Connections with totally skew-symmetric parallel torsion

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(joint work with R. Cleyton and U. Semmelmann)

Dirac operators in differential geometry and global analysis
– in memory of Thomas Friedrich –
Bedlewo, October 7, 2019
Geometries with torsion
The standard decomposition
Geometries with parallel curvature
Parallel g-structures

Thomas Friedrich in 2008 at MFO (© Ilka Agricola).
Geometries with (totally skew-symmetric and parallel) torsion
Some literature


Geometries with torsion

Let $M$ be a smooth manifold and $\nabla$ a connection on $TM$. The torsion of $\nabla$ is the $(2, 1)$-tensor

$$T_{\nabla}^X Y := \nabla_X Y - \nabla_Y X - [X, Y].$$

If $g$ is a Riemannian metric on $M$ $\implies$ unique torsion-free metric connection $\nabla^g$. Every other connection $\nabla$ can be written

$$\nabla = \nabla^g + \tau$$

for some $(2, 1)$-tensor $\tau$. Its torsion is

$$T_{\nabla}^X Y = \tau_X Y - \tau_Y X.$$ 

$\nabla$ is metric ($\nabla g = 0$) $\iff$ $\tau_X$ is skew-symmetric $\forall X$. 

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Using the Riemannian metric $g$ we identify:

- vectors and 1-forms
- skew-symmetric endomorphisms and 2-forms
- totally skew-symmetric tensors of type $(2, 1)$ and 3-forms:

$$g(\tau_X Y, Z) = \tau(X, Y, Z), \quad \forall \ X, Y, Z \in TM.$$
Remark

The torsion of $\nabla^g + \tau$ is totally skew-symmetric $\iff$ $\tau$ is totally skew-symmetric. In this case, the torsion of $\nabla^g + \tau$ is $2\tau$.

Definition

A geometry with parallel skew-symmetric torsion (or simply geometry with torsion) on $M$ is a Riemannian metric $g$ with Levi-Civita connection $\nabla^g$ and a 3-form $\tau \in \Omega^3(M)$ which is parallel with respect to the metric connection $\nabla^\tau := \nabla^g + \tau$, i.e. $\nabla^\tau \tau = 0$. 
Examples of geometries with torsion:

- Naturally reductive homogeneous spaces. The homogeneous connection $\nabla$ has skew-symmetric torsion $T$ and moreover $\nabla T = 0$, $\nabla R = 0$. The converse also holds (Ambrose-Singer).

- Nearly Kähler (NK) manifolds: almost Hermitian manifolds $(M, g, J)$ with $(\nabla X J)X = 0$ $\forall X$. The canonical Hermitian connection
  \[
  \nabla := \nabla^g - \frac{1}{2} J \circ \nabla^g J
  \]
  has $\nabla$-parallel skew-symmetric torsion (Gray, Kirichenko). Examples of NK manifolds:
  - Twistor bundles over positive QK manifolds
  - 3-symmetric spaces with naturally reductive metric
Examples of geometries with torsion

- Sasakian manifolds $(M, g, \xi)$, where $\xi$ is a unit Killing vector field satisfying the condition $\nabla^g_X d\xi = -2X \wedge \xi$, $\forall X$. The metric connection

$$\nabla := \nabla^g + \frac{1}{2} \xi \wedge d\xi$$

has skew-symmetric torsion $T = \xi \wedge d\xi$ which is $\nabla$-parallel (Friedrich). Examples of Sasakian structures: $S^1$-bundles over Hodge manifolds.

- 3-Sasakian manifolds.
Examples of geometries with torsion

- Nearly parallel $G_2$-structures in dimension 7: a positive 3-form $\varphi$ on a 7-dimensional manifold $M$, which induces a Riemannian metric $g$ on $M$ and such that $d\varphi = \lambda \ast \varphi$ for some $\lambda \in \mathbb{R}$. Then

$$\nabla := \nabla^g + \frac{\lambda}{12} \varphi$$

is a metric connection with totally skew-symmetric and $\nabla$-parallel torsion (Friedrich - Ivanov). Examples of nearly parallel $G_2$-manifolds:
- $SO(5)/SO(3)$, where the embedding of $SO(3)$ into $SO(5)$ is given by the 5-dimensional irreducible representation of $SO(3)$
- the Aloff-Wallach spaces $SU(3)/U(1)_{k,l}$
- On any 7-dimensional 3-Sasakian manifold there exists a second Einstein metric defined by a nearly parallel $G_2$-structure (Friedrich, Kath, –, Semmelmann).
The Cleyton-Swann theorem

Key idea: irreducible holonomy group $\Rightarrow$ classification results.

