Dirac operators on hypersurfaces as large mass limits

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Abstract

We show that the eigenvalues of the intrinsic Dirac operator on the boundary of a Euclidean domain can be obtained as the limits of eigenvalues of Euclidean Dirac operators, either in the domain with a MIT-bag type boundary condition or in the whole space, with a suitably chosen zero order mass term.

Contents

1 Introduction

1.1 Problem setting and main results

The aim of the present paper is to make a new link between a number of recent papers on Dirac operators in bounded Euclidean domains with the theory of Dirac operators on manifolds, which is a classical topic in Riemannian geometry. Namely, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Σ. We are going to show that the intrinsic Dirac operator $\not{D}$ on Σ, which acts on sections of the spinor bundle of Σ, can be interpreted as a limit of Euclidean Dirac operators, either in Ω with a suitable boundary condition, or in the whole of $\mathbb{R}^n$, with a suitably chosen term containing a large mass.

For $n \geq 2$ and $N := 2^{[\frac{n+1}{2}]}$ let $\alpha_1, \dots, \alpha_{n+1}$ be anticommuting Hermitian $N \times N$ matrices with $\alpha_j^2 = I_N$, where I_N is the $N \times N$ identity matrix. The associated Dirac operator with a mass $m \in \mathbb{R}$ acts on functions $u : \mathbb{R}^n \to \mathbb{C}^N$ (spinors) by the differential expression

$$D_m u = -\mathrm{i} \sum_{j=1}^{n} \alpha_j \frac{\partial u}{\partial x_j} + m\alpha_{n+1} u, \tag{1}$$

see e.g. [24]. We remark that the expression D_m does not correspond to the intrinsic Dirac operator on $\mathbb{R}^n$ (see Subsection 2.2) and can be interpreted as follows: the intrinsic operator $\widetilde{D}$ in $\mathbb{R}^{n+1}$ is defined as

$$\widetilde{D} v = -\mathrm{i} \sum_{j=1}^{n+1} \alpha_j \frac{\partial v}{\partial x_j},$$

and acts on functions $v : \mathbb{R}^{n+1} \to \mathbb{C}^N$, then assuming that v is of the form $v(x_1, \ldots, x_{n+1}) = e^{\mathrm{i}mx_{n+1}} u(x_1, \ldots, x_n)$ one obtains $\widetilde{D} v = e^{\mathrm{i}mx_{n+1}} D_m u$.

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we define the associated $N \times N$ matrices $\Gamma(x)$ by

$$\Gamma(x) := \sum_{j=1}^{n} x_j \alpha_j. \tag{2}$$

Denote by ν the unit normal at Σ pointing to the exterior of Ω and consider the $N \times N$ matrices

$$\mathcal{B}(s) := -\mathrm{i}\alpha_{n+1} \Gamma\big(\nu(s)\big), \quad s \in \Sigma. \tag{3}$$

By the Dirac operator A_m in Ω with a mass $m \in \mathbb{R}$ and the infinite mass boundary condition (also called MIT Bag boundary condition) we mean the operator in $L^2(\Omega, \mathbb{C}^N)$ given by

$$A_m u = D_m u \text{ on the domain } \mathcal{D}(A_m) = \big\{ u \in H^1(\Omega, \mathbb{C}^N) : \ u = \mathcal{B} u \text{ on } \Sigma \big\},$$

which is self-adjoint with compact resolvent (see Proposition A.2). Remark that in order to have a simpler writing we prefer to use the same symbol u for a function $u \in H^1(\Omega)$ and its zero order boundary trace $\gamma_0 u \in H^{\frac{1}{2}}(\Sigma)$ on Σ, where H^k stands for the usual Sobolev space of order $k \in \mathbb{R}$.

In addition, for $m, M \in \mathbb{R}$ we consider the following operator $B_{m,M}$ in $L^2(\mathbb{R}^n, \mathbb{C}^N)$, which is the Dirac operator in the whole space with the mass m in Ω and the mass M outside Ω, i.e.

$$B_{m,M} = D_0 + \big[m 1_\Omega + M(1 - 1_\Omega) \big] \alpha_{n+1} \equiv D_m + (M - m)(1 - 1_\Omega)\, \alpha_{n+1}$$

with domain $\mathcal{D}(B_{m,M}) = H^1(\mathbb{R}^n, \mathbb{C}^N)$. We are going to show that the eigenvalues of the intrinsic Dirac operator $\not{D}$ on Σ (whose construction is briefly reviewed in Subsection 2.2) and of the Euclidean Dirac operators A_m and $B_{m,M}$, are related to each other for suitable values of m and M. We provide first strict formulations of the main results and then discuss their relation with the existing literature.

For a self-adjoint lower semibounded operator T and $j \in \mathbb{N}$ we denote by $E_j(T)$ the jth eigenvalue of T, if it exists, when enumerated in the non-decreasing order and counted with multiplicities. First we show that the eigenvalues of $\not{D}^2$ on Σ are the limits of the eigenvalues of the square of the MIT Bag Dirac operator A_m on Ω for large negative m:

Theorem 1.1. *For each $j \in \mathbb{N}$ there holds $E_j(\not{D}^2) = \lim_{m \to -\infty} E_j(A_m^2)$.*

Then we show that, in turn, for any fixed m, the MIT Bag Dirac operators A_m on Ω can be viewed as the limits of the Dirac operators $B_{m,M}$ in the whole space with a large mass outside Ω (which justifies the use of the term "infinite mass boundary condition"):

Theorem 1.2. *Let $j \in \mathbb{N}$ and $m \in \mathbb{R}$. Then there exists $M_j > 0$ such that $B_{m,M}^2$ has at least j discrete eigenvalues for $M > M_j$, and $\lim_{M \to +\infty} E_j(B_{m,M}^2) = E_j(A_m^2)$.*

It is easily seen (Proposition A.1) that the essential spectrum of $B_{m,M}$ is equal to $\big(-\infty, -|M|\big] \cup \big[|M|, +\infty\big)$ while the spectrum in $\big(-|M|, |M|\big)$ consists of at most finitely many discrete eigenvalues. The above Theorem 1.2 shows, in particular, that the number of discrete eigenvalues grows unboundedly as M becomes large: the essential spectrum escapes to ∞, and the individual eigenvalues have finite limits.

