# PURELY COCLOSED $G_2$-STRUCTURES ON 2-STEP NILPOTENT LIE GROUPS

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ABSTRACT. We consider left-invariant (purely) coclosed $G_2$-structures on 7-dimensional 2-step nilpotent Lie groups. According to the dimension of the commutator subgroup, we obtain various criteria characterizing the Riemannian metrics induced by left-invariant purely coclosed $G_2$-structures. Then, we use them to determine the isomorphism classes of 2-step nilpotent Lie algebras admitting such type of structures. As an intermediate step, we show that every metric on a 2-step nilpotent Lie algebra admitting coclosed $G_2$-structures is induced by one of them. Finally, we use our results to give the explicit description of the metrics induced by purely coclosed $G_2$-structures on 2-step nilpotent Lie algebras with derived algebra of dimension at most two, up to automorphism.

## 1. INTRODUCTION

A $G_2$-structure on a 7-dimensional manifold $M$ is given by a 3-form $\varphi \in \Omega^3(M)$ whose stabilizer at each point of $M$ is isomorphic to the automorphism group $G_2$ of the octonion algebra $\mathbb{O}$. By [15], $M$ admits $G_2$-structures if and only if its first and second Stiefel-Whitney classes vanish. Any $G_2$-structure $\varphi$ induces a metric $g_\varphi$ and an orientation on $M$, and thus a Hodge duality operator $*_\varphi$.

A $G_2$-structure $\varphi$ is said to be *purely coclosed* if it satisfies the conditions

$$d *_\varphi \varphi = 0, \quad d\varphi \wedge \varphi = 0. \tag{1.1}$$

The equations (1.1) characterize the pure class $\mathcal{W}_3$ in Fernández-Gray's classification of $G_2$-structures [8] (see also [2, 3]), while the condition $d *_\varphi \varphi = 0$ determines the wider class of *coclosed* $G_2$-structures $\mathcal{W}_1 \oplus \mathcal{W}_3$. Remarkably, the latter are known to exist on every compact 7-manifold admitting $G_2$-structures by an $h$-principle argument [5]. However, since this method is not constructive, different techniques are needed to obtain explicit examples. As for purely coclosed $G_2$-structures, no similar existence result is currently available.

The intrinsic torsion of a purely coclosed $G_2$-structure $\varphi$ can be identified with the 3-form $*_\varphi d\varphi$, and so it vanishes identically if and only if $\varphi$ is closed (cf. [2]). When this happens, the Riemannian metric $g_\varphi$ induced by $\varphi$ is Ricci-flat and the corresponding Riemannian holonomy group is a subgroup of $G_2$.

In theoretical physics, purely coclosed $G_2$-structures are closely related to the $G_2$-*Strominger system* of equations that arises considering the Killing spinor equations of 10-dimensional string theory [23] on a compact 7-manifold (see e.g. [4, 17] for more details and for the complete description of this system). Indeed, by [12, 13], the gravitino and dilatino Killing spinor equations with dilaton function $f$ on a compact 7-manifold $M$ are equivalent to the following system of equations for a $G_2$-structure $\varphi$ on $M$:

$$d *_\varphi \varphi = -2\, df \wedge *_\varphi\varphi, \quad d\varphi \wedge \varphi = 0.$$

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Any $G_2$-structure $\varphi$ satisfying them gives rise to a purely coclosed one via the global conformal change $e^{\frac{3}{2}f}\varphi$. Moreover, $\varphi$ itself is purely coclosed whenever $f$ is constant. Therefore, producing examples of compact 7-manifolds admitting purely coclosed $G_2$-structures constitutes an essential step towards the resolution of the $G_2$-Strominger system.

Solutions to the $G_2$-Strominger system have been recently obtained in [4] on $\mathbb{T}^3$-bundles over $K3$ surfaces. Previously, two examples of solutions with constant dilaton function were described in [10]. In these last two examples, the 7-manifold is the compact quotient of a simply connected nilpotent Lie group $N$ by a co-compact discrete subgroup (lattice) $\Gamma \subset N$, i.e., a *nilmanifold*, and the purely coclosed $G_2$-structure on $\Gamma\backslash N$ is induced by a left-invariant one on $N$. Moreover, the Lie group $N$ is 2-step nilpotent, namely its Lie algebra $\mathfrak{n}$ is non-abelian and the corresponding derived algebra is contained in its center. In the non-compact setting, a further solution on a 2-step nilpotent Lie group was given in [9]. It is worth observing that, when working with left-invariant $G_2$-structures on Lie groups, the investigation can be done at the Lie algebra level, as left-invariant $G_2$-structures of a certain class on a Lie group are in one-to-one correspondence with $G_2$-structures of the same type on its Lie algebra.

In this paper, we carry out a systematic study of purely coclosed $G_2$-structures on 7-dimensional 2-step nilpotent Lie algebras, aimed at obtaining a classification of those admitting this type of $G_2$-structures, up to isomorphism, and characterizing the metrics induced by purely coclosed $G_2$-structures.

We begin our investigation focusing on coclosed $G_2$-structures. Our first result is a refinement of the classification obtained in [1]. There, the authors proved through a case-by-case study relying on the classification of 7-dimensional 2-step nilpotent Lie algebras, that each isomorphism class of such Lie algebras admits a coclosed $G_2$-structure, with the exception of $\mathfrak{n}_{7,2,A}$ and $\mathfrak{n}_{7,2,B}$ (see the notation in Appendix A, where we review the classification results obtained in [14]). These last two Lie algebras are irreducible and have 2-dimensional derived algebra. Here, we prove through direct arguments the following more precise statement which also takes into account the metric Lie algebra structure.

**Theorem 1.1.** *Let $\mathfrak{n}$ be a 7-dimensional 2-step nilpotent Lie algebra. If $\mathfrak{n}$ is irreducible and has 2-dimensional derived algebra, then it carries no coclosed $G_2$-structures. If $\mathfrak{n}$ is either reducible, or its derived algebra has dimension different from 2, then every metric on $\mathfrak{n}$ is induced by a coclosed $G_2$-structure.*

It follows from [11, 19, 24] that every 7-dimensional 2-step nilpotent Lie algebra $\mathfrak{n}$ admits a (necessarily unique up to automorphism and scaling) *nilsoliton metric*, i.e., a metric $g$ whose Ricci endomorphism is of the form $\mathrm{Rc}(g) = \lambda\,\mathrm{Id} + D$, for some $\lambda \in \mathbb{R}$ and some derivation $D$ of $\mathfrak{n}$. Nilsolitons correspond to left-invariant Ricci soliton metrics on nilpotent Lie groups [18] and so they constitute a generalization of Einstein metrics, that cannot exist on non-abelian nilpotent Lie groups by [21, Thm. 2.4]. Using the above observation together with Theorem 1.1, we obtain a direct proof of [1, Thm. 6.1, Thm. 6.3].

**Corollary 1.2.** *Any 7-dimensional 2-step nilpotent Lie algebra admitting coclosed $G_2$-structures has a coclosed $G_2$-structure inducing the nilsoliton metric.*

We then focus on purely coclosed $G_2$-structures on 2-step nilpotent Lie algebras. According to the dimension of the derived algebra, we obtain criteria for a given metric $g$ to be induced by such a structure. In order to state them here, we recall that for any 2-step nilpotent metric Lie algebra $(\mathfrak{n}, g)$ with derived algebra $\mathfrak{n}'$ and $g$-orthogonal decomposition $\mathfrak{n} = \mathfrak{r} \oplus \mathfrak{n}'$, the Chevalley-Eilenberg differential $\mathrm{d} : \mathfrak{n}^* \to \Lambda^2\mathfrak{n}^*$ vanishes on $\mathfrak{r}^*$ and defines an injection $j$ from $\mathfrak{n}'$ into $\mathfrak{so}(\mathfrak{r}) \simeq \Lambda^2\mathfrak{r}^*$ (we refer the reader to Sect. 3 for the precise details). In particular, $\dim(\mathfrak{n}') \leq \dim(\mathfrak{so}(\mathfrak{r}))$, so when $\mathfrak{n}$ is 7-dimensional, this implies that the dimension of $\mathfrak{n}'$ is at most 3.

**Theorem 1.3.** *Let $(\mathfrak{n}, g)$ be a 7-dimensional 2-step nilpotent metric Lie algebra with derived algebra $\mathfrak{n}'$, and consider the $g$-orthogonal decomposition $\mathfrak{n} = \mathfrak{r} \oplus \mathfrak{n}'$.*

(*i*) *If $\dim(\mathfrak{n}') = 1$, there exists a purely coclosed $G_2$-structure on $\mathfrak{n}$ inducing the metric $g$ if and only if $\operatorname{tr}^2(j(z)^2) = 4\operatorname{tr}(j(z)^4)$ for every $z \in \mathfrak{n}'$.*

(*ii*) *If $\dim(\mathfrak{n}') = 2$, there exists a purely coclosed $G_2$-structure on $\mathfrak{n}$ inducing the metric $g$ if and only if there exists an oriented 4-dimensional subspace $\tilde{\mathfrak{r}} \subset \mathfrak{r}$ with $d(\mathfrak{n}')^* \subset \Lambda^2\tilde{\mathfrak{r}}^*$ such that for every orthonormal basis $\{\zeta_1, \zeta_2\}$ of $(\mathfrak{n}')^*$, the self-dual components $d\zeta_1^+,\ d\zeta_2^+ \in \Lambda^2_+\tilde{\mathfrak{r}}^*$ of $d\zeta_1,\ d\zeta_2 \in \Lambda^2\tilde{\mathfrak{r}}^*$ are orthogonal and have equal norms.*

(*iii*) *If $\dim(\mathfrak{n}') = 3$, there exists a purely coclosed $G_2$-structure on $\mathfrak{n}$ inducing the metric $g$ if and only if for some orientation of the 4-dimensional space $\mathfrak{r}$, and for every orthonormal basis $\{\zeta_1, \zeta_2, \zeta_3\}$ of $(\mathfrak{n}')^*$, the Gram matrix of the self-dual components of their differentials in $\Lambda^2_+\mathfrak{r}^*$, $(S_{ij}) := (g(d\zeta_i^+, d\zeta_j^+))$, satisfies $\operatorname{tr}^2(S) = 2\operatorname{tr}(S^2)$.*

As a consequence of this this result, we obtain the classification of all 7-dimensional 2-step nilpotent Lie algebras admitting purely coclosed $G_2$-structures, up to isomorphism.

**Theorem 1.4.** *A 7-dimensional 2-step nilpotent Lie algebra $\mathfrak{n}$ admits purely coclosed $G_2$-structures if and only if $\mathfrak{n}$ is not isomorphic to $\mathfrak{h}_3 \oplus \mathbb{R}^4$, $\mathfrak{n}_{7,2,A}$ or $\mathfrak{n}_{7,2,B}$, where $\mathfrak{h}_3$ denotes the 3-dimensional Heisenberg Lie algebra.*

Theorem 1.3, combined with metric classification results from [6, 22], allows us to give the explicit description, up to automorphisms, of the metrics induced by purely coclosed $G_2$-structures on every 2-step nilpotent Lie algebra admitting such structures and with derived algebra of dimension at most two. In the remaining case, where $\dim(\mathfrak{n}') = 3$, the lack of classification of metric Lie algebra structures prevents us from obtaining a similar result. Nevertheless, we show that each of these Lie algebras carries purely coclosed $G_2$-structures but also metrics which are not induced by any of them.

Finally, for each 2-step nilpotent Lie algebra of dimension 7, we are able to determine whether its nilsoliton metric is induced by a purely coclosed $G_2$-structure (see Corollaries 5.3, 5.9, 5.11).

It is worth stressing that all Lie groups corresponding to the Lie algebras carrying purely coclosed $G_2$-structures admit a lattice (cf. [20]). Therefore, the results above provide many new examples of (compact) manifolds where the $G_2$-Strominger system may be investigated.

The paper is organized as follows. In Sect. 2, we review some preliminaries on $G_2$-structures, and in Sect. 3 we recall the main properties of 2-step nilpotent metric Lie algebras. Theorem 1.1 and Theorem 1.3 are proved in Sect. 4. The discussion is divided into three parts according to the dimension of the derived algebra $\mathfrak{n}'$. Finally, in Sect. 5 we describe the metrics induced by purely coclosed $G_2$-structures on 2-step nilpotent metric Lie algebras $\mathfrak{n}$ with $\dim(\mathfrak{n}') \leq 2$, and for each Lie algebra in the remaining case $\dim(\mathfrak{n}') = 3$ we construct purely coclosed $G_2$-structures, as well as metrics which are not compatible with any purely coclosed $G_2$-structure.

## 2. Preliminaries on $G_2$-structures

2.1. **Basic definitions.** A $G_2$-structure on a 7-dimensional vector space $V$ is defined by a 3-form $\varphi \in \Lambda^3V^*$ satisfying the non-degeneracy condition

$$v \lrcorner \varphi \wedge v \lrcorner \varphi \wedge \varphi \neq 0, \quad \forall\, v \in V \setminus \{0\}. \tag{2.1}$$

Since the stabilizer $\mathrm{GL}(V)_\varphi \subset \mathrm{GL}(V)$ of any such 3-form is isomorphic to the exceptional Lie group $\mathrm{G}_2$, the set $\Lambda^3_+V^*$ of all $\mathrm{G}_2$-structures on $V$ is isomorphic to $\mathrm{GL}(7,\mathbb{R})/\mathrm{G}_2$ and thus open in $\Lambda^3V^*$.

A $\mathrm{G}_2$-structure $\varphi \in \Lambda^3_+V^*$ gives rise to a unique inner product $g_\varphi$ and orientation on $V$ with corresponding volume form $\mathrm{vol}_\varphi$ satisfying

$$g_\varphi(v,w)\,\mathrm{vol}_\varphi = \frac{1}{6}\, v\lrcorner\varphi \wedge w\lrcorner\varphi \wedge \varphi. \tag{2.2}$$

Moreover, there exists a $g_\varphi$-orthonormal basis $\mathcal{B} = \{e_1,\dots,e_7\}$ of $V$ with dual basis $\mathcal{B}^* = \{e^1,\dots,e^7\}$ such that

$$\begin{aligned} \varphi &= e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \\ *_\varphi\varphi &= e^{1234} + e^{1256} + e^{3456} + e^{1367} + e^{1457} + e^{2357} - e^{2467}, \end{aligned} \tag{2.3}$$

where $*_\varphi$ is the Hodge operator determined by $g_\varphi$ and $\mathrm{vol}_\varphi$, and $e^{ijk\cdots}$ is a shorthand for the wedge product of covectors $e^i \wedge e^j \wedge e^k \wedge \cdots$. We shall call both $\mathcal{B}$ and $\mathcal{B}^*$ *adapted* bases to the $\mathrm{G}_2$-structure $\varphi$.

On the other hand, given an inner product $g$ on $V$, we can consider a $g$-orthonormal basis $\mathcal{B}$ of $V$ and the $\mathrm{G}_2$-structure $\varphi \in \Lambda^3_+V^*$ having $\mathcal{B}$ as an adapted basis. By (2.2), the metric $g_\varphi$ induced by $\varphi$ coincides with $g$. We shall refer to such $\varphi$ as the $\mathrm{G}_2$-structure *induced* by the basis $\mathcal{B}$. Hence, there is a surjective map

$$\mathcal{G} : \Lambda^3_+V^* \to \mathcal{S}^2_+V^*, \quad \mathcal{G}(\varphi) = g_\varphi,$$

which is not injective, as the set of all $\mathrm{G}_2$-structures inducing the same metric is parametrized by $\mathrm{SO}(7)/\mathrm{G}_2 \cong \mathbb{R}P^7$ (see [2, Remark 4] for an explicit description). The $\mathrm{G}_2$-structures belonging to $\mathcal{G}^{-1}(g)$ will be called *compatible* with $g$.

Consider now a $\mathrm{G}_2$-structure $\varphi$ on $V$, let $z \in V$ be a unit vector and denote by $W$ the 6-dimensional $g_\varphi$-orthogonal complement of $\langle z\rangle \subset V$. Then, the $\mathrm{G}_2$-structure $\varphi$ induces an $\mathrm{SU}(3)$-structure $(h, J, \omega, \psi_+, \psi_-)$ on $W$ by means of the identities

$$\varphi = \omega \wedge z^\flat + \psi_+, \quad *_\varphi\varphi = \frac{1}{2}\,\omega\wedge\omega + \psi_-\wedge z^\flat, \quad g_\varphi = h + z^\flat \otimes z^\flat,$$

where $z^\flat \in V^*$ denotes the $g_\varphi$-dual covector of $z$. Recall that the non-degenerate 2-form $\omega$ and the 3-forms $\psi_+, \psi_-$ satisfy the *compatibility condition* $\omega \wedge \psi_\pm = 0$ and the *normalization condition*

$$\psi_+ \wedge \psi_- = \frac{2}{3}\,\omega^3 = 4\,\mathrm{vol}_h,$$

where $\mathrm{vol}_h$ is the volume form of the inner product $h$. Moreover, the $h$-orthogonal complex structure $J \in \mathrm{End}(W)$ is related to $h$ and $\omega$ via the identity $\omega = h(J\cdot,\cdot)$. Finally, there exists an adapted basis $\mathcal{B} = \{e_1,\dots,e_7\}$ to $\varphi$ with $e_7 = z$ and such that $\{e_1,\dots,e_6\}$ is an $h$-orthonormal basis of $W$ which is *adapted* to the $\mathrm{SU}(3)$-structure, that is to say

$$\omega = e^{12} + e^{34} + e^{56}, \quad \psi_+ = e^{135} - e^{146} - e^{236} - e^{245}, \quad \psi_- = e^{136} + e^{145} + e^{235} - e^{246}, \tag{2.4}$$

and $J(e_{2k-1}) = e_{2k}$, $k = 1,2,3$. As before, any orthonormal basis $\{e_1,\dots,e_6\}$ of $W$ *induces* an $\mathrm{SU}(3)$-structure, namely, the structure defined by (2.4) in the given basis.

This procedure can be reversed, allowing one to obtain a $\mathrm{G}_2$-structure on the 1-dimensional extension of a 6-dimensional vector space $W$ endowed with an $\mathrm{SU}(3)$-structure $(h, J, \omega, \psi_\pm)$. In detail, if $\{e_1,\dots,e_6\}$ is a basis of $W$ which is adapted to the $\mathrm{SU}(3)$-structure, then the 7-dimensional vector space $V = W \oplus \langle z\rangle$ is endowed with a $\mathrm{G}_2$-structure $\varphi$ having $\{e_1,\dots,e_6,z\}$ as an adapted basis.