**Theorem (Berger - Simons)**

Riemannian manifolds with non-generic irreducible holonomy representation of the Levi-Civita connection:

- manifolds with holonomy $U_m$, $SU_m$, $Sp_k$, $Sp_kSp_1$, $G_2$, $Spin_7$.
- irreducible locally symmetric spaces.

**Theorem (Cleyton - Swann)**

Metric connections with parallel skew-symmetric torsion and irreducible holonomy:

- NK 6-dimensional manifolds or nearly parallel $G_2$-manifolds.
- irreducible naturally reductive homogeneous spaces.
In contrast to the Riemannian case, there are two different notions of reducibility for geometries with parallel skew-symmetric torsion:

**Definition**

A geometry with parallel skew-symmetric torsion \((M, g, \tau)\) is:

- **reducible** if the holonomy representation of \(\nabla^\tau\) is reducible, i.e. the tangent bundle of \(M\) decomposes in a (non-trivial) orthogonal direct sum of \(\nabla^\tau\)-parallel distributions \(TM = T_1 \oplus T_2\).

- **decomposable** if it is reducible, \(TM = T_1 \oplus T_2\), and the torsion form satisfies \(\tau = \tau_1 + \tau_2 \in \Lambda^3 T_1 \oplus \Lambda^3 T_2\).
Irreducible case $\Rightarrow$ Cleyton - Swann.

Decomposable case $\Rightarrow$ de Rham-type decomposition theorem:

**Lemma**

Assume that $(M, g, \tau)$ is decomposable, with $\nabla^\tau$-parallel orthogonal decomposition $TM = T_1 \oplus T_2$ and such that $\tau = \tau_1 + \tau_2 \in \Lambda^3 T_1 \oplus \Lambda^3 T_2$. Then $(M, g, \tau)$ is locally isometric to a product of two manifolds with parallel skew-symmetric torsion $(M_i, g_i, \tau_i)$.

Remaining problem: understand reducible and indecomposable geometries with parallel skew-symmetric torsion.
The standard decomposition
The standard decomposition

Assume from now on that \((M, g, \tau)\) is reducible: \(\mathbb{T}M = T_1 \oplus T_2\),

\[ \tau \in \Lambda^3 T_1 \oplus (\Lambda^2 T_1 \otimes T_2) \oplus (T_1 \otimes \Lambda^2 T_2) \oplus \Lambda^3 T_2. \]

\(\mathbb{T}M\) may have several such splittings. However:

Theorem (Cleyton, –, Semmelmann)

There exists a canonically defined \(\nabla^\tau\)-parallel orthogonal decomposition \(\mathbb{T}M = \mathcal{H}M \oplus \mathcal{V}M\) such that \(\tau\) is a section of \(\Lambda^3 \mathcal{H}M \oplus (\Lambda^2 \mathcal{H}M \otimes \mathcal{V}M) \oplus \Lambda^3 \mathcal{V}M\).

Definition

The above decomposition \(\mathbb{T}M = \mathcal{H}M \oplus \mathcal{V}M\) is called the standard decomposition of the reducible geometry with torsion \((M, g, \tau)\).
Proof of the standard decomposition

Lemma (Cleyton, −, Semmelmann)

If $\mathfrak{k} \subset \mathfrak{so}(n)$ is a faithful orthogonal representation of a Lie algebra $\mathfrak{k}$ and $\mathfrak{h} \subset \mathbb{R}^n$ is an irreducible summand such that $\mathfrak{so}(\mathfrak{h}) \cap \mathfrak{k} \neq 0$, then the representation of $\mathfrak{k}$ on $\mathfrak{h} \otimes \Lambda^2(\mathfrak{h}^\perp)$ has no invariant vector.

