in \Gamma$), then $\omega := i \frac{\partial \bar{\partial} \varphi}{\varphi}$ defines an LCK structure on $M := \tilde{M} / \Gamma$, with Lee form $\theta = -d \ln \varphi$.

Andrei Moroianu, CNRS – Univ. Paris-Saclay
(Counter)-examples of LCK manifolds

Example (Compact complex manifolds not admitting LCK metrics)

- $S^{2m-1} \times S^{2n-1}$ for $m, n \geq 2$ (Calabi-Eckmann)
- Some Inoue surfaces (Belgun)
- OT manifolds $(\mathbb{C}^t \times \mathbb{H}^s)/\Gamma$ for $t > s > 1$ (Vuletescu). Conjecturally for $t > 1$. 
Conjectures and open problems
The topology of LCK manifolds

**Definition**
An LCK structure \((g, J, \omega, \theta)\) is **Vaisman** if \(\theta\) is \(\nabla^g\)-parallel.

**Conjecture (Vaisman)**

*The first Betti number of a strict LCK manifold is odd.*

True for Vaisman manifolds (Kashiwada, Vaisman, Tsukada, Ornea-Verbitsky...) and for complex surfaces (Buchdal, Lamari).

False in general (OT).

**Remark**

No topological obstruction for the existence of (strict) LCK metrics is known, except \(b_1 > 0\).
Remark

A product of LCK structures is never LCK (unless they are both Kähler).

Indeed, if $d\omega_i = \theta_i \wedge \omega_i$, $i = 1, 2$, and $d(\omega_1 + \omega_2) = \theta \wedge (\omega_1 + \omega_2)$, then $(\theta_1 - \theta) \wedge \omega_1 = (\theta - \theta_2) \wedge \omega_2$, whence $\theta = \theta_1 = \theta_2 = 0$.

Conjecture (Ornea)

A product of two compact complex manifolds carries an LCK metric if and only if they are both of Kähler type.
Partial results

**Theorem (Tsukada, 1999)**

If $M_1$ and $M_2$ are compact complex manifolds which carry Vaisman metrics, then $M_1 \times M_2$ has no LCK metric.

The proof uses properties of the canonical foliation on Vaisman manifolds. Stronger version:

**Theorem (Istrati, 2018)**

If $M_1$ is compact and carries a Vaisman metric and $M_2$ is any compact complex manifold, then $M_1 \times M_2$ has no LCK metric.

**Remark (Istrati, 2018)**

If $M_1$ and $M_2$ are compact complex manifolds, then $M_1 \times M_2$ has no Vaisman metric.
Ornea’s conjecture thus holds when one of the factors is of Vaisman type. It also holds when one of the factors is Kähler:

**Theorem (Istrati, Vuletescu, 2018)**

If $M_1$ and $M_2$ are compact complex manifolds and $M_1$ is of Kähler type, then $M_1 \times M_2$ has an LCK metric $\iff M_2$ is of Kähler type.

**Idea of the proof:** Assume that $(\omega, \theta)$ is an LCK structure on $M_1 \times M_2$. Its restriction to $M_1$ is GCK (Vaisman). If $k := \dim_{\mathbb{C}}(M_1) \geq 2$, the restriction of $\theta$ to $M_1 \times \{y\}$ is exact for every $y \in M_2$. By Künneth, one can assume $\theta = p_2^* \theta_2$. The LCK condition shows that the push-forward $f := (p_2)_*(\omega^k)$ satisfies $df = k \theta_2 f$, so $\theta_2$ is exact.
If $M_1$ is a complex curve, the argument is more involved (Vuletescu):

- $M_2$ has an LCK metric with potential (Istrati)
- $M_2$ has a complex curve $C$ (Ornea-Verbitsky)
- The restriction of the LCK structure to $M_1 \times C$ is GCK, so the restriction of the Lee form to $M_1 \times C$ is exact, so by Küneth $\theta$ is cohomologous to a pullback $p_2^*\theta_2$. 

$\Box$
Thank you for your attention!
Happy birthday, Liviu!