2.2. **(Purely) coclosed $G_2$-structures.** Let $M$ be a 7-manifold endowed with a $G_2$-structure $\varphi \in \Omega^3(M)$. By [2, Prop. 1], there exist unique differential forms $\tau_0 \in \mathcal{C}^\infty(M)$, $\tau_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2_{14}(M) := \{\alpha \in \Omega^2(M) \mid \alpha \wedge *_\varphi\varphi = 0\}$, and $\tau_3 \in \Omega^3_{27}(M) := \{\gamma \in \Omega^3(M) \mid \gamma \wedge \varphi = 0,\ \gamma \wedge *_\varphi\varphi = 0\}$ such that

$$\mathrm{d}\varphi = \tau_0 *_\varphi \varphi + 3\tau_1 \wedge \varphi + *_\varphi\tau_3,$$
$$\mathrm{d} *_\varphi \varphi = 4\tau_1 \wedge *_\varphi\varphi + \tau_2 \wedge \varphi.$$

These differential forms are called the *torsion forms* of the $G_2$-structure $\varphi$, as they completely determine its intrinsic torsion (see also [8]).

A $G_2$-structure $\varphi$ is said to be *coclosed* if it satisfies the equation

$$\mathrm{d} *_\varphi \varphi = 0.$$

In terms of the torsion forms, the above condition is equivalent to the vanishing of $\tau_1$ and $\tau_2$. Coclosed $G_2$-structures constitute the class $\mathcal{W}_1 \oplus \mathcal{W}_3$ in Fernández-Gray's classification of $G_2$-structures [8]. The "pure" subclasses $\mathcal{W}_1$, $\mathcal{W}_3$ are characterized by the vanishing of $\tau_3$ and $\tau_0$, respectively. In the former case, the coclosed $G_2$-structure is called *nearly parallel* and the associated metric $g_\varphi$ is Einstein with positive scalar curvature $\mathrm{Scal}(g_\varphi) = \frac{21}{8}(\tau_0)^2$. In the latter, the $G_2$-structure is called *purely coclosed*. Notice that the vanishing of $\tau_0$ is equivalent to the condition

$$\mathrm{d}\varphi \wedge \varphi = 0,$$

as $\tau_0 = \frac{1}{7} *_\varphi (\mathrm{d}\varphi \wedge \varphi)$.

Simple examples of 7-manifolds admitting (purely) coclosed $G_2$-structures can be obtained as follows. Let $N$ be a 6-dimensional manifold endowed with an SU(3)-structure $(h, J, \omega, \psi_\pm)$. Then, the product manifold $M = N \times \mathbb{R}$ is endowed with a $G_2$-structure defined by the non-degenerate 3-form

$$\varphi = \omega \wedge \mathrm{d}t + \psi_+,$$

where $\mathrm{d}t$ denotes the global 1-form on $\mathbb{R}$. The Riemannian metric induced by $\varphi$ is $g_\varphi = h + \mathrm{d}t^2$ and the Hodge dual of $\varphi$ is given by $*_\varphi\varphi = \frac{1}{2}\,\omega^2 + \psi_- \wedge \mathrm{d}t$. Now, we have

$$\mathrm{d} *_\varphi \varphi = \mathrm{d}\omega \wedge \omega + \mathrm{d}\psi_- \wedge \mathrm{d}t,$$
$$\mathrm{d}\varphi \wedge \varphi = \mathrm{d}\omega \wedge \mathrm{d}t \wedge \psi_+ + \mathrm{d}\psi_+ \wedge \omega \wedge \mathrm{d}t = -2\,\mathrm{d}\omega \wedge \psi_+ \wedge \mathrm{d}t.$$

Thus, we immediately see that $\varphi$ is coclosed if and only if the SU(3)-structure satisfies the conditions

$$\mathrm{d}\omega \wedge \omega = 0, \quad \mathrm{d}\psi_- = 0. \tag{2.5}$$

Moreover, $\varphi$ is purely coclosed if and only if the SU(3)-structure satisfies the additional condition

$$\mathrm{d}\omega \wedge \psi_+ = 0. \tag{2.6}$$

An SU(3)-structure satisfying the equations (2.5) is called *half-flat* (cf. [3]).

## 3. The structure of 2-step nilpotent metric Lie algebras

We now consider the case when the 7-dimensional manifold is a Lie group $N$ endowed with a left-invariant $G_2$-structure, namely a non-degenerate 3-form $\varphi \in \Omega^3(N)$ that is invariant by left translations of $N$. In this case, the Riemannian metric $g_\varphi$ induced by $\varphi$ is also left-invariant.

The identification of the Lie algebra $\mathfrak{n}$ of $N$ with the tangent space to $N$ at the identity gives rise to a one-to-one correspondence between left-invariant tensors on $N$ and algebraic tensors of the same type defined on $\mathfrak{n}$. In particular, left-invariant Riemannian metrics on $N$ correspond to inner products on $\mathfrak{n}$, and left-invariant $G_2$-structures on $N$ correspond to $\mathrm{G}_2$-structures $\varphi$ on $\mathfrak{n}$, i.e., $\varphi \in \Lambda^3_+\mathfrak{n}^*$. The conditions for $\varphi$ to be purely coclosed read $\mathrm{d} *_\varphi \varphi = 0$ and $\mathrm{d}\varphi \wedge \varphi = 0$, where d denotes the Chevalley-Eilenberg differential of $\mathfrak{n}$.

Along this paper, a *metric Lie algebra* is the data of a real Lie algebra $\mathfrak{n}$ endowed with an inner product $g$. We focus on the case when $\mathfrak{n}$ is 2-step nilpotent, namely when $\mathfrak{n}$ is not abelian and $\mathrm{ad}_x^2 = 0$ for all $x \in \mathfrak{n}$, where ad denotes the adjoint map of $\mathfrak{n}$. Under this assumption, the structure of a metric Lie algebra $(\mathfrak{n}, g)$ can be described as follows (see [7] for more details).

Denote by $\mathfrak{n}' := [\mathfrak{n}, \mathfrak{n}]$ the derived algebra of $\mathfrak{n}$ and by $\mathfrak{z}$ its center. As $\mathfrak{n}$ is 2-step nilpotent, we have $\{0\} \neq \mathfrak{n}' \subset \mathfrak{z}$. Let $\mathfrak{r}$ denote the $g$-orthogonal complement of $\mathfrak{n}'$ in $\mathfrak{n}$, so that $\mathfrak{n} = \mathfrak{r} \oplus \mathfrak{n}'$ as a direct sum of vector spaces. The metric Lie algebra structure of $(\mathfrak{n}, g)$ is encoded into the injective linear map $j : \mathfrak{n}' \to \mathfrak{so}(\mathfrak{r})$ defined via the identity

$$g(j(z)x, y) := g(z, [x, y]), \tag{3.1}$$

for all $z \in \mathfrak{n}'$ and $x, y \in \mathfrak{r}$.

Using the metric $g$, we can always identify $\mathfrak{n}$ with its dual Lie algebra $\mathfrak{n}^*$, and we can see the elements in $\Lambda^2\mathfrak{n}^*$ as skew-symmetric endomorphisms in $\mathfrak{so}(\mathfrak{n})$. Under these identifications, it is straightforward to check that $\mathrm{d}x^\flat = 0$ if $x \in \mathfrak{r}$, where $x^\flat$ denotes the metric dual of $x$. In addition, for any $z \in \mathfrak{n}'$ the 2-form $\mathrm{d}z^\flat$ belongs to the subspace $\Lambda^2\mathfrak{r}^*$ of $\Lambda^2\mathfrak{n}^*$ and it corresponds to the skew-symmetric endomorphism $-j(z) \in \mathfrak{so}(\mathfrak{r})$.

The previous discussion allows us to associate to any 2-step nilpotent metric Lie algebra $(\mathfrak{n}, g)$, the following data: the inner product spaces $(\mathfrak{r}, g_\mathfrak{r})$ and $(\mathfrak{n}', g_{\mathfrak{n}'})$ together with an injection $j : \mathfrak{n}' \to \mathfrak{so}(\mathfrak{r})$. Here and henceforth, we denote by $g_\mathfrak{k}$ the restriction of the metric $g$ to the subspace $\mathfrak{k}$ of $\mathfrak{n}$.

Conversely, given two inner product spaces $(\mathfrak{r}, g_\mathfrak{r})$ and $(\mathfrak{n}', g_{\mathfrak{n}'})$, together with an injective linear map $j : \mathfrak{n}' \to \mathfrak{so}(\mathfrak{r})$, we can define a 2-step nilpotent metric Lie algebra $(\mathfrak{n}, g)$ as follows. We set $\mathfrak{n} := \mathfrak{r} \oplus \mathfrak{n}'$, we endow it with the metric $g = g_\mathfrak{r} + g_{\mathfrak{n}'}$ and we define the Lie bracket on $\mathfrak{n}$ so that the elements in $\mathfrak{n}'$ are in the center, it satisfies $[\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{n}'$ and it is determined by

$$g_{\mathfrak{n}'}(z, [x, y]) := g_\mathfrak{r}(j(z)(x), y), \quad \text{for all } x, y \in \mathfrak{r},\ z \in \mathfrak{n}'.$$

It is easy to verify that $\mathfrak{n}$ is a 2-step nilpotent Lie algebra with derived algebra $\mathfrak{n}'$ (which justifies the initial notation).

For any 2-step nilpotent metric Lie algebra $(\mathfrak{n}, g)$ we have $\mathfrak{n}' \subset \mathfrak{z}$. Let $\mathfrak{a}$ denote the orthogonal complement of $\mathfrak{n}'$ inside $\mathfrak{z}$. It is straightforward to check from (3.1) that $\mathfrak{a}$ is the common kernel of the endomorphisms $j(z) \in \mathfrak{so}(\mathfrak{r})$, when $z$ runs through $\mathfrak{n}'$. In addition, for any $x \in \mathfrak{a}$, the orthogonal complement $\tilde{\mathfrak{n}} := \langle x \rangle^\perp$ is an ideal of $\mathfrak{n}$, which now decomposes as a direct sum of orthogonal ideals $(\mathfrak{n}, g) = (\tilde{\mathfrak{n}}, g_{\tilde{\mathfrak{n}}}) \oplus (\langle x \rangle, g_{\langle x \rangle})$.

Nilpotent Lie algebras of dimension 7 are classified up to isomorphism (see [14]); we recall the classification of those which are real and 2-step nilpotent in Appendix A. Throughout the paper, the structure equations of an $n$-dimensional Lie algebra $\mathfrak{n}$ are written with respect to a basis of covectors $\{f^1, \ldots, f^n\}$ by specifying the $n$-tuple $(\mathrm{d}f^1, \ldots, \mathrm{d}f^n)$.

## 4. When is a metric induced by a (purely) coclosed $\mathrm{G}_2$-structure?

In this section, we will prove Theorem 1.1 and Theorem 1.3. As we already observed, given a 7-dimensional 2-step nilpotent Lie algebra $\mathfrak{n}$, the possible dimensions of its derived algebra $\mathfrak{n}'$ are 1, 2 or 3. We shall discuss each case separately.

4.1. **Case 1:** $\dim(\mathfrak{n}') = 1$. Let $(\mathfrak{n}, g)$ be a 2-step nilpotent metric Lie algebra of dimension 7 with 1-dimensional derived algebra $\mathfrak{n}'$ and consider the $g$-orthogonal splitting $\mathfrak{n} = \mathfrak{r} \oplus \mathfrak{n}'$, where $\mathfrak{r} = (\mathfrak{n}')^{\perp}$. Given a unit vector $z \in \mathfrak{n}'$, the structure equations of $\mathfrak{n}$ are completely determined by the differential of the metric dual $z^\flat$ of $z$, which we denote by $\alpha \coloneqq dz^\flat \in \Lambda^2\mathfrak{r}^*$. Let $A \coloneqq -j(z) \in \mathfrak{so}(\mathfrak{r})$ denote the skew-symmetric endomorphism corresponding to $\alpha \in \Lambda^2\mathfrak{r}^*$.

Notice that, depending on the rank of $A$, $\mathfrak{n}$ is isomorphic to one of $\mathfrak{h}_3 \oplus \mathbb{R}^4$ ($\mathrm{rank}(A) = 2$), $\mathfrak{h}_5 \oplus \mathbb{R}^2$ ($\mathrm{rank}(A) = 4$), $\mathfrak{h}_7$ ($\mathrm{rank}(A) = 6$), where $\mathfrak{h}_i$ denotes the Heisenberg Lie algebra of dimension $i$.

Let $\varphi$ be a $G_2$-structure on $\mathfrak{n}$ such that $g_\varphi = g$. Then the 6-dimensional vector subspace $\mathfrak{r} \subset \mathfrak{n}$ is endowed with an SU(3)-structure $(h, J, \omega, \psi_\pm)$ (see Sect. 2.1). In particular, we can write

$$\varphi = \omega \wedge z^\flat + \psi_+, \quad *_\varphi\varphi = \frac{1}{2}\omega^2 + \psi_- \wedge z^\flat. \tag{4.1}$$

Using the complex structure $J$ of $\mathfrak{r}$, we can give the following characterization.

**Proposition 4.1.** *The $G_2$-structure $\varphi$ in (4.1) is coclosed if and only if $JA = AJ$ and, moreover, it is purely coclosed if and only if $\mathrm{tr}(JA) = 0$.*

*Proof.* Since $\omega$ and $\psi_-$ are forms on $\mathfrak{r}$, their differentials vanish, so the $G_2$-structure $\varphi$ in (4.1) is coclosed if and only if

$$0 = d *_\varphi \varphi = -\psi_- \wedge \alpha.$$

Thus, $\alpha$ belongs to the kernel of the map $\cdot \wedge \psi_- : \Lambda^2\mathfrak{r}^* \to \Lambda^5\mathfrak{r}^*$, which is the space of real 2-forms of type $(1,1)$ with respect to $J$. In terms of the skew-symmetric endomorphism $A \in \mathfrak{so}(\mathfrak{r})$ corresponding to $\alpha$, this is equivalent to $JA = AJ$. Moreover, the intrinsic torsion form $\tau_0$ of $\varphi$ vanishes if and only if

$$0 = d\varphi \wedge \varphi = \omega^2 \wedge \alpha \wedge z^\flat,$$

which is equivalent to the additional constraint $\mathrm{tr}(JA) = 0$. □

Using Proposition 4.1, we can show the following result, which proves Theorem 1.1 for the case $\dim(\mathfrak{n}') = 1$, and part (i) of Theorem 1.3.

**Proposition 4.2.** *Let $(\mathfrak{n}, g)$ be a 7-dimensional 2-step nilpotent metric Lie algebra with* $\dim(\mathfrak{n}') = 1$.

1) *There exists a coclosed $G_2$-structure on $\mathfrak{n}$ inducing the metric $g$.*
2) *Furthermore, there exists a purely coclosed $G_2$-structure on $\mathfrak{n}$ inducing the metric $g$ if and only if $\mathrm{tr}^2(j(z)^2) = 4\,\mathrm{tr}(j(z)^4)$ for every $z \in \mathfrak{n}'$.*

*Proof.*

1) Let $g$ be an inner product on $\mathfrak{n}$ and consider a unit vector $z$ in $\mathfrak{n}'$. Then, $A \coloneqq -j(z) \in \mathfrak{so}(\mathfrak{r})$ and there exists a $g$-orthonormal basis $\{e_1, \ldots, e_6\}$ of $\mathfrak{r}$ such that the matrix of $A$ with respect to this basis has the form

$$A = \begin{pmatrix} 0 & -a & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c \\ 0 & 0 & 0 & 0 & c & 0 \end{pmatrix}, \tag{4.2}$$

where $a, b, c \in \mathbb{R}$ are not all zero. Consider the SU(3)-structure $(h, J, \omega, \psi_\pm)$ on $\mathfrak{r}$ induced by this basis $\{e_1, \ldots, e_6\}$. Then, $\omega$ satisfies (2.4) and the complex structure $J$ satisfies $JA = AJ$. Thus, the 3-form $\varphi = \omega \wedge z^\flat + \psi_+$ verifies (4.1) and, by Proposition 4.1, it defines a coclosed $G_2$-structure on $\mathfrak{n}$ with $g_\varphi = g$.

2) Let $\varphi$ be a purely coclosed $\mathrm{G}_2$-structure on $\mathfrak{n}$ such that $g_\varphi = g$, and let $z$ be a unit vector in $\mathfrak{n}'$. Then $\varphi$ induces an SU(3)-structure $(h, J, \omega, \psi_\pm)$ on $\mathfrak{r}$ satisfying $\varphi = \omega \wedge z^\flat + \psi_+$ and $\omega = g_\mathfrak{r}(J\cdot, \cdot)$ (see Sect. 2.1). So $\varphi$ is of the form (4.1) and, by Proposition 4.1, we obtain that $JA = AJ$ and $\mathrm{tr}(JA) = 0$, for $A = -j(z)$. The eigenspaces of the symmetric endomorphism $AJ$ are preserved by $J$ so they have even dimension. Consequently the spectrum of $AJ$ is of the form $(a, a, b, b, c, c)$ with $a + b + c = 0$, whence

$$\mathrm{tr}(A^2) = -\mathrm{tr}((AJ)^2) = -2(a^2 + b^2 + (a + b)^2) = -4(a^2 + b^2 + ab),$$

and thus

$$\mathrm{tr}(A^4) = \mathrm{tr}((AJ)^4) = 2(a^4 + b^4 + (a + b)^4) = 4(a^2 + b^2 + ab)^2 = \frac{1}{4}\mathrm{tr}^2(A^2).$$

It is clear that for any other vector $z' = \lambda z$ in $\mathfrak{n}'$, the corresponding matrix $A' = -j(z')$ verifies $A' = \lambda A$, so the same equation holds by homogeneity.

