Infinitesimal Einstein Deformations on Nearly Kähler Manifolds

Andrei Moroianu, CNRS – Ecole Polytechnique

(joint work with P.-A. Nagy and U. Semmelmann)

Gemeinsame Jahrestagung DMV – GDM
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Nearly Kähler manifolds

Part I

Basics on nearly Kähler manifolds

Nearly Kähler manifolds

Definition. An almost Hermitian manifold (M, g, J) with fundamental form $\omega := g(J., .)$ is nearly Kähler (or NK) if $(\nabla_X J)X = 0$, $\forall X \in TM$. M is strictly nearly Kähler (SNK) if $\nabla_X J \neq 0$, $\forall X \in TM$.

The projection of the Levi-Civita connection on the space of Hermitian connections $\rightsquigarrow$ canonical Hermitian connection :

$$\bar{\nabla}_X := \nabla_X - \frac{1}{2} J \circ \nabla_X J.$$

Its torsion $T^{\bar{\nabla}} = -J(\nabla J)$ is totally skew-symmetric.

A SNK manifold is neither symplectic nor Hermitian. The obstruction is given by ∇J : $d\omega = 3\nabla J$, $N^J = J(\nabla J)$.

Nearly Kähler manifolds

Properties :

- ∇J (and hence $T^{\bar\nabla}$) is parallel wrt $\bar\nabla$.
- in dimension 4 : NK $=$ Kähler
- in dimension 6 : a NK manifold is either Kähler or SNK and :
  - carries a $\bar\nabla$-parallel SU_3-structure : $(g, J, d\omega)$.
  - carries Killing spinors $\rightsquigarrow$ Einstein with positive scalar curvature.

Nearly Kähler manifolds

Theorem. An almost Hermitian manifold (M^6, g, J) is SNK iff the Riemannian cone $\bar{M} := (M \times \mathbb{R}_+, t^2 g + dt^2)$ satisfies $\mathrm{Hol}(\bar{M}) \subset G_2$.

Idea of the proof : define a generic 3-form (in the sense of Hitchin) :

$$\varphi := t^2 dt \wedge \omega + \frac{1}{3} t^3 d\omega$$

and check that ω is parallel $\iff$ (M^6, g, J) is NK.

Corollary. The sphere (S^6, can) is SNK, wrt the almost complex structure induced by

$$\omega = e^{123} + e^{145} + e^{167} + e^{246} + e^{257} + e^{347} + e^{356}.$$

Examples of NK manifolds

- Kähler manifolds.
- Twistor spaces over QK manifolds with $Scal > 0$, endowed with the non-integrable almost complex structure $\mathcal{J}^-$.
- Naturally reductive 3-symmetric spaces : Homogeneous spaces G/H, with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where H is the fix point set of an automorphism σ of G of order 3 which defines an almost complex structure on $\mathfrak{m}$ by the relation $\sigma_* = -\frac{1}{2} Id + \frac{\sqrt{3}}{2} J$; Example : $G = K \times K \times K$, $H = \Delta \simeq K$ (diagonal).
- In dimension 6 :
  - $S^6 = G2/\mathrm{SU}_3$
  - $\mathbb{C}\mathrm{P}^3 = \mathrm{Sp}_2/S^1 \times \mathrm{Sp}_1$, $F(1,2) = \mathrm{SU}_3/S^1 \times S^1$
  - $S^3 \times S^3 = \mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2/\Delta$

Canonical connection of a G-structure

Let (M^n, g) be an oriented Riemannian manifold and $G \subset \mathrm{SO}_n$ compact. A G-structure on M $\rightsquigarrow$ reduction to G of the ON frame bundle. If G is the stabilizer of some element of a SO_n representation, G-structure $\iff$ section of the associated vector bundle. Example : $G = U_m \subset \mathrm{SO}_{2m}$.

The Levi-Civita connection is not a G-connection in general. Its projection on the space of G-connections : canonical connection $\bar{\nabla}$.

Cleyton and Swann studied special G-structures, i.e. satisfying :

- The holonomy of the canonical connection $\bar{\nabla}$ acts irreducibly on $\mathbb{R}^n$.
- The torsion of $\bar{\nabla}$ is totally skew-symmetric and $\bar{\nabla}$-parallel.