Finally, by an additional construction we find an asymptotic regime in which the eigenvalues of $\not{D}^2$ on Σ are directly recovered as the limits of the eigenvalues of the square of the Dirac operator $B_{m,M}^2$ on the whole space:

Theorem 1.3. *Let $j \in \mathbb{N}$. Then there exist $\mu_j > 0$ and $\delta_j > 0$ such that the operator $B_{m,M}^2$ has at least j discrete eigenvalues for $m < -\mu_j$ and $M > |m|/\delta_j$, and the jth eigenvalue $E_j(B_{m,M}^2)$ converges to $E_j(\not{D}^2)$ as $m \to -\infty$ and $M \to +\infty$ with $m/M \to 0$.*

In the recent paper [2] the operator A_m in three dimensions was considered, and it was shown that for each $j \in \mathbb{N}$ one has $\lim_{m\to-\infty} E_j(A_m^2) = E_j(L)$ for some operator L on Σ given by its sesquilinear form. Hence, Theorem 1.1 extends this result in two directions: first, we consider arbitrary dimensions and, second, we show that the operator L in question is in fact unitarily equivalent to the geometric Dirac operator $\not{D}^2$, which is the central observation. Some analogs of Theorem 1.2 in two and three dimensions were obtained very recently in [1, 5, 23], and we extend them to all dimensions and shorten the existing proofs by making use of the monotone convergence. As for Theorem 1.3, the interpretation of $\not{D}$ using an infinite mass jump on Σ seems to be completely new, as we are not aware of any previous work containing similar results. In a sense, it can be viewed as a potential-induced collapse by analogy with Dirac operators on manifolds converging to a lower-dimensional structure [17, 20]. Nevertheless, our situation appears to have the special feature that the rank of the spin bundle on which $\not{D}$ acts is the half of that for A_m and $B_{m,M}$, which could be viewed as an effect of the boundary condition $u = \mathcal{B}u$ on Σ. (We recall that A_m and $B_{m,M}$ are *not* intrinsic Dirac operators in $\mathbb{R}^n$, see above.) Our results have a direct application to estimating the central gap (i.e. the first eigenvalue) of A_m or $B_{m,M}$: in the respective asymptotic regime one is simply reduced to the eigenvalue estimate for the Dirac operator $\not{D}$, for which a number of results are available in terms of the geometry of Σ, (we refer to the book [12] for a review). We mention explicitly some simple situations. Consider first the case $n \geq 3$:

Corollary 1.4. *Let $n \geq 3$ and $S(\Sigma)$ denote the minimum scalar curvature of Σ. Assume that $S(\Sigma) > 0$, which holds at least if and each maximal connected component of Σ is strictly convex, then*

$$\lim_{m\to-\infty} E_1(A_m^2) \geq \frac{n-1}{4(n-2)}\, S(\Sigma),$$

$$\lim_{m\to-\infty,\, M\to+\infty,\, m/M\to 0} E_1(B_{m,M}^2) \geq \frac{n-1}{4(n-2)}\, S(\Sigma).$$

Proof. As Σ is a compact $(n-1)$-dimensional spin manifold, Friedrich's estimate [11, Sec. 5.1] gives $E_1(\not{D}^2) \geq \frac{n-1}{4(n-2)}\, S(\Sigma)$, and the rest follows from Theorems 1.1 and 1.3. □

The case $n = 2$ allows for a more precise analysis, as the spectrum of $\not{D}$ can be computed explicitly.

Corollary 1.5. *Let $n = 2$. Assume that Σ has k connected components Σ_p, $p = 1, \dots, k$, and ℓ_p stands for the length of Σ_p. Denote*

$$\varepsilon_j := \text{the } j\text{th element of the disjoint union } \bigsqcup_{p=1}^{k} \bigsqcup_{r\in\mathbb{Z}} \Big\{ \frac{\pi^2}{\ell_p^2} (2r-1)^2 \Big\}$$

(when numbered in the non-decreasing order). Then for each $j \in \mathbb{N}$ there holds

$$\lim_{m\to-\infty} E_j(A_m^2) = \varepsilon_j, \qquad \lim_{m\to-\infty,\, M\to+\infty,\, m/M\to 0} E_j(B_{m,M}^2) = \varepsilon_j.$$

Proof. If one denotes by $\not{D}_p$ the intrinsic Dirac operator on Σ_p, then $\not{D} = \bigoplus_{p=1}^{k} \not{D}_p$, and the eigenvalues of $\not{D}_p$ are $(2r-1)\pi/\ell_p$, $r \in \mathbb{Z}$, by a simple explicit computation (see Appendix C). The rest follows again by Theorems 1.1 and 1.3. □

It is well known, see e.g. [12, Theorem 1.3.7], that the spectrum of the intrinsic Dirac operator on a k-dimensional compact spin manifold is symmetric with respect to zero provided $k \notin 4\mathbb{Z} + 3$. In our language (as the dimension of Σ is $n-1$) this means that for $n \notin 4\mathbb{Z}$ one has $\dim\ker(\not{D} - E) = \dim\ker(\not{D} + E)$ for any $E \in \mathbb{R}$. On the other hand, elementary considerations show that the symmetry of the spectrum for $n \notin 4\mathbb{Z}$ holds for and A_m and $B_{m,M}$ with arbitrary m and M as well (see Propositions A.1 and A.2). Hence, in these dimensions, the spectra of $\not{D}$, A_m, $B_{m,M}$ are uniquely recovered from the spectra of their squares: if one denotes by $E_j^\pm$ the jth non-negative/non-positive eigenvalue (numbered in the order of increasing absolute value), then the preceding results imply the convergences

$$E_j^\pm(\not{D}) = \lim_{m\to-\infty} E_j^\pm(A_m), \quad E_j^\pm(A_m) = \lim_{M\to+\infty} E_j^\pm(B_{m,M}),$$
$$E_j^\pm(\not{D}) = \lim_{m\to-\infty,\, M\to+\infty,\, m/M\to 0} E_j^\pm(B_{m,M}).$$

It would be interesting to understand if such a convergence also holds for $n \in 4\mathbb{Z}$. Indeed, the study of the squares of the operators is insufficient then, and a reconsideration of the whole proof strategy using more involved min-max characterizations of eigenvalues in gaps of the essential spectrum [9] could be necessary. This remains an open question for future work.

