ON WEYL-REDUCIBLE CONFORMAL MANIFOLDS AND LCK STRUCTURES

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Abstract. A recent result of M. Kourganoff states that if D is a closed, reducible, non-flat Weyl connection on a compact conformal manifold M, then the universal cover of M, endowed with the metric whose Levi-Civita covariant derivative is the pull-back of D, is isometric to $\mathbb{R}^q \times N$ for some irreducible, incomplete Riemannian manifold N. Moreover, he characterized the case where the dimension of N is 2 by showing that M is then a mapping torus of some Anosov diffeomorphism of the torus $\mathbb{T}^{q+1}$. We show that in this case one necessarily has $q = 1$ or $q = 2$.

1. Weyl-reducible manifolds

Let (M, c) be a compact conformal manifold. A *Weyl structure* on M is a torsion-free linear connection D preserving the conformal structure c, in the sense that for every Riemannian metric $g \in c$, $D_X g = \theta_g(X) g$ for some 1-form θ_g on M called the *Lee form* of D with respect to g. If $g' := e^f g$ is another metric in the conformal class, then

$$\theta_{g'} = \theta_g + \mathrm{d}f.$$

The Weyl structure D is called *closed* if θ_g is closed for one (and thus all) metrics $g \in c$ and *exact* if θ_g is exact for all $g \in c$. From the above formula we see that if D is exact, so that $\theta_g = \mathrm{d}f$ for some $g \in c$, then $\theta_{e^{-f}g} = 0$, thus D is the Levi-Civita connection of the metric $e^{-f} g \in c$.

The manifold (M, c, D) is called *Weyl-reducible* if the Weyl structure D is reducible and non-flat.

Based on some evidence given by the Gallot theorem on Riemannian cones [4], it was conjectured in [2] that every closed, non-exact Weyl structure on a compact conformal manifold is either irreducible or flat. Matveev and Nikolayevsky [7] constructed a counterexample to the general conjecture, but later on Kourganoff proved that a weaker form of this conjecture holds:

Theorem 1. *(cf.* [6, Thm. 1.5]*). A closed non-exact Weyl structure D on a compact conformal manifold M, is either flat or irreducible, or the universal cover $\widetilde{M}$ of M together with the Riemannian metric g_D whose Levi-Civita connection is D, is the Riemannian product of a complete flat space $\mathbb{R}^q$ and an incomplete Riemannian manifold (N, g_N) with irreducible holonomy:*

$$(\widetilde{M}, g_D) = \mathbb{R}^q \times (N, g_N).$$

In [6, Example 1.6] (see also [7]), examples of closed reducible Weyl structures on compact manifolds are constructed using a linear map $A \in \mathrm{SL}_{q+1}(\mathbb{Z})$, such that:

(1) there exists an A-invariant decomposition $\mathbb{R}^{q+1} = E^s \oplus E^u$ with $\dim(E^u) = 1$ and $A|_{E^u} = \lambda^q \mathrm{Id}_{E^u}$ for some real number $\lambda > 1$;
(2) there exists a positive definite symmetric bilinear form b on E^s, such that $\lambda A|_{E^s}$ is orthogonal with respect to b.

Then A induces a diffeomorphism (also denoted by A) of the torus $\mathbb{T}^{q+1}$, whose mapping torus $M_A := \mathbb{T}^{q+1} \times (0, \infty)/(x, t) \sim (Ax, \frac{1}{\lambda}t)$, carries a reducible non-flat closed Weyl structure D_φ obtained by projecting to M_A the Levi-Civita connection of the metric on $\mathbb{T}^{q+1} \times (0, \infty)$ given by:

$$g_\varphi := \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_q^2 + \varphi(t)\mathrm{d}x_{q+1}^2 + \mathrm{d}t^2,$$

where $x_1, \ldots, x_{q+1}$ are the local coordinates with respect to an orthonormal basis $(e_1, \ldots, e_{q+1})$ with $e_1, \ldots, e_q \in E^s$, $e_{q+1} \in E^u$, and $\varphi\colon (0, +\infty) \to (0, +\infty)$ is any smooth function satisfying $\varphi(\lambda t) = \lambda^{2q+2}\varphi(t)$ for every $t \in (0, +\infty)$.

