METRIC CONNECTIONS WITH PARALLEL TWISTOR-FREE TORSION

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Abstract. The torsion of every metric connection on a Riemannian manifold has three components: one totally skew-symmetric, one of vectorial type, and one of twistorial type. In this paper we classify complete simply connected Riemannian manifolds carrying a metric connection whose torsion is parallel, has non-zero vectorial component and vanishing twistorial component.

1. Introduction

The main tool of Riemannian geometry is the Levi-Civita connection, which is the unique torsion-free metric connection on a given Riemannian manifold. However, on Riemannian manifolds carrying additional structures (*e.g.* Sasakian [1], [10], almost Hermitian [6], [13], hyperhermitian [3], G_2 [9], Spin(7) [11], homogeneous [4], etc.), metric connections with torsion are more adapted in order to understand the underlying geometries.

In most of the aforementioned cases, the torsion of the relevant metric connection is totally skew-symmetric and parallel. In [7], Cleyton and Swann obtained the classification of metric connections with parallel skew-symmetric torsion whose holonomy representation is irreducible. They show that in this case, the manifold is either naturally reductive homogeneous, or nearly Kähler in dimension 6, or nearly parallel G_2 in dimension 7.

However, since no analogue of the de Rham decomposition theorem holds for connections with torsion, the reducible case is more involved and not completely classified. A systematic study of this problem in the reducible case has been started recently in [8], where it was shown that every Riemannian manifold carrying a metric connection with parallel skew-symmetric torsion is locally a Riemannian submersion with totally geodesic naturally reductive homogeneous fibers over a lower-dimensional manifold carrying a principal bundle with parallel curvature.

The case of metric connections with torsion of vectorial type was considered by Agricola and Kraus in [2], where it is noticed that the condition of having parallel torsion is too restrictive, and is relaxed by asking that the corresponding 1-form is closed.

In the present paper we study a more general problem, by asking for the torsion of the metric connection to be parallel, while allowing it to have both a skew-symmetric and a vectorial component. Since the third component of the torsion is called twistorial component, such connections will be referred to as having twistor-free torsion. Surprisingly, it turns out that if the vectorial component is non-zero, then the problem is completely solvable.

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The rough idea of the classification is as follows. In Theorem 3.2 we show that the presence of the vectorial component forces the manifold to decompose as a warped product with fiber $\mathbb{R}$, with explicit warping function, and with basis carrying a parallel 3-form τ satisfying the condition $(\tau_X)_*\tau = 0$ for every basic tangent vector X (here τ_X denotes the skew-symmetric endomorphism corresponding to $X \lrcorner \tau$ via the metric on the basis, and acting as a derivation on the exterior bundle).

The condition $(\tau_X)_*\tau = 0$ can be interpreted as the Jacobi identity for the bracket defined on each tangent space by $[X, Y] := \tau_X Y$, and induces a parallel compact type Lie algebra structure on the tangent bundle. We then show in Theorem 5.2 that such structures decompose as products of irreducible bricks of four types: symmetric spaces of type II or IV, 3-dimensional Riemannian manifolds, simple Lie algebras of compact type with bi-invariant metric, and arbitrary Riemannian manifolds with abelian Lie algebra structure.

Using these results, the complete classification of metric connections with parallel twistor-free torsion is given in Theorem 5.3.

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2. Preliminaries

Let us first introduce some notation and conventions used in this paper. On a Riemannian manifold (M, g) vectors and 1-forms or g-skew-symmetric endomorphisms and 2-forms will be as usually identified via the metric g. A 3-form τ on M will be identified with a tensor of type $(2, 1)$ as follows:

$$\tau(X, Y, Z) = g(\tau_X Y, Z), \quad \forall X, Y, Z \in \Gamma(\mathrm{T}M).$$

In this way, the 2-form $X \lrcorner \tau$ is identified with the skew-symmetric endomorphism τ_X for every tangent vector X. The kernel of a 3-form τ at a point $x \in M$ is defined as follows:

$$\ker(\tau) := \{X \in \mathrm{T}_x M \mid \tau_X = 0\}.$$

For every $k \geq 0$, a skew-symmetric endomorphism A of $\mathrm{T}M$ acts as a derivation on the bundle of exterior k-forms by the formula

$$A_*\sigma := \sum_i Ae_i \wedge e_i \lrcorner \sigma, \qquad \forall \sigma \in \Gamma(\Omega^k M), \tag{1}$$

where $\{e_i\}_i$ is a local orthonormal basis of $\mathrm{T}M$. For later use, note that if $k = 2$ and σ is identified with a skew-symmetric endomorphism via the metric, then $A_*\sigma$ is the 2-form corresponding to the commutator $[A, \sigma]$.

Let ∇^g denote the Levi-Civita connection of g. Every other metric connection ∇ on (M, g) can be written as

$$\nabla_X = \nabla^g_X + T_X, \qquad \forall X \in \mathrm{T}M,$$

where $T \in \Gamma(\Lambda^1 M \otimes \mathrm{End}^-(\mathrm{T}M))$ is a 1-form on M with values in the g-skew-symmetric endomorphisms of $\mathrm{T}M$. The tensor T can be identified with the torsion $\widetilde{T}$ of ∇ via the isomorphism

$$\Lambda^1 M \otimes \mathrm{End}^-(\mathrm{T}M) \to \Lambda^2 M \otimes \mathrm{T}M, \qquad T \mapsto \widetilde{T},$$

with $\widetilde{T}(X,Y) := T_X Y - T_Y X$.

Recall that the tensor product of the $\mathrm{SO}(n)$ representations $\mathbb{R}^n$ and $\Lambda^2\mathbb{R}^n$ decomposes as

$$\mathbb{R}^n \otimes \Lambda^2\mathbb{R}^n = \mathbb{R}^n \oplus \mathcal{T}^{2,1}\mathbb{R}^n \oplus \Lambda^3\mathbb{R}^n,$$

where $\mathcal{T}^{2,1}$ is the Cartan summand and is generated by elements of the form $v \otimes \omega \in \mathbb{R}^n \otimes \Lambda^2\mathbb{R}^n$ which satisfy $v \wedge \omega = 0$ and $v \lrcorner \omega = 0$. The inclusions of $\mathbb{R}^n$ and $\Lambda^3\mathbb{R}^n$ into $\mathbb{R}^n \otimes \Lambda^2\mathbb{R}^n$ are given by

$$\xi \mapsto \sum_i e_i \otimes (e_i \wedge \xi), \qquad \tau \mapsto \sum_i e_i \otimes (e_i \lrcorner \tau),$$

where $\{e_i\}_i$ is any orthonormal basis of $\mathbb{R}^n$. Correspondingly, on every Riemannian manifold we have the decomposition

$$\Lambda^1 M \otimes \Lambda^2 M = \Lambda^1 M \oplus \Lambda^{2,1}\mathrm{T}M \oplus \Lambda^3 M.$$

Therefore, the torsion of a metric connection $\nabla = \nabla^g + T$, identified with the tensor $T \in \Gamma(\Lambda^1 M \otimes \Lambda^2 M)$, can be decomposed as $T = T_1 + T_2 + T_3$ with:

$$T_1 \in \mathcal{T}_1(M) := \Gamma(\Lambda^1 M), \qquad T_2 \in \mathcal{T}_2(M) := \Gamma(\Lambda^{2,1} M), \qquad T_3 \in \mathcal{T}_3(M) := \Gamma(\Lambda^3 M).$$

By analogy with the Gray-Hervella classification of almost Hermitian structures, we will say that the torsion of a metric connection $\nabla = \nabla^g + T$ is of type $\mathcal{T}_i$ for $i \in \{1,2,3\}$ if $T_i \neq 0$ and $T_j = 0$ for $j \neq i$. Similarly, for distinct $i, j \in \{1,2,3\}$ we say that the torsion of ∇ is of type $\mathcal{T}_i \oplus \mathcal{T}_j$ if $T_i \neq 0$, $T_j \neq 0$ and $T_k = 0$ for k different from i, j.