The text is organized as follows. In Subsection 1.2 we recall a link between self-adjoint operators and sesquilinear forms, choose a suitable notation, and then recall two important tools of the spectral analysis: the min-max characterization of the eigenvalues and the monotone convergence. In Section 2 we construct the sesquilinear forms for the squares of all the Dirac operators in question, which will allow one to obtain eigenvalue estimates based on the min-max principle: in Subsection 2.1 we recall the definition of various curvatures of Σ and study A_m and $B_{m,M}$, and in Subsection 2.2 we introduce an operator L, which already appeared in [2] for the three-dimensional case, and prove that it is unitarily equivalent to $\not{D}^2$. The unitary equivalence is shown using the Schrödinger-Lichnerowicz formula for extrinsically defined Dirac operators whose elementary proof for our Euclidean setting is given in Appendix B for reader's convenience. In Section 3 we collect some preliminary constructions: in Subsection 3.1 we study the eigenvalues and the eigenfunctions of one-dimensional Laplacians S and S' with a large parameter in the boundary conditions, and in Subsection 3.2 we give some computations in tubular coordinates near Σ.

In Section 4 we prove Theorem 1.1. We first reduce the problem to the spectral analysis in small tubular δ-neighborhoods of Σ, and in order to work in $\Sigma\times(0,\delta)$ we use the computations from Subsection 3.2. The upper bound is obtained by taking as test functions the tensor products of the eigenfunctions of (a small perturbation of) the effective operator L on Σ with the first eigenfunction of the model operator S in the normal direction. For the lower bound we perform a unitary transform, which is just the expansion in eigenfunctions of the second model operator S' in the normal variable, thus transforming the problem into the study of a monotonically increasing sequence of operators. A simple application of the respective machinery presented in Subsection 1.2 then shows that only the projection onto the lowest eigenfunction of S' contributes to the asymptotics of the individual eigenvalues, which induces an effective operator acting on Σ only.

The proof of Theorem 1.2 is presented in Section 5. To establish the upper bound we construct first an extension operator from Σ to the exterior of Ω with a suitable control in terms of the mass M, and then use the corresponding extensions of the eigenfunctions of A_m to construct test functions for $B_{m,M}$ used in the min-max principle. For the lower bound we first decouple the two sides of Ω in order to deal separately with Ω and its exterior, then it is easily seen that the exterior does not contribute to the lowest eigenvalues, while the part in Ω appears to be monotonically increasing in M and then easily handled with the help of the monotone convergence. The overall scheme here is very close to the one used in [23] for the two-dimensional case.

In Section 6 we prove Theorem 1.3. The proof is by combining in a new way various components from the preceding analysis, but we still provide a complete self-contained argument.The upper bound is obtained by taking the eigenfunctions of the operator L on Σ and extending them on both sides of Σ by taking tensor products with the first eigenfunctions of the model operators S and S' in the two normal directions, and then using them as test functions in the min-max principle for $B^2_{m,M}$. For the lower bound we again decouple the two sides of Σ and eliminate the exterior of Ω as in Theorem 1.2. The analysis of the part in Ω is then quite similar to the one in Theorem 1.1: one is first reduced to the analysis in a thin tubular neighborhood of Ω, and then one applies a unitary transform in order obtain a monotone family with an explicit limit operator. We remark that our proof of Theorem 1.3 is independent of the other two theorems, in particular, it does not make any use of the Dirac operator A_m in Ω.

As will be seen from the proofs, the two sides of Σ play symmetric roles, and, as a result, a number of similar convergences hold in other asymptotic regimes. In particular, one can consider an additional operator A'_m in $L^2(\Omega,\mathbb{C}^N)$ given by

$$A'_m u = D_m u \text{ on the domain } \mathcal{D}(A'_m) = \big\{u\in H^1(\Omega,\mathbb{C}^N) : u = -\mathcal{B}u \text{ on } \Sigma\big\},$$

i.e. the only difference from A_m is in the sign in the boundary condition, and which is also self-adjoint with compact resolvent (see Proposition A.2). Then the following complementary results can be proved (the proofs are almost literally the same as for above Theorems 1.1, 1.2, 1.3, one only needs to apply suitable sign changes, see e.g. Remarks 2.2, 2.5, 4.1 in the text).

Theorem 1.6. *For each $j \in \mathbb{N}$ there holds $E_j(\not{D}^2) = \lim_{m\to+\infty} E_j({A'_m}^2)$.*

Theorem 1.7. *Let $j \in \mathbb{N}$ and $m \in \mathbb{R}$. Then there exists $M_j > 0$ such that $B^2_{m,M}$ has at least j discrete eigenvalues for $M < -M_j$, and $\lim_{M\to-\infty} E_j(B^2_{m,M}) = E_j({A'_m}^2)$.*

Finally, the convergence of the eigenvalues of $B^2_{m,M}$ to those of $\not{D}^2$ can be recovered in three additional asymptotic regimes:

