Connections with totally skew-symmetric parallel torsion

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(joint work with R. Cleyton and U. Semmelmann)

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Thomas Friedrich in 2008 at MFO (© Ilka Agricola).

Part I

Geometries with (totally skew-symmetric and parallel) torsion

Some literature

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- R. Cleyton, A. Swann, *Einstein metrics via intrinsic or parallel torsion*, Math. Z. **247** (2004), 513–528.
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Geometries with torsion

Let M be a smooth manifold and ∇ a connection on $\mathrm{T}M$. The torsion of ∇ is the $(2,1)$-tensor

$$T^{\nabla}_X Y := \nabla_X Y - \nabla_Y X - [X, Y].$$

If g is a Riemannian metric on $M \implies$ unique torsion-free metric connection ∇^g. Every other connection ∇ can be written

$$\nabla = \nabla^g + \tau$$

for some $(2,1)$-tensor τ. Its torsion is $T^{\nabla}_X Y = \tau_X Y - \tau_Y X$.

∇ is metric $(\nabla g = 0) \iff \tau_X$ is skew-symmetric $\forall X$.

Geometries with torsion

Using the Riemannian metric g we identify:

- vectors and 1-forms
- skew-symmetric endomorphisms and 2-forms
- totally skew-symmetric tensors of type $(2, 1)$ and 3-forms:

$$g(\tau_X Y, Z) = \tau(X, Y, Z), \qquad \forall\ X, Y, Z \in \mathrm{T}M\ .$$

Geometries with torsion

Remark

The torsion of $\nabla^g + \tau$ is totally skew-symmetric $\iff$ τ is totally skew-symmetric. In this case, the torsion of $\nabla^g + \tau$ is 2τ.

Definition

A geometry with parallel skew-symmetric torsion (or simply geometry with torsion) on M is a Riemannian metric g with Levi-Civita connection ∇^g and a 3-form $\tau \in \Omega^3(M)$ which is parallel with respect to the metric connection $\nabla^\tau := \nabla^g + \tau$, i.e. $\nabla^\tau \tau = 0$.

Examples of geometries with torsion

Examples of geometries with torsion:

- Naturally reductive homogeneous spaces. The homogeneous connection ∇ has skew-symmetric torsion T and moreover $\nabla T = 0$, $\nabla R = 0$. The converse also holds (Ambrose-Singer).
- Nearly Kähler (NK) manifolds: almost Hermitian manifolds (M, g, J) with $(\nabla_X J)X = 0 \ \forall X$. The canonical Hermitian connection

$$\nabla := \nabla^g - \frac{1}{2} J \circ \nabla^g J$$

  has ∇-parallel skew-symmetric torsion (Gray, Kirichenko). Examples of NK manifolds:
  - Twistor bundles over positive QK manifolds
  - 3-symmetric spaces with naturally reductive metric

Examples of geometries with torsion

- Sasakian manifolds (M, g, ξ), where ξ is a unit Killing vector field satisfying the condition $\nabla^g_X d\xi = -2X \wedge \xi, \quad \forall X$. The metric connection

$$\nabla := \nabla^g + \frac{1}{2}\xi \wedge d\xi$$

  has skew-symmetric torsion $T = \xi \wedge d\xi$ which is ∇-parallel (Friedrich). Examples of Sasakian structures: S^1-bundles over Hodge manifolds.
- 3-Sasakian manifolds.

Examples of geometries with torsion

- Nearly parallel G_2-structures in dimension 7: a positive 3-form φ on a 7-dimensional manifold M, which induces a Riemannian metric g on M and such that $\mathrm{d}\varphi = \lambda * \varphi$ for some $\lambda \in \mathbb{R}$. Then

$$\nabla := \nabla^g + \frac{\lambda}{12}\varphi$$

is a metric connection with totally skew-symmetric and ∇-parallel torsion (Friedrich - Ivanov). Examples of nearly parallel G_2-manifolds:
  - $\mathrm{SO}(5)/\mathrm{SO}(3)$, where the embedding of $\mathrm{SO}(3)$ into $\mathrm{SO}(5)$ is given by the 5-dimensional irreducible representation of $\mathrm{SO}(3)$
  - the Aloff-Wallach spaces $\mathrm{SU}(3)/\mathrm{U}(1)_{k,l}$
  - On any 7-dimensional 3-Sasakian manifold there exists a second Einstein metric defined by a nearly parallel G_2-structure (Friedrich, Kath, –, Semmelmann).