Conversely, suppose that $g$ is an inner product on $\mathfrak{n}$ for which $\mathrm{tr}^2(j(z)^2) = 4\,\mathrm{tr}(j(z)^4)$ holds for every $z$ in $\mathfrak{n}'$. Consider again a unit vector $z \in \mathfrak{n}'$ and a $g$-orthonormal basis $\{e_1, \ldots, e_6\}$ of $\mathfrak{r}$ in which the matrix of $A = -j(z)$ is given by (4.2). Since $\mathrm{tr}^2(A^2) = 4\,\mathrm{tr}(A^4)$, we can write

$$\begin{aligned} 0 &= \mathrm{tr}(A^4) - \frac{1}{4}\mathrm{tr}^2(A^2) = 2(a^4 + b^4 + c^4) - (a^2 + b^2 + c^2)^2 \\ &= a^4 + b^4 + c^4 - 2(a^2b^2 + b^2c^2 + c^2a^2) \\ &= (a + b + c)(a + b - c)(a - b + c)(a - b - c). \end{aligned}$$

By permuting $e_1$ with $e_2$ and/or $e_3$ with $e_4$ if necessary, we can change the signs of $a$ and/or $b$, so from the above equation we can assume that $a+b+c = 0$. Let $(h, J, \omega, \psi_\pm)$ be the SU(3)-structure on $\mathfrak{r}$ induced by the basis $\{e_1, \ldots, e_6\}$. Then, we have $JA = AJ = \mathrm{diag}(-a, -a, -b, -b, -c, -c)$, so $\mathrm{tr}(JA) = -2(a + b + c) = 0$, and therefore the 3-form $\varphi = \omega \wedge z^\flat + \psi_+$ defines a purely coclosed $\mathrm{G}_2$-structure on $\mathfrak{n}$ with $g_\varphi = g$.

□

One can prove that the condition $\mathrm{tr}^2(j(z)^2) = 4\,\mathrm{tr}(j(z)^4)$ cannot hold whenever $j(z)$ has rank 2, so we can state the following.

**Corollary 4.3.** *The Lie algebra $\mathfrak{h}_3 \oplus \mathbb{R}^4$ does not admit any purely coclosed* $\mathrm{G}_2$*-structure.*

*Proof.* For every metric on $\mathfrak{h}_3 \oplus \mathbb{R}^4$, the endomorphism $A = -j(z)$ has rank 2, for any non-zero $z \in \mathfrak{n}'$, so its square is proportional to an orthogonal projector on a 2-plane: $A^2 = \lambda P$ with $\lambda \in \mathbb{R} \smallsetminus \{0\}$, $P^2 = P$ and $\mathrm{tr}(P) = 2$. Then $\mathrm{tr}(A^2) = 2\lambda$, $\mathrm{tr}(A^4) = 2\lambda^2$, so the equation $\mathrm{tr}^2(A^2) = 4\,\mathrm{tr}(A^4)$ cannot hold. □

4.2. **Case 2:** $\dim(\mathfrak{n}') = 2$. In this section we consider coclosed $\mathrm{G}_2$-structures on 7-dimensional 2-step nilpotent Lie algebras $\mathfrak{n}$ with $\dim(\mathfrak{n}') = 2$. From the classification reviewed in Appendix A, we know that any such Lie algebra is isomorphic either to one of the decomposable Lie algebras

$$\mathfrak{n}_{5,2} \oplus \mathbb{R}^2, \quad \mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}, \quad \mathfrak{h}_3^{\mathbb{C}} \oplus \mathbb{R}, \quad \mathfrak{n}_{6,2} \oplus \mathbb{R},$$

or to one of the indecomposable Lie algebras $\mathfrak{n}_{7,2,A}, \mathfrak{n}_{7,2,B}$.

We will first show that the existence of coclosed $\mathrm{G}_2$-structures on $\mathfrak{n}$ forces its decomposability. More precisely, we have:

**Proposition 4.4.** *Let $\mathfrak{n}$ be a 7-dimensional 2-step nilpotent Lie algebra with 2-dimensional derived algebra $\mathfrak{n}'$ and let $\varphi$ be a coclosed* $\mathrm{G}_2$*-structure on $\mathfrak{n}$. If $z_1, z_2$ is any $g_\varphi$-orthonormal basis of $\mathfrak{n}'$, then the (unit length) vector $x := \varphi(\cdot, z_1, z_2)^\sharp$ belongs to $\mathfrak{a}$.*

*Proof.* Let $\varphi$ be a $G_2$-structure on $\mathfrak{n}$ and let $\{z_1, z_2\}$ be an orthonormal basis of $\mathfrak{n}'$. As $G_2$ acts transitively on ordered pairs of orthonormal vectors on $\mathbb{R}^7$, there exists a basis $\{e_1, \dots, e_7\}$ of $\mathfrak{n}$ adapted to $\varphi$ such that $z_1 = e_5$ and $z_2 = e_6$. Then, $\varphi$ and $*_\varphi\varphi$ can be written as in (2.3), so $\varphi(\cdot, e_5, e_6) = e^7$. Consequently, the vector $x$ defined above is equal to $e_7$ and the thesis is equivalent to showing that $e_7 \lrcorner \mathrm{d}e^5 = e_7 \lrcorner \mathrm{d}e^6 = 0$.

Consider the subspace $\tilde{\mathfrak{r}} := \langle e_1, \dots, e_4\rangle \subset \mathfrak{r} = \langle e_1, \dots, e_4, e_7\rangle$. Then there exist $\alpha_k \in \Lambda^2\tilde{\mathfrak{r}}^*$ and $\theta_k \in \tilde{\mathfrak{r}}^*$, $k = 5, 6$, such that the structure equations of $\mathfrak{n}$ with respect to the basis $\{e^1, \dots, e^7\}$ of $\mathfrak{n}^*$ are the following

$$\begin{cases} \mathrm{d}e^k = 0, \ k = 1,2,3,4,7 \\ \mathrm{d}e^k = \alpha_k + \theta_k \wedge e^7, \ k = 5,6. \end{cases} \tag{4.3}$$

We thus have $e_7 \lrcorner \mathrm{d}e^k = -\theta_k$, so our aim is to show that $\theta_k = 0$ for $k = 5,\ 6$.

We orient the subspace $\tilde{\mathfrak{r}} \subset \mathfrak{n}$ by the volume form $e^{1234}$. The space $\Lambda^2\tilde{\mathfrak{r}}^*$ can be decomposed into the orthogonal direct sum of the subspaces of self-dual forms $\Lambda^2_+\tilde{\mathfrak{r}}^* := \{\sigma \in \Lambda^2\tilde{\mathfrak{r}}^* \mid *\,\sigma = \sigma\}$ and anti-self-dual forms $\Lambda^2_-\tilde{\mathfrak{r}}^* := \{\sigma \in \Lambda^2\tilde{\mathfrak{r}}^* \mid *\,\sigma = -\sigma\}$, where $*$ denotes the Hodge operator on $\tilde{\mathfrak{r}}$. We choose the following $g_\varphi$-orthogonal basis of $\Lambda^2_+\tilde{\mathfrak{r}}^*$

$$\sigma_1 = e^{13} - e^{24}, \quad \sigma_2 = -e^{14} - e^{23}, \quad \sigma_3 = e^{12} + e^{34}. \tag{4.4}$$

Note that $|\sigma_k| = \sqrt{2}$, for $k = 1,2,3$. By (2.3), we have

$$\begin{aligned} \varphi &= \sigma_1 \wedge e^5 + \sigma_2 \wedge e^6 + \sigma_3 \wedge e^7 + e^{567}, \\ *_\varphi\varphi &= e^{1234} + \sigma_1 \wedge e^{67} + \sigma_2 \wedge e^{75} + \sigma_3 \wedge e^{56}. \end{aligned} \tag{4.5}$$

Now, using (4.3), we obtain

$$\begin{aligned} \mathrm{d} *_\varphi \varphi &= \sigma_1 \wedge \mathrm{d}(e^{67}) + \sigma_2 \wedge \mathrm{d}(e^{75}) + \sigma_3 \wedge \mathrm{d}(e^{56}) \\ &= (\sigma_1 \wedge \alpha_6 - \sigma_2 \wedge \alpha_5) \wedge e^7 + \sigma_3 \wedge \alpha_5 \wedge e^6 - \sigma_3 \wedge \theta_5 \wedge e^{67} \\ &\quad - \sigma_3 \wedge \alpha_6 \wedge e^5 + \sigma_3 \wedge \theta_6 \wedge e^{57}. \end{aligned}$$

Therefore, $\varphi$ is coclosed if and only if the following equations on $\tilde{\mathfrak{r}}$ hold:

$$\sigma_1 \wedge \alpha_6 - \sigma_2 \wedge \alpha_5 = 0 = \sigma_3 \wedge \alpha_5 = \sigma_3 \wedge \alpha_6, \tag{4.6}$$

$$\sigma_3 \wedge \theta_5 = 0 = \sigma_3 \wedge \theta_6. \tag{4.7}$$

As $\sigma_3$ is a non-degenerate 2-form on $\tilde{\mathfrak{r}}$, the equations (4.7) imply that $\theta_5 = \theta_6 = 0$ as claimed. □

**Remark 4.5.** Recall that $\mathfrak{n}_{7,2,A}$ and $\mathfrak{n}_{7,2,B}$ are the only indecomposable 2-step nilpotent Lie algebras of dimension 7 having $\dim(\mathfrak{n}') = 2$. Proposition 4.4 thus provides an alternative proof of the fact that $\mathfrak{n}_{7,2,A}$ and $\mathfrak{n}_{7,2,B}$ do not admit coclosed $G_2$-structures. This result was already proved in [1, Thm. 5.1] by a case by case analysis using the classification of 7-dimensional 2-step nilpotent Lie algebras.

**Corollary 4.6.** *Let $\mathfrak{n}$ be a 7-dimensional 2-step nilpotent Lie algebra with $\dim(\mathfrak{n}') = 2$, let $\{e^1, \dots, e^7\}$ be a basis of $\mathfrak{n}^*$ for which the structure equations of $\mathfrak{n}$ are given by (4.3) and consider the metric $g$ making this frame orthonormal. Endow the 4-dimensional subspace $\tilde{\mathfrak{r}} = \langle e_1, e_2, e_3, e_4\rangle \subset \mathfrak{n}$ with the metric $g_{\tilde{\mathfrak{r}}}$ and the orientation $e^{1234}$. Then, the $G_2$-structure $\varphi$ induced by $\{e^1, \dots, e^7\}$ is coclosed if and only if $\theta_5 = \theta_6 = 0$ and there exist $a, b, c \in \mathbb{R}$ such that the self-dual parts $\alpha_5^+$, $\alpha_6^+$ of the forms $\alpha_5, \alpha_6 \in \Lambda^2\tilde{\mathfrak{r}}^*$ satisfy*

$$\begin{cases} \alpha_5^+ = a\,\sigma_1 + b\,\sigma_2, \\ \alpha_6^+ = b\,\sigma_1 + c\,\sigma_2, \end{cases} \tag{4.8}$$

*where $\sigma_1, \sigma_2$ are defined in (4.4). In addition, $\varphi$ is purely coclosed if and only if $a + c = 0$.*

*Proof.* Let $\{e^1, \ldots, e^7\}$ be a basis of $\mathfrak{n}^*$ for which the structure equations are given by (4.3), and let $g$ denote the metric making this basis orthonormal. Then, the $\mathrm{G}_2$-structure $\varphi$ induced by $\{e^1, \ldots, e^7\}$ satisfies (4.5), where $\{\sigma_1, \sigma_2, \sigma_3\}$ is the basis of self-dual forms on $\tilde{\mathfrak{r}}$ defined in (4.4).

From the proof of Proposition 4.4, we have that $\varphi$ is coclosed if and only if (4.6) and (4.7) hold. The latter is equivalent to the vanishing of $\theta_5$ and $\theta_6$. As $\{\sigma_1, \sigma_2, \sigma_3\}$ is an orthogonal basis of $\Lambda^2_+\tilde{\mathfrak{r}}^*$, (4.6) imposes constraints on the self-dual part $\alpha_k^+$ of $\alpha_k$, $k = 5, 6$. More precisely, the equations in (4.6) hold if and only if the components of $\alpha_5^+$ and $\alpha_6^+$ along $\sigma_3$ vanish and $g_\varphi(\alpha_5^+, \sigma_2) = g_\varphi(\alpha_6^+, \sigma_1)$, that is, if and only if there exist some real numbers $a, b, c$, such that (4.8) is verified.

When $\varphi$ is purely coclosed, there is an additional constraint in the system (4.8). In detail, we have

$$\begin{aligned} \mathrm{d}\varphi \wedge \varphi &= \left(\sigma_1 \wedge \alpha_5 + \sigma_2 \wedge \alpha_6 + \alpha_5 \wedge e^{67} - \alpha_6 \wedge e^{57}\right) \wedge \varphi \\ &= 2\left(\sigma_1 \wedge \alpha_5 + \sigma_2 \wedge \alpha_6\right) \wedge e^{567}. \end{aligned}$$

Thus, $\mathrm{d}\varphi \wedge \varphi = 0$ if and only if $g_\varphi(\sigma_1, \alpha_5) + g_\varphi(\sigma_2, \alpha_6) = 0$, namely if and only if $a + c = 0$ in (4.8). □

Using Corollary 4.6, we can show the following result which constitutes Theorem 1.1 for the case $\dim(\mathfrak{n}') = 2$, and part (ii) of Theorem 1.3.

**Proposition 4.7.** *Let $(\mathfrak{n}, g)$ be a 7-dimensional 2-step nilpotent metric Lie algebra with $\dim(\mathfrak{n}') = 2$.*

1) *There exists a coclosed $\mathrm{G}_2$-structure on $\mathfrak{n}$ inducing the metric $g$ if and only if $\mathfrak{a} \neq 0$.*
2) *Furthermore, there exists a purely coclosed $\mathrm{G}_2$-structure on $\mathfrak{n}$ inducing the metric $g$ if and only if there exists an oriented 4-dimensional subspace $\tilde{\mathfrak{r}} \subset \mathfrak{r}$ with $\mathrm{d}(\mathfrak{n}')^* \subset \Lambda^2\tilde{\mathfrak{r}}^*$ such that for every orthonormal basis $\{\zeta_1, \zeta_2\}$ of $(\mathfrak{n}')^*$, the self-dual components $\mathrm{d}\zeta_1^+,\ \mathrm{d}\zeta_2^+ \in \Lambda^2_+\tilde{\mathfrak{r}}^*$ of $\mathrm{d}\zeta_1,\ \mathrm{d}\zeta_2 \in \Lambda^2\tilde{\mathfrak{r}}^*$ are orthogonal and have equal norms.*

*Proof.* We first prove the direct implication in both cases. Assume that $(\mathfrak{n}, g)$, with $\dim(\mathfrak{n}') = 2$, has a coclosed $\mathrm{G}_2$-structure $\varphi$ such that $g_\varphi = g$. By the transitivity of $\mathrm{G}_2$ on orthonormal pairs of vectors in $\mathfrak{n}$, there exists a basis $\{e_1, \ldots, e_7\}$ of $\mathfrak{n}$ adapted to $\varphi$ with $\mathfrak{n}' = \langle e_5, e_6\rangle$. By Proposition 4.4, $x = e_7$ belongs to $\mathfrak{a} \subset \mathfrak{r}$, so $\mathfrak{a} \neq 0$. Moreover, the orthogonal complement $\tilde{\mathfrak{r}}$ of $x$ in $\mathfrak{r}$ is $\tilde{\mathfrak{r}} \coloneqq \langle e_1, \ldots, e_4\rangle$.

Let us consider the orientation of $\tilde{\mathfrak{r}}$ given by $e^{1234}$. If $\varphi$ is purely coclosed, by Corollary 4.6 there are some real numbers $a, b$ such that the self-dual parts $\alpha_k^+ \in \Lambda^2_+\tilde{\mathfrak{r}}^*$ of $\alpha_k \coloneqq \mathrm{d}e^k$, for $k = 5, 6$, satisfy (4.8) with $c = -a$ (recall that $\sigma_1,\ \sigma_2,\ \sigma_3$ are given by (4.4)). From this system it clearly follows that $\alpha_5^+$ and $\alpha_6^+$ are orthogonal and have the same length.

Every other orthonormal coframe $\{\zeta_1, \zeta_2\}$ of $(\mathfrak{n}')^*$ differs from $\{e_5, e_6\}$ by the action of an orthogonal matrix in $\mathrm{O}(2)$, which also describes the transformation taking the pair $\{\mathrm{d}\alpha_5^+, \mathrm{d}\alpha_6^+\}$ to $\{\mathrm{d}\zeta_1^+, \mathrm{d}\zeta_2^+\}$. Therefore, $\{\mathrm{d}\zeta_1^+, \mathrm{d}\zeta_2^+\}$ are orthogonal and have equal norms.

We now prove the converse statements.

1) Assume that $\mathfrak{a} \neq 0$, let $e_7 \in \mathfrak{a} \subset \mathfrak{r}$ be a unit vector, and choose a $g$-orthonormal basis $\{e_5, e_6\}$ of $\mathfrak{n}'$. Consider the metric induced by $g$, fix some orientation on the orthogonal complement $\tilde{\mathfrak{r}}$ of $e_7$ in $\mathfrak{r}$, and let $\alpha_k \coloneqq \mathrm{d}e^k \in \Lambda^2\tilde{\mathfrak{r}}^*$, $k = 5, 6$.

Let $\mathcal{P} \subset \Lambda^2_+\tilde{\mathfrak{r}}^*$ be a plane containing the self-dual components $\alpha_5^+, \alpha_6^+$ of $\alpha_5$ and $\alpha_6$. Using the polar decomposition, one can find an orthonormal basis in $\mathcal{P}$ with respect to which the matrix of $\alpha_5^+, \alpha_6^+$ is symmetric. Consequently, one can find two orthogonal elements $\{\sigma_1, \sigma_2\}$

in $\Lambda^2_+\tilde{\mathfrak{r}}^*$ with $|\sigma_1| = |\sigma_2| = \sqrt{2}$ such that the self-dual forms $\alpha_5^+, \alpha_6^+$ can be written as

$$\begin{cases} \alpha_5^+ = a\,\sigma_1 + b\,\sigma_2, \\ \alpha_6^+ = b\,\sigma_1 + c\,\sigma_2, \end{cases}$$

for some real numbers $a, b, c$. Since $\mathrm{SO}(4)$ acts transitively on pairs of orthogonal forms of length $\sqrt{2}$ in $\Lambda^2_+\tilde{\mathfrak{r}}^*$, we can always find an oriented orthonormal basis $\{e_1, \dots, e_4\}$ of $\tilde{\mathfrak{r}}$ such that $\sigma_1 = e^{13} - e^{24}$ and $\sigma_2 = -e^{14} - e^{23}$. Then the $\mathrm{G}_2$-structure induced by the orthonormal basis $\{e_1, \dots, e_7\}$ of $(\mathfrak{n}, g)$ is coclosed by Corollary 4.6.