Recall that the twistor operator on Riemannian manifolds acting on 2-forms is the projection of their covariant derivative from $\Gamma(\Lambda^1 M \otimes \Lambda^2 M)$ onto $\Gamma(\Lambda^{2,1}\mathrm{T}M) = \mathcal{T}_2(M)$. We will thus call the $\mathcal{T}_2(M)$-component of a tensor in $\Gamma(\Lambda^1 M \otimes \Lambda^2 M)$ its twistorial component. Correspondingly, a metric connection is said to have twistor-free torsion if its torsion is of type $\mathcal{T}_1 \oplus \mathcal{T}_3$ and twistor-like torsion if its torsion is of type $\mathcal{T}_2$.

The aim of this paper is the classification of complete simply connected Riemannian manifolds admitting a metric connection ∇ whose torsion is twistor-free and ∇-parallel.

3. Metric connections with parallel twistor-free torsion

Let (M, g_M) be a complete simply connected n-dimensional Riemannian manifold with Levi-Civita connection ∇^{g_M}, which is endowed with a metric connection ∇ with twistor-free torsion. Assume moreover that the torsion is also ∇-parallel. By the above considerations, there exists a non-zero vector field $\xi \in \Gamma(\mathrm{T}M)$ and a non-zero 3-form $\nu \in \Omega^3(M)$, such that for all vector fields $X \in \Gamma(\mathrm{T}M)$ the following identity holds:

$$\nabla_X = \nabla^{g_M}_X + X \wedge \xi + \nu_X. \tag{2}$$

The fact that the torsion of ∇ is ∇-parallel is clearly equivalent to $\nabla\xi = 0$ and $\nabla\nu = 0$.

Our first aim is to reduce the problem to the study of a metric connection with parallel skew-symmetric torsion satisfying some further algebraic constraint on an $(n-1)$-dimensional Riemannian manifold. We start with the following technical result:

Lemma 3.1. *With the above notation, the following identities hold:*

$$\nu_\xi = 0, \quad \mathrm{d}\xi = 0, \quad \sum_{i=1}^n \nu_{e_i} \wedge \nu_{e_i} = 0, \quad \mathrm{d}\nu = 3\xi \wedge \nu,$$

where $\{e_i\}_i$ *is a local orthonormal basis of* $\mathrm{T}M$.

Proof. Up to rescaling the metric g, we may assume that the parallel vector field ξ has constant length equal to 1. Applying (2) to ξ yields $\nabla^{g_M}_X \xi = X - \langle X, \xi\rangle\xi - \nu_X\xi$ for every $X \in \mathrm{T}M$. Hence we obtain:

$$\mathrm{d}\xi = \sum_i e_i \wedge \nabla^{g_M}_{e_i}\xi = \sum_i e_i \wedge (e_i - \langle e_i, \xi\rangle\xi - \nu_{e_i}\xi) = \sum_i e_i \wedge (e_i \lrcorner \nu_\xi) = 2\nu_\xi.$$

We compute now the covariant derivative of ν_ξ using (1), (2) and the fact that both ξ and ν are ∇-parallel as follows:

$$\nabla^{g_M}_X \nu_\xi = -\sum_i (X \wedge \xi)e_i \wedge e_i \lrcorner \nu_\xi - \sum_i \nu_X e_i \wedge e_i \lrcorner \nu_\xi = -\xi \wedge \nu_\xi X + \sum_i \nu_{e_i} X \wedge \nu_\xi e_i.$$

Taking the wedge product with X and summing over $X = e_i$, we obtain:

$$\begin{aligned}\mathrm{d}\nu_\xi &= \sum_i e_i \wedge \nabla^{g_M}_{e_i}\nu_\xi = \sum_i e_i \wedge \left(-\xi \wedge \nu_\xi e_i + \sum_j \nu_{e_j} e_i \wedge \nu_\xi e_j\right) = 2\xi \wedge \nu_\xi - 2\sum_i \nu_{e_i} \wedge \nu_{e_i}\xi \\ &= 2\xi \wedge \nu_\xi - \xi \lrcorner \sum_i \nu_{e_i} \wedge \nu_{e_i}.\end{aligned}$$

On the other hand, since $\mathrm{d}\xi = 2\nu_\xi$, it follows that $\mathrm{d}\nu_\xi = 0$. The above computation then implies that

$$2\xi \wedge \nu_\xi = \xi \lrcorner \sum_i \nu_{e_i} \wedge \nu_{e_i}.$$

Taking a further interior product with ξ and using that ξ is nowhere vanishing (being non-zero and ∇-parallel) yields $\nu_\xi = 0$, hence also $\mathrm{d}\xi = 2\nu_\xi = 0$ and

$$\xi \lrcorner \sum_i \nu_{e_i} \wedge \nu_{e_i} = 0. \tag{3}$$

We further compute the covariant derivative of ν and its exterior differential:

$$\begin{aligned}\nabla^{g_M}_X \nu &= -(X \wedge \xi)_*\nu - (\nu_X)_*\nu = -\sum_i (X \wedge \xi)e_i \wedge e_i \lrcorner \nu - \nu_X e_i \wedge e_i \lrcorner \nu \\ &= -\xi \wedge \nu_X + \sum_i \nu_{e_i} X \wedge \nu_{e_i}, \\ \mathrm{d}\nu &= \sum_i e_i \wedge \nabla^{g_M}_{e_i}\nu = -\sum_i e_i \wedge \xi \wedge \nu_{e_i} + \sum_{i,j} e_i \wedge \nu_{e_j} e_i \wedge \nu_{e_j} = 3\xi \wedge \nu + 2\alpha,\end{aligned}$$

where $\alpha := \sum_i \nu_{e_i} \wedge \nu_{e_i}$. It remains to show that α vanishes. The covariant derivative of α and its exterior differential are obtained as follows, where we assume furthermore that the orthonormal basis $\{e_i\}_i$ is parallel with respect to the Levi-Civita connection ∇^{g_M} at the point where the computation is done:

$$\begin{aligned}\nabla^{g_M}_X\alpha &= \sum_i \nabla^{g_M}_X(\nu_{e_i}\wedge\nu_{e_i}) = \sum_i 2\nu_{e_i}\wedge\nabla^{g_M}_X\nu_{e_i} = 2\sum_i \nu_{e_i}\wedge\nabla^{g_M}_X\nu_{e_i}\\ &= -2\sum_{i,j}\nu_{e_i}\wedge(X\wedge\xi)(e_j)\wedge e_j\lrcorner\nu_{e_i} - 2\sum_{i,j}\nu_{e_i}\wedge\nu_X e_j\wedge e_j\lrcorner\nu_{e_i}\\ &= -2\sum_i \nu_{e_i}\wedge\xi\wedge\nu_{e_i}X + 2\sum_i \nu_{e_i}\wedge X\wedge\nu_{e_i}\xi + 2\sum_{i,j}\nu_{e_j}X\wedge\nu_{e_i}e_j\wedge\nu_{e_i}\\ &= -2\sum_i \xi\wedge\nu_{e_i}X\wedge\nu_{e_i} + 2\sum_{i,j}\nu_{e_j}X\wedge\nu_{e_i}e_j\wedge\nu_{e_i},\end{aligned}$$

$$\begin{aligned}\mathrm{d}\alpha &= \sum_i e_i\wedge\nabla^{g_M}_{e_i}\alpha = -2\sum_{ij} e_i\wedge\xi\wedge\nu_{e_j}e_i\wedge\nu_{e_j} + 2\sum_{i,j,k} e_i\wedge\nu_{e_j}e_i\wedge\nu_{e_k}e_j\wedge\nu_{e_k}\\ &= 4\sum_j \xi\wedge\nu_{e_j}\wedge\nu_{e_j} + 4\sum_{j,k}\nu_{e_j}\wedge\nu_{e_k}e_j\wedge\nu_{e_k} = 4\xi\wedge\alpha.\end{aligned}$$

