NON-EXISTENCE OF ORTHOGONAL COORDINATES ON THE COMPLEX AND QUATERNIONIC PROJECTIVE SPACES

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Abstract. DeTurck and Yang have shown that in the neighbourhood of every point of a 3-dimensional Riemannian manifold, there exists a system of orthogonal coordinates (that is, with respect to which the metric has diagonal form). We show that this property does not generalize to higher dimensions. In particular, the complex projective spaces $\mathbb{CP}^m$ and the quaternionic projective spaces $\mathbb{HP}^q$, endowed with their canonical metrics, do not have local systems of orthogonal coordinates for $m, q \geq 2$.

1. Introduction

A Riemannian manifold is said to *admit orthogonal coordinates* if in the neighbourhood of each point there exists a system of coordinates in which the metric has diagonal form, cf. Definition 2.1.

Metrics admitting orthogonal coordinates naturally arise in the theory of orthogonal separable dynamical systems, related to the Hamilton-Jacobi equation, and have been considered by many authors starting with Paul Stäckel [9] and Luther Pfahler Eisenhart [5], Charles Boyer, Ernie Kalnins and Pavel Winternitz [3], Paul Tod [10], and more recently, James Grant and James Vickers [7], Sergio Benenti [1], [2], Konrad Schöbel [8], and others.

Flat, or, more generally, locally conformally flat Riemannian manifolds (in particular every Riemannian surface) clearly admit orthogonal coordinates. In a beautiful paper published in 1984, Dennis M. DeTurck and Deane Yang [4] showed that every Riemannian metric of dimension 3 has orthogonal coordinates. In the same paper, they also observe that the existence issue of orthogonal coordinates on Riemannian manifolds of dimension greater than 3 becomes an overdetermined problem, and therefore one can hardly expect orthogonal coordinates on a generic Riemannian manifold. On the other hand, the existence/non-existence issue of orthogonal coordinates on a given family of Riemannian manifolds has remained a quite interesting question, albeit largely unexplored.

The aim of this paper is to establish the non-existence of orthogonal coordinates on two classical families of Riemannian manifolds, namely the standard complex projective spaces $\mathbb{CP}^m$ for $m \geq 2$ and the standard quaternionic projective spaces $\mathbb{HP}^q$ for $q \geq 2$. The overall

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argument relies on some remarkable feature — already noticed by DeTurck and Yang — of the curvature of Riemannian manifolds admitting orthogonal coordinates, together with some additional specific arguments in dimension 4, for the complex projective plane $\mathbb{CP}^2$.

A list of open questions is proposed at the end of the paper.

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2. Riemannian metrics with orthogonal coordinates

Let (M, g) be any Riemannian manifold of dimension n. Let $x_1, \ldots, x_n$ be any system of local coordinates defined on some open set $\mathcal{U}$ and denote by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ the corresponding frame on $\mathcal{U}$; the restriction to $\mathcal{U}$ of the metric g is then of the form:

$$g = \sum_{i,j=1}^{n} g_{ij} dx_i \otimes dx_j, \tag{1}$$

by setting $g_{ij} := g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$.

Definition 2.1. The system of coordinates $x_1, \ldots, x_n$ is called *orthogonal*, if $g_{ij} = 0$ whenever $i \neq j$, hence if g is of the form

$$g = \sum_{j=1}^{n} a_j^2 \, dx_j \otimes dx_j, \tag{2}$$

for some positive functions $a_1, \ldots, a_n$. We say that a Riemannian manifold (M, g) *has orthogonal coordinates* if every point of M has a neighbourhood on which there exists a system of orthogonal coordinates.

Remark 2.2. If a system of orthogonal coordinates $x_1, \ldots, x_n$ exists, any system of coordinates $y_1, \ldots, y_n$ of the form $y_i = \varphi_i(x_i)$, where φ_i is a real function whose derivative φ_i' has no zero, is orthogonal as well, since

$$g = \sum_{i=1}^{n} b_i^2 \, dy_i \otimes dy_i, \tag{3}$$

with

$$b_i = \frac{a_i}{|\varphi_i'(x_i)|}, \tag{4}$$

for $i = 1, \ldots, n$.