2) Assume that there exists an oriented 4-dimensional subspace $\tilde{\mathfrak{r}} \subset \mathfrak{r}$ with $\mathrm{d}(\mathfrak{n}')^* \subset \Lambda^2\tilde{\mathfrak{r}}^*$ and such that for every orthonormal basis $\{\zeta_1, \zeta_2\}$ of $(\mathfrak{n}')^*$, the self-dual components $\mathrm{d}\zeta_1^+,\ \mathrm{d}\zeta_2^+ \in \Lambda^2_+\tilde{\mathfrak{r}}^*$ of $\mathrm{d}\zeta_1,\ \mathrm{d}\zeta_2 \in \Lambda^2\tilde{\mathfrak{r}}^*$ are orthogonal and have equal norms, say $\rho$.

We define $e_7$ as a unit vector in $\mathfrak{r}$ orthogonal to $\tilde{\mathfrak{r}}$, $e_5, e_6$ as the metric duals of $\zeta_1, \zeta_2$, and $\alpha_5 := \mathrm{d}\zeta_1$, $\alpha_6 := \mathrm{d}\zeta_2$. Like before, the transitivity of $\mathrm{SO}(4)$ on pairs of orthogonal forms of fixed length in $\Lambda^2_+\tilde{\mathfrak{r}}^*$ shows that there exists an oriented orthonormal basis $\{e_1, \dots, e_4\}$ of $\tilde{\mathfrak{r}}$ such that $\alpha_5^+ = \frac{\rho}{\sqrt{2}}(e^{13} - e^{24})$ and $\alpha_6^+ = \frac{\rho}{\sqrt{2}}(e^{14} + e^{23})$. Then, the system (4.8) holds for $b = 0$ and $a = -c = \frac{\rho}{\sqrt{2}}$. Moreover, the structure equations (4.3) hold for $\theta_5 = \theta_6 = 0$ since $\alpha_5, \alpha_6 \in \Lambda^2\tilde{\mathfrak{r}}^*$. Therefore, the $\mathrm{G}_2$-structure induced by the orthonormal basis $\{e_1, \dots, e_7\}$ of $\mathfrak{n}$ is purely coclosed by Corollary 4.6. □

In Section 5 we will use the criterion above together with the recent classification of 2-step nilpotent metric Lie algebras of dimension 6 with 2-dimensional derived algebra [6, 22], in order to study which of them admit compatible purely coclosed $\mathrm{G}_2$-structures.

4.3. **Case 3:** $\dim(\mathfrak{n}') = 3$. To conclude the discussion, we have to investigate the case when $\mathfrak{n}$ is a 7-dimensional 2-step nilpotent Lie algebra with $\dim(\mathfrak{n}') = 3$. We begin by studying $\mathrm{G}_2$-structures calibrating the derived algebra. Then, in Lemma 4.9, we prove that we can always restrict our study of (purely) coclosed $\mathrm{G}_2$-structures inducing a given metric to those calibrating $\mathfrak{n}'$.

By definition, a 3-dimensional subspace of $\mathfrak{n}$ is *calibrated* by $\varphi$ if and only if there exists an orthonormal basis $\{z_1, z_2, z_3\}$ of it such that $\varphi(z_1, z_2, z_3) = \pm 1$. For more information on the theory of calibrations, we refer the reader to [16].

We start by characterizing (purely) coclosed $\mathrm{G}_2$-structures calibrating $\mathfrak{n}'$ in terms of adapted bases. Consider a 7-dimensional 2-step nilpotent metric Lie algebra $(\mathfrak{n}, g)$ with $\dim(\mathfrak{n}') = 3$. Let $\{e_1, \dots, e_7\}$ be a $g$-orthonormal basis of $\mathfrak{n}$ such that $\mathfrak{r} = \langle e_1, e_2, e_3, e_4\rangle$ and $\mathfrak{n}' = \langle e_5, e_6, e_7\rangle$. The structure equations of $\mathfrak{n}$ are given by

$$\begin{cases} \mathrm{d}e^k = 0, & k = 1, 2, 3, 4, \\ \mathrm{d}e^k =: \alpha_{k-4}, & k = 5, 6, 7, \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \Lambda^2\mathfrak{r}^*$. With respect to the orientation of $\mathfrak{r}$ induced by $e^{1234}$, consider again the basis of $\Lambda^2_+\mathfrak{r}^*$:

$$\sigma_1 = e^{13} - e^{24}, \quad \sigma_2 = -(e^{14} + e^{23}), \quad \sigma_3 = e^{12} + e^{34}.$$

**Lemma 4.8.** *The* $\mathrm{G}_2$*-structure* $\varphi$ *induced by the basis* $\{e_1, \dots, e_7\}$ *is coclosed if and only if*

$$g(\sigma_i, \alpha_j) = g(\sigma_j, \alpha_i), \quad i, j \in \{1, 2, 3\}, \tag{4.9}$$

*and it is purely coclosed if and only if the following additional condition holds*

$$\sum_{i=1}^{3} g(\sigma_i, \alpha_i) = 0. \tag{4.10}$$

*Proof.* Consider the $\mathrm{G}_2$-structure $\varphi$ on $\mathfrak{n}$ induced by $\{e_1,\dots,e_7\}$. Then, $\varphi$ is given by (2.3), $g_\varphi = g$ and $\mathfrak{n}'$ is calibrated by $\varphi$. Moreover, we can write

$$*_\varphi\varphi = \frac{1}{2}\,\sigma_3^2 + \sigma_1\wedge e^{67} + \sigma_2\wedge e^{75} + \sigma_3\wedge e^{56},$$

so that

$$\mathrm{d}*_\varphi\varphi = (\sigma_2\wedge\alpha_3 - \sigma_3\wedge\alpha_2)\wedge e^5 + (\sigma_3\wedge\alpha_1 - \sigma_1\wedge\alpha_3)\wedge e^6 + (\sigma_1\wedge\alpha_2 - \sigma_2\wedge\alpha_1)\wedge e^7.$$

Thus $\varphi$ is coclosed if and only if

$$\sigma_i\wedge\alpha_j = \sigma_j\wedge\alpha_i,\quad i,j\in\{1,2,3\},$$

which, taking into account that $\sigma_i$ are self-dual forms on $\mathfrak{r}$, is equivalent to (4.9).

Moreover, since $\varphi = \sigma_1\wedge e^5 + \sigma_2\wedge e^6 + \sigma_3\wedge e^7 + e^{567}$, we readily compute

$$\varphi\wedge\mathrm{d}\varphi = 2(\sigma_1\wedge\mathrm{d}e^5 + \sigma_2\wedge\mathrm{d}e^6 + \sigma_3\wedge\mathrm{d}e^7)\wedge e^{567}, \tag{4.11}$$

so the extra condition for $\varphi$ being purely coclosed is $\sum_{i=1}^3 \sigma_i\wedge\alpha_i = 0$, which is equivalent to (4.9) since $\sigma_i$ are self-dual forms on $\mathfrak{r}$. □

We will now show that the whenever $\mathfrak{n}$ has a (purely) coclosed $\mathrm{G}_2$-structure, it also carries a (purely) coclosed $\mathrm{G}_2$-structure inducing the same metric and calibrating $\mathfrak{n}'$.

**Lemma 4.9.** *Let $(\mathfrak{n}, g)$ be a 7-dimensional 2-step nilpotent metric Lie algebra with $\dim(\mathfrak{n}') = 3$. If there exists a (purely) coclosed $\mathrm{G}_2$-structure on $\mathfrak{n}$ inducing the metric $g$, then there exists a (purely) coclosed $\mathrm{G}_2$-structure on $\mathfrak{n}$ inducing the metric $g$ and calibrating $\mathfrak{n}'$.*

*Proof.* Let $\varphi$ be a $\mathrm{G}_2$-structure on $\mathfrak{n}$ inducing the metric $g$. If $\mathfrak{n}'$ is calibrated by $\varphi$ there is nothing to show, so we can assume for the rest of the proof that $\mathfrak{n}'$ is not calibrated.

As $\mathrm{G}_2$ acts transitively on ordered pairs of orthonormal vectors on $\mathbb{R}^7$, there exists a $g$-orthonormal basis $\{e_1,\dots,e_7\}$ of $\mathfrak{n}$ adapted to $\varphi$ such that $e_6, e_7\in\mathfrak{n}'$. We denote by $\tilde e_5$ a unit vector in $\mathfrak{n}'$ orthogonal to $e_6$ and $e_7$ (determined up to sign). The stabilizer of $e_6$, $e_7$ in $\mathrm{G}_2$ also fixes $e_5$, and acts on $\mathfrak{r}$ as $\mathrm{SU}(2)$. Using the transitivity of the action of $\mathrm{SU}(2)$ on spheres in $\mathbb{R}^4$, one can assume that the $\mathfrak{r}$-component of $\tilde e_5$ is proportional to $e_4$. Consequently, there exist $\lambda,\mu\in\mathbb{R}$ with $\lambda^2+\mu^2=1$ such that $\tilde e_5 = \lambda e_4 + \mu e_5$. We denote $\tilde e_4 := \mu e_4 - \lambda e_5$ and $\tilde e_i := e_i$ for $i=1,2,3,6,7$. Then, $\mathfrak{r} = \langle \tilde e_1,\tilde e_2,\tilde e_3,\tilde e_4\rangle$, the basis $\{\tilde e_1,\dots,\tilde e_7\}$ is also $g$-orthonormal and the $\mathrm{G}_2$-structure $\tilde\varphi$ induced by it calibrates $\mathfrak{n}' = \langle\tilde e_5,\tilde e_6,\tilde e_7\rangle$.

We will show that if $\varphi$ is (purely) coclosed, then $\tilde\varphi$ is (purely) coclosed too. Expressing

$$e_4 = \mu\tilde e_4 + \lambda\tilde e_5,\qquad e_5 = -\lambda\tilde e_4 + \mu\tilde e_5 \tag{4.12}$$

and using (2.3), we have

$$\begin{aligned} *_\varphi\varphi &= e^{1234} + e^{1256} + e^{3456} + e^{1367} + e^{1457} + e^{2357} - e^{2467}\\ &= \mu\tilde e^{1234} + \lambda\tilde e^{1235} - \lambda\tilde e^{1246} + \mu\tilde e^{1256} + \tilde e^{3456} + \tilde e^{1367} + \tilde e^{1457}\\ &\quad -\lambda\tilde e^{2347} + \mu\tilde e^{2357} - \mu\tilde e^{2467} - \lambda\tilde e^{2567}. \end{aligned}$$

Denoting by $\alpha_i := \mathrm{d}\tilde e^{i+4}$ for $i=1,2,3$ and using that $\mathrm{d}\tilde e^i = 0$ for $i=1,2,3,4$, we get

$$\begin{aligned} \mathrm{d}*_\varphi\varphi =\ & -\lambda\tilde e^{123}\wedge\alpha_1 + \lambda\tilde e^{124}\wedge\alpha_2 + \mu\tilde e^{126}\wedge\alpha_1 - \mu\tilde e^{125}\wedge\alpha_2 + \tilde e^{346}\wedge\alpha_1 - \tilde e^{345}\wedge\alpha_2\\ & + \tilde e^{137}\wedge\alpha_2 - \tilde e^{136}\wedge\alpha_3 + \tilde e^{147}\wedge\alpha_1 - \tilde e^{145}\wedge\alpha_3 + \lambda\tilde e^{234}\wedge\alpha_3 + \mu\tilde e^{237}\wedge\alpha_1\\ & -\mu\tilde e^{235}\wedge\alpha_3 - \mu\tilde e^{247}\wedge\alpha_2 + \mu\tilde e^{246}\wedge\alpha_3 + \lambda\tilde e^{267}\wedge\alpha_1 - \lambda\tilde e^{257}\wedge\alpha_2 + \lambda\tilde e^{256}\wedge\alpha_3. \end{aligned}$$

Suppose now that $\varphi$ is coclosed. Since $\mathfrak{n}'$ is not calibrated by $\varphi$, we have $\lambda\neq 0$. Taking the interior product with $\tilde e_5$ and $\tilde e_6$ in the previous relation, i.e., $\tilde e_5\lrcorner\tilde e_6\lrcorner\mathrm{d}*_\varphi\varphi$, yields $\tilde e^2\wedge\alpha_3 = 0$.

Similarly, taking the interior product with $\tilde e_6$ and $\tilde e_7$, or with $\tilde e_5$ and $\tilde e_7$, we get $\tilde e^2\wedge\alpha_1 = \tilde e^2\wedge\alpha_2 = 0$. The above relation thus simplifies to

$$0 = \tilde e^{346}\wedge\alpha_1 - \tilde e^{345}\wedge\alpha_2 + \tilde e^{137}\wedge\alpha_2 - \tilde e^{136}\wedge\alpha_3 + \tilde e^{147}\wedge\alpha_1 - \tilde e^{145}\wedge\alpha_3. \tag{4.13}$$

Denoting as before

$$\sigma_1 = \tilde e^{13} - \tilde e^{24},\quad \sigma_2 = -(\tilde e^{14} + \tilde e^{23}),\quad \sigma_3 = \tilde e^{12} + \tilde e^{34},$$

and using again that $e^2\wedge\alpha_i = 0$ for $i = 1,2,3$, we readily obtain that (4.13) is equivalent to $g(\sigma_i,\alpha_j) = g(\sigma_j,\alpha_i)$ for all $i,j\in\{1,2,3\}$. By Lemma 4.8, $\tilde\varphi$ is thus coclosed.

Finally, we show that $\tilde\varphi\wedge \mathrm{d}\tilde\varphi = \varphi\wedge \mathrm{d}\varphi$. Since $\{\tilde e^5,\tilde e^6,\tilde e^7\}$ spans $\mathfrak{n}'$, (4.11) gives

$$\tilde\varphi\wedge \mathrm{d}\tilde\varphi = 2(\sigma_1\wedge\alpha_1 + \sigma_2\wedge\alpha_2 + \sigma_3\wedge\alpha_3)\wedge\tilde e^{567}. \tag{4.14}$$

The argument above shows that $\mathrm{d}(e^2\wedge\alpha_i) = -e^2\wedge \mathrm{d}\alpha_i = 0$, for $i = 1,2,3$. Using (4.12), we obtain $\mathrm{d}e^4 = \mathrm{d}(\mu\tilde e^4 + \lambda\tilde e^5) = \lambda\alpha_1$ and $\mathrm{d}e^5 = \mathrm{d}(-\lambda\tilde e^4 + \mu\tilde e^5) = \mu\alpha_1$, so we can write

$$\begin{aligned}
\varphi\wedge \mathrm{d}\varphi &= \varphi\wedge \mathrm{d}(e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245})\\
&= (e^{347} + e^{567} + e^{135} - e^{146})\wedge \mathrm{d}(e^{347} + e^{567} + e^{135} - e^{146})\\
&= (e^{347} + e^{567} + e^{135} - e^{146})\wedge(-\lambda e^{37}\wedge\alpha_1 + e^{34}\wedge\alpha_3 + \mu e^{67}\wedge\alpha_1\\
&\quad -e^{57}\wedge\alpha_2 + e^{56}\wedge\alpha_3 + \mu e^{13}\wedge\alpha_1 + \lambda e^{16}\wedge\alpha_1 - e^{14}\wedge\alpha_2)\\
&= 2(\lambda e^{13467}\wedge\alpha_1 + e^{34567}\wedge\alpha_3 + \mu e^{13567}\wedge\alpha_1 - e^{14567}\wedge\alpha_2).
\end{aligned}$$

Since $\lambda e^4 + \mu e^5 = \tilde e^5$ and $e^4\wedge e^5 = \tilde e^4\wedge\tilde e^5$, the above relation reads

$$\begin{aligned}
\varphi\wedge \mathrm{d}\varphi &= 2(\tilde e^{13567}\wedge\alpha_1 - \tilde e^{14567}\alpha_2 + \tilde e^{34567}\wedge\alpha_3)\\
&= 2(\tilde e^{13}\wedge\alpha_1 - \tilde e^{14}\wedge\alpha_2 + \tilde e^{34}\wedge\alpha_3)\wedge\tilde e^{567}\\
&= 2(\sigma_1\wedge\alpha_1 + \sigma_2\wedge\alpha_2 + \sigma_3\wedge\alpha_3)\wedge\tilde e^{567}.
\end{aligned}$$

By (4.14) we thus have $\tilde\varphi\wedge \mathrm{d}\tilde\varphi = \varphi\wedge \mathrm{d}\varphi$. So, if $\varphi$ is purely coclosed, then $\tilde\varphi$ is purely coclosed too. □

We are now ready to prove the remaining parts of the theorems stated in the Introduction. The next Proposition 4.10 gives the proof of Theorem 1.1 for the case $\dim(\mathfrak{n}') = 3$, and part (iii) of Theorem 1.3.

**Proposition 4.10.** *Let $(\mathfrak{n}, g)$ be a 7-dimensional 2-step nilpotent metric Lie algebra with* $\dim(\mathfrak{n}') = 3$.

1) *There exists a coclosed* $G_2$*-structure on $\mathfrak{n}$ inducing the metric $g$.*
2) *Furthermore, there exists a purely coclosed* $G_2$*-structure on $\mathfrak{n}$ inducing the metric $g$ if and only if for some orientation of the 4-dimensional space $\mathfrak{r}$, and for every orthonormal basis $\{\zeta_1,\zeta_2,\zeta_3\}$ of $(\mathfrak{n}')^*$, the $3\times 3$ Gram matrix $S$ of the self-dual components of their differentials in $\Lambda^2_+\mathfrak{r}^*$, defined by $S_{ij} := g(\mathrm{d}\zeta_i^+, \mathrm{d}\zeta_j^+)$, satisfies* $\mathrm{tr}^2(S) = 2\,\mathrm{tr}(S^2)$.

*Proof.*
1) Consider any $g$-orthonormal bases $\{e_1,e_2,e_3,e_4\}$ and $\{e_5,e_6,e_7\}$ of $\mathfrak{r}$ and $\mathfrak{n}'$, respectively, and denote by $\{e^5,e^6,e^7\}$ the dual basis of $(\mathfrak{n}')^*$. Fix the orientation $e^{1234}$ of $\mathfrak{r}$ and let $\{\sigma_1,\sigma_2,\sigma_3\}$ be the basis (4.4) of the space self-dual forms $\Lambda^2_+\mathfrak{r}^*$. Let $M$ be the $3\times 3$ matrix with entries $M_{ij} := g(\sigma_i, \mathrm{d}e^{j+4})$. We will show that the matrix $M$ can be made symmetric after a change of basis in $\mathfrak{n}'$.