Theorem 1.8. *Let $j \in \mathbb{N}$. Then there exist μ_j and $\varepsilon_j > 0$ such that if one of the following three conditions is satisfied:*

$$\begin{aligned} (i) &\quad M > \mu_j \text{ and } m < -M/\delta_j,\\ (ii) &\quad m > \mu_j \text{ and } M < -m/\delta_j,\\ (iii) &\quad M < -\mu_j \text{ and } m > |M|/\delta_j, \end{aligned}$$

then the operator $B^2_{m,M}$ has at least j discrete eigenvalues. Furthermore, the eigenvalue $E_j(B^2_{m,M})$ converges to $E_j(\not{D}^2)$ in the following asymptotic regimes:

$$\begin{aligned} (I) &\quad m \to -\infty \text{ and } M \to +\infty \text{ with } M/m \to 0,\\ (II) &\quad m \to +\infty \text{ and } M \to -\infty \text{ with } m/M \to 0,\\ (III) &\quad m \to +\infty \text{ and } M \to -\infty \text{ with } M/m \to 0. \end{aligned}$$

Our approach based on the monotone convergence was chosen on purpose in order to obtain the main terms in a transparent way and to be able to concentrate on the geometric aspects. We expect that, in some form, the above convergence results should also hold for a suitable class of unbounded hypersurfaces Σ. However, a rigorous study of this situation has to include highly non-trivial (essential) self-adjointness aspects of the associated operators $\not{D}$ and A_m, and the language of eigenvalues does not seem to be most adapted: the resolvents of these operators are non compact anymore, and their discrete spectra can be empty. A more precise analysis involving detailed remainder estimates and reformulation in terms of suitable operator convergences should be possible in the spirit of recent works in specific dimensions, e.g. [1, 2, 5, 14], but a rigorous implementation requires a considerably higher technical effort, and we prefer to discuss these aspects in separate forthcoming works.

1.2 Notation, min-max principle, monotone convergence

The most part of the subsequent spectral analysis is based on the min-max principle for the eigenvalues of self-adjoint operators and uses rather sesquilinear forms than operators (in particular, most operators are introduced just through their sesquilinear forms, while the action and the domain of the operators are not specified explicitly). The correspondence between operators and sesquilinear forms is well known, and it is based on classical representation theorems, see e.g. the monographs by Kato [15, Ch. 6, §1–2] or Reed–Simon [18, Sec. VIII.6]. In order to avoid potential confusions (due to various conventions used by different communities), we recall here some basic facts of the theory and introduce some notation.

In this text we only work with complex Hilbert spaces. Let $\mathcal{G}$ be a Hilbert space, then by $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ we denote the scalar product in $\mathcal{G}$, which is assumed antilinear with respect to the *first* argument, and the associated norm is denoted $\|\cdot\|_{\mathcal{G}}$. A sesquilinear form t in $\mathcal{G}$ defined on a dense subspace $\mathcal{D}(t)$ of $\mathcal{G}$ is a map

$$\mathcal{D}(t) \times \mathcal{D}(t) \ni (u,v) \mapsto t(u,v) \in \mathbb{C}$$

which is antilinear with respect to the first argument and linear with respect to the second one, and it is called Hermitian if $t(v,u) = \overline{t(u,v)}$ for all $u, v \in \mathcal{D}(t)$. As a consequence of the polar identity, a Hermitian sesquilinear form t is uniquely determined by its diagonal values $t(u,u)$ with $u \in \mathcal{D}(t)$. An Hermitian sesquilinear form t is called lower semibounded if there is $c \in \mathbb{R}$ such that $t(u,u) \geq c\|u\|^2_{\mathcal{G}}$ for all $u \in \mathcal{D}(t)$. Such a form is then called closed if $\mathcal{D}(t)$ endowed with the scalar product $\langle u,v\rangle_t := t(u,v) + (1-c)\langle u,v\rangle_{\mathcal{G}}$ is a Hilbert space. With such a sesquilinear form t one associates a self-adjoint operator T in $\mathcal{G}$ uniquely defined by the following two conditions: (a) the domain $\mathcal{D}(T)$ of T is contained in $\mathcal{D}(t)$ and (b) $t(u,v) = \langle u, Tv\rangle_{\mathcal{G}}$ all $u, v \in \mathcal{D}(T)$, and we then say that T is the *self-adjoint operator generated by the form* t. It is worth noting that $\mathcal{D}(T) \neq \mathcal{D}(t)$ in general.

On the other hand, let T be a self-adjoint operator in $\mathcal{G}$ with domain $\mathcal{D}(T)$. It is called lower semibounded if for some $c \in \mathbb{R}$ one has $\langle u, Tu\rangle_{\mathcal{G}} \geq c\|u\|^2_{\mathcal{G}}$ for all $u \in \mathcal{D}(T)$, or $T \geq c$ for short. In such a case, the completion of $\mathcal{D}(T)$ with respect to the scalar product $\langle u,v\rangle_{\mathcal{Q}(T)} := \langle u, Tv\rangle_{\mathcal{G}} + (1-c)\langle u,v\rangle_{\mathcal{G}}$ is called the *form domain* of T and is denoted by $\mathcal{Q}(T)$. The map $\mathcal{D}(T) \times \mathcal{D}(T) \ni (u,v) \mapsto \langle u, Tv\rangle_{\mathcal{G}}$ then uniquely extends to a closed lower semibounded Hermitian sesquilinear form t with domain $\mathcal{D}(t) = \mathcal{Q}(T)$, which will be called the *sesquilinear form generated by the operator* T. In turn, T is exactly the self-adjoint operator generated by this form t. To have a shorter writing (and to reduce the number of symbols in use), we will write

$$T[u,v] := t(u,v) \text{ for } u, v \in \mathcal{Q}(T),$$

in particular, one has the simple equality $T[u,v] = \langle u, Tv\rangle_{\mathcal{G}}$ if $v \in \mathcal{D}(T)$. We further recall that due to the spectral theorem we have