The Cleyton-Swann theorem

Recall the following :

Theorem. (Berger, Simons) Simply connected Riemannian manifolds with irreducible holonomy :

- manifolds with holonomy U_m, SU_m, Sp_k, $\mathrm{Sp}_k\mathrm{Sp}_1$, G_2, Spin_7.
- Symmetric spaces with irreducible isotropy.

Theorem. (Cleyton, Swann) The special G-structures are :

- manifolds with weak holonomy SU_3 or G_2.
- homogeneous spaces with irreducible isotropy.

Decomposition of NK manifolds

A NK manifold is a U_m-structure which automatically satisfies the second condition of special G-structures. The first condition (irreducibility) is essential in the Cleyton-Swann theorem.

Theorem. (Nagy) Every NK manifold is locally a Riemannian product of

- Kähler manifolds.
- Twistor spaces over positive QK manifolds.
- Naturally reductive 3-symmetric spaces.
- 6-dimensional NK manifolds.

Homogeneous NK manifolds

Corollary. The classification of homogeneous NK manifolds reduces to the dimension 6.

Theorem. (Butruille) Every homogeneous SNK manifold is a naturally reductive 3-symmetric space.

Algebraic problem. Difficulty : when the isotropy is small, the system becomes very complex. Most difficult case : $S^3 \times S^3$ (quadratic system involving 72 unknowns)... Recall the analogous result in QK setting

Theorem. (Wolf, Alekseevski) Every homogeneous QK manifold is symmetric.

Further results

Theorem. (Belgun-M., Nagy) A compact SNK manifold whose canonical Hermitian connection has complex reducible holonomy is isomorphic to $\mathbb{C}\mathrm{P}^3$, $F(1,2)$ (or to a twistor space over a positive QK manifold in dimension greater than 6).

Theorem. (M.-Nagy-Semmelmann) A compact 6-dimensional SNK manifold carrying unit Killing vector fields is isomorphic to $S^3 \times S^3$.

NK versus QK

Similarities between NK and QK geometries :

	nearly Kähler M^6	quaternion Kähler M^{4k}
curvature	$R = R^{CY} + R^{S^6}$	$R = R^{hyper} + R^{\mathbb{HP}^k}$
Einstein	yes	yes
homogeneous classification	3-symmetric (Butruille)	symmetric (Wolf, Alekseevski)
other examples	no	no
rigidity	... (M.-N.-S.)	yes (LeBrun-Salamon)

Deformations of nearly Kähler structures

Part II

Deformations of nearly Kähler structures

SU_3 structures

Definition. Let M be a smooth 6-dimensional manifold. A SU_3 structure on M is a reduction of the frame bundle of M to SU_3. It consists of a 5-tuple $s := (g, J, \omega, \psi^+, \psi^-)$ where g is a Riemannian metric, J is a compatible almost complex structure, ω is the corresponding fundamental 2-form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ and $\psi^+ + i\psi^-$ is a complex volume form of type $(3,0)$.

Definition. Infinitesimal SU_3 deformation $(\dot{g}, \dot{J}, \dot{\omega}, \dot{\psi}^+, \dot{\psi}^-)$ of s : tangent vector at 0 to a path of SU_3 structures s_t with $s_0 = s$.

Lemma. Let $\mathrm{Sym}^- M$ denote the bundle of g-symmetric endomorphisms of M anti-commuting with J. Natural isomorphism $\mathrm{Sym}^- M \to \Lambda_0^{(2,1)+(1,2)} M : S \mapsto S_\star \psi^+$ where $_\star$ denotes the Lie algebra extension of an endomorphism of TM to the tensor bundle.