Example 2.3. The standard flat metric g_0 on $M = \mathbb{R}^n$ is of the form

$$g_0 = \sum_{i=1}^{n} dx_i \otimes dx_i, \tag{5}$$

where the x_i's denote the natural coordinates of $\mathbb{R}^n$. Conversely, a Riemannian metric g is flat whenever, in the neighbourhood of any point, there exists a system of coordinates such that g is of this form.

Example 2.4. Denote by $\mathbb{S}^n$ the n-dimensional standard unit sphere

$$\mathbb{S}^n = \{u = (u_0, \dots, u_n) \mid \sum_{i=0}^{n} u_i^2 = 1\},$$

and by g_S the standard Riemannian metric of sectional curvature 1, induced by the standard flat metric of $\mathbb{R}^{n+1}$. Denote by N the point $(1, 0, \dots, 0)$ of $\mathbb{S}^n$ and by $\mathcal{U}$ the open set $\mathbb{S}^n \setminus \{N\}$. Then, on $\mathcal{U}$, the metric g_S is of the form:

$$g_S = \frac{4 \sum_{j=1}^{n} dx_j \otimes dx_j}{(1 + \sum_{j=1}^{n} x_j^2)^2}, \tag{6}$$

by setting

$$x_j = \frac{u_j}{1 - u_0}, \qquad j = 1, \dots, n. \tag{7}$$

More generally, any locally conformally flat metric, in particular, any Riemannian metric in dimension 2, can be locally written on the form

$$g = a^2 \sum_{j=1}^{n} dx_j \otimes dx_j, \tag{8}$$

i.e. on the form (2), with $a_j = a$, for every $j = 1, \dots, n$.

Assume from now on that (M, g) is a Riemannian manifold of dimension n, with $n \geq 4$. We assume that $x_1, \dots, x_n$ is an orthogonal system of coordinates, as defined above, and we denote by $\{e_1, \dots, e_n\}$ the *associated orthonormal frame*, with

$$e_j := a_j^{-1} \frac{\partial}{\partial x_j}, \qquad j = 1, \dots, n. \tag{9}$$

Notice that this frame remains unchanged if the orthogonal system $x_1, \dots, x_n$ is replaced by $y_1, \dots, y_n$ as in Remark 2.2. We denote by ∇ the Levi-Civita connection of g and by R its curvature, defined by

$$R_{X,Y} Z = \nabla_{[X,Y]} Z - \nabla_X (\nabla_Y Z) + \nabla_Y (\nabla_X Z), \tag{10}$$

for any vector fields X, Y, Z on M.

Proposition 2.5. *Let (M,g) be a Riemannian manifold of dimension $n \geq 4$, equipped with orthogonal coordinates on some open set $\mathcal{U}$, where the metric is of the form* (2). *Denote by $\{e_1,\dots,e_n\}$ the associated orthonormal frame as defined above. Then,*

$$\nabla_{e_i} e_j = a_i^{-1} da_i(e_j)\, e_i - \delta_{ij}\, a_j^{-1} (da_j)^\sharp, \quad i,j = 1,\dots,n, \tag{11}$$

where $(da_j)^\sharp$ denotes the vector field dual to da_j with respect to g and δ_{ij} the usual Kronecker symbol. Moreover,

$$\begin{aligned} g(R_{e_i,e_j} e_k, e_\ell) = {} & \delta_{i\ell}\, a_i^{-1} (\nabla_{e_j} da_i)(e_k) - \delta_{j\ell}\, a_j^{-1} (\nabla_{e_i} da_j)(e_k) \\ & - \delta_{ik}\, a_k^{-1} (\nabla_{e_j} da_k)(e_\ell) + \delta_{jk}\, a_k^{-1} (\nabla_{e_i} da_k)(e_\ell) \\ & + (\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{i\ell})\, a_i^{-1} a_j^{-1}\, g(da_i, da_j), \end{aligned} \tag{12}$$

for any quadruple $i,j,k,\ell = 1,\dots,n$. In particular, for any triple i,j,k with $i \neq j \neq k \neq i$, we have:

$$R_{e_i,e_j} e_k = a_i^{-1} (\nabla_{e_j} da_i)(e_k)\, e_i - a_j^{-1} (\nabla_{e_i} da_j)(e_k)\, e_j, \tag{13}$$