The Cleyton-Swann theorem

Key idea: irreducible holonomy group $\Longrightarrow$ classification results.

Theorem (Berger - Simons)

Riemannian manifolds with non-generic irreducible holonomy representation of the Levi-Civita connection:

- *manifolds with holonomy* U_m, SU_m, Sp_k, $\mathrm{Sp}_k\mathrm{Sp}_1$, G_2, Spin_7.
- *irreducible locally symmetric spaces.*

Theorem (Cleyton - Swann)

Metric connections with parallel skew-symmetric torsion and irreducible holonomy:

- *NK 6-dimensional manifolds or nearly parallel* G_2*-manifolds.*
- *irreducible naturally reductive homogeneous spaces.*

Reducibility versus decomposability

In contrast to the Riemannian case, there are two different notions of reducibility for geometries with parallel skew-symmetric torsion:

Definition

A geometry with parallel skew-symmetric torsion (M, g, τ) is:

- reducible if the holonomy representation of ∇^τ is reducible, i.e. the tangent bundle of M decomposes in a (non-trivial) orthogonal direct sum of ∇^τ-parallel distributions $\mathrm{T}M = T_1 \oplus T_2$.
- decomposable if it is reducible, $\mathrm{T}M = T_1 \oplus T_2$, and the torsion form satisfies $\tau = \tau_1 + \tau_2 \in \Lambda^3 T_1 \oplus \Lambda^3 T_2$.

Irreducible case $\Longrightarrow$ Cleyton - Swann.

Decomposable case $\Longrightarrow$ de Rham-type decomposition theorem:

Lemma

Assume that (M, g, τ) is decomposable, with ∇^τ-parallel orthogonal decomposition $\mathrm{T}M = T_1 \oplus T_2$ and such that $\tau = \tau_1 + \tau_2 \in \Lambda^3 T_1 \oplus \Lambda^3 T_2$. Then (M, g, τ) is locally isometric to a product of two manifolds with parallel skew-symmetric torsion (M_i, g_i, τ_i).

Remaining problem: understand reducible and indecomposable geometries with parallel skew-symmetric torsion.

Part II

The standard decomposition

The standard decomposition

Assume from now on that (M, g, τ) is reducible: $\mathrm{T}M = T_1 \oplus T_2$,

$$\tau \in \Lambda^3 T_1 \oplus (\Lambda^2 T_1 \otimes T_2) \oplus (T_1 \otimes \Lambda^2 T_2) \oplus \Lambda^3 T_2.$$

$\mathrm{T}M$ may have several such splittings. However:

Theorem (Cleyton, –, Semmelmann)

There exists a canonically defined ∇^τ-parallel orthogonal decomposition $\mathrm{T}M = \mathcal{H}M \oplus \mathcal{V}M$ such that τ is a section of $\Lambda^3\mathcal{H}M \oplus (\Lambda^2\mathcal{H}M \otimes \mathcal{V}M) \oplus \Lambda^3\mathcal{V}M$.

Definition

The above decomposition $\mathrm{T}M = \mathcal{H}M \oplus \mathcal{V}M$ is called the standard decomposition of the reducible geometry with torsion (M, g, τ).

Proof of the standard decomposition

Lemma (Cleyton, –, Semmelmann)

If $\mathfrak{k} \subset \mathfrak{so}(n)$ is a faithful orthogonal representation of a Lie algebra $\mathfrak{k}$ and $\mathfrak{h} \subset \mathbb{R}^n$ is an irreducible summand such that $\mathfrak{so}(\mathfrak{h}) \cap \mathfrak{k} \neq 0$, then the representation of $\mathfrak{k}$ on $\mathfrak{h} \otimes \Lambda^2(\mathfrak{h}^\perp)$ has no invariant vector.

Let $\mathfrak{k}$ be the holonomy algebra of ∇^τ. The holonomy representation of $\mathfrak{k}$ on $\mathbb{R}^n$ decomposes into an orthogonal sum of irreducible $\mathfrak{k}$-modules $\mathfrak{h}_\alpha$ and $\mathfrak{v}_j$ with $\mathfrak{so}(\mathfrak{h}_\alpha) \cap \mathfrak{k} \neq 0$ and $\mathfrak{so}(\mathfrak{v}_j) \cap \mathfrak{k} = 0$. We define $\mathfrak{h} := \oplus_\alpha \mathfrak{h}_\alpha$ and $\mathfrak{v} := \oplus_j \mathfrak{v}_j$.