Indeed, it follows from the polar decomposition of $M$ that there exists $P\in \mathrm{O}(3)$ such that $MP$ is symmetric. Using the matrix $P$, we construct a new orthonormal basis $\{\tilde e^5,\tilde e^6,\tilde e^7\}$ of $(\mathfrak{n}')^*$ as follows: for $j = 1,2,3$ we define $\tilde e^{j+4} = \sum_{k=1}^3 P_{kj} e^{k+4}$, and set $\alpha_j := \mathrm{d}\tilde e^{j+4}$. A straightforward computation shows that $g(\sigma_i,\alpha_j) = (MP)_{ij}$. Therefore, by Lemma 4.8, the $G_2$-structure induced by the $g$-orthonormal basis $\{e^1,e^2,e^3,e^4,\tilde e^5,\tilde e^6,\tilde e^7\}$ is coclosed.

2) Suppose that $\mathfrak{n}$ admits a purely coclosed $G_2$-structure inducing $g$. Then, by Lemma 4.9, there also exists a purely coclosed $G_2$-structure $\varphi$ calibrating $\mathfrak{n}'$ and inducing $g$. Therefore, one can find an orthonormal basis $\{e_1, \dots, e_7\}$ adapted to $\varphi$ such that $e_5, e_6, e_7$ span the derived algebra $\mathfrak{n}'$. We denote by $\{e^1, \dots, e^7\}$ the dual basis of $\mathfrak{n}^*$ and we let $\alpha_i := de^{i+4}$, for $i = 1, 2, 3$.

Consider the orientation of $\mathfrak{r}$ determined by $e^{1234}$ and the self-dual forms $\sigma_i \in \Lambda^2_+\mathfrak{r}^*$ given by (4.4). By Lemma 4.8, the matrix $M$ with entries $M_{ij} := g(\sigma_i, \alpha_j)$ is symmetric and trace-free. Since $\left\{\frac{1}{\sqrt{2}}\sigma_1, \frac{1}{\sqrt{2}}\sigma_2, \frac{1}{\sqrt{2}}\sigma_3\right\}$ forms an orthonormal basis of $\Lambda^2_+\mathfrak{r}^*$, the Gram matrix $(S_{ij}) := (g(\alpha_i^+, \alpha_j^+))$ is given by $S = \frac{1}{2}M^2$. The fact that it verifies the required condition $\mathrm{tr}^2(S) = 2\,\mathrm{tr}(S^2)$ is a consequence of the "only if" part of Lemma 4.11 below, with $P = \mathrm{I}_3$.

Moreover, for any other orthonormal coframe $\{\zeta_1, \zeta_2, \zeta_3\}$ of $(\mathfrak{n}')^*$, the Gram matrix $S'$ with entries $g(d\zeta_i^+, d\zeta_j^+)$ differs from the matrix $S$ above by conjugation with the $3\times3$ matrix expressing $\{\zeta_1, \zeta_2, \zeta_3\}$ in terms of $\{e^5, e^6, e^7\}$, so the trace condition $\mathrm{tr}^2(S') = 2\,\mathrm{tr}(S'^2)$ still holds.

Conversely, suppose that for some orientation of the 4-dimensional space $\mathfrak{r}$, and for every orthonormal basis $\{\zeta_1, \zeta_2, \zeta_3\}$ of $(\mathfrak{n}')^*$, the Gram matrix $S$ with entries $S_{ij} := g(d\zeta_i^+, d\zeta_j^+)$ satisfies $\mathrm{tr}^2(S) = 2\,\mathrm{tr}(S^2)$.

Let $\{e^1, \dots, e^4\}$ be an oriented $g$-orthonormal basis of $\mathfrak{r}^*$ with respect to the given orientation, and let $\{\zeta_1, \zeta_2, \zeta_3\}$ be a $g$-orthonormal basis of $(\mathfrak{n}')^*$. Consider the basis (4.4) of the space self-dual forms $\Lambda^2_+\mathfrak{r}^*$, and let $M$ be the matrix with entries $M_{ij} := g(\sigma_i, d\zeta_j)$. Since the 2-forms $\sigma_i$ are mutually orthogonal and have norm $\sqrt{2}$, the Gram matrix of the vectors $d\zeta_j^+ \in \Lambda^2_+\mathfrak{r}^*$, for $j = 1, 2, 3$, satisfies $S = \frac{1}{2}M^*M$, where $M^*$ denotes the transpose of $M$. By hypothesis, $S$ verifies $\mathrm{tr}^2(S) = 2\,\mathrm{tr}(S^2)$ which, by Lemma 4.11 below, implies that there exists $P \in \mathrm{O}(3)$ such that $MP$ is symmetric and trace-free.

As in the first part of the proof, we define a new orthonormal basis $\{e^5, e^6, e^7\}$ of $(\mathfrak{n}')^*$ by setting $e^{j+4} = \sum_{k=1}^3 P_{kj}\zeta_k$, for $j = 1, 2, 3$. Denoting $\alpha_j := de^{j+4}$, it is straightforward to check that $g(\sigma_i, \alpha_j) = (MP)_{ij}$, so that the $G_2$-structure induced by the $g$-orthonormal basis $\{e^1, \dots, e^7\}$ satisfies $g_\varphi = g$, and is purely coclosed by Lemma 4.8. $\square$

We now give the algebraic result required in the proof of Proposition 4.10.

**Lemma 4.11.** *Let $M \in M_3(\mathbb{R})$ and let $S := \frac{1}{2}M^*M$, where $M^*$ denotes the transpose of $M$. Then, there exists $P \in \mathrm{O}(3)$ such that $MP$ is symmetric and trace-free, if and only if $\mathrm{tr}^2(S) = 2\,\mathrm{tr}(S^2)$.*

*Proof.* Assume that $A := MP$ is symmetric and trace-free for some $P \in \mathrm{O}(3)$. Its eigenvalues are $a$, $b$ and $-(a+b)$, for some $a, b \in \mathbb{R}$. Then, $S = \frac{1}{2}M^*M = \frac{1}{2}PA^2P^*$, whence

$$\mathrm{tr}(S) = \frac{1}{2}\,\mathrm{tr}(A^2) = \frac{1}{2}(a^2 + b^2 + (a+b)^2) = a^2 + b^2 + ab,$$

and

$$\mathrm{tr}(S^2) = \frac{1}{4}\,\mathrm{tr}(A^4) = \frac{1}{4}(a^4 + b^4 + (a+b)^4) = \frac{1}{2}(a^2 + b^2 + ab)^2 = \frac{1}{2}\,\mathrm{tr}^2(S).$$

Conversely, assume that $\mathrm{tr}^2(S) = 2\,\mathrm{tr}(S^2)$ and let $\frac{1}{2}a^2, \frac{1}{2}b^2, \frac{1}{2}c^2$ denote the eigenvalues of $S$, with $a, b, c \in \mathbb{R}_{\geq 0}$. We have

$$\begin{aligned} 0 &= 8\,\mathrm{tr}(S^2) - 4\,\mathrm{tr}^2(S) = 2(a^4 + b^4 + c^4) - (a^2 + b^2 + c^2)^2 \\ &= a^4 + b^4 + c^4 - 2(a^2b^2 + b^2c^2 + c^2a^2) \\ &= (a+b+c)(a+b-c)(a-b+c)(a-b-c). \end{aligned}$$

Up to a permutation, one can thus assume that $c = a + b$. On the other hand, the polar decomposition of $M$ gives a matrix $Q \in O(3)$ such that $B := MQ$ is symmetric and positive semi-definite. Since $B^2 = (MQ)^*MQ = Q^*M^*MQ = 2Q^*SQ$, the eigenvalues of $B$ are exactly $a, b, c$, and so there exists $R \in O(3)$ such that $B = R\,\mathrm{diag}(a, b, c)R^*$. We can thus write

$$BR\,\mathrm{diag}(1, 1, -1)R^* = R\,\mathrm{diag}(a, b, -c)R^*.$$

This shows that $MP$ is symmetric and trace-free for $P := QR\,\mathrm{diag}(1, 1, -1)R^* \in O(3)$. □

Propositions 4.10, 4.7 and 4.2 complete the proofs of Theorem 1.1 and Theorem 1.3 in all cases. Notice that Theorem 1.1 can also be rephrased as follows.

**Theorem 4.12.** *Let $\mathfrak{n}$ be 7-dimensional 2-step nilpotent Lie algebra not isomorphic to $\mathfrak{n}_{7,2,A}$ or $\mathfrak{n}_{7,2,B}$. Then, for any metric $g$ on $\mathfrak{n}$ there exists a coclosed $G_2$-structure $\varphi \in \Lambda^3_+\mathfrak{n}^*$ such that $g_\varphi = g$.*

As discussed in Sect. 2.2, there is a close interplay between coclosed $G_2$-structures and half-flat SU(3)-structures. In particular, on a 7-dimensional decomposable 2-step nilpotent metric Lie algebra $(\mathfrak{n}, g) = (\tilde{\mathfrak{n}}, g_{\tilde{\mathfrak{n}}}) \oplus (\mathbb{R}, g_{\mathbb{R}})$ there exists a coclosed $G_2$-structure $\varphi$ such that $g_\varphi = g_{\tilde{\mathfrak{n}}} + g_{\mathbb{R}}$ if and only if there is a half-flat SU(3)-structure on $\tilde{\mathfrak{n}}$ with corresponding metric $g_{\tilde{\mathfrak{n}}}$. This fact allows us to deduce the following consequence of the previous result.

**Corollary 4.13.** *Every metric on a 6-dimensional 2-step nilpotent Lie algebra is induced by a half-flat* SU(3)*-structure.*

We conclude this section with a remark on calibrations. We have proved in Lemma 4.9 that if there is a coclosed $G_2$-structure inducing a given metric on a 2-step nilpotent Lie algebra with 3-dimensional derived algebra $\mathfrak{n}'$, then there is also one inducing the same metric and calibrating $\mathfrak{n}'$. In fact, with the exception of the Lie algebras $\mathfrak{n}$ isomorphic to $\mathfrak{n}_{7,3,A}$, *every* coclosed $G_2$-structure on $\mathfrak{n}$ calibrates $\mathfrak{n}'$, as the next result shows.

**Proposition 4.14.** *If $\mathfrak{n}$ is a 7-dimensional 2-step nilpotent Lie algebra with $\dim(\mathfrak{n}') = 3$ and not isomorphic to $\mathfrak{n}_{7,3,A}$, then every coclosed $G_2$-structure $\varphi$ on $\mathfrak{n}$ calibrates $\mathfrak{n}'$.*

*Proof.* Recall that a $G_2$-structure $\varphi$ calibrates $\mathfrak{n}'$ if and only if $\varphi(z_1, z_2, z_3) = \pm 1$ for some (and thus every) $g_\varphi$-orthonormal basis $\{z_1, z_2, z_3\}$ of $\mathfrak{n}'$. It is easy to check that this is equivalent to having $z_1 \lrcorner z_2 \lrcorner z_3 \lrcorner *_\varphi \varphi = 0$.

Let $\mathfrak{n}$ be a 7-dimensional 2-step nilpotent Lie algebra with $\dim(\mathfrak{n}') = 3$ and let $\varphi$ be a coclosed $G_2$-structure on it. We claim that if $\varphi$ does not calibrate $\mathfrak{n}'$, then there exits a non-zero covector $\xi \in \mathfrak{n}^*$ such that that for every $\zeta \in \mathfrak{n}^*$, $\xi \wedge \mathrm{d}\zeta = 0$.

Let $\{z_1, z_2, z_3\}$ be a $g_\varphi$-orthonormal basis of $\mathfrak{n}'$ and denote by $\{z^1, z^2, z^3\}$ the dual basis of $(\mathfrak{n}')^*$. Since $\mathfrak{n}' \subset \mathfrak{z}$, the Cartan formula shows that $z \lrcorner \mathrm{d}\alpha = -\mathrm{d}(z \lrcorner \alpha)$, for every $z \in \mathfrak{n}'$ and $\alpha \in \Lambda^k\mathfrak{n}^*$. We thus have

$$\mathrm{d}(z_1 \lrcorner z_2 \lrcorner *_\varphi \varphi) = -z_1 \lrcorner \mathrm{d}(z_2 \lrcorner *_\varphi \varphi) = z_1 \lrcorner z_2 \lrcorner (\mathrm{d} *_\varphi \varphi) = 0. \tag{4.15}$$

The 2-form $z_1 \lrcorner z_2 \lrcorner *_\varphi \varphi$ vanishes on $z_1$ and $z_2$, so decomposing $\mathfrak{n} = \mathfrak{r} \oplus \mathfrak{n}'$ it can be written as

$$z_1 \lrcorner z_2 \lrcorner *_\varphi \varphi = z^3 \wedge \xi + \gamma, \tag{4.16}$$

with $\xi \in \mathfrak{r}^*$ and $\gamma \in \Lambda^2\mathfrak{r}^*$. Then, $\mathrm{d}\xi = 0$ and $\mathrm{d}\gamma = 0$, and so (4.15)–(4.16) give

$$0 = \mathrm{d}(z^3 \wedge \xi + \gamma) = \mathrm{d}z^3 \wedge \xi.$$

Similarly, we obtain $\xi \wedge \mathrm{d}z^1 = \xi \wedge \mathrm{d}z^2 = 0$. Since the differential vanishes on $\mathfrak{r} = (\mathfrak{n}')^\perp$, this implies $\xi \wedge \mathrm{d}\zeta = 0$ for every $\zeta \in \mathfrak{n}^*$. Moreover, if $\varphi$ does not calibrate $\mathfrak{n}'$, then $\xi \neq 0$. Indeed,

$$\xi = z_3 \lrcorner (z^3 \wedge \xi + \gamma) = z_1 \lrcorner z_2 \lrcorner z_3 \lrcorner *_\varphi \varphi \neq 0,$$

thus proving our claim.

To finish the proof, it is enough to observe that if $\mathfrak{n}$ is not isomorphic to $\mathfrak{n}_{7,3,A}$ then, according to the classification given in Appendix A for $\dim(\mathfrak{n}') = 3$, there is no non-zero covector $\xi$ whose wedge product with all the differentials of covectors in $\mathfrak{n}^*$ vanishes. □

In the case $\mathfrak{n}_{7,3,A}$ not covered by this result, one can actually construct examples of coclosed $\mathrm{G}_2$-structures not calibrating $\mathfrak{n}'$.

**Example 4.15.** Consider the 7-dimensional 2-step nilpotent Lie algebra

$$\mathfrak{n}_{7,3,A} = (0,0,0,0,f^{12},f^{23},f^{24}).$$

For any non-zero real numbers $a,b,c$, we define a new coframe $\{e^1,\ldots,e^7\}$ as follows

$$e^i = f^i, \text{ for } i = 1,2,3, \text{ and } e^4 = af^7,\ e^5 = f^4,\ e^6 = -bf^6,\ e^7 = cf^5.$$

With respect to this coframe, the structure equations become $\mathrm{d}e^4 = ae^{25}$, $\mathrm{d}e^6 = -be^{23}$ and $\mathrm{d}e^7 = ce^{12}$, and the derived algebra is spanned by $e_4, e_6, e_7$. It is straightforward to check that the $\mathrm{G}_2$-structure $\varphi$ on $\mathfrak{n}_{7,3,A}$ induced by the basis $\{e_1,\ldots,e_7\}$ is always coclosed and that it is purely coclosed if and only if $a+b+c=0$. However, $\mathfrak{n}'_{7,3,A}$ is not calibrated by $\varphi$, as $\varphi(e_4,e_6,e_7)=0$.

## 5. Classification results for metrics induced by purely coclosed $\mathrm{G}_2$-structures

We now use the characterizations obtained in Section 4 to study the existence of purely coclosed $\mathrm{G}_2$-structures inducing a given metric on explicit examples of 2-step nilpotent metric Lie algebras $(\mathfrak{n}, g)$. As before, the discussion will be made according to the dimension of the derived algebra $\mathfrak{n}'$.

Along this section, we denote the symmetric product of two covectors $f^i, f^j \in \mathfrak{n}^*$ by $f^i \odot f^j := \frac{1}{2}(f^i \otimes f^j + f^j \otimes f^i)$.

### 5.1. Case 1: $\dim(\mathfrak{n}') = 1$.

We begin by describing the metrics on the Lie algebras $\mathfrak{h}_3 \oplus \mathbb{R}^4$, $\mathfrak{h}_5 \oplus \mathbb{R}^2$ and $\mathfrak{h}_7$ up to equivalence. Recall that two metrics $g, g'$ on a Lie algebra $\mathfrak{n}$ are equivalent if there exists an automorphism $F$ of $\mathfrak{n}$ such that $F^*g' = g$.

**Proposition 5.1.** *Let $\mathfrak{n}$ be a 7-dimensional 2-step nilpotent Lie algebra with $\dim(\mathfrak{n}')=1$.*

- *If $\mathfrak{n} = \mathfrak{h}_3 \oplus \mathbb{R}^4 = (0,0,0,0,0,0,f^{12})$, then any metric on $\mathfrak{n}$ is equivalent to*

$$g = r^2 f^1 \odot f^1 + f^2 \odot f^2 + f^3 \odot f^3 + f^4 \odot f^4 + f^5 \odot f^5 + f^6 \odot f^6 + f^7 \odot f^7, \tag{5.1}$$

  *for some $r > 0$.*
- *If $\mathfrak{n} = \mathfrak{h}_5 \oplus \mathbb{R}^2 = (0,0,0,0,0,0,f^{12}+f^{34})$, then any metric on $\mathfrak{n}$ is equivalent to*

$$g = r^2 f^1 \odot f^1 + f^2 \odot f^2 + s^2 f^3 \odot f^3 + f^4 \odot f^4 + f^5 \odot f^5 + f^6 \odot f^6 + f^7 \odot f^7, \tag{5.2}$$

  *for some $0 < r \leq s$.*
- *If $\mathfrak{n} = \mathfrak{h}_7 = (0,0,0,0,0,0,f^{12}+f^{34}+f^{56})$, then any metric on $\mathfrak{n}$ is equivalent to*

$$g = r^2 f^1 \odot f^1 + f^2 \odot f^2 + s^2 f^3 \odot f^3 + f^4 \odot f^4 + t^2 f^5 \odot f^5 + f^6 \odot f^6 + f^7 \odot f^7, \tag{5.3}$$

  *for some $0 < r \leq s \leq t$.*

*Moreover, the metrics belonging to the same family are pairwise non-equivalent for different values of the parameters.*

*Proof.* Let $(\mathfrak{n}, g)$ be a 7-dimensional 2-step nilpotent metric Lie algebra with 1-dimensional derived algebra. Consider the orthogonal decomposition $\mathfrak{n} = \mathfrak{r} \oplus \mathfrak{n}'$ and fix a generator $e_7 \in \mathfrak{n}'$ of unit length, with metric dual $e^7$. Let $A$ denote the skew-symmetric endomorphism of $\mathfrak{r}$ determined by $\mathrm{d}e^7$, and let $\{e_1, \dots, e_6\}$ be an orthonormal basis of $\mathfrak{r}$ such that

$$A = \begin{pmatrix} 0 & -a_1 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_3 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_5 \\ 0 & 0 & 0 & 0 & a_5 & 0 \end{pmatrix}, \tag{5.4}$$

for some $a_1 \geq a_3 \geq a_5 \geq 0$, with $a_1 \neq 0$. Let $\{e^1, \dots, e^7\}$ be the dual basis of $\{e_1, \dots, e_7\}$. This implies that $\mathrm{d}e^7 = a_1 e^{12} + a_3 e^{34} + a_5 e^{56}$, and we easily see that the Lie algebra $\mathfrak{n}$ is isomorphic to $\mathfrak{h}_3 \oplus \mathbb{R}^4$ when $a_3 = a_5 = 0$, to $\mathfrak{h}_5 \oplus \mathbb{R}^2$ when $a_3 \neq 0$ and $a_5 = 0$, and to $\mathfrak{h}_7$ if all of these coefficients are non-zero.