and, as observed in [4], *for quadruple i,j,k,ℓ with i,j,k,ℓ mutually distinct:*

$$g(R_{e_i,e_j} e_k, e_\ell) = 0. \tag{14}$$

Proof. For any i,j, we have $[e_i, e_j] = [a_i^{-1} \frac{\partial}{\partial x_i}, a_j^{-1} \frac{\partial}{\partial x_j}]$, hence

$$[e_i, e_j] = a_i^{-1} da_i(e_j)\, e_i - a_j^{-1} da_j(e_i)\, e_j, \tag{15}$$

whereas the usual *Koszul formula* for the Levi-Civita connection is here reduced to

$$2g(\nabla_{e_i} e_j, e_k) = g([e_i, e_j], e_k) + g([e_k, e_i], e_j) + g(e_i, [e_k, e_j]). \tag{16}$$

We easily infer:

$$\begin{aligned} \nabla_{e_i} e_j &= a_i^{-1} da_i(e_j)\, e_i, \qquad i \neq j, \\ \nabla_{e_j} e_j &= -\sum_{i \neq j} a_j^{-1} da_j(e_i)\, e_i, \end{aligned} \tag{17}$$

hence (11). A straightforward computation then gives (12), and (13)–(14) follow readily. □

Remark 2.6. Equation (11) can equivalently be written as

$$\nabla_{e_i} e_j^\flat = e_j \lrcorner (\alpha_i \wedge e_i^\flat), \quad i,j = 1,\dots,n, \tag{18}$$

where $\alpha_i := a_i^{-1} da_i$. Conversely, a (local) orthonormal frame satisfying (18) for some 1-forms α_i is necessarily induced by a system of orthogonal coordinates. Indeed, using (18) we can write

$$de_j^\flat = \sum_{i=1}^n e_i^\flat \wedge \nabla_{e_i} e_j^\flat = \sum_{i=1}^n e_i^\flat \wedge (\alpha_i(e_j) e_i^\flat - \delta_{ij} \alpha_i) = \alpha_i \wedge e_i^\flat,$$

whence $e_j^\flat \wedge de_j^\flat = 0$ for every $j = 1,\dots,n$. The Frobenius theorem shows that there exist functions x_i and b_i (defined on some smaller neighbourhood) such that $e_j^\flat = b_j dx_j$ for every $j = 1, \dots, n$. Changing the sign of x_j if necessary, one can assume that each b_j is a positive function. Then $x_1, \dots, x_n$ is a system of orthogonal coordinates with associated orthonormal frame $e_1, \dots, e_n$.

3. The complex projective spaces

We now consider the complex projective space $M = \mathbb{CP}^m$, $m \geq 2$, equipped with the Fubini-Study metric, g_{FS}, of constant holomorphic sectional curvature c, whose curvature, R, is given by:

$$R^{FS}_{X,Y} Z = \frac{c}{4}\big(g_{FS}(X,Z)\,Y - g_{FS}(Y,Z)\,X + \omega(X,Z)\,JY - \omega(Y,Z)\,JX + 2\omega(X,Y)\,JZ\big) \tag{19}$$

for any vector fields X, Y, Z, where J denotes the complex structure of $\mathbb{CP}^m$ and $\omega = g_{FS}(J\cdot, \cdot)$ its Kähler form. For convenience and without loss of generality, we assume that $c = 4$. Our aim is to show that $\mathbb{CP}^m$, equipped with the Fubini-Study metric, admits no orthogonal system of coordinates. Since the case when $m = 2$ requires a specific argument, see below Proposition 3.2, we first show:

Proposition 3.1. *For $m \geq 3$, the complex projective space $\mathbb{CP}^m$, equipped with the standard Fubini-Study metric, admits no orthogonal system of coordinates.*

Proof. Suppose, for a contradiction, that $\mathbb{CP}^m$ admits local orthogonal coordinates, i.e. that g_{FS} is of the form (2) for some local coordinates $x_1, \dots, x_n$, $n = 2m$, on some open set $\mathcal{U}$, and consider the corresponding orthonormal frame $\{e_1, \dots, e_n\}$ as in Proposition 2.5.