We now obtain:

$$0 = \mathrm{d}^2\nu = \mathrm{d}(3\xi\wedge\nu + 2\alpha) = -3\xi\wedge\mathrm{d}\nu + 2\mathrm{d}\alpha = -6\xi\wedge\alpha + 8\xi\wedge\alpha = 2\xi\wedge\alpha,$$

showing that $\alpha = 0$, because $\xi\lrcorner\alpha = \xi\lrcorner\sum_i \nu_{e_i}\wedge\nu_{e_i} = 0$ by (3). This concludes the proof of the lemma. □

We can now state our reduction result:

Theorem 3.2. *A complete simply connected Riemannian manifold (M, g_M) carries a metric connection with parallel twistor-free torsion if and only if (M, g_M) is homothetic to a warped product $(N\times\mathbb{R}, e^{2t}g_N + \mathrm{d}t^2)$, where (N, g_N) is a complete simply connected Riemannian manifold carrying a parallel 3-form $\tau\in\Omega^3(N)$ which satisfies $\tau_X\tau = 0$, for all $X\in\mathrm{T}N$.*

Proof. Let ∇ be a metric connection on (M, g_M) with parallel twistor-free torsion given as:

$$\nabla_X = \nabla^{g_M}_X + X\wedge\xi + \nu_X, \tag{4}$$

with $\nabla\xi = 0$ and $\nabla\nu = 0$. After rescaling the metric if necessary, one can assume that ξ has unit length. Let $\eta := g_M(\xi,\cdot)$ denote the metric dual of ξ. By Lemma 3.1 one has $\mathrm{d}\eta = 0 = \nu_\xi$, so applying (4) to ξ yields for all vector fields X on M:

$$\nabla^{g_M}_X\xi = X - \eta(X)\xi. \tag{5}$$

By assumption, M is simply connected, so there exists a function $t : M\to\mathbb{R}$ with $\mathrm{d}t = \eta$. We denote by $N := t^{-1}(0)$ the level hypersurface of t at 0, endowed with the induced Riemannian metric g_N. Consider the new metric on M given by $\tilde g := e^{-2t}g_M$. From the standard conformal change formulas we have for any vector fields X, Y on M:

$$\begin{aligned}\nabla^{\tilde g}_X Y &= \nabla^{g_M}_X Y - X(t)Y - Y(t)X + g_M(X,Y)\mathrm{grad}^{g_M}t\\ &= \nabla^{g_M}_X Y - \eta(X)Y - \eta(Y)X + g_M(X,Y)\xi.\end{aligned}$$

Applying this formula to $Y := e^t\xi$ and using (5) yields for every vector field X:

$$\begin{aligned}\nabla^{\tilde g}_X(e^t\xi) &= \nabla^{g_M}_X(e^t\xi) - \eta(X)e^t\xi - e^tX + g_M(X, e^t\xi)\xi\\ &= e^t(X(t)\xi + \nabla^{g_M}_X\xi - \eta(X)\xi - X + \eta(X)\xi) = 0.\end{aligned}$$

Consequently, the vector field $e^t\xi$ is parallel and has unit length on $(M, e^{-2t}g_M)$. This shows that $(M, e^{-2t}g_M)$ is globally isometric to $(N, g_N) \times (\mathbb{R}, \mathrm{d}s^2)$, where s is determined by the fact that $\mathrm{d}s$ is the metric dual of $e^t\xi$ with respect to $\tilde g$, *i.e.* $\mathrm{d}s = \tilde g(e^t\xi, \cdot) = e^{-t}\eta = e^{-t}\mathrm{d}t$. This shows that $M = N \times \mathbb{R}$ and $g_M = e^{2t}(g_N + \mathrm{d}s^2) = e^{2t}g_N + \mathrm{d}t^2$.

Using Cartan's formula and Lemma 3.1 we compute:

$$\mathcal{L}_\xi \nu = \mathrm{d}\nu_\xi + \xi \lrcorner d\nu = 3\xi\lrcorner(\xi\wedge\nu) = 3\nu.$$

Since $\xi = \frac{\partial}{\partial t}$, there exists $\tau \in \Omega^3(N)$ such that $\nu = e^{3t}\tau$.

Let $\{e_i\}_i$ be an orthonormal basis of T_xM, for some $x \in M$. By Lemma 3.1, for every $X \in \mathrm{T}_xM$ we have

$$(\nu_X)_*\nu = \sum_i \nu_X e_i \wedge \nu_{e_i} = -\sum_i \nu_{e_i}X \wedge \nu_{e_i} = -\frac{1}{2}X\lrcorner\sum_i \nu_{e_i}\wedge\nu_{e_i} = 0. \tag{6}$$

This shows in particular that $(\tau_X)_*\tau = 0$, for all $X \in \mathrm{T}N$.

It remains to check that $\nabla^{g_N}\tau = 0$. Let $x \in N$ and $X, Y, Z, W \in \Gamma(\mathrm{T}N)$ which are ∇^{g_N}-parallel at x. We extend these vector fields to M arbitrarily. By (5), the second fundamental form of the hypersurface $N \subset M$ is the identity of $\mathrm{T}N$, so at x we have $\nabla^{g_M}_X Y = -g_M(X, Y)\xi$. Using (6) together with the fact that $\xi\lrcorner\nu = 0$ and $\nabla\nu = 0$, we can compute at x:

$$\begin{aligned}(\nabla^{g_N}_X\tau)(Y, Z, W) &= X(\tau(Y, Z, W)) = X(\nu(Y, Z, W)) = (\nabla^{g_M}_X\nu)(Y, Z, W)\\ &= -((X\wedge\xi + \nu_X)_*\nu)(Y, Z, W) = (\xi\wedge\nu_X - (\nu_X)_*\nu)(Y, Z, W) = 0.\end{aligned}$$

The converse statement follows in a straightforward way, by reversing the above computation. □

4. Examples on symmetric spaces

In this section we investigate the condition given in the conclusion of Theorem 3.2 in the framework of symmetric spaces. We first show that examples of complete simply connected Riemannian manifolds carrying a non-zero parallel 3-form τ satisfying $\tau_X\tau = 0$ for all tangent vectors X are provided by symmetric spaces of type II and IV, and then we prove that these are the only examples in the irreducible case.

Let G/H be an irreducible Riemannian symmetric space, where G is a simply connected Lie group and H is a compact subgroup of G. By definition, there is a decomposition of the Lie algebra of G as: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H,

$$[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}, \tag{7}$$

and $\mathfrak{m}$ carries an $\mathfrak{h}$-invariant scalar product $\langle\cdot,\cdot\rangle_\mathfrak{m}$. We denote by $\lambda\colon \mathfrak{h} \to \mathrm{so}(\mathfrak{m}) \simeq \Lambda^2\mathfrak{m}$ the differential at the identity of the isotropy representation of H.