Consider a new basis $\{f^1, \dots, f^7\}$ of $\mathfrak{n}^*$ defined as follows: $f^i := e^i$, for $i = 2, 4, 6, 7$, and

$$f^i := \begin{cases} a_i e^i, & \text{if } a_i \neq 0, \\ e^i, & \text{if } a_i = 0, \end{cases}$$

for $i = 1, 3, 5$. Then, we have $\mathrm{d}f^7 = f^{12} + \varepsilon_3 f^{34} + \varepsilon_5 f^{56}$, where $\varepsilon_3, \varepsilon_5 \in \{0, 1\}$, $\varepsilon_3 \geq \varepsilon_5$, and $\varepsilon_k$ vanishes whenever $a_k$ does. We can thus assume that this basis is the one defining the Lie algebra structure. It is now immediate to check that the expression of the metric $g$ with respect to the basis $\{f^1, \dots, f^7\}$ is the one given in the statement of the proposition, where $r, s, t$ coincide with $(a_1)^{-1}, (a_3)^{-1}, (a_5)^{-1}$, respectively, when $a_1, a_3, a_5$ are non-zero, and they are equal to one otherwise.

To conclude the proof, we need to show that the metrics belonging to the same family are pairwise non-equivalent for different values of the parameters. Consider the basis $\{f_1, \dots, f_7\}$ of $\mathfrak{n}$ with dual basis $\{f^1, \dots, f^7\}$. Let $g, g'$ be two metrics on $\mathfrak{n}$ belonging to the same family, and let $F$ be an automorphism of $\mathfrak{n}$ such that $F^*g' = g$. Then, $F(f_7) = \pm f_7$ and $F$ preserves the subspace $U := \langle f_1, \dots, f_6 \rangle$. Moreover, the endomorphisms $A, A'$ of $U$ corresponding to $\mathrm{d}f^7$ by means of $g$ and $g'$, respectively, verify $FAF^{-1} = \pm A'$. Hence, $A^2$ and $(A')^2$ have the same eigenvalues, which correspond to the parameters of the metric. □

Now, for each 2-step nilpotent metric Lie algebra $(\mathfrak{n}, g)$ with $\dim(\mathfrak{n}') = 1$ and $g$ as in Proposition 5.1, we study the existence of a purely coclosed G$_2$-structure $\varphi$ inducing $g$.

**Proposition 5.2.** *A metric Lie algebra $(\mathfrak{n}, g)$ with $\dim(\mathfrak{n}') = 1$ and $g$ as in Proposition 5.1 admits a purely coclosed G$_2$-structure $\varphi$ such that $g_\varphi = g$ if and only if one of the following conditions holds:*

*a)* $\mathfrak{n} \cong \mathfrak{h}_5 \oplus \mathbb{R}^2$ *and $g$ is as in* (5.2) *with $r = s$;*
*b)* $\mathfrak{n} \cong \mathfrak{h}_7$ *and $g$ is as in* (5.3) *with $\frac{1}{r} = \frac{1}{s} + \frac{1}{t}$.*

*Proof.* We already know from Corollary 4.3 that $\mathfrak{h}_3 \oplus \mathbb{R}^4$ does not admit any purely coclosed G$_2$-structure. For the remaining two cases, we use the criterion obtained in Proposition 4.2 to prove the assertion.

Suppose that $\mathfrak{n}$ is isomorphic to $\mathfrak{h}_7$ and that $g$ has the form (5.3) with respect to a suitable basis $\{f^1, \dots, f^7\}$ of $\mathfrak{n}^*$ such that $\mathrm{d}f^7 = f^{12} + f^{34} + f^{56}$.

Then, the matrix associated to $\mathrm{d}f^7$ with respect to the basis $\{f_1,\dots,f_6\}$ is given by

$$A=\begin{pmatrix} 0 & -r^{-2} & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -s^{-2} & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -t^{-2}\\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

and we have

$$\mathrm{tr}(A^4)-\frac{1}{4}\,\mathrm{tr}^2(A^2)=\left(\frac{1}{r}+\frac{1}{s}+\frac{1}{t}\right)\left(\frac{1}{r}+\frac{1}{s}-\frac{1}{t}\right)\left(\frac{1}{r}-\frac{1}{s}+\frac{1}{t}\right)\left(\frac{1}{r}-\frac{1}{s}-\frac{1}{t}\right).$$

Since $0<r\leq s\leq t$, we see that the expression above is zero if and only if $\frac{1}{r}=\frac{1}{s}+\frac{1}{t}$.

A similar discussion shows that on $\mathfrak{h}_5\oplus\mathbb{R}^2$ there exists a purely coclosed $\mathrm{G}_2$-structure inducing the metric given in (5.2) if and only if $r=s$. □

This classification allows us to determine whether the nilsoliton metrics on 2-step nilpotent Lie algebras with 1-dimensional derived algebra are induced by purely coclosed $\mathrm{G}_2$-structures. Indeed, from [19, Thm. 5.1], the nilsoliton metrics on $\mathfrak{h}_5\oplus\mathbb{R}^2$ correspond to $r=s$ in (5.2), and by [11], the nilsoliton metrics on $\mathfrak{h}_7$ correspond to $r=s=t$ in (5.3). We thus obtain the following consequence of Proposition 5.2.

**Corollary 5.3.** *On $\mathfrak{h}_5\oplus\mathbb{R}^2$ there exist purely coclosed $G_2$-structures inducing a nilsoliton metric, while the nilsoliton metrics on $\mathfrak{h}_7$ are not induced by any purely coclosed $G_2$-structure.*

5.2. **Case 2:** $\dim(\mathfrak{n}')=2$**.** Let $(\mathfrak{n},g)$ be a 7-dimensional 2-step nilpotent metric Lie algebra with $\dim(\mathfrak{n}')=2$, and suppose that there is a unit vector $x\in\mathfrak{a}$ so that $\mathfrak{n}$ splits into the orthogonal direct sum $(\mathfrak{n},g)=(\tilde{\mathfrak{n}},g_{\tilde{\mathfrak{n}}})\oplus(\langle x\rangle,g_{\langle x\rangle})$ (cf. Proposition 4.4). In this case, the 4-dimensional orthogonal complement of $\mathfrak{n}'\oplus\langle x\rangle$ in $\mathfrak{n}$ is in fact the orthogonal complement $\tilde{\mathfrak{r}}$ of $\tilde{\mathfrak{n}}'=\mathfrak{n}'$ in $\tilde{\mathfrak{n}}$. Moreover, orthonormal coframes of $((\mathfrak{n}')^*,g_{\mathfrak{n}'})$ correspond to orthonormal coframes of $((\tilde{\mathfrak{n}}')^*,g_{\tilde{\mathfrak{n}}'})$, and their differentials coincide as 2-forms in $\Lambda^2\tilde{\mathfrak{r}}^*$.

Therefore, by Proposition 4.7 2), determining the 2-step nilpotent metric Lie algebras $(\mathfrak{n},g)$ with $\dim(\mathfrak{n}')=2$ admitting purely coclosed $\mathrm{G}_2$-structures inducing $g$ is equivalent to determining the 6-dimensional 2-step nilpotent metric Lie algebras $(\tilde{\mathfrak{n}},g_{\tilde{\mathfrak{n}}})$ admitting an orientation of $\tilde{\mathfrak{r}}$ and an orthogonal coframe $\{\zeta_1,\zeta_2\}$ of $((\tilde{\mathfrak{n}}')^*,g_{\tilde{\mathfrak{n}}'})$ for which the self-dual parts of their differentials are orthogonal and have equal norms in $\Lambda^2_+\tilde{\mathfrak{r}}^*$.

Any 6-dimensional 2-step nilpotent Lie algebra is isomorphic to one of $\mathfrak{h}_3^{\mathbb{C}}$, $\mathfrak{h}_3\oplus\mathfrak{h}_3$, $\mathfrak{n}_{6,2}$ and $\mathfrak{n}_{5,2}\oplus\mathbb{R}$. The metrics on these Lie algebras were classified, up to automorphism, by Di Scala in [6] (for $\mathfrak{h}_3^{\mathbb{C}}$) and by Reggiani and Vittone in [22] (for the remaining cases). We recall their classification results here.

**Proposition 5.4** ([6, 22])**.** *Let $\mathfrak{n}$ be a 6-dimensional 2-step nilpotent Lie algebra with $\dim(\mathfrak{n}')=2$.*

- *If $\mathfrak{n}=\mathfrak{h}_3^{\mathbb{C}}=(0,0,0,0,f^{13}-f^{24},f^{14}+f^{23})$, then any metric on $\mathfrak{n}$ is equivalent to*
$$g=f^1\odot f^1+rf^2\odot f^2+f^3\odot f^3+sf^4\odot f^4+Ef^5\odot f^5+2Ff^5\odot f^6+Gf^6\odot f^6, \quad (5.5)$$
*for some $0<s\leq r\leq 1$ and $E,F,G\geq 0$ with $EG-F^2>0$.*
- *If $\mathfrak{n}=\mathfrak{h}_3\oplus\mathfrak{h}_3=(0,0,0,0,f^{12},f^{34})$, then any metric on $\mathfrak{n}$ is equivalent to*
$$g=\sum_{i=1}^{4}f^i\odot f^i+2af^1\odot f^3+2bf^2\odot f^4+Ef^5\odot f^5+2Ff^5\odot f^6+Gf^6\odot f^6, \quad (5.6)$$
*for some $0\leq a\leq b<1$ and $E,F,G\geq 0$ with $EG-F^2>0$.*

- *If* $\mathfrak{n} = \mathfrak{n}_{6,2} = (0,0,0,0,f^{12},f^{14}+f^{23})$, *then any metric on* $\mathfrak{n}$ *is equivalent to*

$$g = \sum_{i=1}^{3} f^i \odot f^i + r f^4 \odot f^4 + E f^5 \odot f^5 + 2F f^5 \odot f^6 + G f^6 \odot f^6, \tag{5.7}$$

*for some* $0 < r \leq 1$, *and* $E, F, G \geq 0$ *with* $EG - F^2 > 0$.
- *If* $\mathfrak{n} = \mathfrak{n}_{5,2} \oplus \mathbb{R} = (0,0,0,0,f^{12},f^{13})$, *then any metric on* $\mathfrak{n}$ *is equivalent to*

$$g = \sum_{i=1}^{4} f^i \odot f^i + E f^5 \odot f^5 + G f^6 \odot f^6, \tag{5.8}$$

*for some* $0 < E \leq G$.

*Moreover, the metrics belonging to the same family are pairwise non-equivalent for different values of the parameters.*

Thus, up to automorphism, we may assume that $(\tilde{\mathfrak{n}}, g_{\tilde{\mathfrak{n}}})$ is one of the metric Lie algebras described in the previous proposition, and using part 2) of Proposition 4.7, we can characterize the existence of purely coclosed $G_2$-structures inducing the given metric $g = g_{\tilde{\mathfrak{n}}} + x^\flat \odot x^\flat$ on $\mathfrak{n} = \tilde{\mathfrak{n}} \oplus \langle x \rangle$ in terms of the parameters appearing in $g_{\tilde{\mathfrak{n}}}$.

**Proposition 5.5.** *The metric Lie algebra* $(\mathfrak{h}_3^{\mathbb{C}} \oplus \mathbb{R}, g)$ *with* $g_{\mathfrak{h}_3^{\mathbb{C}}}$ *as in Proposition 5.4 admits a purely coclosed* $G_2$*-structure* $\varphi$ *such that* $g_\varphi = g$ *if and only if one of the following set of conditions hold*

*a)* $r = s = 1$ *and any choice of* $E, F, G \geq 0$ *with* $EG - F^2 > 0$;

*b)* $0 < r = s < 1$ *with* $F = 0$ *and* $G = E\left(\frac{\sqrt{rs}+1}{\sqrt{r}+\sqrt{s}}\right)^2$;

*c)* $0 < s < r \leq 1$ *with* $F = 0$ *and either* $G = E\left(\frac{\sqrt{rs}+1}{\sqrt{r}+\sqrt{s}}\right)^2$ *or* $G = E\left(\frac{\sqrt{rs}-1}{\sqrt{r}-\sqrt{s}}\right)^2$.

*Proof.* Starting with the basis $\{f^1, \ldots, f^6\}$ of $\tilde{\mathfrak{n}}^* := (\mathfrak{h}_3^{\mathbb{C}})^*$ given in Proposition 5.4, we obtain the following $g$-orthonormal basis:

$$e^1 = f^1, \quad e^2 = \sqrt{r}\, f^2, \quad e^3 = f^3, \quad e^4 = \sqrt{s}\, f^4,$$
$$e^5 = \sqrt{E}\, f^5 + \frac{F}{\sqrt{E}}\, f^6, \quad e^6 = \sqrt{\frac{EG - F^2}{E}}\, f^6,$$

where $\{e^5, e^6\}$ is an orthonormal coframe of $(\tilde{\mathfrak{n}}')^*$ and $\tilde{\mathfrak{r}}^* = \langle e^1, e^2, e^3, e^4 \rangle$. Consequently, the expressions of $\alpha_k := de^k$, for $k = 5, 6$, are the following:

$$\begin{aligned} \alpha_5 &= \sqrt{E}\, e^{13} + \frac{F}{\sqrt{Es}}\, e^{14} + \frac{F}{\sqrt{Er}}\, e^{23} - \sqrt{\frac{E}{rs}}\, e^{24}, \\ \alpha_6 &= \sqrt{\frac{EG - F^2}{Es}}\, e^{14} + \sqrt{\frac{EG - F^2}{Er}}\, e^{23}. \end{aligned} \tag{5.9}$$

With respect to the orientation $e^{1234}$ of $\tilde{\mathfrak{r}}^*$ and the oriented coframe $\{e^1, e^2, e^3, e^4\}$, the basis of self-dual 2-forms on $\tilde{\mathfrak{r}}^*$ introduced in (4.4) is

$$\sigma_1^+ = e^{13} - e^{24}, \quad \sigma_2^+ = -e^{14} - e^{23}, \quad \sigma_3^+ = e^{12} + e^{34}. \tag{5.10}$$

When the opposite orientation of $\tilde{\mathfrak{r}}^*$ is considered, the basis (4.4) with respect to the oriented coframe $\{-e^1, e^2, e^3, e^4\}$ reads

$$\sigma_1^- = -e^{13} - e^{24}, \quad \sigma_2^- = e^{14} - e^{23}, \quad \sigma_3^- = -e^{12} + e^{34}. \tag{5.11}$$

For $k=5,6$, we denote by $\alpha_k^+$ and $\alpha_k^-$ the self-dual parts of $\alpha_k$ with respect to the orientation $e^{1234}$ of $\tilde{\mathfrak{r}}^*$ and its opposite, respectively. By using (5.9)–(5.11) we obtain

$$\alpha_5^+ = B_1^+\,\sigma_1^+ + B_2^+\,\sigma_2^+,\quad \alpha_5^- = B_1^-\,\sigma_1^- + B_2^-\,\sigma_2^-,\quad \alpha_6^+ = A_2^+\,\sigma_2^+,\quad \alpha_6^- = A_2^-\,\sigma_2^-,$$

where

$$B_1^+ = \frac{1}{2}\frac{\sqrt{E}(\sqrt{rs}+1)}{\sqrt{rs}},\quad B_2^+ = -\frac{1}{2}\frac{F(\sqrt{r}+\sqrt{s})}{\sqrt{Ers}},\quad A_2^+ = -\frac{1}{2}\frac{(\sqrt{r}+\sqrt{s})\sqrt{EG-F^2}}{\sqrt{Ers}},$$
$$B_1^- = -\frac{1}{2}\frac{\sqrt{E}(\sqrt{rs}-1)}{\sqrt{rs}},\quad B_2^- = \frac{1}{2}\frac{F(\sqrt{r}-\sqrt{s})}{\sqrt{Ers}},\quad A_2^- = \frac{1}{2}\frac{(\sqrt{r}-\sqrt{s})\sqrt{EG-F^2}}{\sqrt{Ers}}.$$

Therefore, $\alpha_5^+$ and $\alpha_6^+$ are orthogonal with the same norm if and only if

$$\begin{cases} A_2^+B_2^+ = 0,\\ |A_2^+| = \sqrt{(B_1^+)^2+(B_2^+)^2}. \end{cases}$$

As $r\geq s>0$ and $EG-F^2>0$, the first equation holds if and only if $F=0$. Substituting this value in the second equation gives $G = E\left(\frac{\sqrt{rs}+1}{\sqrt{r}+\sqrt{s}}\right)^2$.