Choose any pair e_i, e_j such that $\omega(e_i, e_j) \neq 0$, and any e_k orthogonal to e_i and e_j. In view of (19), with $c = 4$, we have

$$R^{FS}_{e_i,e_j} e_k = \omega(e_i, e_k)\,Je_j - \omega(e_j, e_k)\,Je_i + 2\omega(e_i, e_j)\,Je_k, \tag{20}$$

whereas, by (13), we should have:

$$R^{FS}_{e_i,e_j} e_k = f_i\,e_i - f_j\,e_j, \tag{21}$$

with $f_i := a_i^{-1}(\nabla_{e_j} da_i)(e_k)$, $f_j := a_j^{-1}(\nabla_{e_i} da_j)(e_k)$, so that:

$$2\omega(e_i, e_j)\,e_k = -\omega(e_i, e_k)\,e_j + \omega(e_j, e_k)\,e_i - f_i\,Je_i + f_j\,Je_j. \tag{22}$$

Since e_k is orthogonal to e_i, e_j, the functions f_i, f_j are necessarily given by $f_i = -\frac{\omega(e_i,e_k)}{\omega(e_i,e_j)}$ and $f_j = \frac{\omega(e_j,e_k)}{\omega(e_i,e_j)}$, whence

$$e_k = \frac{\omega(e_i, e_k)}{2(\omega(e_i, e_j))^2}\,(-\omega(e_i, e_j)\,e_j + Je_i) + \frac{\omega(e_j, e_k)}{2(\omega(e_i, e_j))^2}\,(\omega(e_i, e_j)\,e_i + Je_j).$$

Since e_k may be any element in the orthonormal frame $e_1, \dots, e_n$ distinct from e_i, e_j, this means that the $(2m-2)$-dimensional space orthogonal to the 2-dimensional space generated by e_i, e_j would be contained in the 2-dimensional space generated by $-\omega(e_i, e_j)\,e_j + Je_i$ and $\omega(e_i, e_j)\,e_i + Je_j$. This clearly cannot hold unless $m = 2$. $\square$

We now show:

Proposition 3.2. *The complex projective plane $\mathbb{CP}^2$, equipped with the standard Fubini-Study metric, admits no orthogonal system of coordinates.*

Proof. Again, assume for a contradiction, that $\mathbb{CP}^2$ admits local orthogonal coordinates x_1, x_2, x_3, x_4 on some open set $\mathcal{U}$ and denote by e_1, e_2, e_4, e_4 the corresponding orthonormal frame. As for any direct orthonormal frame relative to the orientation induced by the natural complex structure J of $\mathbb{CP}^2$, we have

$$\omega(e_1, e_2)\omega(e_3, e_4) - \omega(e_1, e_3)\omega(e_2, e_4) + \omega(e_1, e_4)\omega(e_2, e_3) = 1, \tag{23}$$

since the volume form of g_{FS} for the chosen orientation is $\frac{\omega\wedge\omega}{2}$. By (19), with $c = 4$, it follows that

$$\begin{aligned} g_{FS}(R^{FS}_{e_1,e_2}e_3, e_4) &= \omega(e_1, e_3)\omega(e_2, e_4) - \omega(e_1, e_4)\omega(e_2, e_3) \\ &\quad + 2\omega(e_1, e_2)\omega(e_3, e_4) \\ &= -1 + 3\omega(e_1, e_2)\omega(e_3, e_4) \\ &= -1 + 3\big(\omega(e_1, e_2)\big)^2, \end{aligned} \tag{24}$$

as ω is self-dual. In view of (14) in Proposition 2.5 and of (24), we have

$$\big(\omega(e_i, e_j)\big)^2 = \frac{1}{3}, \tag{25}$$

for any i, j, $i \neq j$. Up to possibly changing J into $-J$, we may then arrange that

$$\begin{aligned} &\omega(e_1, e_2) = \omega(e_3, e_4) = \omega(e_1, e_3) \\ &= -\omega(e_2, e_4) = \omega(e_1, e_4) = \omega(e_2, e_3) = \frac{1}{\sqrt{3}}, \end{aligned} \tag{26}$$

i.e. that

$$\begin{aligned} Je_1 &= \frac{e_2 + e_3 + e_4}{\sqrt{3}}, \quad Je_2 = \frac{-e_1 + e_3 - e_4}{\sqrt{3}}, \\ Je_3 &= \frac{-e_1 - e_2 + e_4}{\sqrt{3}}, \quad Je_4 = \frac{-e_1 + e_2 - e_3}{\sqrt{3}}. \end{aligned} \tag{27}$$