Let $B_{\mathfrak{g}} \in \mathrm{Sym}^2(\mathfrak{g}^*)$ denote the Killing form of $\mathfrak{g}$, defined by $B_{\mathfrak{g}}(X,Y) := \mathrm{Tr}(\mathrm{ad}(X)\circ\mathrm{ad}(Y))$ for every X, Y in $\mathfrak{g}$. Clearly $B_{\mathfrak{g}}(X,Y) = 0$ for every $X \in \mathfrak{h}$ and $Y \in \mathfrak{m}$ (since by (7), $\mathrm{ad}(X)\circ\mathrm{ad}(Y)$ maps $\mathfrak{m}$ to $\mathfrak{h}$ and $\mathfrak{h}$ to $\mathfrak{m}$). Recall that G/H is called of compact type if $B_{\mathfrak{g}}$ is positive definite on $\mathfrak{m}$ and of non-compact type if $B_{\mathfrak{g}}$ is negative definite on $\mathfrak{m}$. Every irreducible Riemannian symmetric space is either of compact type, or of non-compact type [12, Prop. 7.4].

Lemma 4.1. *The scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{m}}$ on $\mathfrak{m}$ can be extended to an $\mathfrak{h}$-invariant scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$, such that $\mathfrak{m}$ and $\mathfrak{h}$ are orthogonal and such that for all $X, Y \in \mathfrak{m}$ and $A \in \mathfrak{h}$ the following identity holds:*

$$\langle [X,Y], A\rangle_{\mathfrak{g}} = \varepsilon\langle Y, [X,A]\rangle_{\mathfrak{g}}, \tag{8}$$

where $\varepsilon = -1$, if G/H is of compact type, and $\varepsilon = 1$, if G/H is of non-compact type.

Proof. If G/H is of compact type, $B_{\mathfrak{g}}$ is negative definite on $\mathfrak{g}$, and $\mathrm{ad}(\mathfrak{g})$-invariant. By Schur's lemma, there exists a positive constant λ such that $B_{\mathfrak{g}}|_{\mathfrak{m}} = -\lambda\langle\cdot,\cdot\rangle_{\mathfrak{m}}$. Then $\langle\cdot,\cdot\rangle_{\mathfrak{g}} := -\frac{1}{\lambda}B_{\mathfrak{g}}$ is an $\mathfrak{h}$-invariant scalar product extending $\langle\cdot,\cdot\rangle_{\mathfrak{m}}$ to $\mathfrak{g}$, making $\mathfrak{m}$ and $\mathfrak{h}$ orthogonal, and satisfying (8) for $\varepsilon = -1$ thanks to the $\mathrm{ad}(\mathfrak{g})$-invariance of $B_{\mathfrak{g}}$.

If G/H is of non-compact type, then the real subspace $\mathfrak{g}' := \mathfrak{h}\oplus i\mathfrak{m}$ of $\mathfrak{g}^{\mathbb{C}}$ is a Lie subalgebra of compact type, and the above splitting makes $(\mathfrak{g}', \mathfrak{h})$ a symmetric pair of compact type. By the first part of the proof, the $\mathfrak{h}$-invariant scalar product on $i\mathfrak{m}$ defined by

$$\langle iX, iY\rangle_{i\mathfrak{m}} := \langle X, Y\rangle_{\mathfrak{m}}$$

extends to an $\mathfrak{h}$-invariant scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}'}$ on $\mathfrak{g}'$, making $i\mathfrak{m}$ and $\mathfrak{h}$ orthogonal, and satisfying (8) for $\varepsilon = -1$. If $\langle\cdot,\cdot\rangle_{\mathfrak{h}}$ denotes the restriction of $\langle\cdot,\cdot\rangle_{\mathfrak{g}'}$ to $\mathfrak{h}$, then $\langle\cdot,\cdot\rangle_{\mathfrak{g}} := \langle\cdot,\cdot\rangle_{\mathfrak{h}} + \langle\cdot,\cdot\rangle_{\mathfrak{m}}$ is an $\mathfrak{h}$-invariant scalar product extending $\langle\cdot,\cdot\rangle_{\mathfrak{m}}$ to $\mathfrak{g}$, making $\mathfrak{m}$ and $\mathfrak{h}$ orthogonal, and for every $X, Y \in \mathfrak{m}$ and $A \in \mathfrak{h}$ we have

$$\begin{aligned}\langle [X,Y], A\rangle_{\mathfrak{g}} &= \langle [X,Y], A\rangle_{\mathfrak{h}} = \langle [X,Y], A\rangle_{\mathfrak{g}'} = -\langle [iX, iY], A\rangle_{\mathfrak{g}'} = \langle iY, [iX, A]\rangle_{\mathfrak{g}'} \\ &= \langle iY, i[X,A]\rangle_{i\mathfrak{m}} = \langle Y, [X,A]\rangle_{\mathfrak{m}} = \langle Y, [X,A]\rangle_{\mathfrak{g}}.\end{aligned}$$

□

Recall that the canonical 3-form of a Lie algebra $\mathfrak{h}$ of compact type is defined as follows:

$$\omega(X,Y,Z) := B_{\mathfrak{h}}([X,Y], Z), \quad \forall X, Y, Z \in \mathfrak{h}, \tag{9}$$

where $B_{\mathfrak{h}}$ denotes the Killing form of $\mathfrak{h}$. Since $B_{\mathfrak{h}}$ is $\mathrm{ad}(\mathfrak{h})$-invariant, the Jacobi identity shows that ω is $\mathrm{ad}(\mathfrak{h})$-invariant and satisfies $(\omega_X)_*\omega = 0$, for all $X \in \mathfrak{h}$.

Example 4.2. Let G/H be either an irreducible simply connected symmetric space of type II, *i.e.* $G := H \times H$, H is embedded diagonally in G and $\mathfrak{g} = \Delta^+ \oplus \Delta^-$, with $\mathfrak{h} \simeq \Delta^+ := \{(X,X) \,|\, X \in \mathfrak{h}\}$ and $\mathfrak{m} = \Delta^- := \{(X,-X) \,|\, X \in \mathfrak{h}\}$, or an irreducible simply connected symmetric space of type IV, *i.e.* $G := H^{\mathbb{C}}$ and $\mathfrak{g} = \mathfrak{h} \oplus i\mathfrak{h}$.

In both cases, there exists an $\mathfrak{h}$-invariant isomorphism, $\psi\colon \mathfrak{h} \to \mathfrak{m}$, defined in the type II case by $\psi(X) := (X, -X)$ and in the type IV case by $\psi(X) := iX$. The pull-back of the canonical form ω of $\mathfrak{h}$ through ψ^{-1} thus defines an $\mathfrak{h}$-invariant 3-form τ on $\mathfrak{m}$, which also satisfies $(\tau_X)_*\tau = 0$, for all $X \in \mathfrak{m}$. Hence, in both cases, τ defines a parallel 3-form on G/H, also denoted by τ, such that $(\tau_X)_*\tau = 0$, for all tangent vectors $X \in \Gamma(G/H)$.

Conversely, we show:

Theorem 4.3. *Let G/H be an irreducible Riemannian symmetric space carrying a parallel non-zero 3-form τ which satisfies $(\tau_X)_*\tau = 0$, for all tangent vectors X. Then G/H is an irreducible symmetric space of type II or IV and τ is, up to a constant multiple, equal to the above constructed 3-form.*

Proof. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the decomposition of the Lie algebra of G satisfying (7), and let $\langle\cdot,\cdot\rangle_\mathfrak{m}$ denote the $\mathfrak{h}$-invariant scalar product on $\mathfrak{m}$ induced by the Riemannian metric on G/H. We extend it to an $\mathfrak{h}$-invariant scalar product $\langle\cdot,\cdot\rangle_\mathfrak{g}$ on $\mathfrak{g}$ satisfying (8) by Lemma 4.1. We identify vectors and covectors on $\mathfrak{g}$ using this scalar product.