$$\begin{gathered}\mathcal{Q}(T) = \mathcal{D}\left(\sqrt{T-c}\right) = \mathcal{D}(\sqrt{|T|}),\\ T[u,v] \equiv t(u,v) = \langle \sqrt{T-c}\,u, \sqrt{T-c}\,v\rangle_{\mathcal{G}} + c\langle u,v\rangle_{\mathcal{G}}, \quad u,v \in \mathcal{Q}(T),\end{gathered}$$

and the operator T has compact resolvent iff its form domain $\mathcal{Q}(T)$ endowed with the above scalar product $\langle\cdot,\cdot\rangle_t \equiv \langle\cdot,\cdot\rangle_Q$ is compactly embedded into $\mathcal{G}$. It follows from the preceding discussion that a lower semibounded self-adjoint operator T is uniquely determined by the knowledge of its form domain $\mathcal{Q}(T)$ and of the diagonal values $T[u,u]$ of its sesquilinear form for all $u \in \mathcal{Q}(T)$. Many operators appearing in the subsequent discussion will be introduced in this way.

Using the above convention let us recall the min-max characterization of eigenvalues. Let T be a lower semibounded self-adjoint operator in an infinite-dimensional Hilbert space $\mathcal{G}$. For $j \in \mathbb{N}$ we denote

$$E_j(T) := \inf_{\substack{\mathcal{L} \subset \mathcal{Q}(T)\\ \dim \mathcal{L} = j}} \sup_{\substack{u \in \mathcal{L}\\ u \neq 0}} \frac{T[u,u]}{\|u\|^2_{\mathcal{G}}},$$

The classical min-max principle states that $E_j(T) \leq \inf \operatorname{spec}_{\text{ess}} T$ for any j, and, if the inequality is strict, that $E_j(T)$ is the jth eigenvalue of T when enumerated in the non-decreasing order and counted with multiplicities, see e.g. [19, Section XIII.1]. In particular, one always has $E_1(T) = \inf \operatorname{spec} T$, and if T has compact resolvent, then $E_j(T)$ is the jth eigenvalue of T for any $j \in \mathbb{N}$. The main well-known consequence of the min-max principle we are going to use is as follows (the proof directly follows from the definition):

Proposition 1.9. *Let T and T' be lower semibounded self-adjoint operators in infinite-dimensional Hilbert spaces $\mathcal{G}$ and $\mathcal{G}'$ respectively. Assume that there exists a linear map $J : \mathcal{Q}(T) \to \mathcal{Q}(T')$ such that $\|Ju\|_{\mathcal{G}'} = \|u\|_{\mathcal{G}}$ and $T'[Ju, Ju] \le T[u, u]$ for all $u \in \mathcal{Q}(T)$, then $E_j(T') \le E_j(T)$ for any $j \in \mathbb{N}$.*

We will also use some classical results on the monotone convergence of operators. The following particular case will be sufficient for our purposes:

Proposition 1.10. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}_\infty$ be a closed subspace of $\mathcal{H}$ endowed with the induced scalar product. Let*

- *T_n with $n \in \mathbb{N}$ be lower semibounded self-adjoint operators with compact resolvents in $\mathcal{H}$,*
- *T_∞ be a lower semibounded self-adjoint operator with compact resolvent in $\mathcal{H}_\infty$*

such that the following conditions are satisfied:

- *the sequence (T_n) is monotonically increasing, i.e.*

$$\mathcal{Q}(T_n) \supset \mathcal{Q}(T_{n+1}), \quad T_n[u, u] \le T_{n+1}[u, u] \quad \text{for all } n \in \mathbb{N} \text{ and } u \in \mathcal{Q}(T_{n+1}),$$

- *one has the equalities*

$$\mathcal{Q}(T_\infty) = \Big\{u \in \bigcap_{n\in\mathbb{N}} \mathcal{Q}(T_n) : \quad \sup T_n[u, u] < \infty\Big\},$$
$$T_\infty[u, u] = \lim_{n\to+\infty} T_n[u, u] \text{ for each } u \in \mathcal{Q}(T_\infty),$$

then for each $j \in \mathbb{N}$ there holds $E_j(T_\infty) = \lim_{n\to+\infty} E_j(T_n)$.

The result follows, for example, from the constructions of [25, Abs. 3]: Satz 3.1 establishes a (generalized) strong resolvent convergence of T_n to T_∞ and Satz 3.2 gives the convergence of the eigenvalues. An interested reader may refer to the papers [6, 22, 25] dealing with the monotone convergence in a more general framework, i.e. beyond densely defined operators with compact resolvents.

2 Sesquilinear forms

This section is about preliminary computations for the various Dirac operators we are interested in. In particular, its aim is to obtain explicit expressions for the sesquilinear forms of the squares of these operators in terms of geometric properties of the hypersurface Σ. After recalling why A_m and $B_{m,M}$ are self-adjoint operators on their respective domains, Subsection 2.1 gives explicit expressions for the sesquilinear form of A_m^2 and $B_{m,M}^2$. In particular, the influence of the hypersurface Σ is encoded in a boundary term involving the mean curvature. The objective of Subsection 2.2 is to relate the sesquilinear form of the square of the intrinsic Dirac operator $\not{D}$ on Σ to a sesquilinear form which naturally arises when performing the asymptotic spectral analysis involved in the main theorems.

2.1 Sesquilinear forms for the squares of Euclidean Dirac operators

For the rest of the text we denote

$$\Omega^c := \mathbb{R}^n \setminus \overline{\Omega},$$

and recall that ν stands for the unit normal on $\Sigma \equiv \partial\Omega$ pointing to the exterior of Ω. The shape operator $W : T\Sigma \to T\Sigma$ is given by

$$WX := \nabla_X \nu$$

with ∇ being the gradient in $\mathbb{R}^n$, and its eigenvalues $h_1, \dots, h_{n-1}$ are the principal curvatures of Σ. For $k = 1, \dots, n-1$ we will denote by H_k the *k-th mean curvature of Σ with respect to ν* defined by

$$H_k = \sum_{1\le j_1<\dots<j_k\le n-1} h_{j_1} \cdot \ldots \cdot h_{j_k},$$

in particular, $H_1 = h_1 + \ldots + h_{n-1} = \operatorname{tr} W$ is the mean curvature, $R = 2H_2 \equiv H_1^2 - |W|^2$ with $|W|^2 := \operatorname{tr}(W^2)$ is the scalar curvature. We set formally $H_k = 0$ for $k \ge n$.

