ON WEYL-REDUCIBLE CONFORMAL MANIFOLDS
AND LCK STRUCTURES

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Abstract. A recent result of M. Kourganoff states that if $D$ is
a closed, reducible, non-flat Weyl connection on a compact con-
formal manifold $M$, then the universal cover of $M$, endowed with
the metric whose Levi-Civita covariant derivative is the pull-back
of $D$, is isometric to $\mathbb{R}^q \times N$ for some irreducible, incomplete Rie-
mannian manifold $N$. Moreover, he characterized the case where
the dimension of $N$ is 2 by showing that $M$ is then a mapping
torus of some Anosov diffeomorphism of the torus $\mathbb{T}^{q+1}$. We show
that in this case one necessarily has $q = 1$ or $q = 2$.

1. WEYL-REDUCIBLE MANIFOLDS

Let $(M, c)$ be a compact conformal manifold. A Weyl structure on $M$
is a torsion-free linear connection $D$ preserving the conformal structure $c$, in the sense that for every Riemannian metric $g \in c$, $D_X g = \theta_g(X)g$
for some 1-form $\theta_g$ on $M$ called the Lee form of $D$ with respect to $g$. If $g' := e^f g$ is another metric in the conformal class, then

$$\theta_{g'} = \theta_g + df.$$ 

The Weyl structure $D$ is called closed if $\theta_g$ is closed for one (and thus all) metrics $g \in c$ and exact if $\theta_g$ is exact for all $g \in c$. From the above formula we see that if $D$ is exact, so that $\theta_g = df$ for some $g \in c$, then $\theta_{e^{-f}g} = 0$, thus $D$ is the Levi-Civita connection of the metric $e^{-f}g \in c$.

The manifold $(M, c, D)$ is called Weyl-reducible if the Weyl structure $D$ is reducible and non-flat.

Based on some evidence given by the Gallot theorem on Riemannian cones [4], it was conjectured in [2] that every closed, non-exact Weyl structure on a compact conformal manifold is either irreducible or flat. Matveev and Nikolayevsky [7] constructed a counterexample to the general conjecture, but later on Kourganoff proved that a weaker form of this conjecture holds:
Theorem 1. (cf. [6, Thm. 1.5]). A closed non-exact Weyl structure $D$ on a compact conformal manifold $M$, is either flat or irreducible, or the universal cover $\tilde{M}$ of $M$ together with the Riemannian metric $g_D$ whose Levi-Civita connection is $D$, is the Riemannian product of a complete flat space $\mathbb{R}^q$ and an incomplete Riemannian manifold $(N, g_N)$ with irreducible holonomy:

$$(\tilde{M}, g_D) = \mathbb{R}^q \times (N, g_N).$$

In [6, Example 1.6] (see also [7]), examples of closed reducible Weyl structures on compact manifolds are constructed using a linear map $A \in SL_{q+1}(\mathbb{Z})$, such that:

1. there exists an $A$-invariant decomposition $\mathbb{R}^{q+1} = E^s \oplus E^u$ with $\dim(E^u) = 1$ and $A|_{E^u} = \lambda E^u$ for some real number $\lambda > 1$;
2. there exists a positive definite symmetric bilinear form $b$ on $E^s$, such that $\lambda A|_{E^s}$ is orthogonal with respect to $b$.

Then $A$ induces a diffeomorphism (also denoted by $A$) of the torus $\mathbb{Z}^{q+1}$, whose mapping torus $M_A := \mathbb{Z}^{q+1} \times (0, \infty)/(x, t) \sim (Ax, \frac{1}{\lambda}t)$, carries a reducible non-flat closed Weyl structure $D_A$ obtained by projecting to $M_A$ the Levi-Civita connection of the metric on $\mathbb{Z}^{q+1} \times (0, \infty)$ given by:

$$g_A := dx_1^2 + \cdots + dx_q^2 + \varphi(t)dx_{q+1}^2 + dt^2,$$

where $x_1, \ldots, x_{q+1}$ are the local coordinates with respect to an orthonormal basis $(e_1, \ldots, e_q)$ with $e_1, \ldots, e_q \in E^s$, $e_{q+1} \in E^u$, and $\varphi : (0, +\infty) \to (0, +\infty)$ is any smooth function satisfying $\varphi(\lambda t) = \lambda^2 \varphi(t)$ for every $t \in (0, +\infty)$.

Moreover, Kourganoff proved that these are, up to diffeomorphism, the only examples of Weyl-reducible manifolds when the incomplete factor $N$ is 2-dimensional:

Theorem 2. [6, Theorem 1.7] Assume that $D$ is a closed non-exact Weyl structure $D$ on a compact conformal manifold $M$ which is neither flat nor irreducible. If the irreducible manifold $N$ given by Theorem 1 is 2-dimensional, then $(M, D)$ is isomorphic to one of the Riemannian manifolds $(M_A, D_A)$.