A parallel 3-form τ on the symmetric space G/H is determined by an $\mathfrak{h}$-invariant 3-form in $\Lambda^3\mathfrak{m}$, which we further denote by τ. Note that the $\mathfrak{h}$-invariance of τ, viewed as a linear map $\tau\colon \Lambda^1\mathfrak{m} \to \Lambda^2\mathfrak{m}$, reads

$$[\lambda(A), \tau(X)] = \tau([A, X]), \qquad \forall\ A \in \mathfrak{h},\ X \in \mathfrak{m}, \tag{10}$$

where the first bracket is the commutator in $\mathrm{so}(\mathfrak{m}) \simeq \Lambda^2\mathfrak{m}$.

Applying Lemma 6.1 (proved in the Appendix) to the $\mathfrak{h}$-representation $V := \mathfrak{m}$, we obtain the inclusion $\lambda(\mathfrak{h}) \subseteq \tau(\mathfrak{m}) \subseteq \Lambda^2\mathfrak{m}$. The fact that G/H is irreducible and τ is parallel imply that $\tau_X \neq 0$, for all $X \neq 0$. Thus $\tau\colon \Lambda^1\mathfrak{m} \to \Lambda^2\mathfrak{m}$ is injective, so one can define an injective $\mathfrak{h}$-invariant map $\varphi\colon \mathfrak{h} \to \mathfrak{m}$, $\varphi := \tau^{-1} \circ \lambda$, where $\tau^{-1}\colon \tau(\mathfrak{m}) \to \mathfrak{m}$. Since $\mathfrak{m}$ is an irreducible $\mathfrak{h}$ representation, φ is bijective, so in particular $\mathfrak{h}$ is simple.

By Schur's Lemma, and the $\mathfrak{h}$-invariance of φ, the pull-back of $\langle\cdot,\cdot\rangle_\mathfrak{m}$ through φ is a constant multiple of the restriction to $\mathfrak{h}$ of $\langle\cdot,\cdot\rangle_\mathfrak{g}$. Hence, up to rescaling τ, we may assume that φ is an isometry.

We claim that φ fulfills the following identities for all $A, B \in \mathfrak{h}$:

$$[\varphi(A), B] = \varphi([A, B]) = [A, \varphi(B)], \tag{11}$$

$$[\varphi(A), \varphi(B)] = -\varepsilon[A, B], \tag{12}$$

where $\varepsilon = -1$, if G/H is of compact type, and $\varepsilon = 1$, if G/H is of non-compact type.

Using the $\mathfrak{h}$-invariance of τ given by (10) we compute:

$$\begin{aligned} \tau(\varphi([A, B])) &= \lambda([A, B]) = [\lambda(A), \lambda(B)] = [\tau(\varphi(A)), \lambda(B)] = -[\lambda(B), \tau(\varphi(A))] \\ &= -\tau([B, \varphi(A)]) = \tau([\varphi(A), B]), \end{aligned}$$

so the injectivity of τ yields (11). The identity (12) is a consequence of (8) and (11) together with the following computation, which holds for all $A, B, C \in \mathfrak{h}$:

$$\begin{aligned} \langle[\varphi(A), \varphi(B)], C\rangle_\mathfrak{g} &= \varepsilon\langle\varphi(B), [\varphi(A), C]\rangle_\mathfrak{g} = \varepsilon\langle\varphi(B), \varphi([A, C])\rangle_\mathfrak{g} = \varepsilon\langle B, [A, C]\rangle_\mathfrak{g} \\ &= -\varepsilon\langle[A, B], C\rangle_\mathfrak{g}. \end{aligned}$$

Let us now define the following maps. If $\varepsilon = -1$, then

$$\Psi_-\colon \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{m}, \quad \Psi_-(A, B) := \frac{1}{2}\left(A + B + \varphi(A - B)\right),$$

and if $\varepsilon = 1$, then
$$\Psi_+ \colon \mathfrak{h}^{\mathbb{C}} \to \mathfrak{h} \oplus \mathfrak{m}, \quad \Psi_+(A + iB) := A + \varphi(B).$$

We claim that Ψ_- and Ψ_+ are isomorphisms of Lie algebras. Using (11) and (12) we compute for all $A_1, A_2, B_1, B_2 \in \mathfrak{h}$:

$$\begin{aligned}
[\Psi_-(A_1, B_1), \Psi_-(A_2, B_2)] &= \frac{1}{4}[A_1 + B_1 + \varphi(A_1 - B_1), A_2 + B_2 + \varphi(A_2 - B_2)] \\
&= \frac{1}{4}([A_1 + B_1, A_2 + B_2] + [\varphi(A_1 - B_1), A_2 + B_2] \\
&\quad + [A_1 + B_1, \varphi(A_2 - B_2)] + [\varphi(A_1 - B_1), \varphi(A_2 - B_2)]) \\
&\overset{(11),(12)}{=} \frac{1}{4}([A_1 + B_1, A_2 + B_2] + \varphi([A_1 - B_1, A_2 + B_2]) \\
&\quad + \varphi([A_1 + B_1, A_2 - B_2]) + [A_1 - B_1, A_2 - B_2]) \\
&= \frac{1}{2}\left([A_1, A_2] + [B_1, B_2] + \varphi([A_1, A_2] - [B_1, B_2])\right) \\
&= \Psi_-([A_1, A_2], [B_1, B_2]) = \Psi_-([(A_1, B_1), (A_2, B_2)])
\end{aligned}$$

and similarly

$$\begin{aligned}
[\Psi_+(A_1 + iB_1), \Psi_+(A_2 + iB_2)] &= [A_1 + \varphi(B_1), A_2 + \varphi(B_2)] = \\
&= [A_1, A_2] + [\varphi(B_1), A_2] + [A_1, \varphi(B_2)] + [\varphi(B_1), \varphi(B_2)] \\
&\overset{(11),(12)}{=} [A_1, A_2] + \varphi([B_1, A_2]) + \varphi([A_1, B_2]) - [B_1, B_2] \\
&= \Psi_+([A_1, A_2] - [B_1, B_2] + i([A_1, B_2] + [B_1, A_2])) \\
&= \Psi_+([A_1 + iB_1, A_2 + iB_2]).
\end{aligned}$$

Hence, if G/H is of compact type, then it is isometric to the irreducible type II symmetric space $H \times H/H$ and if G/H is of non-compact type, then it is isometric to the irreducible type IV symmetric space $H^{\mathbb{C}}/H$. □

5. The classification

We now consider a manifold (N, g_N, τ) satisfying the conclusion of Theorem 3.2. In order to keep notation as simple as possible, we denote by $g := g_N$. Thus (N, g) is a complete simply connected Riemannian manifold endowed with a metric connection

$$\nabla = \nabla^g + \tau$$

with skew-symmetric torsion $\tau \in \Omega^3(\mathrm{T}N)$ satisfying

$$\nabla^g \tau = 0 \tag{13}$$

and such that for all vectors $X \in \mathrm{T}N$:

$$(\tau_X)_* \tau = 0. \tag{14}$$

Since τ is ∇^g-parallel, its kernel $\mathrm{Ker}(\tau_x) := \{X \in \mathrm{T}_x N \mid \tau_X = 0\} \subset \mathrm{T}_x N$ defines a ∇^g-parallel distribution on N. Hence, the manifold (N, g) splits as a product $(N', g') \times (N'', g'')$, where τ acts trivially on N' and its restriction to N'' still satisfies (13) and (14) and has, moreover, trivial kernel. In order to keep notation simple, we further denote $(N'', g'', \tau|_{N''})$ by (N, g, τ). Let us denote by $\mathfrak{g} := \mathrm{T}_x N$, for some fixed point $x \in N$. The condition (14) implies that for all $X, Y \in \mathfrak{g}$ the following equality holds:

$$[\tau_X, \tau_Y] = \tau_{\tau_X Y}. \tag{15}$$

The bracket on $\mathfrak{g}$ defined by $[X, Y] := \tau_X Y$, for all $X, Y \in \mathfrak{g}$ satisfies the Jacobi identity thanks to (15), and tautologically τ becomes a morphism of Lie algebras from $\mathfrak{g}$ to $\Lambda^2 \mathrm{T}_x N$. Because τ is totally skew-symmetric, the metric g on $\mathrm{T}_x N$ is $\mathrm{ad}(\mathfrak{g})$-invariant, so the Lie algebra $\mathfrak{g}$ is of compact type. Moreover, as τ has no kernel, $\mathfrak{g}$ is semisimple.