We will need some identities related to the operator A_m, which is known to be self-adjoint with compact resolvent and essentially self-adjoint on $C^\infty(\overline{\Omega}, \mathbb{C}^N) \cap \mathcal{D}(A_m)$ (see Proposition A.2).

Lemma 2.1. *For all $u \in \mathcal{D}(A_m)$ there holds*

$$\langle A_m u, A_m u\rangle_{L^2(\Omega,\mathbb{C}^N)} = \int_\Omega \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x + \int_\Sigma \Big(m + \frac{H_1}{2}\Big)\,|u|^2\,\mathrm{d}s. \tag{4}$$

Proof. As A_m is essentially self-adjoint on $C^\infty(\overline{\Omega}, \mathbb{C}^N) \cap \mathcal{D}(A_m)$, it is sufficient to show (4) for $u \in C^\infty(\overline{\Omega}, \mathbb{C}^N) \cap \mathcal{D}(A_m)$. We use a partial integration formula in the form given in [13, Section 3, Eq. (13)].[1] The map Γ induces the extrinsically defined Dirac operator $\widetilde{D}^\Sigma$ in $L^2(\Sigma, \mathbb{C}^N)$ given by

$$\widetilde{D}^\Sigma \psi := \frac{H_1}{2}\,\psi - \Gamma(\nu)\sum_{j=1}^{n-1}\Gamma(e_j)\nabla_{e_j}\psi$$

with $(e_1, \dots, e_{n-1})$ being an orthonormal frame tangent to Σ. For $u \in C^\infty(\Omega, \mathbb{C}^N)$ one has the integral identity, see [13, Section 3, Eq. (13)],

$$\int_\Omega |D_0 u|^2\,\mathrm{d}x = \int_\Omega |\nabla u|^2\,\mathrm{d}x + \int_\Sigma \Big(\frac{H_1}{2}\,|u|^2 - \langle \widetilde{D}^\Sigma u, u\rangle\Big)\,\mathrm{d}s,$$

where D_0 is given by (1) with $m = 0$. Therefore, for $u \in \mathcal{D}_\infty(A_m)$ one has

[1] It was pointed out by one of the anonymous reviewers that some of the results of [13] on the self-adjointness of Dirac operators with boundary conditions depend on implicit assumptions, as the respective proofs of [13] use previous works which were found to contain flaws. We stress that we do not use any of the self-adjointness results stated in [13], and we only use some constructions of Sections 1–3 in [13], which are not concerned by the issues mentioned.

$$\begin{aligned}
&\langle A_m u, A_m u\rangle_{L^2(\Omega,\mathbb{C}^N)} \equiv \big\langle \big(D_0 + m\alpha_{n+1}\big)u, \big(D_0 + m\alpha_{n+1}\big)u\big\rangle_{L^2(\Omega,\mathbb{C}^N)}\\
&\quad= \langle D_0 u, D_0 u\rangle_{L^2(\Omega,\mathbb{C}^N)} + 2m\Re\langle D_0 u, \alpha_{n+1}u\rangle_{L^2(\Omega,\mathbb{C}^N)}) + m^2\langle \alpha_{n+1}u, \alpha_{n+1}u\rangle_{L^2(\Omega,\mathbb{C}^N)}\\
&\qquad\qquad= \int_\Omega \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x + \int_\Sigma \Big(\frac{H_1}{2}\,|u|^2 - \langle \widetilde{D}_\Sigma u, u\rangle\Big)\,\mathrm{d}s\\
&\qquad\qquad\qquad\qquad\qquad\qquad + 2m\Re\langle D_0 u, \alpha_{n+1}u\rangle_{L^2(\Omega,\mathbb{C}^N)}.
\end{aligned}\tag{5}$$

The operator $\widetilde{D}^\Sigma$ anticommutes with $\Gamma(\nu)$, see [13, Proposition 1]. As the matrix α_{n+1} anticommutes with all $\Gamma(x)$, it commutes with $\widetilde{D}^\Sigma$ by construction. Therefore, using the boundary condition for u we have the pointwise equalities

$$\begin{aligned}
\langle \widetilde{D}^\Sigma u, u\rangle &= \big\langle \widetilde{D}^\Sigma\big[-\mathrm{i}\alpha_{n+1}\Gamma(\nu)\big]u, u\big\rangle\\
&= \big\langle \mathrm{i}\alpha_{n+1}\Gamma(\nu)\widetilde{D}^\Sigma u, u\big\rangle = \big\langle \widetilde{D}^\Sigma u, -\mathrm{i}\Gamma(\nu)\alpha_{n+1}u\big\rangle\\
&= \big\langle \widetilde{D}^\Sigma u, \mathrm{i}\alpha_{n+1}\Gamma(\nu)u\big\rangle = -\langle \widetilde{D}^\Sigma u, u\rangle,
\end{aligned}$$

implying

$$\langle \widetilde{D}^\Sigma u, u\rangle = 0 \text{ on } \Sigma. \tag{6}$$

It remains to transform the third summand on the right-hand side of (5). Recall that due to the integration by parts for any $v, w \in H^1(\Omega, \mathbb{C}^N)$ we have

$$\int_\Omega \sum_{j=1}^n \langle \alpha_j\partial_j v, w\rangle_{\mathbb{C}^N}\,\mathrm{d}x = -\int_\Omega \sum_{j=1}^n \langle v, \alpha_j\partial_j w\rangle_{\mathbb{C}^N}\,\mathrm{d}x + \int_\Sigma \sum_{j=1}^n \langle \alpha_j\nu_j v, w\rangle_{\mathbb{C}^N}\,\mathrm{d}s,$$