$\mathcal{H}M$ and $\mathcal{V}M$ are the associated bundles to $\mathfrak{h}$ and $\mathfrak{v}$. The ∇^τ-parallel torsion τ defines a $\mathfrak{k}$-invariant vector of $\Lambda^3\mathbb{R}^n$, whose projection to $\mathfrak{h} \otimes \Lambda^2\mathfrak{v}$ vanishes by the above lemma.

Denote by $\tau^{\mathfrak{h}} \in \Lambda^3\mathcal{H}M$, $\tau^{\mathfrak{v}} \in \Lambda^3\mathcal{V}M$ and $\tau^m \in \Lambda^2\mathcal{H}M \otimes \mathcal{V}M$ the projections of τ wrt the standard decomposition.

Theorem (Cleyton, –, Semmelmann)

- *The distribution $\mathcal{V}M$ is the vertical distribution of a locally defined Riemannian submersion $(M, g) \xrightarrow{\pi} (N, g^N)$ with totally geodesic fibers, called the standard submersion.*
- *The horizontal part $\tau^{\mathfrak{h}}$ of τ is projectable to the base N of the standard submersion: $\tau^{\mathfrak{h}} = \pi^*\sigma$.*
- *The metric connection $\nabla^\sigma := \nabla^{g^N} + \sigma$ on N has parallel skew-symmetric torsion.*
- *The restriction of the curvature tensor $R^\tau : \Lambda^2\,\mathrm{T}M \to \Lambda^2\,\mathrm{T}M$ to $\Lambda^2\mathcal{V}M$ is ∇^τ-parallel. In particular, the fibres of the standard submersion are naturally reductive homogeneous spaces.*

Remark

If one of the summands in the standard decomposition $\mathrm{T}M = \mathcal{H}M \oplus \mathcal{V}M$ is trivial, then either $\mathcal{H}M = 0$ and (M, g) is locally a naturally reductive homogeneous space, or $\mathcal{V}M = 0$, in which case (M, g) is locally a product of irreducible geometries with torsion. By Cleyton - Swann, each factor is either naturally reductive homogeneous, or has a nearly Kähler structure in dimension 6, or a nearly parallel G_2-structure in dimension 7.

We will thus implicitly assume from now on that the standard decomposition $\mathrm{T}M = \mathcal{H}M \oplus \mathcal{V}M$ is non-trivial.

We have seen that the base space N of the standard submersion $M \to N$ of a manifold M with parallel skew-symmetric torsion inherits a geometry with parallel skew-symmetric torsion. This geometry carries an additional structure, which can be understood in terms of principal bundles:

Fix some orthonormal frame u on M and denote with K the holonomy group of ∇^τ at u, with $\mathfrak{k}$ its Lie algebra, and with $\pi_M : Q \to M$ the reduction of the frame bundle of M to a principal K-fibre bundle. Denote by $\mathbb{R}^n = \mathfrak{h} \oplus \mathfrak{v}$ the above $\mathfrak{k}$-invariant decomposition of $\mathbb{R}^n$.

The connection form of ∇^τ is denoted by $\alpha \in \Omega^1(Q, \mathfrak{k})$, and the soldering form is denoted by $\theta \in \Omega^1(Q, \mathbb{R}^n)$.

Any $A \in \mathfrak{k}$ induces a fundamental vertical vector field A^* on Q.

As Q is a subbundle of the frame bundle of M, any $\xi \in \mathbb{R}^n$ induces a standard horizontal vector field ξ^* defined at $u \in Q$ by $\xi^*_u := \widetilde{u\xi}$.

For $A, B \in \mathfrak{k}$ and $\xi \in \mathbb{R}^n$ we have

$$[A^*, B^*] = [A, B]^*, \qquad [A^*, \xi^*] = (A\xi)^* \ .$$

Key idea: define a Lie algebra structure on $\mathfrak{l} := \mathfrak{k} \oplus \mathfrak{v}$ induced from the Lie algebra structure on the space of vector fields on Q by the injective map

$$\Phi : \mathfrak{l} = \mathfrak{k} \oplus \mathfrak{v} \to \Gamma(\mathrm{T}Q), \quad A + \xi \mapsto A^* + \xi^*,$$

for $A \in \mathfrak{k}$ and ξ in $\mathfrak{v}$.