It turns out, however, that matrices $A \in SL_{q+1}(\mathbb{Z})$ satisfying the conditions (1) and (2) above, only exist for $q = 1$ or $q = 2$. This is the object of the next section.
2. A number-theoretical result

**Proposition 3.** Let \( q \in \mathbb{N}^\ast \) and \( A \in \text{SL}_{q+1}(\mathbb{Z}) \), such that there is a direct sum decomposition \( \mathbb{R}^{q+1} = E^s \oplus E^u \) invariant by \( A \), with \( \dim(E^u) = 1 \). If there exists a positive definite symmetric bilinear form \( b \) on \( E^s \) and a real number \( \lambda > 1 \), such that \( \lambda A|_{E^s} \) is orthogonal with respect to \( b \), then \( q \in \{1, 2\} \).

**Proof.** Let \( C \) be a symmetric positive definite matrix, such that \( b = \langle C^2 \cdot \cdot \cdot \cdot \rangle \), where \( \langle \cdot \cdot \cdot \rangle \) is the standard Euclidean scalar product. Then the following equivalence holds:

\[
\lambda A|_{E^s} \in O(E^s, b) \iff C \cdot (\lambda A|_{E^s}) \cdot C^{-1} \in O(q).
\]

In particular, each eigenvalue of \( \text{Spec}(\lambda A|_{E^s}) \) has modulus 1 and the characteristic polynomial of \( A \) denoted by \( \mu_A \) is given by:

\[
\mu_A(X) = (X - \lambda^0) \prod_{j=1}^{q} \left( X - \frac{z_j}{\lambda} \right),
\]

where \( z_j \) are complex numbers with \( |z_j| = 1 \) for all \( j \in \{1, \ldots, q\} \), and \( \prod_{j=1}^{q} z_j = 1 \). Note that \( \mu_A \) is irreducible in \( \mathbb{Z}[X] \), since if it were a product of two non-constant polynomials with integer coefficients, one of them would have all roots of modulus less than 1, which is impossible.

We distinguish the following two cases:

**Case 1.** If \( q = 2p \) is even, denoting \( \mu_A(X) = \sum_{j=0}^{2p+1} a_j X^j \) with \( a_j \in \mathbb{Z} \) and \( a_{2p+1} = 1 \), \( a_0 = -1 \), we get

\[
\lambda^{2p} + \frac{1}{\lambda} \sum_{j=1}^{2p} z_j = -a_{2p}, \quad \lambda^{-2p} + \lambda \sum_{j=1}^{2p} \frac{1}{z_j} = a_1.
\]

This shows that the sum \( s := \sum_{j=1}^{2p} z_j \) is real, and since \( |z_j| = 1 \) for all \( j \in \{1, \ldots, 2p\} \), \( s \) is also equal to \( \sum_{j=1}^{2p} \frac{1}{z_j} \). Eliminating \( s \) from the two equations above, yields

\[
\lambda^{4p+2} + a_{2p}\lambda^{2p+2} + a_1\lambda^{2p} - 1 = 0.
\]

Consequently, \( \lambda^2 \) is root of the polynomial

\[
Q(X) := X^{2p+1} + a_{2p}X^{p+1} + a_1X^p - 1.
\]

Denote by \( r_1, \ldots, r_{2p} \) the other complex roots of \( Q \). Newton’s relations show that there exists a monic polynomial \( \tilde{Q} \in \mathbb{Z}[X] \) whose roots are \( \lambda^{2p}, r_1^p, \ldots, r_{2p}^p \). The monic polynomials \( \mu_A \) and \( \tilde{Q} \in \mathbb{Z}[X] \) have both degree \( 2p + 1 \) and \( \lambda^{2p} \) is a common root. Since \( \mu_A \) is irreducible, they
must coincide, so up to a permutation, one can assume that $r_j^p = \frac{z_j}{\lambda}$ for all $j \in \{1, \ldots, 2p\}$. This shows that $\lambda^{\frac{1}{p}}r_j$ are complex numbers of modulus one for all $j \in \{1, \ldots, 2p\}$.