which then gives

$$\begin{aligned}
\big\langle D_0 u, \alpha_{n+1}u\big\rangle_{L^2(\Omega,\mathbb{C}^N)} &= \int_\Omega \big\langle D_0 u, \alpha_{n+1}u\big\rangle_{\mathbb{C}^N}\,\mathrm{d}x\\
&= \int_\Omega \big\langle u, D_0\alpha_{n+1}u\big\rangle_{\mathbb{C}^N}\,\mathrm{d}x + \int_\Sigma \sum_{j=1}^n \langle -\mathrm{i}\alpha_j\nu_j u, \alpha_{n+1}u\rangle_{\mathbb{C}^N}\,\mathrm{d}s\\
&\qquad= -\int_\Omega \big\langle \alpha_{n+1}u, D_0 u\big\rangle_{\mathbb{C}^N}\,\mathrm{d}x + \int_\Sigma \big\langle -\mathrm{i}\Gamma(\nu)u, \alpha_{n+1}u\big\rangle_{\mathbb{C}^N}\,\mathrm{d}s.
\end{aligned}\tag{7}$$

Therefore, using the boundary condition,

$$\begin{aligned}
2m\Re\big\langle D_0 u, \alpha_{n+1}u\big\rangle_{L^2(\Omega,\mathbb{C}^N)} &= m\Big(\big\langle D_0 u, \alpha_{n+1}u\big\rangle_{L^2(\Omega,\mathbb{C}^N)} + \big\langle \alpha_{n+1}u, D_0 u\big\rangle_{L^2(\Omega,\mathbb{C}^N)}\Big)\\
&= m\int_\Sigma \big\langle -\mathrm{i}\Gamma(\nu)u, \alpha_{n+1}u\big\rangle_{\mathbb{C}^N}\,\mathrm{d}s = m\int_\Sigma \big\langle -\mathrm{i}\alpha_{n+1}\Gamma(\nu)u, u\big\rangle_{\mathbb{C}^N}\,\mathrm{d}s\\
&= m\int_\Sigma \langle \mathcal{B}u, u\rangle_{\mathbb{C}^N}\,\mathrm{d}s = m\int_\Sigma |u|^2_{\mathbb{C}^N}\,\mathrm{d}s.
\end{aligned}\tag{8}$$

Using the last equality and (6) in the right-hand side of (5) one arrives at the result. □

Remark 2.2. For the operator A'_m, only a sign change in (8) is needed, which results in

$$\langle A'_m u, A'_m u\rangle_{L^2(\Omega,\mathbb{C}^N)} = \int_\Omega \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x + \int_\Sigma \Big(-m + \frac{H_1}{2}\Big)\,|u|^2\,\mathrm{d}s. \tag{9}$$

With the help of the Fourier transform one easily establishes the self-adjointness of $B_{m,M}$ as well as the essentially self-adjointness on $C_c^\infty(\mathbb{R}^n, \mathbb{C}^N)$, see Proposition A.1. Let us obtain a suitable expression for the sesquilinear form of $B_{m,M}^2$.

Lemma 2.3. *For all $u \in \mathcal{D}(B_{m,M})$ there holds*

$$\langle B_{m,M}u, B_{m,M}u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)} = \int_\Omega \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x + \int_{\Omega^c} \big(|\nabla u|^2 + M^2|u|^2\big)\,\mathrm{d}x \\ + (M-m)\Big(\int_\Sigma |\mathcal{P}_- u|^2\,\mathrm{d}s - \int_\Sigma |\mathcal{P}_+ u|^2\,\mathrm{d}s\Big), \quad (10)$$

where $\mathcal{P}_\pm(s) := \dfrac{I_N \pm \mathcal{B}(s)}{2}$ *for* $s \in \Sigma$.

Proof. As $B_{m,M}$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^n, \mathbb{C}^N)$, it is sufficient to obtain (10) for $u \in C_c^\infty(\mathbb{R}^n, \mathbb{C}^N)$. Representing $B_{m,M}u = D_M u + (m-M)1_\Omega \alpha_{n+1} u$ we have

$$\begin{aligned}
&\langle B_{m,M}u, B_{m,M}u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\\
&\quad= \langle D_M u + (m-M)1_\Omega\alpha_{n+1}u, D_M u + (m-M)1_\Omega\alpha_{n+1}u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\\
&\quad= \langle D_M u, D_M u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)} + (m-M)^2\langle 1_\Omega\alpha_{n+1}u, 1_\Omega\alpha_{n+1}u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\\
&\qquad + (m-M)\Big(\langle D_M u, 1_\Omega\alpha_{n+1}u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)} + \langle 1_\Omega\alpha_{n+1}u, D_M u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\Big)\\
&\quad= \int_{\mathbb{R}^n} \big(|\nabla u|^2 + M^2|u|^2\big)\,\mathrm{d}x + (m-M)^2\int_\Omega |u|^2\,\mathrm{d}x\\
&\qquad + (m-M)\Big(\langle D_M u, 1_\Omega\alpha_{n+1}u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)} + \langle 1_\Omega\alpha_{n+1}u, D_M u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\Big).
\end{aligned}$$

