Infinitesimal Einstein Deformations on Nearly Kähler Manifolds

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Nearly Kähler manifolds

Part I

Basics on nearly Kähler manifolds
Nearly Kähler manifolds

**Definition.** An almost Hermitian manifold \((M, g, J)\) with fundamental form \(\omega := g(J., .)\) is nearly Kähler (or NK) if \((\nabla_X J)X = 0, \forall X \in TM.\) \(M\) is strictly nearly Kähler (SNK) if \(\nabla_X J \neq 0, \forall X \in TM.\)

The projection of the Levi-Civita connection on the space of Hermitian connections \(\rightsquigarrow\) canonical Hermitian connection :

\[
\tilde{\nabla}_X := \nabla_X - \frac{1}{2} J \circ \nabla_X J.
\]

Its torsion \(T^{\tilde{\nabla}} = -J(\nabla J)\) is totally skew-symmetric.

A SNK manifold is neither symplectic nor Hermitian. The obstruction is given by \(\nabla J : d\omega = 3\nabla J, N^J = J(\nabla J).\)
Nearly Kähler manifolds

Properties:

- $\nabla J$ (and hence $T\tilde{\nabla}$) is parallel wrt $\tilde{\nabla}$.
- In dimension 4: $\text{NK} = \text{Kähler}$
- In dimension 6: a NK manifold is either Kähler or SNK and:
  - carries a $\tilde{\nabla}$-parallel $\text{SU}_3$-structure: $(g, J, d\omega)$.
  - carries Killing spinors $\leadsto$ Einstein with positive scalar curvature.
**Theorem.** An almost Hermitian manifold \((M^6, g, J)\) is SNK iff the Riemannian cone \(\bar{M} := (M \times \mathbb{R}_+, t^2 g + dt^2)\) satisfies \(\text{Hol}(\bar{M}) \subset G_2\).

Idea of the proof: define a **generic** 3-form (in the sense of Hitchin):

\[\varphi := t^2 dt \wedge \omega + \frac{1}{3} t^3 d\omega\]

and check that \(\omega\) is parallel \(\iff\) \((M^6, g, J)\) is NK.

**Corollary.** The sphere \((S^6, can)\) is SNK, wrt the almost complex structure induced by

\[\omega = e^{123} + e^{145} + e^{167} + e^{246} + e^{257} + e^{347} + e^{356} .\]
Examples of NK manifolds

- **Kähler** manifolds.
- **Twistor spaces** over QK manifolds with $\text{Scal} > 0$, endowed with the non-integrable almost complex structure $J^-$. 
- **Naturally reductive 3-symmetric spaces**: Homogeneous spaces $G/H$, with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $H$ is the fix point set of an automorphism $\sigma$ of $G$ of order 3 which defines an almost complex structure on $\mathfrak{m}$ by the relation $\sigma_* = -\frac{1}{2} \text{Id} + \frac{\sqrt{3}}{2} J$;  
  **Example**: $G = K \times K \times K$, $H = \Delta \simeq K$ (diagonal).
- **In dimension 6**: 
  - $S^6 = G2/\text{SU}_3$
  - $\mathbb{C}P^3 = \text{Sp}_2/S^1 \times \text{Sp}_1$, $F(1, 2) = \text{SU}_3/S^1 \times S^1$
  - $S^3 \times S^3 = \text{SU}_2 \times \text{SU}_2 \times \text{SU}_2/\Delta$
Let \((M^n, g)\) be an oriented Riemannian manifold and \(G \subset SO_n\) compact. A \(G\)-structure on \(M \rightsquigarrow \) reduction to \(G\) of the ON frame bundle. If \(G\) is the stabilizer of some element of a \(SO_n\) representation, \(G\)-structure \(\iff\) section of the associated vector bundle. Example: \(G = U_m \subset SO_{2m}\).

The Levi-Civita connection is not a \(G\)-connection in general. Its projection on the space of \(G\)-connections: canonical connection \(\nabla\).

Cleyton and Swann studied special \(G\)-structures, i.e. satisfying:
- The holonomy of the canonical connection \(\nabla\) acts irreducibly on \(\mathbb{R}^n\).
- The torsion of \(\nabla\) is totally skew-symmetric and \(\nabla\)-parallel.
The Cleyton-Swann theorem

Recall the following:

**Theorem.** (Berger, Simons) Simply connected Riemannian manifolds with irreducible holonomy:

- manifolds with holonomy $U_m, SU_m, Sp_k, Sp_kSp_1, G_2, Spin_7$.
- Symmetric spaces with irreducible isotropy.