If $p \geq 2$, the coefficients of $X^{2p}$ and $X$ in the polynomial $Q$ vanish, so

$$\lambda^2 + \sum_{j=1}^{2p} r_j = 0 = \frac{1}{\lambda^2} + \sum_{j=1}^{2p} \frac{1}{r_j}.$$ 

Thus $\sum_{j=1}^{2p} r_j = -\lambda^2$ and as $|\lambda^{\frac{1}{p}}r_j| = 1$ for all $j$,

$$-\lambda^{-2} = \sum_{j=1}^{2p} \frac{1}{r_j} = \lambda^2 p \sum_{j=1}^{2p} r_j = -\lambda^2 \lambda^2.$$

This contradicts the fact that $\lambda > 1$, showing that $p = 1$ and therefore $q = 2$ (see also [1, Lemma 3.5]).

**Case 2.** If $q$ is odd, then $\mu_A$ has at least one further real root, so either $\frac{1}{\lambda}$ or $-\frac{1}{\lambda}$ is a root of $\mu_A$. Up to reordering the subscripts one thus has $z_1 = \pm 1$. Assume that $z_1 = 1$ (the argument for $z_1 = -1$ is the same). The monic polynomial $P \in \mathbb{Z}[X]$ defined by $P(X) := X^{q+1}\mu_A\left(\frac{1}{X}\right)$ satisfies $P(0) = 1$, and its roots are $\{\lambda^{-q}, \lambda, \frac{\lambda}{z_2}, \ldots, \frac{\lambda}{z_q}\}$.

By Newton’s identities again, there exists a monic polynomial $\tilde{P} \in \mathbb{Z}[X]$ with $\tilde{P}(0) = 1$, whose roots are $\{\lambda^{-q}, \lambda^q, (\frac{\lambda}{z_2})^q, \ldots, (\frac{\lambda}{z_q})^q\}$.

Since the monic polynomials $\mu_A$ and $\tilde{P}$ have $\lambda^q$ as common root, and $\mu_A$ is irreducible, they must coincide. In particular $\lambda^{-q} = 1$. On the other hand every root of $\mu_A$ has complex modulus equal to either $\lambda^q$ or $\frac{1}{\lambda}$. Since $\lambda > 1$, we obtain $q = 1$.

**Remark 4.** As pointed out by V. Vuletescu, for odd $q$, Proposition 3 also follows from a more general result of Ferguson [3], whose proof, however, is rather involved.

### 3. Applications

Our main application concerns locally conformally Kähler manifolds. Recall that a Hermitian manifold $(M, g, J)$ of complex dimension $n \geq 2$ is called *locally conformally Kähler* (in short, lK) if around every point in $M$ the metric $g$ can be conformally rescaled to a Kähler metric. This condition is equivalent to the existence of a closed 1-form $\theta$, such that

$$d\Omega = \theta \wedge \Omega,$$
where $\Omega := g(J \cdot, \cdot)$ denotes the fundamental 2-form. Let now $\tilde{M}$ be the universal cover of an lcK manifold $(M, J, g, \theta)$, endowed with the pull-back lcK structure $(\tilde{J}, \tilde{g}, \tilde{\theta})$. Since $\tilde{M}$ is simply connected, $\tilde{\theta} = d\varphi$, and by the above considerations, the metric $g^K := e^{-\varphi} \tilde{g}$ is Kähler.

The group $\pi_1(M)$ acts on $(\tilde{M}, \tilde{J}, g^K)$ by holomorphic homotheties. Furthermore, we assume that the lcK structure is strict, in the sense that $\pi_1(M)$ is not a subgroup of the isometry group of $(\tilde{M}, g^K)$. In particular, the Levi-Civita connection of the Kähler metric $g^K$ projects to a closed, non-exact Weyl structure on $M$, called the standard Weyl structure. Its Lee form with respect to $g$ is exactly $\theta$.

Due to the fact that the real dimension of an lcK manifold is even, applying Proposition 3 to the special case of a compact strict lcK manifold whose standard Weyl structure is reducible, we obtain the following:

**Proposition 5.** Let $M$ be a compact Weyl-reducible strict lcK manifold. If the irreducible factor $N$ in the splitting of the universal cover $(\tilde{M}, g^K)$ as a Riemannian product $\mathbb{R}^q \times N$ given by Theorem 1 is 2-dimensional, then $q = 2$ and thus $M$ is an Inoue surface $S^0$, cf. [5].

Let us remark that if in Proposition 5 we drop the assumption on the dimension of the irreducible factor, then there are many more examples of Weyl-reducible lcK structures. They are obtained on lcK manifolds constructed by Oeljeklaus and Toma [9], for every integer $s \geq 1$, on certain compact quotients $M_\Gamma$ of $\mathbb{C} \times \mathbb{H}^s$, where $\mathbb{H}$ denotes the upper complex half-plane, $\Gamma$ are certain groups whose action on $\mathbb{C} \times \mathbb{H}^s$ is cocompact and properly discontinuous (for the precise definition of $\Gamma$ and its action see [9]). We will briefly review them here.