By making explicit the identities $\nabla_{e_i} Je_1 = J\nabla_{e_i} e_1$, $i = 1, 2, 3, 4$, via (11) and (27) we easily get:

$$da_1(e_2) = da_1(e_3) = da_1(e_4), \tag{28}$$

$$da_2(e_1) = -da_2(e_3) = da_2(e_4), \tag{29}$$

$$da_3(e_1) = da_3(e_2) = -da_3(e_4), \tag{30}$$

$$da_4(e_1) = -da_4(e_2) = da_4(e_3). \tag{31}$$

From (28) we infer that the vector fields $e_2 - e_3$, and $e_2 - e_4$ both belong to the kernel of da_1; it follows that their bracket $-[e_2, e_4] + [e_2, e_3] + [e_3, e_4]$, which, by (11) is equal to $2a_2^{-1}da_2(e_3)\,e_2 + 2a_3^{-1}da_3(e_4)\,e_3 + 2a_4^{-1}da_4(e_2)\,e_4$, also belongs to the kernel of da_1, so that: $a_2^{-1}da_2(e_3)da_1(e_2) + a_3^{-1}da_3(e_4)da_1(e_3) + a_4^{-1}da_4(e_2)da_1(e_4) = 0$. By introducing the notation

$$\begin{aligned} c_1 &:= a_1^{-1}da_1(e_2) = a_1^{-1}da_1(e_3) = a_1^{-1}da_1(e_4), \\ c_2 &:= a_2^{-1}da_2(e_1) = -a_2^{-1}da_2(e_3) = a_2^{-1}da_2(e_4), \\ c_3 &:= a_3^{-1}da_3(e_1) = a_3^{-1}da_3(e_2) = -a_3^{-1}da_3(e_4), \\ c_4 &:= a_4^{-1}da_4(e_1) = -a_4^{-1}da_4(e_2) = a_4^{-1}da_4(e_3), \end{aligned} \tag{32}$$

and by using (28) again, this can be rewritten as $(c_2 + c_3 + c_4)\,c_1 = 0$. We thus get the following alternative:

$$\text{either} \quad c_2 + c_3 + c_4 = 0 \quad \text{or} \quad c_1 = 0. \tag{33}$$

By considering (29), (30) and (31), we similarly obtain the following three alternatives:

$$\text{either} \quad c_1 + c_3 - c_4 = 0 \quad \text{or} \quad c_2 = 0, \tag{34}$$

$$\text{either} \quad c_1 - c_2 + c_4 = 0 \quad \text{or} \quad c_3 = 0, \tag{35}$$

$$\text{either} \quad c_1 + c_2 - c_3 = 0 \quad \text{or} \quad c_4 = 0. \tag{36}$$

Since the matrix $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$ is invertible, the left hand sides of (33), (34), (35), (36) cannot be all equal to zero, unless all c_i are zero, which would imply that each a_j is a function of x_j only, hence that the Fubini-Study metric g_{FS} is flat. It then follows that $c_i = 0$, for some i. As just observed, this implies that a_i is a function of x_i only, and we can then consider that a_i is constant. By (13), this implies that $R^{FS}_{e_i,e_j}e_k = -a_j^{-1}(\nabla_{e_i}da_j)(e_k)e_j$ for any $j \neq k$, both distinct from i; in particular, we then have:

$$g_{FS}(R^{FS}_{e_i,e_j}e_k, e_i) = 0. \tag{37}$$

On the other hand, by (19), with $c = 4$, we have that

$$R^{FS}_{e_i,e_j}e_k = \omega(e_i, e_k)\,Je_j - \omega(e_j, e_k)\,Je_i + 2\omega(e_i, e_j)\,Je_k, \tag{38}$$

hence $g_{FS}(R^{FS}_{e_i,e_j}e_k, e_i) = -3\omega(e_i, e_j)\,\omega(e_i, e_k)$; from (25), we then infer:

$$g_{FS}(R^{FS}_{e_i,e_j}e_k, e_i) = \pm 1, \tag{39}$$

which evidently contradicts (37). □

Remark 3.3. Since every Riemannian metric in dimension 2 is Kähler and conformally flat, any product of two Riemannian surfaces is Kähler and admits local orthogonal coordinates. D. Johnson has raised the question whether, conversely, any 4-dimensional Kähler manifold admitting local orthogonal coordinates is locally a product of two Riemannian surfaces. It turns out that the answer is negative, as the following example shows.