Consider the g-orthogonal splitting of $\mathfrak{g} = \mathrm{T}_x N$ into simple Lie algebras: $\mathfrak{g} = \overset{\ell}{\underset{i=1}{\oplus}} \mathfrak{g}_i$. If $\mathfrak{hol}$ denotes the holonomy algebra of the Levi-Civita connection ∇^g at the fixed point $x \in N$, then the following result holds:

Lemma 5.1. *Each summand $\mathfrak{g}_i$ is $\mathfrak{hol}$-invariant.*

Proof. Let $A \in \mathfrak{hol}$ and $X \in \mathfrak{g}_i$, for some $i \in \{1, \ldots, \ell\}$. We need to show that $AX \in \mathfrak{g}_i$. Since τ is parallel, $\mathfrak{hol}$ acts trivially on τ, so $A_* \tau_X = \tau_{AX}$. For any $j \in \{1, \ldots, \ell\} \setminus \{i\}$ and $Y \in \mathfrak{g}_j$, we have $\tau(X, Y) = 0$, so we obtain:

$$[AX, Y] = \tau_{AX} Y = A \tau_X Y - \tau_X AY = -\tau_X AY = -[X, AY]. \tag{16}$$

The left hand side of (16), namely $[AX, Y]$, belongs to $\mathfrak{g}_j$, whereas its right hand side belongs to $\mathfrak{g}_i$, showing that both sides have to vanish. Thus $[AX, Y] = 0$, for all $Y \in \mathfrak{g}_i^\perp$, so AX belongs to the commutator of $\mathfrak{g}_i^\perp$ in $\mathfrak{g}$, which coincides with the simple Lie algebra $\mathfrak{g}_i$. Hence, AX belongs to $\mathfrak{g}_i$. ☐

Lemma 5.1 implies that the tangent bundle of N decomposes as a sum of ∇^g-parallel distributions, defined by the parallel transport of $\mathfrak{g}_i$, for $i \in \{1, \ldots, \ell\}$. Thus (N, g) splits as a product $\prod_{i=1}^{\ell} (N_i, g_i)$ and $\tau = \sum_{i=1}^{\ell} \tau_i$, where each $\tau_i \in \Lambda^3(\mathrm{T}N_i)$ has trivial kernel, satisfies (13) and (14), and $\mathfrak{g}_i = \tau_i(\mathrm{T}_x N_i)$ is a simple Lie algebra. Let us fix an $i \in \{1, \ldots, \ell\}$ and again simplify the notation and denote (N_i, g_i, τ_i) by (N, g, τ).

We are ready for the main step in the classification:

Theorem 5.2. *Let (N, g) be a complete simply connected Riemannian manifold carrying a metric connection with parallel skew-symmetric torsion τ which satisfies $(\tau_X)_* \tau = 0$, for all $X \in \Gamma(\mathrm{T}N)$, $\ker(\tau) = 0$ and $\mathfrak{g} := \mathrm{T}_x N$ is a simple Lie algebra, for some $x \in N$. Then one of the following cases holds:*

(1) (N, g) is an oriented 3-dimensional Riemannian manifold and τ is a constant multiple of its Riemannian volume form vol_g.
(2) (N, g) is a simple Lie algebra with an ad-invariant metric g and τ is a constant multiple of its canonical 3-form defined in (9).
(3) (N, g) is an irreducible symmetric space of type II or of type IV and τ is a constant multiple of the 3-form constructed in Example 4.2.

Proof. Let us consider the de Rham decomposition $N = N_0 \times N_1 \times \cdots \times N_k$, where N_0 denotes the flat factor and each N_α is de Rham irreducible, for $\alpha \in \{1, \dots, k\}$. We may assume that $k \geq 1$, since otherwise $N = N_0$ is a simple Lie algebra, g is an ad-invariant metric and τ is proportional to its canonical 3-form, which is case (2).

We denote by $\mathfrak{hol}$ the holonomy algebra at the fixed point x and by $D_\alpha := \mathrm{T}_x N_\alpha$ the $\mathfrak{hol}$-invariant subspaces of $\mathrm{T}_x N$, for all $\alpha \in \{0, 1, \dots, k\}$.

Let $\alpha \in \{1, \dots, k\}$ be fixed. We denote by $V_1 := D_\alpha$, $V_2 := D_\alpha^\perp$, $\mathfrak{h} := \mathfrak{hol}$, and by $\rho_i : \mathfrak{h} \to \mathfrak{so}(V_i)$ the restrictions of the holonomy representation to V_i for $i = 1, 2$. The de Rham decomposition theorem gives the existence of an element $A \in \mathfrak{h}$ acting non-trivially on V_1 and trivially on V_2. From Lemma 6.2 (proved below in the Appendix), we obtain that $\tau(D_\alpha, D_\alpha^\perp, D_\alpha^\perp) = 0$, whence

$$\tau(D_\alpha, D_\beta, D_\gamma) = 0, \qquad \forall\, \alpha \in \{1, \dots, k\},\ \forall\, \beta, \gamma \in \{0, \dots, k\} \setminus \{\alpha\}. \tag{17}$$

From (17), we immediately obtain:

$$\tau \in \Lambda^3 D_0 \oplus \bigoplus_{\alpha=1}^{k} (\Lambda^1 D_0 \otimes \Lambda^2 D_\alpha) \oplus (\bigoplus_{\alpha=1}^{k} \Lambda^3 D_\alpha). \tag{18}$$

We consider the following two cases:

1. D_0 **is trivial.** If $k \geq 2$, then $\tau \in \bigoplus_{\alpha=1}^{k} \Lambda^3 D_\alpha$, which implies that each D_α is a Lie subalgebra of $\mathfrak{g} = \bigoplus_{\alpha=1}^{k} D_\alpha$, contradicting the assumption that $\mathfrak{g}$ is simple. Therefore $k = 1$ and the above splitting has only one non-trivial component, meaning that (N, g) is de Rham irreducible. Let us denote by $n := \dim(N) = \dim(\mathfrak{g})$. By the Berger-Simons holonomy theorem, (N, g) is either an irreducible symmetric space, or its holonomy belongs to the list of Berger: $\mathrm{SO}(n)$, $\mathrm{U}(n/2)$, $\mathrm{SU}(n/2)$, $\mathrm{Sp}(n/4)$, $\mathrm{Sp}(n/4) \cdot \mathrm{Sp}(1)$, G_2 for $n = 7$, or $\mathrm{Spin}(7)$ for $n = 8$ (see [5], p. 301).

In the former case, Theorem 4.3 implies that (N, g) is an irreducible symmetric space of type II and IV, so we are in case (3).

In the latter case, we remark that for $n \geq 8$, all holonomy groups in the list of Berger have dimension strictly larger than n. On the other hand, the hypothesis $\ker(\tau) = 0$ together with the inclusion $\mathfrak{hol} \subseteq \tau(\mathfrak{g})$ proved in Lemma 6.1 below, show that $\dim(\mathfrak{hol}) \leq n$. Thus $n \leq 7$, and since $\mathfrak{g}$ is a simple Lie algebra of dimension n, the only possible case is $n = 3$. Then the parallel 3-form τ has to be a constant multiple of the Riemannian volume form of (N, g), so we are in case (1).