SU_3 structures

Proposition. 1-1 correspondence between infinitesimal SU_3 deformations and sections of $TM \oplus \mathrm{Sym}^- M \oplus \Lambda^{(1,1)} M \oplus \mathbb{R}$, given by $(\xi, S, \varphi, \mu) \mapsto (\dot g, \dot J, \dot\omega, \dot\psi^+, \dot\psi^-)$:

$$(1) \quad \begin{cases} \dot g = g((h+S)\cdot, \cdot) \\ \dot J = JS + \psi^+_\xi \\ \dot\omega = \varphi + \xi \lrcorner \psi^+ \\ \dot\psi^+ = -\xi \wedge \omega + \lambda\psi^+ + \mu\psi^- - \frac{1}{2} S_\star \psi^+ \\ \dot\psi^- = -J\xi \wedge \omega - \mu\psi^+ + \lambda\psi^- - \frac{1}{2} S_\star \psi^- \end{cases}$$

with $g(h\cdot, \cdot) = \varphi(\cdot, J\cdot)$, $\lambda = \frac{1}{4}\mathrm{tr}(h)$ and $g(\psi^+_\xi \cdot, \cdot); = \psi^+(\xi, \cdot, \cdot)$.

Gray structures

A Gray structure on a 6-dimensional manifold M is a SNK structure with $\mathrm{scal} = 30$.

Proposition. (Hitchin, Reyes-Carrión) A Gray structure is a SU_3 structure $(g, J, \omega, \psi^+, \psi^-)$ which satisfies the exterior differential system

$$(2) \quad \begin{cases} d\omega = 3\psi^+ \\ d\psi^- = -2\omega \wedge \omega \end{cases}$$

At the infinitesimal level, the tangent vector at $t = 0$ to a curve of Gray structures $(g_t, J_t, \omega_t, \psi_t^+, \psi_t^-)$ is an infinitesimal SU_3 deformation $(\dot{g}, \dot{J}, \dot{\omega}, \dot{\psi}^+, \dot{\psi}^-)$ which satisfies the linearized system :

$$(3) \quad \begin{cases} d\dot{\omega} = 3\dot{\psi}^+, \\ d\dot{\psi}^- = -4\dot{\omega} \wedge \omega. \end{cases}$$

Infinitesimal Gray deformations

Definition. The set of $(\dot{g}, \dot{J}, \dot{\omega}, \dot{\psi}^+, \dot{\psi}^-)$ satisfying (1) and (3) is called virtual tangent space of Gray structures.

Infinitesimal Gray deformations : the quotient of the virtual tangent space by the action (by Lie derivative) of vector fields.

Lemma. The Laplace operator Δ leaves invariant the space of *co-closed* primitive $(1,1)$-forms $\Lambda_0^{(1,1)}M$.

Denote by $E(\lambda)$ the λ-eigenspace of its restriction.

Theorem. Assume that (M, g) is a Gray manifold not isometric to the round sphere (S^6, can). Then the space of infinitesimal Gray deformations is isomorphic to $E(12)$.

Idea of the proof

- Since M is not the sphere, the full Gray structure is determined by the metric.
- By the Ebin slice theorem, infinitesimal Gray deformations correspond to solutions of (1) and (3) with $\delta\dot{g} = 0$ and $\mathrm{tr}(\dot{g}) = 0$.
- Show that $\mu = 0$ and $\xi = 0$. The system (1), (3) becomes

$$\begin{cases} d\varphi = -\frac{3}{2} S_\star \psi^+, \\ \delta(S_\star\psi^+) = *d(S_\star\psi^-) = -2 * d\dot{\psi}^- = 8 * (\dot{\omega} \wedge \omega) = -8\varphi \end{cases}$$

- Thus $\varphi \in E(12)$.
- The converse follows from the fact that if φ is a co-closed form in $\Omega_0^{(1,1)} M$ then $d\varphi \in \Omega_0^{(2,1)+(1,2)} M$.

Einstein deformations of NK manifolds

Part III

Einstein deformations of NK manifolds

The curvature operator

Let ∇ be a connection (not necessarily torsion-free) on some manifold M. For every 2-form α, the curvature tensor of ∇ maps α to an endomorphism $R(\alpha)$ of TM, which induces an endomorphism $R(\alpha)_\star$ of the tensor bundles of M as before. On every tensor bundle one defines the curvature endomorphism

$$q(R) := \sum_k \alpha_{k\star} R(\alpha_k)_\star,$$

where α_k is an ON basis of $\Lambda^2 M$. If for some $G \subset \mathrm{SO}_n$ ∇ happens to be a G-connection whose curvature tensor satisfies the symmetry by pairs, the curvature endomorphism commutes with all G-equivariant endomorphisms between tensor bundles, so in particular it leaves invariant all ∇-parallel sub-bundles.