Moreover, Kourganoff proved that these are, up to diffeomorphism, the only examples of Weyl-reducible manifolds when the incomplete factor N is 2-dimensional:

Theorem 2. [6, Theorem 1.7] *Assume that D is a closed non-exact Weyl structure D on a compact conformal manifold M which is neither flat nor irreducible. If the irreducible manifold N given by Theorem 1 is 2-dimensional, then (M, D) is isomorphic to one of the Riemannian manifolds (M_A, D_φ).*

It turns out, however, that matrices $A \in \mathrm{SL}_{q+1}(\mathbb{Z})$ satisfying the conditions (1) and (2) above, only exist for $q = 1$ or $q = 2$. This is the object of the next section.

2. A number-theoretical result

Proposition 3. *Let $q \in \mathbb{N}^*$ and $A \in \mathrm{SL}_{q+1}(\mathbb{Z})$, such that there is a direct sum decomposition $\mathbb{R}^{q+1} = E^s \oplus E^u$ invariant by A, with $\dim(E^u) = 1$. If there exists a positive definite symmetric bilinear form b on E^s and a real number $\lambda > 1$, such that $\lambda A|_{E^s}$ is orthogonal with respect to b, then $q \in \{1, 2\}$.*

Proof. Let C be a symmetric positive definite matrix, such that $b = \langle C^2 \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product. Then the following equivalence holds:

$$\lambda A|_{E^s} \in \mathrm{O}(E^s, b) \Longleftrightarrow C \cdot (\lambda A|_{E^s}) \cdot C^{-1} \in \mathrm{O}(q).$$

In particular, each eigenvalue of $\mathrm{Spec}(\lambda \mathrm{A}|_{\mathrm{E^s}})$ has modulus 1 and the characteristic polynomial of A denoted by μ_A is given by:

$$\mu_A(X) = (X - \lambda^q) \prod_{j=1}^{q} \left(X - \frac{z_j}{\lambda} \right),$$

where z_j are complex numbers with $|z_j| = 1$ for all $j \in \{1, \ldots, q\}$, and $\prod_{j=1}^{q} z_j = 1$. Note that μ_A is irreducible in $\mathbb{Z}[X]$, since if it were a product of two non-constant polynomials with integer coefficients, one of them would have all roots of modulus less than 1, which is impossible. We distinguish the following two cases:

Case 1. If $q = 2p$ is even, denoting $\mu_A(X) = \sum_{j=0}^{2p+1} a_j X^j$ with $a_j \in \mathbb{Z}$ and $a_{2p+1} = 1$, $a_0 = -1$, we get

$$\lambda^{2p} + \frac{1}{\lambda} \sum_{j=1}^{2p} z_j = -a_{2p}, \qquad \lambda^{-2p} + \lambda \sum_{j=1}^{2p} \frac{1}{z_j} = a_1.$$

This shows that the sum $s := \sum_{j=1}^{2p} z_j$ is real, and since $|z_j| = 1$ for all $j \in \{1, \ldots, 2p\}$, s is also equal to $\sum_{j=1}^{2p} \frac{1}{z_j}$. Eliminating s from the two equations above, yields

$$\lambda^{4p+2} + a_{2p} \lambda^{2p+2} + a_1 \lambda^{2p} - 1 = 0.$$

Consequently, λ^2 is root of the polynomial

$$Q(X) := X^{2p+1} + a_{2p} X^{p+1} + a_1 X^p - 1.$$

Denote by $r_1, \ldots, r_{2p}$ the other complex roots of Q. Newton's relations show that there exists a monic polynomial $\widetilde{Q} \in \mathbb{Z}[X]$ whose roots are $\lambda^{2p}, r_1^p, \ldots, r_{2p}^p$. The monic polynomials μ_A and $\widetilde{Q} \in \mathbb{Z}[X]$ have both degree $2p+1$ and λ^{2p} is a common root. Since μ_A is irreducible, they must coincide, so up to a permutation, one can assume that $r_j^p = \frac{z_j}{\lambda}$ for all $j \in \{1, \ldots, 2p\}$. This shows that $\lambda^{\frac{1}{p}} r_j$ are complex numbers of modulus one for all $j \in \{1, \ldots, 2p\}$.

If $p \geq 2$, the coefficients of X^{2p} and X in the polynomial Q vanish, so

$$\lambda^2 + \sum_{j=1}^{2p} r_j = 0 = \frac{1}{\lambda^2} + \sum_{j=1}^{2p} \frac{1}{r_j}.$$

Thus $\sum_{j=1}^{2p} r_j = -\lambda^2$ and as $|\lambda^{\frac{1}{p}} r_j| = 1$ for all j,

$$-\lambda^{-2} = \sum_{j=1}^{2p} \frac{1}{r_j} = \lambda^{\frac{2}{p}} \sum_{j=1}^{2p} r_j = -\lambda^{\frac{2}{p}} \lambda^2.$$

This contradicts the fact that $\lambda > 1$, showing that $p = 1$ and therefore $q = 2$ (see also [1, Lemma 3.5]).