The self-dual parts of $\alpha_5$ and $\alpha_6$ with respect to the orientation $-e^{1234}$ are orthogonal and have the same norm if and only if

$$\begin{cases} A_2^-B_2^- = 0,\\ |A_2^-| = \sqrt{(B_1^-)^2+(B_2^-)^2}. \end{cases}$$

The first equation holds if either $F=0$ and $r\neq s$, or $r=s$. In the former case, we see that the second equation holds if and only if $G = E\left(\frac{\sqrt{rs}-1}{\sqrt{r}-\sqrt{s}}\right)^2$. In the latter case, the second equation holds if and only if $r=s=1$. Notice that $r=s=1$ implies $\alpha_5^- = \alpha_6^- = 0$. □

**Proposition 5.6.** *The metric Lie algebra $(\mathfrak{h}_3\oplus\mathfrak{h}_3\oplus\mathbb{R}, g)$ with $g_{\mathfrak{h}_3\oplus\mathfrak{h}_3}$ as in Proposition 5.4 admits a purely coclosed $\mathrm{G}_2$-structure inducing $g$ if and only if $(ab\pm\sqrt{(1-a^2)(1-b^2)})^2<1$, $G=E$ and $F=-E\left(ab\pm\sqrt{(1-a^2)(1-b^2)}\right)$. In particular, there are no purely coclosed $\mathrm{G}_2$-structures inducing a metric $g$ for which the decomposition $\mathfrak{h}_3\oplus\mathfrak{h}_3\oplus\mathbb{R}$ is orthogonal.*

*Proof.* Consider the following $g$-orthonormal basis of $\tilde{\mathfrak{n}}^* := (\mathfrak{h}_3\oplus\mathfrak{h}_3)^*$

$$e^1 = f^1 + a\,f^3,\quad e^2 = f^2 + b\,f^4,\quad e^3 = \sqrt{1-a^2}\,f^3,\quad e^4 = \sqrt{1-b^2}\,f^4,$$
$$e^5 = \sqrt{E}\,f^5 + \frac{F}{\sqrt{E}}\,f^6,\quad e^6 = \sqrt{\frac{EG-F^2}{E}}\,f^6,$$

where $\{e^5,e^6\}$ is an orthonormal coframe of $(\tilde{\mathfrak{n}}')^*$ and $\tilde{\mathfrak{r}}^* = \langle e^1,e^2,e^3,e^4\rangle$. The expressions of $\alpha_k := de^k$, for $k=5,6$, are the following:

$$\alpha_5 = \sqrt{E}\,e^{12} - b\sqrt{\frac{E}{1-b^2}}\,e^{14} + a\sqrt{\frac{E}{1-a^2}}\,e^{23} + \frac{abE+F}{\sqrt{E(1-a^2)(1-b^2)}}\,e^{34},$$
$$\alpha_6 = \sqrt{\frac{EG-F^2}{E(1-a^2)(1-b^2)}}\,e^{34}.$$

As in the previous proposition, depending on the two possible orientations and the corresponding oriented coframes of $\tilde{\mathfrak{r}}$, the bases of self-dual forms are given by (5.10) or (5.11).

From the expressions of $\alpha_5$ and $\alpha_6$, we thus obtain that their self-dual parts in the two cases are

$$\alpha_5^+ = B_2^+\,\sigma_2^+ + B_3^+\,\sigma_3^+, \quad \alpha_5^- = B_2^-\,\sigma_2^- + B_3^-\,\sigma_3^-, \quad \alpha_6^+ = A_3^+\,\sigma_3^+, \quad \alpha_6^- = A_3^-\,\sigma_3^-,$$

where

$$A_3^+ = \frac{1}{2}\sqrt{\frac{EG-F^2}{E(1-a^2)(1-b^2)}} = A_3^-,$$

and

$$B_2^+ = \frac{1}{2}\frac{\sqrt{E}(b\sqrt{1-a^2}-a\sqrt{1-b^2})}{\sqrt{(1-a^2)(1-b^2)}}, \quad B_3^+ = \frac{1}{2}\frac{E(\sqrt{(1-a^2)(1-b^2)}+ab)+F}{\sqrt{E(1-a^2)(1-b^2)}},$$
$$B_2^- = -\frac{1}{2}\frac{\sqrt{E}(b\sqrt{1-a^2}+a\sqrt{1-b^2})}{\sqrt{(1-a^2)(1-b^2)}}, \quad B_3^- = -\frac{1}{2}\frac{E(\sqrt{(1-a^2)(1-b^2)}-ab)-F}{\sqrt{E(1-a^2)(1-b^2)}}.$$

Thus, we see that $\alpha_5^+$ and $\alpha_6^+$ are orthogonal and have the same norm if and only if

$$\begin{cases} A_3^+B_3^+ = 0, \\ |A_3^+| = \sqrt{(B_2^+)^2+(B_3^+)^2}, \end{cases} \iff \begin{cases} F + E\left(\sqrt{(1-a^2)(1-b^2)}+ab\right) = 0, \\ \sqrt{EG-F^2} = E\left|\left(b\sqrt{1-a^2}-a\sqrt{1-b^2}\right)\right|. \end{cases}$$

Now, the expression of $F$ can be deduced from the first equation. We know that $0 \le a \le b < 1$. If $b = a$, the right hand side of the second equation would be zero. Assuming then $b > a$ and plugging the value of $F$ in the second equation and making some computations gives $G = E$. Finally, the condition $EG - F^2 > 0$ is equivalent to $(ab + \sqrt{(1-a^2)(1-b^2)})^2 < 1$.

If we consider the opposite orientation of $\tilde{\mathfrak{r}}^*$, we see that the self-dual parts of $\alpha_5$ and $\alpha_6$ are orthogonal with the same norm if and only if

$$\begin{cases} A_3^-B_3^- = 0, \\ |A_3^-| = \sqrt{(B_2^-)^2+(B_3^-)^2}, \end{cases} \iff \begin{cases} E\left(\sqrt{(1-a^2)(1-b^2)}-ab\right) - F = 0, \\ \sqrt{EG-F^2} = E\left(b\sqrt{1-a^2}+a\sqrt{1-b^2}\right), \end{cases}$$

and the thesis follows.

To conclude the proof, it is sufficient to observe that any metric for which the decomposition $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R}$ is orthogonal must satisfy $a = b = 0$. This is not possible under either of the constraints $(ab \pm \sqrt{(1-a^2)(1-b^2)})^2 < 1$. □

**Proposition 5.7.** *The metric Lie algebra* $(\mathfrak{n}_{6,2} \oplus \mathbb{R}, g)$ *with* $g_{\mathfrak{n}_{6,2}}$ *as in Proposition 5.4 admits a purely coclosed* G$_2$*-structure* $\varphi$ *such that* $g_\varphi = g$ *if and only if* $F = 0$ *and either* $G = E\frac{r}{(\sqrt{r}+1)^2}$ *and* $0 < r \le 1$ *or* $G = E\frac{r}{(\sqrt{r}-1)^2}$ *with* $0 < r < 1$.

*Proof.* Consider the following $g$-orthonormal basis of $\tilde{\mathfrak{n}}^* := (\mathfrak{n}_{6,2})^*$

$$e^1 = f^1, \quad e^2 = f^2, \quad e^3 = f^3, \quad e^4 = \sqrt{r}\,f^4,$$
$$e^5 = \sqrt{E}\,f^5 + \frac{F}{\sqrt{E}}\,f^6, \quad e^6 = \sqrt{\frac{EG-F^2}{E}}\,f^6.$$

The pair $\{e^5, e^6\}$ is an orthonormal coframe of $(\tilde{\mathfrak{n}}')^*$ and $\tilde{\mathfrak{r}}^* = \langle e^1, e^2, e^3, e^4\rangle$. The expressions of $\alpha_k := \mathrm{d}e^k$, for $k = 5, 6$, are the following:

$$\alpha_5 = \sqrt{E}\,e^{12} + \frac{F}{\sqrt{Er}}\,e^{14} + \frac{F}{\sqrt{E}}\,e^{23}, \quad \alpha_6 = \sqrt{\frac{EG-F^2}{Er}}\,e^{14} + \sqrt{\frac{EG-F^2}{E}}\,e^{23}.$$

The self-dual parts of $\alpha_5$ and $\alpha_6$ with respect to the orientations $e^{1234}$ and $-e^{1234}$ of $\tilde{\mathfrak{r}}$, and the corresponding oriented bases (5.10) and (5.11), are the following

$$\alpha_5^+ = B_2^+\,\sigma_2^+ + B_3^+\,\sigma_3^+\,,\quad \alpha_5^- = B_2^-\,\sigma_2^- + B_3^-\,\sigma_3^-\,,\quad \alpha_6^+ = A_2^+\,\sigma_2^+\,,\quad \alpha_6^- = A_2^-\,\sigma_2^-\,,$$

where

$$B_2^+ = -\frac{1}{2}\frac{F(\sqrt{r}+1)}{\sqrt{Er}},\quad B_3^+ = \frac{1}{2}\sqrt{E},\quad A_2^+ = -\frac{1}{2}\frac{(\sqrt{r}+1)\sqrt{EG-F^2}}{\sqrt{Er}},$$
$$B_2^- = -\frac{1}{2}\frac{F(\sqrt{r}-1)}{\sqrt{Er}},\quad B_3^- = -\frac{1}{2}\sqrt{E},\quad A_2^- = -\frac{1}{2}\frac{(\sqrt{r}-1)\sqrt{EG-F^2}}{\sqrt{Er}}.$$

Now, $\alpha_5^+$ and $\alpha_6^+$ are orthogonal with the same length if and only if

$$\begin{cases} A_2^+ B_2^+ = 0, \\ |A_2^+| = \sqrt{(B_2^+)^2 + (B_3^+)^2}. \end{cases}$$

Since $0 < r \leq 1$, the first equation gives $F = 0$. From the second equation we then get $G = E\frac{r}{(\sqrt{r}+1)^2}$.

The forms $\alpha_5^-$ and $\alpha_6^-$ are orthogonal with the same length if and only if

$$\begin{cases} A_2^- B_2^- = 0, \\ |A_2^-| = \sqrt{(B_2^-)^2 + (B_3^-)^2}. \end{cases}$$

The first equation is satisfied if either $F = 0$ or $r = 1$. In the first case, solving the system we obtain $r \neq 1$ and $G = E\frac{r}{(\sqrt{r}-1)^2}$. In the second case, we get $E = 0$, a contradiction. □

**Proposition 5.8.** *The metric Lie algebra* $(\mathfrak{n}_{5,2}\oplus\mathbb{R}^2, g)$ *with* $g_{\mathfrak{n}_{5,2}\oplus\mathbb{R}}$ *as in Proposition 5.4 admits a purely coclosed* $\mathrm{G}_2$*-structure inducing the metric* $g$ *if and only if* $G = E$*.*

*Proof.* We choose the following $g$-orthonormal basis of $\tilde{\mathfrak{n}}^* := (\mathfrak{n}_{5,2}\oplus\mathbb{R})^*$

$$e^1 = f^1,\quad e^2 = f^2,\quad e^3 = f^3,\quad e^4 = f^4,\quad e^5 = \sqrt{E}\,f^5,\quad e^6 = \sqrt{G}\,f^6,$$

where $\{e^5, e^6\}$ is an orthonormal coframe of $(\tilde{\mathfrak{n}}')^*$ and $\tilde{\mathfrak{r}}^* = \langle e^1, e^2, e^3, e^4\rangle$. Then,

$$\alpha_5 = \sqrt{E}\,e^{12},\quad \alpha_6 = \sqrt{G}\,e^{13}.$$

Depending on the two possible orientations of $\tilde{\mathfrak{r}}$, we see that the self-dual parts of $\alpha_5$ and $\alpha_6$, with respect to the bases given by (5.10) and (5.11), are

$$\alpha_5^+ = \frac{\sqrt{E}}{2}\,\sigma_3^+\,,\quad \alpha_5^- = -\frac{\sqrt{E}}{2}\,\sigma_3^-\,,\quad \alpha_6^+ = \frac{\sqrt{G}}{2}\,\sigma_1^+\,,\quad \alpha_6^- = -\frac{\sqrt{G}}{2}\,\sigma_1^-\,.$$

Both $\alpha_5^+, \alpha_6^+$ and $\alpha_5^-, \alpha_6^-$ are orthogonal. Moreover, they have the same norm if and only if $G = E$. □

By [24, Thm. 3.1], the nilsoliton metrics correspond (up to automorphism and scaling) to the following values of the parameters in Proposition 5.4:

- $\mathfrak{h}_3^{\mathbb{C}}$: $r = s = E = G = 1$, $F = 0$;
- $\mathfrak{h}_3 \oplus \mathfrak{h}_3$: $a = b = F = 0$, $E = G = 1$;
- $\mathfrak{n}_{6,2}$: $r = E = G = 1$, $F = 0$;
- $\mathfrak{n}_{5,2}\oplus\mathbb{R}$: $E = G = 1$.

Consequently, we have the following.

**Corollary 5.9.** *On the* 2*-step nilpotent Lie algebras* $\mathfrak{h}_3^{\mathbb{C}}\oplus\mathbb{R}$ *and* $\mathfrak{n}_{5,2}\oplus\mathbb{R}^2$ *there exist purely coclosed* $\mathrm{G}_2$*-structures inducing a nilsoliton metric, while the Lie algebras* $\mathfrak{h}_3\oplus\mathfrak{h}_3\oplus\mathbb{R}$ *and* $\mathfrak{n}_{6,2}\oplus\mathbb{R}$ *do not admit any such structure.*

We conclude this subsection by giving explicit examples of purely coclosed $G_2$-structures on each of the 7-dimensional decomposable 2-step nilpotent Lie algebras with 2-dimensional derived algebra. The given structures satisfy the criteria obtained in Propositions 5.5, 5.6, 5.7 and 5.8.

**Example 5.10.** On the Lie algebras listed below, the $G_2$-structures induced by the following coframes are purely coclosed:

- $\mathfrak{h}_3^{\mathbb{C}} \oplus \mathbb{R} = \big(0,0,0,0,f^{13}-f^{24},f^{14}+f^{23},0\big)$ with the coframe
$$\left\{f^1,\ \sqrt{r}\,f^2,\ f^3,\ \sqrt{s}f^4,\ \sqrt{E}\,f^5,\ \sqrt{E}\frac{\sqrt{rs}+1}{\sqrt{r}+\sqrt{s}}\,f^6,\ f^7\right\},$$
for any $0<s<r\leq 1$ and $E>0$.
- $\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R} = \big(0,0,0,0,f^{12},f^{34},0\big)$ with the coframe
$$\left\{f^1+a\,f^3,\ -\sqrt{1-a^2}\,f^3,\ f^2+b\,f^4,\ \sqrt{1-b^2}\,f^4,\ \sqrt{E}\,f^5+\frac{F}{\sqrt{E}}\,f^6,\ \sqrt{\frac{E^2-F^2}{E}}\,f^6,\ f^7\right\},$$
where $(ab+\sqrt{(1-a^2)(1-b^2)})^2<1$, and $F=-E\left(ab+\sqrt{(1-a^2)(1-b^2)}\right)$.
- $\mathfrak{n}_{6,2} \oplus \mathbb{R} = \big(0,0,0,0,f^{12},f^{14}+f^{23},0\big)$ with the coframe
$$\left\{f^1,\ -f^3,\ f^2,\ \sqrt{r}\,f^4,\ \frac{\sqrt{Er}}{\sqrt{r}+1}\,f^6, -\sqrt{E}\,f^5, f^7\right\},$$
for any $0<r\leq 1$ and $E>0$.
- $\mathfrak{n}_{5,2} \oplus \mathbb{R}^2 = \big(0,0,0,0,f^{12},f^{13},0\big)$ with the coframe
$$\left\{f^1,\ f^4,\ f^3,\ f^2,\ \sqrt{E}\,f^6,\ \sqrt{E}\,f^5,\ f^7\right\},$$
with $E>0$.

In each case, the fact that the induced $G_2$-structure is purely coclosed, follows from Corollary 4.6.

5.3. **Case 3:** $\dim(\mathfrak{n}')=3$**.** Currently, no classification of the equivalence classes of metrics on 7-dimensional 2-step nilpotent Lie algebras with 3-dimensional derived algebra is available. We will therefore restrict ourselves to constructing, on each such algebra, a purely coclosed $G_2$-structure, as well as a metric which is not compatible with any purely coclosed $G_2$-structure. For the existence part we will use Lemma 4.8, and for the non-existence part we will apply Proposition 4.10.

Let $\mathfrak{n}$ be a 7-dimensional 2-step nilpotent Lie algebra with 3-dimensional derived algebra $\mathfrak{n}'$. We recall the following notation used in Section 4.3. Assume that $\mathcal{B}=\{e^1,\dots,e^7\}$ is a coframe of $\mathfrak{n}^*$ such that $e^1,e^2,e^3,e^4$ vanish on $\mathfrak{n}'$. We denote by $g$ the metric in which this coframe is orthonormal and by $\varphi$ the $G_2$-structure induced by $\mathcal{B}$ via (2.3). If $\{e_1,\dots,e_7\}$ denotes the basis of $\mathfrak{n}$ dual to $\mathcal{B}$, then $\mathfrak{r}=\langle e_1,e_2,e_3,e_4\rangle$ and $\mathfrak{n}'=\langle e_5,e_6,e_7\rangle$. In particular, $\mathfrak{n}'$ is calibrated by $\varphi$ by construction.

We let $\alpha_i := de^{i+4}\in\Lambda^2\mathfrak{r}^*$, for $i=1,2,3$, and we define the $3\times 3$ matrices $M^{\mathcal{B}}$ and $S_\pm^{\mathcal{B}}$ with coefficients $(M^{\mathcal{B}})_{ij}:=g(\sigma_i,\alpha_j)$ and $(S_\pm^{\mathcal{B}})_{ij}:=g(\alpha_i^\pm,\alpha_j^\pm)$, where
$$\sigma_1=e^{13}-e^{24},\quad \sigma_2=-(e^{14}+e^{23}),\quad \sigma_3=e^{12}+e^{34},$$
and $\alpha_i^+$ and $\alpha_i^-$ denote the self-dual and anti-self-dual parts of $\alpha_i$ with respect to the metric and orientation of $\mathfrak{r}$ for which $\{e^1,e^2,e^3,e^4\}$ is an oriented orthonormal coframe. Notice that the self-dual forms with respect to the opposite orientation $-e^{1234}$ of $\mathfrak{r}$ are the anti-self-dual forms with respect to the orientation $e^{1234}$.

By Lemma 4.8, $\varphi$ is purely coclosed if and only if $M^{\mathcal{B}}$ is symmetric and trace-free. Moreover, from Proposition 4.10 we know that if

$$\mathrm{tr}^2\left(S_+^{\mathcal{B}}\right) \neq 2\,\mathrm{tr}\left((S_+^{\mathcal{B}})^2\right), \qquad \text{and} \qquad \mathrm{tr}^2\left(S_-^{\mathcal{B}}\right) \neq 2\,\mathrm{tr}\left((S_-^{\mathcal{B}})^2\right), \tag{5.12}$$

then $g$ is not compatible with any purely coclosed $\mathrm{G}_2$-structure on $\mathfrak{n}$ calibrating $\mathfrak{n}'$. Even more, by Lemma 4.9, $g$ is not compatible with any purely coclosed $\mathrm{G}_2$-structure on $\mathfrak{n}$.