Lemma

The image of the map Φ is closed under the bracket of vector fields.

Proof: Use the above properties of the torsion and curvature of ∇^τ.

The decomposition $\mathrm{T}Q = \mathrm{T}^{hor}Q \oplus \mathrm{T}^{\mathfrak{k}}Q$ of the tangent bundle of Q given by the connection α can be refined as

$$\mathrm{T}Q = \mathrm{T}^{\mathfrak{h}}Q \oplus \mathrm{T}^{\mathfrak{v}}Q \oplus \mathrm{T}^{\mathfrak{k}}Q \ ,$$

where $\mathrm{T}^{\mathfrak{h}}Q_u = \{\eta^*_u \mid \eta \in \mathfrak{h}\}$, $\mathrm{T}^{\mathfrak{v}}Q_u = \{\xi^*_u \mid \xi \in \mathfrak{v}\}$, and $\mathrm{T}^{\mathfrak{k}}Q_u = \{\mathrm{A}^*_u \mid A \in \mathfrak{k}\}$.

The map $\Phi : \mathfrak{l} \to \Gamma(\mathrm{T}Q)$ is by definition a Lie algebra homomorphism, i.e. it defines a structure of infinitesimal $\mathfrak{l}$-principal bundle on Q over some locally defined manifold N, whose fibers are the leaves of the integrable distribution $\Phi(\mathfrak{l}) = \mathrm{T}^{\mathfrak{v}}Q \oplus \mathrm{T}^{\mathfrak{k}}Q$. Since $(\pi_M)^{-1}_*(\mathcal{V}M) = \mathrm{T}^{\mathfrak{v}}Q \oplus \mathrm{T}^{\mathfrak{k}}Q$, this locally defined manifold N is the same as the locally defined manifold N introduced in the previous section.

Lemma

The 1-form $\beta := \alpha + \theta^{\mathfrak{v}} \in \Omega^1(Q, \mathfrak{l})$ is a connection form on Q with respect to the infinitesimal $\mathfrak{l}$–principal bundle structure, i.e. it satisfies $\beta(B^*) = B$ for every $B \in \mathfrak{l}$ and

$$(\mathcal{L}_{B^*}\beta)(U) = -[B, \beta(U)], \qquad \forall\ B \in \mathfrak{l},\ \forall\ U \in \Gamma(\mathrm{T}Q)\ .$$

Some components of the curvature form of β, viewed as a 2-form with values in the adjoint bundle, are parallel but not all of them (the structure group L is too large, and contains unnecessary information). After a reduction procedure $\Longrightarrow$ a principal fibre bundle over N with parallel curvature form, containing enough information in order to recover the geometry of M.

Part III

Geometries with parallel curvature

The principal bundle with parallel curvature

Consider the linear map

$$\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{v} \to \mathfrak{so}(\mathfrak{v}) \oplus \mathfrak{v} := \mathfrak{g}.$$

$\exists!$ Lie algebra structure on $\mathfrak{g}$ making this map a Lie algebra morphism. Let L and G be the simply connected Lie groups with Lie algebras $\mathfrak{l}$ and $\mathfrak{g}$ respectively, and $\lambda : L \to G$ the corresponding group morphism. The associated G-principal bundle

$$P := Q \times_\lambda G$$

over N carries a connection 1-form $\gamma \in \Omega^1(P, \mathfrak{g})$ (induced by β).

Theorem (Cleyton, –, Semmelmann)

The section R^γ of $\Lambda^2\mathrm{T}N \otimes \mathrm{ad}(P)$ is parallel wrt $\nabla^\sigma \otimes \nabla^\gamma$ and satisfies some extra conditions.
Conversely, given a geometry with parallel skew-symmetric torsion (N, g^N, σ) and a G-principal bundle with parallel curvature form (+ some extra conditions), one obtains a geometry with parallel skew-symmetric torsion on quotients of P by compact subgroups of G.