**Theorem.** (Cleyton, Swann) The special $G$-structures are:

- manifolds with *weak holonomy* $SU_3$ or $G_2$.
- homogeneous spaces with irreducible isotropy.
A NK manifold is a $U_m$-structure which automatically satisfies the second condition of special $G$-structures. The first condition (irreducibility) is essential in the Cleyton-Swann theorem.

**Theorem.** (Nagy) Every NK manifold is locally a Riemannian product of
- Kähler manifolds.
- Twistor spaces over positive QK manifolds.
- Naturally reductive 3-symmetric spaces.
- 6-dimensional NK manifolds.
Corollary. The classification of homogeneous NK manifolds reduces to the dimension 6.

Theorem. (Butruille) Every homogeneous SNK manifold is a naturally reductive 3-symmetric space.

Algebraic problem. Difficulty: when the isotropy is small, the system becomes very complex. Most difficult case: $S^3 \times S^3$ (quadratic system involving 72 unknowns)... Recall the analogous result in QK setting

Theorem. (Wolf, Alekseevski) Every homogeneous QK manifold is symmetric.
**Theorem.** (Belgun-M., Nagy) A compact SNK manifold whose canonical Hermitian connection has complex reducible holonomy is isomorphic to $\mathbb{CP}^3$, $F(1, 2)$ (or to a twistor space over a positive QK manifold in dimension greater than 6).

**Theorem.** (M.-Nagy-Semmelmann) A compact 6-dimensional SNK manifold carrying unit Killing vector fields is isomorphic to $S^3 \times S^3$. 
## NK versus QK

Similarities between NK and QK geometries:

<table>
<thead>
<tr>
<th></th>
<th>nearly Kähler $M^6$</th>
<th>quaternion Kähler $M^{4k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>curvature</td>
<td>$R = R^{CY} + R^{S^6}$</td>
<td>$R = R^{\text{hyper}} + R^{HP^k}$</td>
</tr>
<tr>
<td>Einstein</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>homogeneous classification</td>
<td>3-symmetric (Butruille)</td>
<td>symmetric (Wolf, Alekseevski)</td>
</tr>
<tr>
<td>other examples</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>rigidity</td>
<td>... (M.-N.-S.)</td>
<td>yes (LeBrun-Salamon)</td>
</tr>
</tbody>
</table>
Deformations of nearly Kähler structures

Part II
Deformations of nearly Kähler structures
**Definition.** Let $M$ be a smooth 6-dimensional manifold. A $\text{SU}_3$ structure on $M$ is a reduction of the frame bundle of $M$ to $\text{SU}_3$. It consists of a 5-tuple $s := (g, J, \omega, \psi^+, \psi^-)$ where $g$ is a Riemannian metric, $J$ is a compatible almost complex structure, $\omega$ is the corresponding fundamental 2-form $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$ and $\psi^+ + i\psi^-$ is a complex volume form of type $(3,0)$.

**Definition.** Infinitesimal $\text{SU}_3$ deformation $(\dot{g}, \dot{J}, \dot{\omega}, \dot{\psi}^+, \dot{\psi}^-)$ of $s$ : tangent vector at 0 to a path of $\text{SU}_3$ structures $s_t$ with $s_0 = s$.

**Lemma.** Let $\text{Sym}^- M$ denote the bundle of $g$-symmetric endomorphisms of $M$ anti-commuting with $J$. Natural isomorphism $\text{Sym}^- M \to \Lambda_0^{(2,1)+(1,2)} M : S \mapsto S \star \psi^+$ where $\star$ denotes the Lie algebra extension of an endomorphism of $TM$ to the tensor bundle.
**Proposition.** 1-1 correspondence between infinitesimal $SU_3$ deformations and sections of $TM \oplus \text{Sym}^{-}M \oplus \Lambda^{(1,1)}M \oplus \mathbb{R}$, given by $(\xi, S, \varphi, \mu) \mapsto (\dot{g}, \dot{J}, \dot{\omega}, \dot{\psi}^+, \dot{\psi}^-)$:

\[
\begin{align*}
\dot{g} &= g((h + S)\cdot, \cdot) \\
\dot{J} &= JS + \psi^+ \\
\dot{\omega} &= \varphi + \xi \wedge \psi^+ \\
\dot{\psi}^+ &= -\xi \wedge \omega + \lambda \psi^+ + \mu \psi^- - \frac{1}{2} S_\ast \psi^+ \\
\dot{\psi}^- &= -J\xi \wedge \omega - \mu \psi^+ + \lambda \psi^- - \frac{1}{2} S_\ast \psi^-
\end{align*}
\]

with $g(h\cdot, \cdot) = \varphi(\cdot, J\cdot)$, $\lambda = \frac{1}{4} \text{tr}(h)$ and $g(\psi^+_{\xi} \cdot, \cdot) = \psi^+(\xi, \cdot, \cdot)$. 

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Infinitesimal Einstein Deformations
Gray structures

A Gray structure on a 6-dimensional manifold $M$ is a SNK structure with $\text{scal} = 30$.

**Proposition.** (Hitchin, Reyes-Carrión) A Gray structure is a $SU_3$ structure $(g, J, \omega, \psi^+, \psi^-)$ which satisfies the exterior differential system

$$
\begin{align*}
\{ d\omega &= 3\psi^+ \\
   d\psi^- &= -2\omega \wedge \omega
\}
\end{align*}
$$

At the infinitesimal level, the tangent vector at $t = 0$ to a curve of Gray structures $(g_t, J_t, \omega_t, \psi^+_t, \psi^-_t)$ is an infinitesimal $SU_3$ deformation $(\dot{g}, \dot{J}, \dot{\omega}, \dot{\psi}^+, \dot{\psi}^-)$ which satisfies the linearized system :

$$
\begin{align*}
\{ d\dot{\omega} &= 3\dot{\psi}^+ \\
   d\dot{\psi}^- &= -4\dot{\omega} \wedge \omega.
\}
\end{align*}
$$
Infinitesimal Gray deformations

**Definition.** The set of \((\dot{g}, \dot{J}, \dot{\omega}, \dot{\psi}^+, \dot{\psi}^-)\) satisfying (1) and (3) is called **virtual tangent space** of Gray structures.

**Infinitesimal Gray deformations:** the quotient of the virtual tangent space by the action (by Lie derivative) of vector fields.

**Lemma.** The Laplace operator \(\Delta\) leaves invariant the space of **co-closed** primitive \((1, 1)\)-forms \(\Lambda^{(1,1)}_0 M\).

Denote by \(E(\lambda)\) the \(\lambda\)-eigenspace of its restriction.

**Theorem.** Assume that \((M, g)\) is a Gray manifold not isometric to the round sphere \((S^6, can)\). Then the space of infinitesimal Gray deformations is isomorphic to \(E(12)\).
Idea of the proof

- Since $M$ is not the sphere, the full Gray structure is determined by the metric.
- By the Ebin slice theorem, infinitesimal Gray deformations correspond to solutions of (1) and (3) with $\delta \dot{g} = 0$ and $tr(\dot{g}) = 0$.
- Show that $\mu = 0$ and $\xi = 0$. The system (1), (3) becomes

$$\begin{cases}
    d\varphi = -\frac{3}{2} S_*\psi^+, \\
    \delta(S_*\psi^+) = *d(S_*\psi^-) = -2 * d\dot{\psi}^- = 8 * (\dot{\omega} \wedge \omega) = -8\varphi
\end{cases}$$

- Thus $\varphi \in E(12)$.
- The converse follows from the fact that if $\varphi$ is a co-closed form in $\Omega^{(1,1)}_0 M$ then $d\varphi \in \Omega^{(2,1)+(1,2)}_0 M$. 
Part III

Einstein deformations of NK manifolds
The curvature operator

Let $\nabla$ be a connection (not necessarily torsion-free) on some manifold $M$. For every 2-form $\alpha$, the curvature tensor of $\nabla$ maps $\alpha$ to an endomorphism $R(\alpha)$ of $TM$, which induces an endomorphism $R(\alpha)_\star$ of the tensor bundles of $M$ as before. On every tensor bundle one defines the curvature endomorphism

$$q(R) := \sum_k \alpha_k \star R(\alpha_k)_\star,$$

where $\alpha_k$ is an ON basis of $\Lambda^2 M$. If for some $G \subset SO_n$ $\nabla$ happens to be a $G$-connection whose curvature tensor satisfies the symmetry by pairs, the curvature endomorphism commutes with all $G$-equivariant endomorphisms between tensor bundles, so in particular it leaves invariant all $\nabla$-parallel sub-bundles.
The Lichnerowicz Laplacian

**Example**: If $\nabla$ is the Levi-Civita connection, $q(R) = \text{Ric}$ on $\Lambda^1$.

Let $(M^6, g, J)$ be a Gray manifold. Aim: Characterize the infinitesimal Einstein deformations of the metric $g$.