Example 3.4. Let (Σ, g) be a 3-dimensional compact Sasakian manifold with non-constant sectional curvature. The Riemannian cone $(\bar{\Sigma} := \mathbb{R} \times \Sigma, \bar{g} := e^{2t}(dt^2 + g))$ is Kähler, and locally irreducible as Riemannian manifold, according to a result of S. Gallot [6]. In particular $(\bar{\Sigma}, \bar{g})$ is not locally a product of Riemannian surfaces.

On the other hand, since the 3-dimensional Riemannian manifold (Σ, g) admits local orthogonal coordinates by DeTurck and Yang's result [4], $(\bar{\Sigma}, \bar{g})$, which is conformal to the Riemannian product $(\Sigma, g) \times \mathbb{R}$, also admits local orthogonal coordinates.

4. The quaternionic projective space

In this section, we consider the quaternionic projective space $\mathbb{HP}^q$, $q \geq 2$, equipped with its standard quaternionic Kähler structure, determined by the Riemannian metric g and a rank 3 subbundle, Q, of the bundle of skew-symmetric endomorphisms of $T\mathbb{HP}^q$, preserved by the Levi-Civita of g and locally generated by triplets of almost complex structures, J_1, J_2, J_3, such that $J_1J_2J_3 = -\mathrm{Id}$. For any such triplet, we set $\omega_\alpha := g(J_\alpha \cdot, \cdot)$, $\alpha = 1, 2, 3$.

If $q = 1$, $\mathbb{HP}^1$ is isometric, up to scaling, to the standard round sphere $\mathbb{S}^4$ and therefore does admit orthogonal coordinates, cf. Example 2.4. We have however:

Proposition 4.1. *For $q \geq 2$, the quaternionic projective space $\mathbb{HP}^q$ admits no local orthogonal coordinates.*

Proof. Up to scaling, the curvature, R, of $\mathbb{HP}^q$, viewed as a symmetric endomorphism of $\Lambda^2 T\mathbb{HP}^q$, is locally given by:

$$\mathrm{R}(X \wedge Y) = X \wedge Y + \sum_{\alpha=1}^{3} J_\alpha X \wedge J_\alpha Y + 2\sum_{\alpha=1}^{3} \omega_\alpha(X, Y)\, \omega_\alpha^{\sharp_g}, \tag{40}$$

for any vector fields X, Y, where $\omega_\alpha^{\sharp_g}$ denotes the section of $\Lambda^2 T\mathbb{HP}^q$ determined by ω_α by Riemannian duality.

Assume for a contradiction, that $\mathbb{HP}^q$ admits an orthogonal system of coordinates, $\{x_1, \ldots, x_{4q}\}$, on some connected open set $\mathcal{U}$ where Q is trivialized by a triplet J_1, J_2, J_3 as above, where R is then given by (40), and denote by $\{e_1, \ldots, e_{4q}\}$ the corresponding orthonormal frame, as defined by (9). For convenience, we introduce the notation:

$$a_{ijk\ell} := \sum_{\alpha=1}^{3} \omega_\alpha(e_i, e_j)\omega_\alpha(e_k, e_\ell). \tag{41}$$