Case 2. If q is odd, then μ_A has at least one further real root, so either $\frac{1}{\lambda}$ or $-\frac{1}{\lambda}$ is a root of μ_A. Up to reordering the subscripts one thus has $z_1 = \pm 1$. Assume that $z_1 = 1$ (the argument for $z_1 = -1$ is the same). The monic polynomial $P \in \mathbb{Z}[X]$ defined by $P(X) := X^{q+1} \mu_A(\frac{1}{X})$ satisfies $P(0) = 1$, and its roots are $\{\lambda^{-q}, \lambda, \frac{\lambda}{z_2}, \ldots, \frac{\lambda}{z_q}\}$.

By Newton's identities again, there exists a monic polynomial $\widetilde{P} \in \mathbb{Z}[X]$ with $\widetilde{P}(0) = 1$, whose roots are $\{\lambda^{-q^2}, \lambda^q, (\frac{\lambda}{z_2})^q, \ldots, (\frac{\lambda}{z_q})^q\}$.

Since the monic polynomials μ_A and $\widetilde{P} \in \mathbb{Z}[X]$ (of same degree) have λ^q as common root, and μ_A is irreducible, they must coincide. In particular λ^{-q^2} is a root of μ_A. On the other hand every root of μ_A has complex modulus equal to either λ^q or $\frac{1}{\lambda}$. Since $\lambda > 1$, we obtain $q = 1$. □

Remark 4. As pointed out by V. Vuletescu, for odd q, Proposition 3 also follows from a more general result of Ferguson [3], whose proof, however, is rather involved.

3. Applications

Our main application concerns locally conformally Kähler manifolds. Recall that a Hermitian manifold (M, g, J) of complex dimension $n \geq 2$ is called *locally conformally Kähler* (in short, lcK) if around every point in M the metric g can be conformally rescaled to a Kähler metric. This condition is equivalent to the existence of a closed 1-form θ, such that

$$\mathrm{d}\Omega = \theta \wedge \Omega,$$

where $\Omega := g(J\cdot,\cdot)$ denotes the fundamental 2-form. Let now $\widetilde{M}$ be the universal cover of an lcK manifold (M, J, g, θ), endowed with the pull-back lcK structure $(\tilde{J}, \tilde{g}, \tilde{\theta})$. Since $\widetilde{M}$ is simply connected, $\tilde{\theta}$ is exact, *i.e.* $\tilde{\theta} = \mathrm{d}\varphi$, and by the above considerations, the metric $g^K := e^{-\varphi}\tilde{g}$ is Kähler.

The group $\pi_1(M)$ acts on $(\widetilde{M}, \tilde{J}, g^K)$ by holomorphic homotheties. Furthermore, we assume that the lcK structure is strict, in the sense that $\pi_1(M)$ is not a subgroup of the isometry group of $(\widetilde{M}, g^K)$. In particular, the Levi-Civita connection of the Kähler metric g^K projects to a closed, non-exact Weyl structure on M, called the *standard Weyl structure*. Its Lee form with respect to g is exactly θ.

Due to the fact that the real dimension of an lcK manifold is even, applying Proposition 3 to the special case of a compact strict lcK manifold whose standard Weyl structure is reducible, we obtain the following:

Proposition 5. *Let M be a compact Weyl-reducible strict lcK manifold. If the irreducible factor N in the splitting of the universal cover $(\widetilde{M}, g^K)$ as a Riemannian product $\mathbb{R}^q \times N$ given by Theorem 1 is* 2*-dimensional, then $q = 2$ and thus M is an Inoue surface S^0, cf.* [5].

Let us remark that if in Proposition 5 we drop the assumption on the dimension of the irreducible factor, then there are many more examples of Weyl-reducible lcK structures. They are obtained on lcK manifolds constructed by Oeljeklaus and Toma [9], for every integer $s \geq 1$, on certain compact quotients M_Γ of $\mathbb{C} \times \mathbb{H}^s$, where $\mathbb{H}$ denotes the upper complex half-plane, Γ are certain groups whose action on $\mathbb{C} \times \mathbb{H}^s$ is cocompact and properly discontinuous (for the precise definition of Γ and its action see [9]). We will briefly review them here.