In order to define the lcK structure on the quotient $M_\Gamma$, Oeljeklaus and Toma consider the function

$$F : \mathbb{C} \times \mathbb{H}^s \to \mathbb{R}, \quad F(z, z_1, \ldots, z_s) := |z|^2 + \frac{1}{y_1 \cdots y_s},$$

with $z_k = x_k + iy_k$ and claim that it is a global Kähler potential on $\mathbb{C} \times \mathbb{H}^s$ (note a small sign error in [9]). To check this, we introduce

$$u : \mathbb{H}^s \to \mathbb{R}, \quad u(z_1, \ldots, z_s) := \frac{1}{y_1 \cdots y_s} = \frac{(2i)^s}{\prod_{j=1}^{s}(z_j - \bar{z}_j)},$$

and compute

$$\bar{\partial} u = u \sum_{j=1}^{s} \frac{d\bar{z}_j}{z_j - \bar{z}_j}, \quad \partial u = -u \sum_{j=1}^{s} \frac{dz_j}{z_j - \bar{z}_j}, \quad (1)$$
\[ \partial \bar{\partial} u = \partial u \wedge \sum_{j=1}^{s} \frac{dz_j}{z_j - \bar{z}_j} - u \sum_{j=1}^{s} \frac{dz_j \wedge d\bar{z}_j}{(z_j - \bar{z}_j)^2} \]

\[ = -u \sum_{j,k=1}^{s} \frac{1 + \delta_{jk}}{(z_j - \bar{z}_j)(z_k - \bar{z}_k)} dz_j \wedge d\bar{z}_k, \]

whence

\[ \partial \bar{\partial} u = \frac{u}{4} \sum_{j,k=1}^{s} \frac{1 + \delta_{jk}}{y_j y_k} dz_j \wedge d\bar{z}_k. \]

This shows that \( i \partial \bar{\partial} u \) is the fundamental 2-form of a Kähler metric \( h \) on \( \mathbb{H}^s \) whose coefficients are

\[ h_{jk} = \frac{u^{1+\delta_{jk}}}{y_j y_k}. \]

**Proposition 6.** The Kähler metric on \( \mathbb{H}^s \) with Kähler potential \( u \) is irreducible.

**Proof.** The matrix \( (h_{jk}) \) can be written as the product of 3 matrices

\[ (h_{jk}) = \frac{u}{4} \left( \begin{array}{cccc} \frac{1}{y_1} & 0 & \ldots & 0 \\ 0 & \frac{1}{y_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{y_s} \end{array} \right) \left( \begin{array}{cccc} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \ldots & 2 \end{array} \right) \left( \begin{array}{cccc} \frac{1}{y_1} & 0 & \ldots & 0 \\ 0 & \frac{1}{y_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{y_s} \end{array} \right), \]

so its determinant equals

\[ \det(h_{jk}) = \left( \frac{u}{4} \right)^s (s + 1) \frac{1}{(y_1 \ldots y_s)^2} = \frac{(s + 1)u^{s+2}}{4^s}. \]

The usual formula for the Ricci form \( \rho \) of \( h \) (cf. e.g. [8, Eq. (12.6)]) together with (1) and (2) gives

\[ \rho = -i \partial \bar{\partial} \ln(\det(h_{jk})) = -i(s + 2) \partial \bar{\partial} \ln(u) = -i(s + 2) \partial(\frac{1}{u} \partial \bar{\partial} u) \]

\[ = -i(s + 2) \left( \frac{1}{u} \partial \bar{\partial} u - \frac{1}{u^2} \partial u \wedge \bar{\partial} u \right) \]

\[ = \frac{-i(s + 2)}{4} \sum_{j,k=1}^{s} 2 + \delta_{jk} dy_j \wedge dz_k \wedge d\bar{z}_k. \]

This shows that the Ricci tensor of \( h \) is negative definite on \( \mathbb{H}^s \), so \( h \) is irreducible. \( \Box \)

As a consequence of Proposition 6, the Kähler metric on \( \mathbb{C} \times \mathbb{H}^s \) with fundamental 2-form \( \Omega = i \partial \bar{\partial} F = idz \wedge d\bar{z} + i \partial \bar{\partial} u \) is the product of the flat metric on \( \mathbb{C} \) with an irreducible Kähler metric on \( \mathbb{H}^s \). Therefore, the
induced lcK structure on the compact quotient $M_{\Gamma}$ is Weyl-reducible, and the irreducible factor of the universal cover given by Theorem 1 is exactly $N = \mathbb{H}^s$, so it has dimension $2s$.

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References


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