From (14) and (40), we should have

$$0 = g(\mathrm{R}(e_i \wedge e_j), e_k \wedge e_\ell) = a_{ikj\ell} + a_{kji\ell} + 2a_{ijk\ell}, \tag{42}$$

for any pairwise distinct 4-uplets i, j, k, ℓ. For any such 4-uplet, we then infer $a_{ikj\ell} + a_{kji\ell} + a_{jik\ell} = 3a_{jik\ell}$. Since the left hand side of this identity is invariant by circular permutation of i, k, j, we thus obtain:

$$a_{ikj\ell} = a_{kji\ell} = a_{jik\ell}, \tag{43}$$

for any pairwise distinct 4-uplets i, j, k, ℓ. From the first equality in (44), we infer that $\sum_{\alpha=1}^{3} \omega_\alpha(e_i, e_k) J_\alpha e_j + \sum_{\alpha=1}^{3} \omega_\alpha(e_j, e_k) J_\alpha e_i$ is orthogonal to ℓ, for any ℓ distinct from i, j, k, so that

$$\sum_{\alpha=1}^{3} \omega_\alpha(e_i, e_k) J_\alpha e_j + \sum_{\alpha=1}^{3} \omega_\alpha(e_j, e_k) J_\alpha e_i \in \mathrm{span}(e_i, e_j, e_k), \tag{44}$$

for any pairwise distinct triplets i, j, k.

We now fix i, k such that $\omega_1(e_i, e_k) \neq 0$ (for any fixed i, we can obviously chose such a k). Denote $b_\alpha := \omega_\alpha(e_i, e_k)$. Then $b_1 \neq 0$ and the endomorphism

$$J := \frac{b_1 J_1 + b_2 J_2 + b_3 J_3}{\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

is a well-defined section of Q on $\mathcal{U}$. From (44), we get:

$$J e_j \in \mathrm{span}(e_i, e_j, e_k, J_1 e_i, J_2 e_i, J_3 e_i), \tag{45}$$

for any j distinct from i, k.

At this point of the argument, we use the following easy general fact:

Lemma 4.2. *Let (E, J) be a complex vector space of any dimension, V a* real *subspace of E and v an element of E such that Jv belongs to $\mathbb{R}\, v + V$. Then, v belongs to $V + JV$.*

Proof. By hypothesis, $Jv = a\,v + w$, for some real number a and some element w of V. If $a = 0$, then $v = -Jw$ belongs to JV. If $a \neq 0$, then $v = a^{-1} Jv - a^{-1} w$, hence $Jv = -a^{-1} v - a^{-1} Jw$. Since we also have $Jv = a\,v + w$, it follows that $(a + a^{-1}) v = -w - a^{-1} Jw$. As $(a + a^{-1}) \neq 0$, we infer that $v = -(a + a^{-1})^{-1} (w + a^{-1} w)$ belongs to $V + JV$. This concludes the proof of the lemma. □

By using Lemma 4.2 for $V := \mathrm{span}(e_i, e_k, J_1 e_i, J_2 e_i, J_3 e_i)$, we readily infer from (45) that e_j belongs to $V + JV$, for any j distinct from i, k, so actually for any j, as e_i and e_k already belong to V. We would then eventually get:

$$V + JV = T\mathbb{HP}^q. \tag{46}$$

On the other hand, $V + JV$ is generated by $e_i, e_k, J_1 e_i, J_2 e_i, J_3 e_i, J e_k$, hence is of dimension at most equal to 6, whereas the dimension of $T\mathbb{HP}^q$, is equal to $4q \geq 8$. This contradiction completes the proof of Proposition 4.1. □

5. Open questions

While writing these notes, we have encountered several natural questions about metrics admitting orthogonal coordinates whose answers are unknown to us. We list some of them below:

– Is there any topological obstruction for the existence of metrics with orthogonal coordinates, or does every smooth manifold carry such metrics?

– A Riemannian product of Riemannian manifolds with orthogonal coordinates also has orthogonal coordinates. Conversely, if a Riemannian product has orthogonal coordinates, does this hold for the two factors?

– In view of Remark 3.3 and Example 3.4, one can ask whether there exist any 4-dimensional Kähler metrics with orthogonal coordinates which are not locally conformal to a Riemannian product.

– For a given Riemannian metric, can one find obstructions (in terms of the curvature tensor) to the existence of orthogonal coordinates, other than those given by (14)? Note that the Fubini-Study metric on $\mathbb{CP}^2$ carries local orthonormal frames satisfying (14), but no orthogonal coordinates (by Proposition 3.2).

– Is every locally symmetric space carrying orthogonal coordinates locally conformally flat? The results in this paper constitute some evidence in favor of a positive answer to this question.

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