The Lichnerowicz Laplacian

Example : If ∇ is the Levi-Civita connection, $q(R) = \mathrm{Ric}$ on Λ^1.

Let (M^6, g, J) be a Gray manifold. Aim : Characterize the infinitesimal Einstein deformations of the metric g.

Theorem. (Lichnerowicz) Let (M^n, g) be a compact Einstein manifold with scalar curvature scal. The moduli space of infinitesimal Einstein deformations of g is isomorphic to the set of symmetric endomorphisms H of TM such that $\mathrm{tr}(H) = 0$, $\delta H = 0$ and $\Delta_L H = 2\mathrm{scal}/nH$, where Δ_L is the Lichnerowicz Laplacian, related to the curvature operator $q(R)$ by :

$$\Delta_L = \nabla^*\nabla + q(R).$$

The statement

Theorem. Let (M^6, g, J) be a compact Gray manifold. Then the moduli space of infinitesimal Einstein deformations of g is isomorphic to the direct sum $E(2) \oplus E(6) \oplus E(12)$. Recall that $E(\lambda)$ denotes the space of primitive co-closed $(1,1)$-eigenforms of the Laplace operator for the eigenvalue λ.

Outline of the proof. Main tool : the canonical Hermitian connection $\bar{\nabla}$.

Step 1. Lichnerowicz theorem $\rightsquigarrow$ the moduli space of Einstein deformations of g is isomorphic to the set of $H \in \mathrm{Sym} M$ with $\delta H = 0 = \mathrm{tr} H$ such that

$$(\nabla^*\nabla + q(R))H = 10H. \qquad (*)$$

Main steps of the proof

Step 2. Let $h := \mathrm{pr}_+ H$ and $S := \mathrm{pr}_- H$ denote the projections of H onto $\mathrm{Sym}^\pm M$. Define the primitive $(1,1)$-form $\varphi(.,.) := g(Jh.,.)$ and the 3-form $\sigma := S_\star \psi^+$. Express the equation $(*)$ in terms of $\bar\nabla$ (need to compute the difference of the rough Laplacians and the difference $q(R) - q(\bar R)$).

Step 3. Using the naturality of $q(\bar R)$, transform the equation into a system involving φ and σ :

$$\begin{cases} (\bar\nabla^*\bar\nabla + q(\bar R))\varphi = 4\varphi - \delta\sigma, \\ (\bar\nabla^*\bar\nabla + q(\bar R))\sigma = 6\sigma - 4d\varphi, \\ \delta\varphi = 0. \end{cases}$$

Main steps of the proof

Step 4. Transform this back into an exterior differential system :

$$\begin{cases} \Delta\varphi = 4\varphi - \delta\sigma, \\ \Delta\sigma = 6\sigma - 4d\varphi, \\ \delta\varphi = 0. \end{cases}$$

Step 5. Define an explicit isomorphism from the space of solutions of this system to $E(2) \oplus E(6) \oplus E(12)$.

$$(\varphi, \sigma) \overset{\Psi}{\mapsto} (8\varphi + \delta\sigma, *d\sigma, 2\varphi - \delta\sigma) \in E(2) \oplus E(6) \oplus E(12)$$

with inverse

$$(\alpha, \beta, \gamma) \in E(2) \oplus E(6) \oplus E(12) \overset{\Phi}{\mapsto} \left(\frac{\alpha + \gamma}{10}, \frac{3d\alpha - 5 * d\beta - 2d\gamma}{30}\right).$$

Possible developments

In order to apply this result, one should try to compute the spectrum of the Laplacian on 2-forms on some explicit compact nearly Kähler 6-dimensional manifolds. Besides the sphere S^6 – which has no infinitesimal Einstein deformations – the only known examples are the two twistor spaces $\mathbb{C}\mathrm{P}^3$ and $F(1,2)$ and the homogeneous space $S^3 \times S^3 \times S^3/S^3$. Preliminary computations on this last manifold seem to indicate that 12 belongs to the spectrum of the Laplacian. There is therefore some hope for obtaining nearly Kähler deformations of this homogeneous metric, at least at the infinitesimal level.