Open problems in LCK geometry

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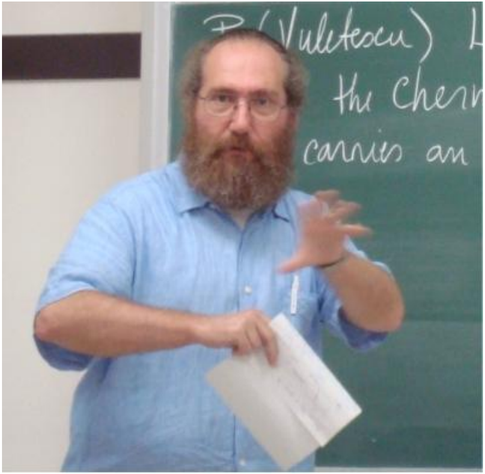
Conformal structures in geometry
– On the occasion of Liviu Ornea's 60th birthday –

Zoom, July 16, 2020

Prologue

The many facets of Liviu Ornea

The many facets of Liviu Ornea



Professor

The many facets of Liviu Ornea

Theater critic

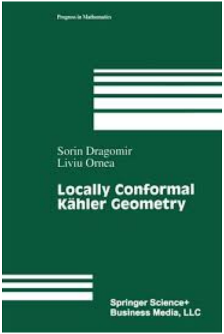
The many facets of Liviu Ornea

Columnist at *Observatorul Cultural*

The many facets of Liviu Ornea

Actor (*Aferim*, 2015)

The many facets of Liviu Ornea



Founder of LCK geometry in Romania

Part I

LCK structures:

Definition and first properties

Kähler structures

Definition

A Kähler metric on a complex manifold (M, J) is a Riemannian metric g such that

- g is Hermitian : $g(J\cdot, \cdot) = -g(\cdot, J\cdot)$
- the associated 2-form $\omega := g(J\cdot, \cdot)$ is closed : $\mathrm{d}\omega = 0$.

Remark

If M is compact $\implies$ topological obstructions:

- odd degree Betti numbers are even : $b_{2k+1} \in 2\mathbb{Z}$.
- even degree Betti numbers are non-zero : $b_{2k} > 0$, $\forall k \leq \dim_{\mathbb{C}} M$.

LCK structures

Corollary

No Kähler metrics on simple complex manifolds, e.g. $S^1 \times S^{2n-1}$ *for* $n \geq 2$.

Definition (Vaisman, 80's)

An LCK metric on (M, J) is a Hermitian metric which is conformal to a Kähler metric around each point.
Equivalently, the fundamental 2-form satisfies $\mathrm{d}\omega = \theta \wedge \omega$ for some closed 1-form θ (the Lee form).

Remark

The Lee form $\theta = 0 \iff \omega$ is Kähler.

LCK structures

Remark

Conformal invariance: (ω, θ) LCK $\iff$ $(e^f \omega, \theta + \mathrm{d}f)$ LCK.

Corollary

θ exact $\Longrightarrow$ g is globally conformally Kähler (GCK). The converse holds if the complex dimension is at least 2.

Theorem (Vaisman)

If (M, J) satisfies the $\partial\bar{\partial}$-Lemma (in particular if it carries a Kähler metric), then any LCK metric on (M, J) is GCK.

Part II

Examples

Examples of LCK manifolds

Example (Compact complex manifolds admitting LCK metrics)

- Hopf manifolds: $\mathbb{Z}$-quotients of $\mathbb{C}^n \setminus \{0\}$, diffeomorphic to $S^1 \times S^{2n-1}$ (Vaisman)
- Most complex surfaces (Gauduchon-Ornea, LeBrun, Belgun...)
- Some OT manifolds (Oeljeklaus-Toma): quotients of $\mathbb{C}^t \times \mathbb{H}^s$ by co-compact lattices sitting in the ring of algebraic integers $\mathcal{O}_{\mathbb{K}}$ of a number field $\mathbb{K}$ with s real embeddings and $2t$ complex embeddings. OT manifolds are LCK for $t = 1$.
- LCK metrics with potential: If $(\tilde{M}, J)$ has a positive PSH function φ which is automorphic wrt the action of a discrete co-compact group Γ of holomorphisms of $\tilde{M}$ (i.e. $\gamma^*\varphi = c_\gamma \varphi$, $\forall \gamma \in \Gamma$), then $\omega := i\frac{\partial\bar{\partial}\varphi}{\varphi}$ defines an LCK structure on $M := \tilde{M}/\Gamma$, with Lee form $\theta = -\mathrm{d}\ln\varphi$.