In order to define the lcK structure on the quotient M_Γ, Oeljeklaus and Toma consider the function

$$F : \mathbb{C} \times \mathbb{H}^s \to \mathbb{R}, \qquad F(z, z_1, \ldots, z_s) := |z|^2 + \frac{1}{y_1 \ldots y_s},$$

with $z_k = x_k + iy_k$ and claim that it is a global Kähler potential on $\mathbb{C} \times \mathbb{H}^s$ (note a small sign error in [9]). To check this, we introduce

$$u : \mathbb{H}^s \to \mathbb{R}, \qquad u(z_1, \ldots, z_s) := \frac{1}{y_1 \ldots y_s} = \frac{(2i)^s}{\prod_{j=1}^s (z_j - \bar{z}_j)},$$

and compute

$$\bar{\partial} u = u \sum_{j=1}^s \frac{\mathrm{d}\bar{z}_j}{z_j - \bar{z}_j}, \qquad \partial u = -u \sum_{j=1}^s \frac{\mathrm{d}z_j}{z_j - \bar{z}_j}, \tag{1}$$

$$\begin{aligned}\partial\bar\partial u &= \partial u \wedge \sum_{j=1}^{s} \frac{\mathrm{d}\bar z_j}{z_j - \bar z_j} - u \sum_{j=1}^{s} \frac{\mathrm{d}z_j \wedge \mathrm{d}\bar z_j}{(z_j - \bar z_j)^2} \\ &= -u \sum_{j,k=1}^{s} \frac{1+\delta_{jk}}{(z_j - \bar z_j)(z_k - \bar z_k)} \mathrm{d}z_j \wedge \mathrm{d}\bar z_k,\end{aligned}$$

whence

$$\partial\bar\partial u = \frac{u}{4} \sum_{j,k=1}^{s} \frac{1+\delta_{jk}}{y_j y_k} \mathrm{d}z_j \wedge \mathrm{d}\bar z_k. \tag{2}$$

This shows that $i\partial\bar\partial u$ is the fundamental 2-form of a Kähler metric h on $\mathbb{H}^s$ whose coefficients are $h_{j\bar k} = \frac{u}{4}\frac{1+\delta_{jk}}{y_j y_k}$.

Proposition 6. *The Kähler metric on $\mathbb{H}^s$ with Kähler potential u is irreducible.*

Proof. The matrix $(h_{j\bar k})$ can be written as the product of 3 matrices

$$(h_{j\bar k}) = \frac{u}{4} \begin{pmatrix} \frac{1}{y_1} & 0 & \dots & 0 \\ 0 & \frac{1}{y_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{y_s} \end{pmatrix} \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{y_1} & 0 & \dots & 0 \\ 0 & \frac{1}{y_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{y_s} \end{pmatrix},$$

so its determinant equals

$$\det(h_{j\bar k}) = \left(\frac{u}{4}\right)^s (s+1) \frac{1}{(y_1 \dots y_s)^2} = \frac{(s+1)u^{s+2}}{4^s}.$$

The usual formula for the Ricci form ρ of h (cf. e.g. [8, Eq. (12.6)]) together with (1) and (2) gives

$$\begin{aligned}\rho &= -i\partial\bar\partial \ln(\det(h_{j\bar k})) = -i(s+2)\partial\bar\partial \ln(u) = -i(s+2)\partial(\frac{1}{u}\bar\partial u) \\ &= -i(s+2)\left(\frac{1}{u}\partial\bar\partial u - \frac{1}{u^2}\partial u \wedge \bar\partial u\right) \\ &= -\frac{i(s+2)}{4} \sum_{j,k=1}^{s} \frac{2+\delta_{jk}}{y_j y_k} \mathrm{d}z_j \wedge \mathrm{d}\bar z_k.\end{aligned}$$

This shows that the Ricci tensor of h is negative definite on $\mathbb{H}^s$, so h is irreducible. ☐

As a consequence of Proposition 6, the Kähler metric on $\mathbb{C} \times \mathbb{H}^s$ with fundamental 2-form $\Omega = i\partial\bar\partial F = i\mathrm{d}z \wedge \mathrm{d}\bar z + i\partial\bar\partial u$ is the product of the flat metric on $\mathbb{C}$ with an irreducible Kähler metric on $\mathbb{H}^s$. Therefore, the induced lcK structure on the compact quotient M_Γ is Weyl-reducible, and the irreducible factor of the universal cover given by Theorem 1 is exactly $N = \mathbb{H}^s$, so it has dimension $2s$.

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