2. D_0 **is not trivial.** We will show that $k = 1$. If we denote by $r := \dim(D_0) \geq 1$, then using (18), the 3-form τ can be decomposed as follows:

$$\tau = \eta + \sum_{i=1}^{r} \sum_{\alpha=1}^{k} \xi_i \wedge \omega_{i\alpha} + \sum_{\alpha=1}^{k} \sigma_\alpha, \tag{19}$$

where $\eta \in \Lambda^3 D_0$, $\{\xi_i\}_{i=\overline{1,r}}$ is a basis of D_0, and $\omega_{i\alpha} \in \Lambda^2 D_\alpha$, $\sigma_\alpha \in \Lambda^3 D_\alpha$, for $i \in \{1, \dots, r\}$ and $\alpha \in \{1, \dots, k\}$. Note that the holonomy algebra $\mathfrak{hol}$ acts trivially on the two-forms

$\omega_{j\alpha} \in \Lambda^2 D_\alpha$ so by the $\mathfrak{hol}$-irreducibility of D_α, each $\omega_{j\alpha}$ is proportional to a complex structure on D_α.

For any $j \in \{1, \ldots, r\}$ we have $\tau_{\xi_j} = \eta_{\xi_j} + \sum_{\alpha=1}^{k} \omega_{j\alpha}$. Thus the condition that $(\tau_X)_*\tau = 0$ applied to $X = \xi_j$ yields:

$$0 = (\tau_{\xi_j})_*\tau = (\eta_{\xi_j})_*\eta + \sum_{i=1}^{r}\sum_{\alpha=1}^{k} \eta_{\xi_j}\xi_i \wedge \omega_{i\alpha} + \sum_{i=1}^{r}\sum_{\alpha=1}^{k} \xi_i \wedge [\omega_{j\alpha}, \omega_{i\alpha}] + \sum_{\alpha=1}^{k} (\omega_{j\alpha})_*\sigma_\alpha,$$

which, by comparing the types of the forms, is equivalent to the following set of identities, for all $j, \ell \in \{1, \ldots, r\}$ and $\alpha \in \{1, \ldots, k\}$:

$$\begin{cases} (\eta_{\xi_j})_*\eta = 0, \\ [\omega_{j\alpha}, \omega_{\ell\alpha}] + \sum_{i=1}^{r} \eta(\xi_j, \xi_i, \xi_\ell)\omega_{i\alpha} = 0, \\ (\omega_{j\alpha})_*\sigma_\alpha = 0. \end{cases} \tag{20}$$

We first prove that all 3-forms σ_α vanish, for $\alpha \in \{1, \ldots, k\}$. Assuming, by contradiction, that there exists some $\alpha \in \{1, \ldots, k\}$ with $\sigma_\alpha \neq 0$, then the last identity in (20) implies that $\omega_{j\alpha} = 0$, for all $j \in \{1, \ldots, r\}$ (indeed, the action of complex structures on odd-dimensional exterior forms is injective). Hence, by (19), $\tau \in \Lambda^3 D_\alpha \oplus \Lambda^3 D_\alpha^\perp$, which yields that $\mathfrak{g}$ splits in a direct sum of Lie subalgebras, $\mathfrak{g} = D_\alpha \oplus D_\alpha^\perp$, contradicting the assumption that $\mathfrak{g}$ is simple. This shows that $\sigma_\alpha = 0$, for all $\alpha \in \{1, \ldots, k\}$. Consequently, (19) reads:

$$\tau = \eta + \sum_{i=1}^{r}\sum_{\alpha=1}^{k} \xi_i \wedge \omega_{i\alpha}. \tag{21}$$

The first identity in (20) ensures that $\mathfrak{g}_0 := D_0 = \mathrm{T}_x N_0$ carries a Lie algebra structure, with bracket $[X, Y] := \eta_X Y$, such that η is a Lie morphism from $\mathfrak{g}_0$ to $\Lambda^2 D_0$. The second set of identities in (20) shows that for each $\alpha \in \{1, \ldots, k\}$, the linear map

$$\rho_\alpha \colon \mathfrak{g}_0 \to \Lambda^2 D_\alpha, \quad \rho_\alpha(\xi_i) := \omega_{i\alpha}, \text{ for } i \in \{1, \ldots, r\}$$

is a representation of the Lie algebra $\mathfrak{g}_0$ on D_α. We next prove that

$$\tau \in \Lambda^3(\mathrm{im}(\rho_1^*) \oplus D_1) \oplus \Lambda^3\left(\ker(\rho_1) \oplus \left(\mathop{\oplus}_{\alpha=2}^{k} D_\alpha\right)\right). \tag{22}$$

For this, we consider the decomposition $\mathfrak{g}_0 = \ker(\rho_1) \oplus \mathrm{im}(\rho_1^*)$ and we claim that the following inclusions hold:

$$\mathrm{im}(\rho_1^*) \subseteq \ker(\rho_\alpha), \quad \forall \alpha \in \{2, \ldots, k\}. \tag{23}$$

Indeed, if $\{e_i\}_i$ is a basis of $\mathrm{T}_x N$, the 4-form $\sum_{i=1}^{n} \tau_{e_i} \wedge \tau_{e_i}$, vanishes by Lemma 3.1. In particular, for any $\alpha \in \{2, \ldots, k\}$ the projection of this form onto $\Lambda^2(D_1) \otimes \Lambda^2(D_\alpha)$ vanishes, which can be written as

$$0=\sum_{j=1}^{r}\omega_{j1}\wedge\omega_{j\alpha}=\sum_{j=1}^{r}\rho_1(\xi_j)\wedge\rho_\alpha(\xi_j).$$

Taking, for any $\xi\in\mathfrak{g}_0$, the interior product (the metric adjoint of the wedge product) with $\rho_1(\xi)$, we obtain that

$$\sum_{j=1}^{r}\langle\rho_1(\xi),\rho_1(\xi_j)\rangle\rho_\alpha(\xi_j)=0,$$

which means that $\rho_1^*\rho_1(\xi)=\sum_{j=1}^{r}\langle\rho_1(\xi),\rho_1(\xi_j)\rangle\xi_j\in\ker(\rho_\alpha)$. As $\mathrm{im}(\rho_1^*\rho_1)=\mathrm{im}(\rho_1^*)$, we obtain that $\mathrm{im}(\rho_1^*)\subseteq\ker(\rho_\alpha)$, thus proving our claim (23).

Since ρ_1 is a representation, $\ker(\rho_1)$ is an ideal of $\mathfrak{g}_0$. Moreover, since the metric on $\mathfrak{g}_0$ is ad-invariant, its orthogonal complement $\mathrm{im}(\rho_1^*)$ is an ideal too. Thus the canonical 3-form η of the metric Lie algebra $\mathfrak{g}_0$ decomposes as $\eta=\eta_1+\eta_2$, with $\eta_1\in\Lambda^3(\mathrm{im}(\rho_1^*))$, and $\eta_2\in\Lambda^3(\ker(\rho_1))$. Let $\{\xi_i\}_{i=\overline{1,d_1}}$ and $\{\zeta_j\}_{j=\overline{1,d_2}}$ be orthonormal bases of $\mathrm{im}(\rho_1^*)$ and $\ker(\rho_1)$ respectively. According to (21) and using the inclusions (23), the 3-form τ can be decomposed as follows:

$$\tau=\underbrace{\eta_1+\sum_{i=1}^{d_1}\xi_i\wedge\rho_1(\xi_i)}_{\in\Lambda^3(\mathrm{im}(\rho_1^*)\oplus D_1)}+\underbrace{\eta_2+\sum_{\alpha=2}^{k}\sum_{j=1}^{d_2}\zeta_j\wedge\rho_\alpha(\zeta_j)}_{\in\Lambda^3\left(\ker(\rho_1)\oplus\left(\overset{k}{\underset{\alpha=2}{\oplus}}D_\alpha\right)\right)},$$

thus proving the splitting (22). Since $\mathfrak{g}$ is simple and $D_1\neq0$, it follows that $\ker(\rho_1)=0$ and $\overset{k}{\underset{\alpha=2}{\oplus}}D_\alpha=0$, whence $k=1$, $\mathfrak{g}=\mathfrak{g}_0\oplus D_1$, and $N=N_0\times N_1$.