Let $\mathfrak{k}$ be the holonomy algebra of $\nabla^\tau$. The holonomy representation of $\mathfrak{k}$ on $\mathbb{R}^n$ decomposes into an orthogonal sum of irreducible $\mathfrak{k}$-modules $\mathfrak{h}_\alpha$ and $\mathfrak{v}_j$ with $\mathfrak{so}(\mathfrak{h}_\alpha) \cap \mathfrak{k} \neq 0$ and $\mathfrak{so}(\mathfrak{v}_j) \cap \mathfrak{k} = 0$. We define $\mathfrak{h} := \bigoplus_\alpha \mathfrak{h}_\alpha$ and $\mathfrak{v} := \bigoplus_j \mathfrak{v}_j$.

$\mathcal{H}M$ and $\mathcal{V}M$ are the associated bundles to $\mathfrak{h}$ and $\mathfrak{v}$. The $\nabla^\tau$-parallel torsion $\tau$ defines a $\mathfrak{k}$-invariant vector of $\Lambda^3\mathbb{R}^n$, whose projection to $\mathfrak{h} \otimes \Lambda^2\mathfrak{v}$ vanishes by the above lemma.
Denote by $\tau^h \in \Lambda^3 \mathcal{H}M$, $\tau^v \in \Lambda^3 \mathcal{V}M$ and $\tau^m \in \Lambda^2 \mathcal{H}M \otimes \mathcal{V}M$ the projections of $\tau$ wrt the standard decomposition.

**Theorem (Cleyton, –, Semmelmann)**

- The distribution $\mathcal{V}M$ is the vertical distribution of a locally defined Riemannian submersion $(M, g) \xrightarrow{\pi} (N, g^N)$ with totally geodesic fibers, called the standard submersion.

- The horizontal part $\tau^h$ of $\tau$ is projectable to the base $N$ of the standard submersion: $\tau^h = \pi^* \sigma$.

- The metric connection $\nabla^\sigma := \nabla^N + \sigma$ on $N$ has parallel skew-symmetric torsion.

- The restriction of the curvature tensor $R^\tau : \Lambda^2 TM \to \Lambda^2 TM$ to $\Lambda^2 \mathcal{V}M$ is $\nabla^\tau$-parallel. In particular, the fibres of the standard submersion are naturally reductive homogeneous spaces.
Remark

If one of the summands in the standard decomposition $TM = \mathcal{H}M \oplus \mathcal{V}M$ is trivial, then either $\mathcal{H}M = 0$ and $(M, g)$ is locally a naturally reductive homogeneous space, or $\mathcal{V}M = 0$, in which case $(M, g)$ is locally a product of irreducible geometries with torsion. By Cleyton - Swann, each factor is either naturally reductive homogeneous, or has a nearly Kähler structure in dimension 6, or a nearly parallel $G_2$-structure in dimension 7.

We will thus implicitly assume from now on that the standard decomposition $TM = \mathcal{H}M \oplus \mathcal{V}M$ is non-trivial.
We have seen that the base space $N$ of the standard submersion $M \to N$ of a manifold $M$ with parallel skew-symmetric torsion inherits a geometry with parallel skew-symmetric torsion. This geometry carries an additional structure, which can be understood in terms of principal bundles:

Fix some orthonormal frame $u$ on $M$ and denote with $K$ the holonomy group of $\nabla^T$ at $u$, with $\mathfrak{k}$ its Lie algebra, and with $\pi_M : Q \to M$ the reduction of the frame bundle of $M$ to a principal $K$-fibre bundle. Denote by $\mathbb{R}^n = \mathfrak{k} \oplus \mathfrak{v}$ the above $\mathfrak{k}$-invariant decomposition of $\mathbb{R}^n$.

The connection form of $\nabla^T$ is denoted by $\alpha \in \Omega^1(Q, \mathfrak{k})$, and the soldering form is denoted by $\theta \in \Omega^1(Q, \mathbb{R}^n)$. 
Any $A \in \mathfrak{k}$ induces a fundamental vertical vector field $A^*$ on $Q$.

As $Q$ is a subbundle of the frame bundle of $M$, any $\xi \in \mathbb{R}^n$ induces a standard horizontal vector field $\xi^*$ defined at $u \in Q$ by $\xi_u^* := \tilde{u}\xi$.