In what follows, for each 7-dimensional 2-step nilpotent Lie algebra $\mathfrak{n}$ with $\dim(\mathfrak{n}') = 3$ we will give an example of basis $\mathcal{B}$ such that the metric making it orthonormal is not compatible with any purely coclosed $\mathrm{G}_2$-structure, and a basis $\mathcal{C}$ inducing a purely coclosed $\mathrm{G}_2$-structure. We will explicit the computations in the first case, and sketch the remaining cases.

5.3.1. $\mathfrak{n}_{6,3}\oplus\mathbb{R} = \left(0,0,0,0,f^{12},f^{13},f^{23}\right)$. With respect to the coframe $\mathcal{B} = \{f^1,\dots,f^7\}$, we have

$$\alpha_1 = \mathrm{d}f^5 = f^{12}, \quad \alpha_2 = \mathrm{d}f^6 = f^{13}, \quad \alpha_3 = \mathrm{d}f^7 = f^{23},$$

whence

$$\alpha_1^{\pm} = \tfrac{1}{2}(f^{12} \pm f^{34}), \quad \alpha_2^{\pm} = \tfrac{1}{2}(f^{13} \mp f^{24}), \quad \alpha_3^{\pm} = \tfrac{1}{2}(f^{14} \pm f^{23}).$$

Consequently $S_+^{\mathcal{B}} = S_-^{\mathcal{B}} = \frac{1}{2}\mathrm{I}_3$, so the inequalities (5.12) hold, showing that the metric for which $\mathcal{B}$ is orthonormal is not compatible with any purely coclosed $\mathrm{G}_2$-structure.

Consider now the coframe $\mathcal{C} = \{f^1,f^2,f^3,f^4,f^6,2f^7,f^5\}$. With respect to this coframe, we have

$$\alpha_1 = \mathrm{d}f^6 = f^{13}, \quad \alpha_2 = \mathrm{d}(2f^7) = 2f^{23}, \quad \alpha_3 = \mathrm{d}f^5 = f^{12},$$

showing that

$$M^{\mathcal{C}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is symmetric and trace-free. Thus the $\mathrm{G}_2$-structure induced by $\mathcal{C}$ is purely coclosed.

5.3.2. $\mathfrak{n}_{7,3,A} = \left(0,0,0,0,f^{12},f^{23},f^{24}\right)$. With respect to the coframes $\mathcal{B} = \{f^1,\dots,f^7\}$ and $\mathcal{C} = \{f^1,f^2,f^3,f^4,f^7,f^6,2f^5\}$, we have $S_+^{\mathcal{B}} = S_-^{\mathcal{B}} = \frac{1}{2}\mathrm{I}_3$ and

$$M^{\mathcal{C}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Therefore, the metric making $\mathcal{B}$ orthonormal is not compatible with any purely coclosed $\mathrm{G}_2$-structure, whereas the $\mathrm{G}_2$-structure induced by $\mathcal{C}$ is purely coclosed.

5.3.3. $\mathfrak{n}_{7,3,B} = \left(0,0,0,0,f^{12},f^{23},f^{34}\right)$. With respect to the coframes

$$\mathcal{B} = \left\{f^1,f^2,f^3,f^4,f^5,\frac{1}{\sqrt{2}}\,f^6,f^7\right\} \qquad \text{and} \qquad \mathcal{C} = \{f^1,f^2,f^3,f^4,f^5-f^7,2f^6,f^5+f^7\},$$

we have

$$S_+^{\mathcal{B}} = \tfrac{1}{4}\begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}, \qquad S_-^{\mathcal{B}} = \tfrac{1}{4}\begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 2 \end{pmatrix}, \qquad M^{\mathcal{C}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

and one easily checks that

$$\tfrac{25}{16} = \mathrm{tr}^2\left(S_{\pm}^{\mathcal{B}}\right) \neq 2\,\mathrm{tr}\left((S_{\pm}^{\mathcal{B}})^2\right) = \tfrac{17}{8}.$$

Thus, the metric making $\mathcal{B}$ orthonormal is not compatible with any purely coclosed $\mathrm{G}_2$-structure, whereas the $\mathrm{G}_2$-structure induced by $\mathcal{C}$ is purely coclosed. Notice that the coframe $\{f^1,\dots,f^7\}$ is also of the same type as $\mathcal{B}$, i.e., the metric making it orthonormal is not compatible with any purely coclosed $\mathrm{G}_2$-structure. The interesting property of the coframe

$\mathcal{B}$ given above is that it is orthonormal with respect to a nilsoliton metric on $\mathfrak{n}_{7,3,B}$ (see Sect. 5.4).

5.3.4. $\mathfrak{n}_{7,3,B_1} = \left(0,0,0,0,f^{12}-f^{34},f^{13}+f^{24},f^{14}\right)$. Consider the coframes $\mathcal{B} = \{f^1,\ldots,f^7\}$ and $\mathcal{C} = \{-f^1,f^2,f^3,f^4,f^6,4f^7,f^5\}$. We compute

$$S_+^{\mathcal{B}} = \tfrac{1}{2}\begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&1\end{pmatrix}, \qquad S_-^{\mathcal{B}} = \tfrac{1}{2}\begin{pmatrix} 4&0&0\\ 0&4&0\\ 0&0&1\end{pmatrix}, \qquad M^{\mathcal{C}} = \begin{pmatrix} -2&0&0\\ 0&4&0\\ 0&0&-2\end{pmatrix},$$

and we get

$$\tfrac{1}{4} = \operatorname{tr}^2\left(S_+^{\mathcal{B}}\right) \neq 2\operatorname{tr}\left((S_+^{\mathcal{B}})^2\right) = \tfrac{1}{2}, \qquad \text{and} \qquad \tfrac{25}{4} = \operatorname{tr}^2\left(S_-^{\mathcal{B}}\right) \neq 2\operatorname{tr}\left((S_-^{\mathcal{B}})^2\right) = \tfrac{17}{2}.$$

Again, this shows that the metric making $\mathcal{B}$ orthonormal is not compatible with any purely coclosed G$_2$-structure, and that the G$_2$-structure induced by $\mathcal{C}$ is purely coclosed.

5.3.5. $\mathfrak{n}_{7,3,C} = \left(0,0,0,0,f^{12}+f^{34},f^{23},f^{24}\right)$. In contrast to the previous cases, the metric making $\{f^1,\ldots,f^7\}$ orthonormal turns out to be compatible with a purely coclosed G$_2$-structure. To see this, consider the coframes $\mathcal{B} = \{f^1,f^2,f^3,f^4,f^5,f^6,2f^7\}$ and $\mathcal{C} = \{f^1,f^2,f^3,f^4,f^7,f^6,f^5\}$. Then, we have

$$S_+^{\mathcal{B}} = \tfrac{1}{2}\begin{pmatrix} 4&0&0\\ 0&1&0\\ 0&0&2\end{pmatrix}, \qquad S_-^{\mathcal{B}} = \tfrac{1}{2}\begin{pmatrix} 0&0&0\\ 0&1&0\\ 0&0&2\end{pmatrix}, \qquad M^{\mathcal{C}} = \begin{pmatrix} -1&0&0\\ 0&-1&0\\ 0&0&2\end{pmatrix}.$$

So

$$\tfrac{49}{4} = \operatorname{tr}^2\left(S_+^{\mathcal{B}}\right) \neq 2\operatorname{tr}\left((S_+^{\mathcal{B}})^2\right) = \tfrac{21}{2}, \qquad \text{and} \qquad \tfrac{9}{4} = \operatorname{tr}^2\left(S_-^{\mathcal{B}}\right) \neq 2\operatorname{tr}\left((S_-^{\mathcal{B}})^2\right) = \tfrac{9}{2},$$

showing that the metric making $\mathcal{B}$ orthonormal is not compatible with any purely coclosed G$_2$-structure, whereas the G$_2$-structure induced by $\mathcal{C}$ is purely coclosed.

5.3.6. $\mathfrak{n}_{7,3,D} = \left(0,0,0,0,f^{12}+f^{34},f^{13},f^{24}\right)$. With respect to the coframes $\mathcal{B} = \{f^1,\ldots,f^7\}$ and $\mathcal{C} = \{f^1,f^2,f^3,f^4,\frac{1}{\sqrt{2}}(f^7-f^6),\frac{1}{\sqrt{2}}(f^7+f^6),\frac{1}{\sqrt{2}}f^5\}$ we have

$$S_+^{\mathcal{B}} = \tfrac{1}{2}\begin{pmatrix} 4&0&0\\ 0&1&-1\\ 0&-1&1\end{pmatrix}, \qquad S_-^{\mathcal{B}} = \tfrac{1}{2}\begin{pmatrix} 0&0&0\\ 0&1&1\\ 0&1&1\end{pmatrix}, \qquad M^{\mathcal{C}} = \sqrt{2}\begin{pmatrix} -1&0&0\\ 0&0&0\\ 0&0&1\end{pmatrix},$$

and we get

$$9 = \operatorname{tr}^2\left(S_+^{\mathcal{B}}\right) \neq 2\operatorname{tr}\left((S_+^{\mathcal{B}})^2\right) = 10, \qquad \text{and} \qquad 1 = \operatorname{tr}^2\left(S_-^{\mathcal{B}}\right) \neq 2\operatorname{tr}\left((S_-^{\mathcal{B}})^2\right) = 2.$$

Thus, the metric making $\mathcal{B}$ orthonormal is not compatible with any purely coclosed G$_2$-structure, whereas the G$_2$-structure induced by $\mathcal{C}$ is purely coclosed. Here again, the reason for the choice of the coframe $\mathcal{C}$ is that it is orthonormal with respect to a nilsoliton metric on $\mathfrak{n}_{7,3,D}$ (see Sect. 5.4).

5.3.7. $\mathfrak{n}_{7,3,D_1} = \left(0,0,0,0,f^{12}-f^{34},f^{13}+f^{24},f^{14}-f^{23}\right)$. Also in this case, the standard metric turns out to be compatible with a purely coclosed G$_2$-structure. However, in order to construct a metric not compatible with any purely coclosed G$_2$-structure we need to modify the standard one in some $\mathfrak{r}$ direction, in contrast to 5.3.5 where the modification was only necessary on $\mathfrak{n}'$. This makes the computation slightly more tricky, since the self-dual and anti-self-dual parts are different from the standard case.

In detail, with respect to the coframe $\mathcal{C} = \{f^1,\ldots,f^7\}$, the matrix $M^{\mathcal{C}}$ vanishes, so it is trivially symmetric and trace-free, showing that the G$_2$-structure induced by $\mathcal{C}$ is purely coclosed.

Consider now the coframe $\mathcal{B} = \{2f^1,f^2,f^3,f^4,f^5,f^6,f^7\}$. The decomposition of

$$\alpha_1 := \mathrm{d}f^5 = f^{12}-f^{34}, \quad \alpha_2 := \mathrm{d}f^6 = f^{13}+f^{24}, \quad \alpha_3 := \mathrm{d}f^7 = f^{14}-f^{23},$$

into self-dual and anti-self-dual parts with respect to the metric and orientation of $\mathfrak{r}$ determined by the fact that $\{2f^1, f^2, f^3, f^4\}$ is an oriented orthonormal coframe reads

$$\begin{aligned}
\alpha_1 &= -\tfrac{1}{4}(2f^{12}+f^{34})+\tfrac{3}{4}(2f^{12}-f^{34}),\\
\alpha_2 &= -\tfrac{1}{4}(2f^{13}-f^{24})+\tfrac{3}{4}(2f^{13}+f^{24}),\\
\alpha_3 &= -\tfrac{1}{4}(2f^{14}+f^{23})+\tfrac{3}{4}(2f^{14}-f^{23}).
\end{aligned}$$

Since the forms in the brackets are mutually orthogonal and have norm $\sqrt{2}$, the matrices $S^{\mathcal{B}}_{\pm}$ are given by $S^{\mathcal{B}}_{+} = \frac{1}{8}\mathrm{I}_3$, $S^{\mathcal{B}}_{-} = \frac{9}{2}\mathrm{I}_3$, so the inequalities (5.12) hold, showing that the metric making $\mathcal{B}$ orthonormal is not compatible with any purely coclosed $\mathrm{G}_2$-structure.

5.4. **Nilsoliton metrics induced by purely coclosed $\mathbf{G_2}$-structures.** Let $\mathfrak{n}$ be one of the 7-dimensional 2-step nilpotent Lie algebras with $\dim(\mathfrak{n}')=3$, and let $\{f^1,\ldots,f^7\}$ be the corresponding basis of $\mathfrak{n}^*$ given in Appendix A.

From [11], we know that, up to automorphism and scaling, a basis of $\mathfrak{n}^*$ which is orthonormal with respect to the nilsoliton metric is given by $\{f^1,\ldots,f^7\}$ for the Lie algebras $\mathfrak{n}_{6,3}\oplus\mathbb{R}$, $\mathfrak{n}_{7,3,A}$, $\mathfrak{n}_{7,3,B_1}$, $\mathfrak{n}_{7,3,C}$ and $\mathfrak{n}_{7,3,D_1}$, by $\left\{f^1,f^2,f^3,f^4,\frac{1}{\sqrt{2}}f^5,f^6,f^7\right\}$ for $\mathfrak{n}_{7,3,D}$, and by $\left\{f^1,f^2,f^3,f^4,f^5,\frac{1}{\sqrt{2}}f^6,f^7\right\}$ for $\mathfrak{n}_{7,3,B}$.

From the discussion in the previous subsections, we can explicitly state which of these nilsoliton metrics are induced by a purely coclosed $\mathrm{G}_2$-structure.

**Corollary 5.11.** *On $\mathfrak{n}_{7,3,C}$, $\mathfrak{n}_{7,3,D}$ and $\mathfrak{n}_{7,3,D_1}$ there exist purely coclosed $\mathrm{G}_2$-structures inducing a nilsoliton metric, while the nilsoliton metrics on $\mathfrak{n}_{6,3}\oplus\mathbb{R}$, $\mathfrak{n}_{7,3,A}$, $\mathfrak{n}_{7,3,B}$ and $\mathfrak{n}_{7,3,B_1}$ are not induced by any purely coclosed $\mathrm{G}_2$-structure.*

## Appendix A. The classification of 7-dimensional 2-step nilpotent Lie algebras

The isomorphism classes of 7-dimensional nilpotent Lie algebras were determined in [14]. Here, we recall the classification of those that are real and 2-step nilpotent.

The notation we use is consistent with [14]: $\mathfrak{n}_{n,t}$ or $\mathfrak{n}_{n,t,\bullet}$ means that the Lie algebra has dimension $n$ and derived algebra of dimension $t$, while different capital letters in the third argument are used to distinguish non-isomorphic Lie algebras whose derived algebras have the same dimension. We also denote by $\mathfrak{h}_n$ the Heisenberg Lie algebra of dimension $n$ and by $\mathfrak{h}_3^{\mathbb{C}}$ the real Lie algebra underlying the complex Heisenberg Lie algebra.

For each Lie algebra $\mathfrak{n}$, the structure equations are written with respect to a basis $\{f^1,\ldots,f^7\}$ of the dual Lie algebra $\mathfrak{n}^*$.

- 7-dimensional 2-step nilpotent Lie algebras $\mathfrak{n}$ with $\dim(\mathfrak{n}')=1$:

$$\begin{aligned}
\mathfrak{h}_3\oplus\mathbb{R}^4 &= \left(0,0,0,0,0,0,f^{12}\right),\\
\mathfrak{h}_5\oplus\mathbb{R}^2 &= \left(0,0,0,0,0,0,f^{12}+f^{34}\right),\\
\mathfrak{h}_7 &= \left(0,0,0,0,0,0,f^{12}+f^{34}+f^{56}\right).
\end{aligned}$$

The Heisenberg Lie algebra $\mathfrak{h}_7$ is the only indecomposable one in the above list.

- 7-dimensional 2-step nilpotent Lie algebras $\mathfrak{n}$ with $\dim(\mathfrak{n}') = 2$:

$$\begin{aligned}
\mathfrak{n}_{5,2} \oplus \mathbb{R}^2 &= \left(0,0,0,0,f^{12},f^{13},0\right),\\
\mathfrak{h}_3 \oplus \mathfrak{h}_3 \oplus \mathbb{R} &= \left(0,0,0,0,f^{12},f^{34},0\right),\\
\mathfrak{h}_3^{\mathbb{C}} \oplus \mathbb{R} &= \left(0,0,0,0,f^{13}-f^{24},f^{14}+f^{23},0\right),\\
\mathfrak{n}_{6,2} \oplus \mathbb{R} &= \left(0,0,0,0,f^{12},f^{14}+f^{23},0\right),\\
\mathfrak{n}_{7,2,A} &= \left(0,0,0,0,0,f^{12},f^{14}+f^{35}\right),\\
\mathfrak{n}_{7,2,B} &= \left(0,0,0,0,0,f^{12}+f^{34},f^{15}+f^{23}\right).
\end{aligned}$$

The only indecomposable Lie algebras in the above list are $\mathfrak{n}_{7,2,A}$ and $\mathfrak{n}_{7,2,B}$.

- 7-dimensional 2-step nilpotent Lie algebras $\mathfrak{n}$ with $\dim(\mathfrak{n}') = 3$:

$$\begin{aligned}
\mathfrak{n}_{6,3} \oplus \mathbb{R} &= \left(0,0,0,0,f^{12},f^{13},f^{23}\right),\\
\mathfrak{n}_{7,3,A} &= \left(0,0,0,0,f^{12},f^{23},f^{24}\right),\\
\mathfrak{n}_{7,3,B} &= \left(0,0,0,0,f^{12},f^{23},f^{34}\right),\\
\mathfrak{n}_{7,3,B_1} &= \left(0,0,0,0,f^{12}-f^{34},f^{13}+f^{24},f^{14}\right)\\
\mathfrak{n}_{7,3,C} &= \left(0,0,0,0,f^{12}+f^{34},f^{23},f^{24}\right),\\
\mathfrak{n}_{7,3,D} &= \left(0,0,0,0,f^{12}+f^{34},f^{13},f^{24}\right),\\
\mathfrak{n}_{7,3,D_1} &= \left(0,0,0,0,f^{12}-f^{34},f^{13}+f^{24},f^{14}-f^{23}\right).
\end{aligned}$$

The only decomposable Lie algebra in the above list is $\mathfrak{n}_{6,3} \oplus \mathbb{R}$.

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