Part IV

Parallel $\mathfrak{g}$-structures

Example

Let (M^{2k+1}, g, ξ) be the Sasakian S^1-bundle over a Hodge manifold (N, g^N, ω), such that $\mathrm{d}\xi^\flat = \omega$. Then (generically):

- the holonomy group of the connection $\nabla = \nabla^g + \frac{1}{2}\xi \wedge \mathrm{d}\xi$ is $K = \mathrm{U}(k) \subset \mathrm{SO}(2k+1)$,
- the standard decomposition is $\mathcal{V}M =< \xi >$, $\mathcal{H}M = \xi^\perp$,
- the standard Riem. submersion is just the fibration $M \to N$,
- the extended Lie algebra $\mathfrak{l} := \mathfrak{u}(k) \oplus \mathfrak{u}(1) \to 0 \oplus \mathfrak{u}(1) =: \mathfrak{g}$,
- the K-principal bundle Q over M is the holonomy bundle of ∇, also seen as $(\mathrm{U}(k) \times U(1))$-bundle over N,
- the G-principal bundle with parallel curvature $P := M \to N$.

Definition

A geometry with torsion (M, g, τ) is special if $\mathfrak{v}$ is a trivial $\mathfrak{k}$-representation (i.e. $\mathcal{V}M$ is spanned by ∇-parallel vector fields ξ_i).

Remark

Any ∇-parallel vector field ξ is Killing, since $0 = \nabla\xi = \nabla^g\xi + \tau_\xi$.

In this case the projection of the holonomy algebra $\mathfrak{k}$ on $\mathfrak{so}(\mathfrak{v})$ vanishes, so the Lie algebra $\mathfrak{g} = \mathfrak{v}$. In fact it is easy to see directly that the set of ∇-parallel vector fields is closed under Lie bracket:

$$[\xi_i, \xi_j] = \nabla^g_{\xi_i}\xi_j - \nabla^g_{\xi_j}\xi_i = -2\tau(\xi_i, \xi_j)$$

is also ∇-parallel. Like in the previous example, M is (locally) identified to a principal bundle over the space of leaves of $\mathcal{V}M$.

Parallel $\mathfrak{g}$-structures

Definition

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. A parallel $\mathfrak{g}$-structure on a manifold N is given by:

1. a Riemannian metric g^N on N;
2. a locally defined G-principal bundle $P \to N$ with adjoint bundle $\mathrm{ad}(P)$;
3. an $\mathrm{ad}_{\mathfrak{g}}$-invariant scalar product $\langle .,. \rangle$ on $\mathfrak{g}$, thus on $\mathrm{ad}(P)$;
4. a connection form $\gamma \in \Omega^1(P, \mathfrak{g})$ with parallel curvature tensor $R^\gamma : \Lambda^2 \mathrm{T}N \to \mathrm{ad}(P)$, s.t. the metric adjoint of $-R^\gamma$ is a Lie algebra bundle morphism $\psi : \mathrm{ad}(P) \to \Lambda^2 \mathrm{T}N$.

Theorem (Cleyton, –, Semmelmann)

There is a 1-1 correspondence between special geometries with torsion and parallel $\mathfrak{g}$-structures.

There are several types of natural operations that one can make with parallel $\mathfrak{g}$-structures: products, reductions to ideals of the Lie algebra, restrictions to Riemannian factors of the manifold N, or Whitney products.

Definition

A parallel $\mathfrak{g}$-structure is non-degenerate if it is not locally a product of parallel $\mathfrak{g}$-structures.

The classification

Theorem (Cleyton, –, Semmelmann)

Let $\mathfrak{g}$ be a Lie algebra of compact type and $(g^N, P, \mathfrak{g}, \gamma, \psi)$ a non-degenerate parallel $\mathfrak{g}$-structure on a manifold N. Then either:

- *N is quaternion-Kähler with positive scalar curvature, $\mathfrak{g} = \mathfrak{sp}(1)$ and P is the Konishi bundle, or*
- *$N = L/H$ is an irreducible locally symmetric space of compact type, $\mathfrak{g}$ is isomorphic to a semi-simple factor of $\mathfrak{h}$, or*
- *N is locally a Riemannian product $N = N_1 \times \ldots \times N_p \times S_1 \times \ldots \times S_q$ with N_α Kähler, $S_\beta = L_\beta/\mathrm{U}(1)H_\beta$ Hermitian symmetric of compact type, and $\mathfrak{g} = \mathfrak{u}(1)^m \oplus \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_q$ with $\mathfrak{k}_\beta$ a non-zero factor of $\mathfrak{h}_\beta$.*