For $A, B \in \mathfrak{k}$ and $\xi \in \mathbb{R}^n$ we have

$$[A^*, B^*] = [A, B]^*, \quad [A^*, \xi^*] = (A\xi)^*.$$
Key idea: define a Lie algebra structure on $l := \mathfrak{k} \oplus \mathfrak{v}$ induced from the Lie algebra structure on the space of vector fields on $Q$ by the injective map

$$\Phi : l = \mathfrak{k} \oplus \mathfrak{v} \to \Gamma(TQ), \quad A + \xi \mapsto A^* + \xi^*,$$

for $A \in \mathfrak{k}$ and $\xi$ in $\mathfrak{v}$.

**Lemma**

*The image of the map $\Phi$ is closed under the bracket of vector fields.*

Proof: Use the above properties of the torsion and curvature of $\nabla^\tau$. 
The decomposition $TQ = T^{\text{hor}} Q \oplus T^k Q$ of the tangent bundle of $Q$ given by the connection $\alpha$ can be refined as

$$TQ = T^h Q \oplus T^v Q \oplus T^k Q,$$

where $T^h Q_u = \{ \eta_u^* \mid \eta \in \mathfrak{h} \}$, $T^v Q_u = \{ \xi_u^* \mid \xi \in \mathfrak{v} \}$, and $T^k Q_u = \{ A_u^* \mid A \in \mathfrak{k} \}$.

The map $\Phi : \mathfrak{l} \to \Gamma(TQ)$ is by definition a Lie algebra homomorphism, i.e. it defines a structure of infinitesimal $\mathfrak{l}$-principal bundle on $Q$ over some locally defined manifold $N$, whose fibers are the leaves of the integrable distribution $\Phi(\mathfrak{l}) = T^v Q \oplus T^k Q$. Since $(\pi_M)^{-1}(\mathcal{V}M) = T^v Q \oplus T^k Q$, this locally defined manifold $N$ is the same as the locally defined manifold $N$ introduced in the previous section.
The 1-form $\beta := \alpha + \theta^v \in \Omega^1(Q, \mathfrak{l})$ is a connection form on $Q$ with respect to the infinitesimal $\mathfrak{l}$–principal bundle structure, i.e. it satisfies $\beta(B^*) = B$ for every $B \in \mathfrak{l}$ and

$$(\mathcal{L}_{B^*}\beta)(U) = -[B, \beta(U)], \quad \forall \ B \in \mathfrak{l}, \ \forall \ U \in \Gamma(TQ).$$

Some components of the curvature form of $\beta$, viewed as a 2-form with values in the adjoint bundle, are parallel but not all of them (the structure group $L$ is too large, and contains unnecessary information). After a reduction procedure $\implies$ a principal fibre bundle over $N$ with parallel curvature form, containing enough information in order to recover the geometry of $M$. 
Part III

Geometries with parallel curvature

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Connections with skew-symmetric parallel torsion
Consider the linear map

\[ l = \mathfrak{k} \oplus \mathfrak{v} \to \mathfrak{so}(\mathfrak{v}) \oplus \mathfrak{v} := \mathfrak{g}. \]

\( \exists! \) Lie algebra structure on \( \mathfrak{g} \) making this map a Lie algebra morphism. Let \( L \) and \( G \) be the simply connected Lie groups with Lie algebras \( \mathfrak{l} \) and \( \mathfrak{g} \) respectively, and \( \lambda : L \to G \) the corresponding group morphism. The associated \( G \)-principal bundle

\[ P := Q \times_{\lambda} G \]

over \( N \) carries a connection 1-form \( \gamma \in \Omega^1(P, \mathfrak{g}) \) (induced by \( \beta \)).
Theorem (Cleyton, –, Semmelmann)

The section $R^\gamma$ of $\Lambda^2 TN \otimes \text{ad}(P)$ is parallel wrt $\nabla^\sigma \otimes \nabla^\gamma$ and satisfies some extra conditions.