**Theorem.** (Lichnerowicz) Let $(M^n, g)$ be a compact Einstein manifold with scalar curvature $\text{scal}$. The moduli space of infinitesimal Einstein deformations of $g$ is isomorphic to the set of symmetric endomorphisms $H$ of $TM$ such that $\text{tr}(H) = 0$, $\delta H = 0$ and $\Delta_L H = 2\text{scal}/nH$, where $\Delta_L$ is the Lichnerowicz Laplacian, related to the curvature operator $q(R)$ by:

$$\Delta_L = \nabla^* \nabla + q(R).$$
**The statement**

**Theorem.** Let $(M^6, g, J)$ be a compact Gray manifold. Then the moduli space of infinitesimal Einstein deformations of $g$ is isomorphic to the direct sum $E(2) \oplus E(6) \oplus E(12)$. Recall that $E(\lambda)$ denotes the space of primitive co-closed $(1, 1)$-eigenforms of the Laplace operator for the eigenvalue $\lambda$.

**Outline of the proof.** Main tool : the canonical Hermitian connection $\bar{\nabla}$.

**Step 1.** Lichnerowicz theorem $\rightsquigarrow$ the moduli space of Einstein deformations of $g$ is isomorphic to the set of $H \in \text{Sym} M$ with $\delta H = 0 = \text{tr} H$ such that

$$(\nabla^* \nabla + q(R))H = 10H. \quad (*)$$
Main steps of the proof

Step 2. Let $h := \text{pr}_+ H$ and $S := \text{pr}_- H$ denote the projections of $H$ onto $\text{Sym}^\pm M$. Define the primitive $(1, 1)$-form $\varphi(., .) := g(Jh., .)$ and the $3$-form $\sigma := S_\ast \psi^+$. Express the equation ($\ast$) in terms of $\bar{\nabla}$ (need to compute the difference of the rough Laplacians and the difference $q(R) − q(\bar{R})$).

Step 3. Using the naturality of $q(\bar{R})$, transform the equation into a system involving $\varphi$ and $\sigma$:

$$
\begin{cases}
(\bar{\nabla}_\ast \bar{\nabla} + q(\bar{R}))\varphi = 4\varphi − \delta\sigma, \\
(\bar{\nabla}_\ast \bar{\nabla} + q(\bar{R}))\sigma = 6\sigma − 4d\varphi, \\
\delta\varphi = 0.
\end{cases}
$$
Main steps of the proof

Step 4. Transform this back into an exterior differential system:

\[
\begin{cases}
\Delta \varphi = 4 \varphi - \delta \sigma, \\
\Delta \sigma = 6 \sigma - 4 d \varphi, \\
\delta \varphi = 0.
\end{cases}
\]

Step 5. Define an explicit isomorphism from the space of solutions of this system to \( E(2) \oplus E(6) \oplus E(12) \).

\[
(\varphi, \sigma) \mapsto (8 \varphi + \delta \sigma, *d \sigma, 2 \varphi - \delta \sigma) \in E(2) \oplus E(6) \oplus E(12)
\]

with inverse

\[
(\alpha, \beta, \gamma) \in E(2) \oplus E(6) \oplus E(12) \mapsto \left( \frac{\alpha + \gamma}{10}, \frac{3d\alpha - 5 * d\beta - 2d\gamma}{30} \right).
\]
In order to apply this result, one should try to compute the spectrum of the Laplacian on 2-forms on some explicit compact nearly Kähler 6-dimensional manifolds. Besides the sphere $S^6$ – which has no infinitesimal Einstein deformations – the only known examples are the two twistor spaces $\mathbb{CP}^3$ and $F(1, 2)$ and the homogeneous space $S^3 \times S^3 \times S^3 / S^3$. Preliminary computations on this last manifold seem to indicate that 12 belongs to the spectrum of the Laplacian. There is therefore some hope for obtaining nearly Kähler deformations of this homogeneous metric, at least at the infinitesimal level.