We now use again the inclusion $\mathfrak{hol}\subseteq\tau(\mathfrak{g})$ provided by Lemma 6.1 in the Appendix. Since $\mathfrak{hol}$ preserves D_1 and $\tau=\eta+\sum_{i=1}^{r}\xi_i\wedge\omega_{i1}\in\Lambda^3D_0\oplus(\Lambda^1D_0\otimes\Lambda^2D_1)$, this implies that each element of the holonomy algebra is of the form $\tau(\xi)$, for some $\xi\in\mathfrak{g}_0$, *i.e.* $\mathfrak{hol}\subseteq\tau(\mathfrak{g}_0)$. On the other hand, for all $j\in\{1,\dots,r\}$, ω_{j1} is a parallel 2-form, so $[\mathfrak{hol},\tau(\mathfrak{g}_0)]=0$, which shows in particular that $\mathfrak{hol}$ is a commutative Lie algebra. As the holonomy representation of $\mathfrak{hol}$ on D_1 is irreducible, the commutativity of $\mathfrak{hol}$ implies that $\dim(D_1)=2$ and $\dim(\mathfrak{g}_0)=1$, because $\rho_1\colon\mathfrak{g}_0\to\Lambda^2D_1$ is injective. Thus N is 3-dimensional, which is again case (3). □

Summing up, we have shown the following result:

Theorem 5.3. *Let ξ be a non-zero vector field and let ν be a 3-form on a complete simply connected Riemannian manifold (M,g_M). Then the metric connection $\nabla_X:=\nabla^{g_M}_X+X\wedge\xi+\nu_X$ has ∇-parallel twistor-free torsion if and only if (M,g_M) is homothetic to a warped product $(N\times\mathbb{R},e^{2t}g_N+\mathrm{d}t^2)$, with $\xi=\frac{\partial}{\partial t}$ and $\nu=e^{3t}\tau$, where (N,g_N,τ) is a Riemannian product of complete simply connected Riemannian manifolds (N_i,g_i) endowed with 3-forms τ_i, such that each (N_i,g_i,τ_i) is of one of the following types:*

(1) (N_i, g_i) *is a* 3*-dimensional oriented Riemannian manifold and* τ_i *is a constant multiple of its Riemannian volume form.*
(2) N_i *is a simple Lie algebra with an* ad*-invariant metric* g_i *and* τ_i *is a constant multiple of its canonical* 3*-form defined in* (9)*.*
(3) (N_i, g_i) *is an irreducible symmetric space of type II or of type IV and* τ_i *is a constant multiple of the* 3*-form constructed in Example 4.2.*
(4) (N_i, g_i) *is a Riemannian manifold and* $\tau_i = 0$*.*

6. Appendix. Some representation theory

We finally prove the two representation theoretical results that have been used above.

Lemma 6.1. *Let* $\mathfrak{h}$ *be a Lie algebra and let* $\rho\colon \mathfrak{h} \to \mathfrak{so}(V) \simeq \Lambda^2 V$ *be an orthogonal representation of* $\mathfrak{h}$ *on a finite dimensional Euclidean vector space* $(V, \langle\cdot,\cdot\rangle)$*. Assume that* $\tau \in \Lambda^3 V$ *is an* $\mathfrak{h}$*-invariant* 3*-form, such that* $\tau_X \neq 0$ *for all* $X \in V \setminus \{0\}$ *and* $(\tau_X)_*\tau = 0$ *for all* $X \in V$*. Then the following inclusion holds:*

$$\rho(\mathfrak{h}) \subseteq \tau(V).$$

Proof. Let us denote by W the orthogonal complement of $\tau(V)$ in $(\tau(V) + \rho(\mathfrak{h}))$. We claim that $W = \{0\}$.

Let $A \in W$. We first notice that $A_*\tau = 0$, because $A \in \tau(V) + \rho(\mathfrak{h})$ and τ is $\mathfrak{h}$-invariant and satisfies also $(\tau_X)_*\tau = 0$, for all $X \in V$. Then for any $X \in V$ we have:

$$A_*\tau_X = \tau_{AX}. \tag{24}$$

For any $Y \in V$ we compute

$$\langle A_*\tau_X, \tau_Y\rangle = \langle [A, \tau_X], \tau_Y\rangle = \langle A, [\tau_X, \tau_Y]\rangle = \langle A, \tau_{\tau_X Y}\rangle = 0,$$

where for the last equality we used that $A \in \tau(V)^\perp$. Thus, we obtained that $A_*\tau_X \in \tau(V)^\perp$, which together with (24) implies that $0 = A_*\tau_X = \tau_{AX}$. The hypothesis on τ yields $AX = 0$, for all $X \in V$, and thus $A = 0$.

This shows that $W = \{0\}$, whence $\tau(V) = \tau(V) + \rho(\mathfrak{h})$ and thus $\rho(\mathfrak{h}) \subseteq \tau(V)$. □

The second result is an avatar of Lemma 3.4 in [8]:

Lemma 6.2. *Let* $\mathfrak{h}$ *be a Lie algebra of compact type and let* $\rho_j\colon \mathfrak{h} \to \mathfrak{so}(V_j)$*, for* $j \in \{1, 2\}$*, be two orthogonal representations of* $\mathfrak{h}$*, such that* V_1 *is irreducible and there exists* $A \in \mathfrak{h}$ *with* $\rho_1(A) \neq 0$ *and* $\rho_2(A) = 0$*. If* τ *is an* $\mathfrak{h}$*-invariant* 3*-form on* $V_1 \oplus V_2$*, then* $\tau(X_1, Y_2, Z_2) = 0$*, for every* $X_1 \in V_1$*, and* $Y_2, Z_2 \in V_2$*.*

Proof. The subspace $V := \rho_1(\ker(\rho_2))(V_1)$ of V_1 is non-zero since it contains the image of $\rho_1(A)$. We claim that V is $\mathfrak{h}$-invariant. Indeed, $\ker(\rho_2)$ is an ideal in $\mathfrak{h}$, so for all $C \in \mathfrak{h}$, $B \in \ker(\rho_2)$ and $X_1 \in V_1$ we have $[C, B] \in \ker(\rho_2)$ and thus

$$\rho_1(C)\rho_1(B)X_1 = \rho_1([C, B])X_1 + \rho_1(B)\rho_1(C)X_1 \in V_1.$$

Therefore $V = V_1$ by the irreducibility of ρ_1. Let $X_1 \in V_1$ and $Y_2, Z_2 \in V_2$. Since $V = V_1$, there exists $X_1' \in V_1$ and $B \in \ker(\rho_2)$, such that $X_1 = \rho_1(B)X_1'$. Using the $\mathfrak{h}$-invariance of τ, we compute:

$$\tau(X_1, Y_2, Z_2) = \tau(\rho_1(B)X_1', Y_2, Z_2) = -\tau(X_1', \rho_2(B)Y_2, Z_2) - \tau(X_1', Y_2, \rho_2(B)Z_2) = 0.$$

□

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