Conversely, given a geometry with parallel skew-symmetric torsion $(N, g^N, \sigma)$ and a $G$-principal bundle with parallel curvature form (+ some extra conditions), one obtains a geometry with parallel skew-symmetric torsion on quotients of $P$ by compact subgroups of $G$.
Parallel $g$-structures
Example

Let \((M^{2k+1}, g, \xi)\) be the Sasakian \(S^1\)-bundle over a Hodge manifold \((N, g^N, \omega)\), such that \(d\xi^\flat = \omega\). Then (generically):

- The holonomy group of the connection \(\nabla = \nabla^g + \frac{1}{2} \xi \wedge d\xi\) is \(K = U(k) \subset SO(2k + 1)\),
- The standard decomposition is \(VM = \langle \xi \rangle, HM = \xi^\perp\),
- The standard Riem. submersion is just the fibration \(M \to N\),
- The extended Lie algebra \(\mathfrak{l} := u(k) \oplus u(1) \to 0 \oplus u(1) =: \mathfrak{g}\),
- The \(K\)-principal bundle \(Q\) over \(M\) is the holonomy bundle of \(\nabla\), also seen as \((U(k) \times U(1))\)-bundle over \(N\),
- The \(G\)-principal bundle with parallel curvature \(P := M \to N\).
**Definition**

A geometry with torsion \((M, g, \tau)\) is **special** if \(\mathfrak{v}\) is a trivial \(\mathfrak{k}\)-representation (i.e. \(\mathcal{V}M\) is spanned by \(\nabla\)-parallel vector fields \(\xi_i\)).

**Remark**

Any \(\nabla\)-parallel vector field \(\xi\) is Killing, since 
\[
0 = \nabla \xi = \nabla^g \xi + \tau \xi.
\]

In this case the projection of the holonomy algebra \(\mathfrak{k}\) on \(\mathfrak{so}(\mathfrak{v})\) vanishes, so the Lie algebra \(\mathfrak{g} = \mathfrak{v}\). In fact it is easy to see directly that the set of \(\nabla\)-parallel vector fields is closed under Lie bracket:

\[
[\xi_i, \xi_j] = \nabla^g_{\xi_i} \xi_j - \nabla^g_{\xi_j} \xi_i = -2\tau(\xi_i, \xi_j)
\]

is also \(\nabla\)-parallel. Like in the previous example, \(M\) is (locally) identified to a principal bundle over the space of leaves of \(\mathcal{V}M\).
Parallel $g$-structures

Definition

Let $G$ be a compact Lie group with Lie algebra $g$. A parallel $g$-structure on a manifold $N$ is given by:

1. a Riemannian metric $g^N$ on $N$;
2. a locally defined $G$-principal bundle $P \to N$ with adjoint bundle $\text{ad}(P)$;
3. an $\text{ad}_g$-invariant scalar product $\langle . , . \rangle$ on $g$, thus on $\text{ad}(P)$;
4. a connection form $\gamma \in \Omega^1(P, g)$ with parallel curvature tensor $R^\gamma : \Lambda^2 TN \to \text{ad}(P)$, s.t. the metric adjoint of $-R^\gamma$ is a Lie algebra bundle morphism $\psi : \text{ad}(P) \to \Lambda^2 TN$. 

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Connections with skew-symmetric parallel torsion
Theorem (Cleyton, –, Semmelmann)

There is a 1-1 correspondence between special geometries with torsion and parallel $g$-structures.

There are several types of natural operations that one can make with parallel $g$-structures: products, reductions to ideals of the Lie algebra, restrictions to Riemannian factors of the manifold $N$, or Whitney products.

Definition

A parallel $g$-structure is non-degenerate if it is not locally a product of parallel $g$-structures.
The classification

Theorem (Cleyton, –, Semmelmann)

Let \( \mathfrak{g} \) be a Lie algebra of compact type and \((g^N, P, \mathfrak{g}, \gamma, \psi)\) a non-degenerate parallel \( \mathfrak{g} \)-structure on a manifold \( N \). Then either:

1. \( N \) is quaternion-Kähler with positive scalar curvature, \( \mathfrak{g} = \mathfrak{sp}(1) \) and \( P \) is the Konishi bundle, or
2. \( N = L/H \) is an irreducible locally symmetric space of compact type, \( \mathfrak{g} \) is isomorphic to a semi-simple factor of \( \mathfrak{h} \), or
3. \( N \) is locally a Riemannian product \( N = N_1 \times \ldots \times N_p \times S_1 \times \ldots \times S_q \) with \( N_\alpha \) Kähler, \( S_\beta = L_\beta/U(1)H_\beta \) Hermitian symmetric of compact type, and \( \mathfrak{g} = \mathfrak{u}(1)^m \oplus \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_q \) with \( \mathfrak{k}_\beta \) a non-zero factor of \( \mathfrak{h}_\beta \).