(Counter)-examples of LCK manifolds

Example (Compact complex manifolds not admitting LCK metrics)

- $S^{2m-1} \times S^{2n-1}$ for $m, n \geq 2$ (Calabi-Eckmann)
- Some Inoue surfaces (Belgun)
- OT manifolds $(\mathbb{C}^t \times \mathbb{H}^s)/\Gamma$ for $t > s > 1$ (Vuletescu). Conjecturally for $t > 1$.

Part III

Conjectures and open problems

The topology of LCK manifolds

Definition

An LCK structure (g, J, ω, θ) is Vaisman if θ is ∇^g-parallel.

Conjecture (Vaisman)

The first Betti number of a strict LCK manifold is odd.

True for Vaisman manifolds (Kashiwada, Vaisman, Tsukada, Ornea-Verbitsky...) and for complex surfaces (Buchdal, Lamari).

False in general (OT).

Remark

No topological obstruction for the existence of (strict) LCK metrics is known, except $b_1 > 0$.

LCK structures on products of compact complex manifolds

Remark

A product of LCK structures is never LCK (unless they are both Kähler).

Indeed, if $\mathrm{d}\omega_i = \theta_i \wedge \omega_i$, $i = 1, 2$, and $\mathrm{d}(\omega_1 + \omega_2) = \theta \wedge (\omega_1 + \omega_2)$, then $(\theta_1 - \theta) \wedge \omega_1 = (\theta - \theta_2) \wedge \omega_2$, whence $\theta = \theta_1 = \theta_2 = 0$.

Conjecture (Ornea)

A product of two compact complex manifolds carries an LCK metric if and only if they are both of Kähler type.

Partial results

Theorem (Tsukada, 1999)

If M_1 and M_2 are compact complex manifolds which carry Vaisman metrics, then $M_1 \times M_2$ has no LCK metric.

The proof uses properties of the canonical foliation on Vaisman manifolds. Stronger version:

Theorem (Istrati, 2018)

If M_1 is compact and carries a Vaisman metric and M_2 is any compact complex manifold, then $M_1 \times M_2$ has no LCK metric.

Remark (Istrati, 2018)

If M_1 and M_2 are compact complex manifolds, then $M_1 \times M_2$ has no Vaisman metric.

Partial results

Ornea's conjecture thus holds when one of the factors is of Vaisman type. It also holds when one of the factors is Kähler:

Theorem (Istrati, Vuletescu, 2018)

If M_1 and M_2 are compact complex manifolds and M_1 is of Kähler type, then $M_1 \times M_2$ has an LCK metric $\iff$ M_2 is of Kähler type.

Idea of the proof: Assume that (ω, θ) is an LCK structure on $M_1 \times M_2$. Its restriction to M_1 is GCK (Vaisman). If $k := \dim_{\mathbb{C}}(M_1) \geq 2$, the restriction of θ to $M_1 \times \{y\}$ is exact for every $y \in M_2$. By Künneth, one can assume $\theta = p_2^* \theta_2$. The LCK condition shows that the push-forward $f := (p_2)_*(\omega^k)$ satisfies $\mathrm{d}f = k\theta_2 f$, so θ_2 is exact.

Partial results

If M_1 is a complex curve, the argument is more involved (Vuletescu):

- M_2 has an LCK metric with potential (Istrati)
- M_2 has a complex curve C (Ornea-Verbitsky)
- The restriction of the LCK structure to $M_1 \times C$ is GCK, so the restriction of the Lee form to $M_1 \times C$ is exact, so by Künneth θ is cohomologous to a pullback $p_2^*\theta_2$.

$\square$

Thank you for your attention!

Happy birthday, Liviu!