Now using $D_M u = D_0 u + M\alpha_{n+1}u$ we transform the last summand as follows:

$$\begin{aligned}
&(m-M)\Big[\langle D_M u, 1_\Omega\alpha_{n+1}u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)} + \langle 1_\Omega\alpha_{n+1}u, D_M u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\Big]\\
&\quad= (m-M)\Big[\langle D_0 u + M\alpha_{n+1}u, 1_\Omega\alpha_{n+1}u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\\
&\qquad\quad + \langle 1_\Omega\alpha_{n+1}u, D_0 u + M\alpha_{n+1}u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\Big]\\
&\quad= 2M(m-M)\int_\Omega |u|^2\,\mathrm{d}x\\
&\qquad\quad + (m-M)\Big(\langle D_0 u, 1_\Omega\alpha_{n+1}u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)} + \langle 1_\Omega\alpha_{n+1}u, D_0 u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\Big)\\
&\quad= 2M(m-M)\int_\Omega |u|^2\,\mathrm{d}x + (m-M)\int_\Sigma \langle \mathcal{B}u, u\rangle_{\mathbb{C}^N}\,\mathrm{d}s,
\end{aligned}$$

where we used the equality (7) in the last step. This gives

$$\langle B_{m,M}u, B_{m,M}u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)} = \int_\Omega \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x \\ + \int_{\Omega^c} \big(|\nabla u|^2 + M^2|u|^2\big)\,\mathrm{d}x - (M-m)\int_\Sigma \langle \mathcal{B}u, u\rangle_{\mathbb{C}^N}\,\mathrm{d}s,$$

and it remains to remark that

$$\langle \mathcal{B}u, u\rangle_{\mathbb{C}^N} = \frac{1}{2}\big\langle (1+\mathcal{B})u, u\big\rangle_{\mathbb{C}^N} - \frac{1}{2}\big\langle (1-\mathcal{B})u, u\big\rangle_{\mathbb{C}^N} \\ = \langle \mathcal{P}_+ u, u\rangle_{\mathbb{C}^N} - \langle \mathcal{P}_- u, u\rangle_{\mathbb{C}^N} \equiv |\mathcal{P}_+ u|^2_{\mathbb{C}^N} - |\mathcal{P}_- u|^2_{\mathbb{C}^N},$$

where in the last step we used the fact that $\mathcal{P}_\pm$ are orthogonal projectors. □

2.2 Dirac operators on Euclidean hypersurfaces

The definition of the intrinsic Dirac operator $\not{D}$ on Σ with a detailed presentation of preliminary constructions can be found in the monographs [7,11,12]. Recall that if $\mathbb{S}\Sigma$ is the intrinsic spinor bundle over Σ with the associated spin connection $\not\nabla$ and carrying the natural Hermitian and Clifford module structures, then $\not{D}$ acts on smooth sections ψ of $\mathbb{S}\Sigma$ by $\not{D}\psi = \sum_{j=1}^{n-1} e_j \cdot \not\nabla_{e_j}\psi$, where $(e_1, \dots, e_{n-1})$ is an orthonormal frame tangent to Σ and $\cdot$ is the Clifford multiplication. For our situation, the study of $\not{D}$ is easier to approach through the extrinsically defined Dirac operators, which will be more suitable for the subsequent asymptotic analysis, and we explain this link in the present section.

For $n \geq 2$ and $K := 2^{[\frac{n}{2}]}$ let $\beta_1, \dots, \beta_n$ be anticommuting Hermitian $K \times K$ matrices with $\beta_j^2 = I_K$, and for $x = (x_1, \dots, x_n)$ we denote $\beta(x) = \sum_{j=1}^n \beta_j x_j$. Recall that the intrinsic Dirac operator $D^{\mathbb{R}^n}$ in $\mathbb{R}^n$ acts then by

$$D^{\mathbb{R}^n} = -\mathrm{i} \sum_{j=1}^{n} \beta_j \frac{\partial}{\partial x_j},$$

and it is a self-adjoint operator in $L^2(\mathbb{R}^n, \mathbb{C}^K)$ with domain $H^1(\mathbb{R}^n, \mathbb{C}^K)$. In this explicit case the Clifford multplication $x\cdot$ is realized as the multiplication by the matrix $-\mathrm{i}\beta(x)$ where $\beta(e_j) = \beta_j$ and $\nabla_{e_j} = \partial_{x_j}$ for the canonical basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$. We remark that the expression D_0 given in the introduction *does not* correspond to the intrinsic Dirac operator on $\mathbb{R}^n$, as $N \neq K$ in general. The *extrinsically defined* Dirac operator D^Σ on Σ is a self-adjoint operator in $L^2(\Sigma, \mathbb{C}^K)$ with domain $H^1(\Sigma, \mathbb{C}^K)$ and given by

$$D^\Sigma = \frac{H_1}{2} - \beta(\nu) \sum_{j=1}^{n-1} \beta(e_j) \nabla_{e_j}, \tag{11}$$

where $(e_1, \dots, e_{n-1})$ is an orthonormal frame tangent to Σ. It is a fundamental result that D^Σ is unitarily equivalent to $\not{D}$ for odd n and to $\not{D} \oplus (-\not{D})$ for even n; in the latter case, the operator $\not{D}$ can be identified with the restriction of $\beta(\nu) D^\Sigma$ on $\ker \big(1 - \beta(\nu)\big)$, see e.g. [7, Section 2.4]. In other words, the study of the eigenvalues of $(D^\Sigma)^2$ is equivalent to that of $\not{D}^2$, modulo the multiplicities for even n.

In turn, a classical tool for the analysis of the eigenvalues of $(D^\Sigma)^2$ is provided by the Schrödinger-Lichnerowicz formula $(D^\Sigma)^2 = (\nabla^\Sigma)^* \nabla^\Sigma + \frac{1}{2} H_2 I$ (whose proof we recall in Appendix B), where ∇^Σ is the induced spin connection

$$\nabla_X^\Sigma = \nabla_X + \frac{1}{2} \beta(\nu) \beta(WX) : C^\infty(\Sigma, \mathbb{C}^K) \to C^\infty(\Sigma, \mathbb{C}^K), \quad X \in T\Sigma.$$

In other words, for $u \in H^1(\Sigma, \mathbb{C}^K)$ one has

$$\langle D^\Sigma u, D^\Sigma u \rangle_{L^2(\Sigma, \mathbb{C}^K)} = \int_\Sigma \Big(|\nabla^\Sigma u|^2 + \frac{H_2 |u|^2}{2} \Big) \, \mathrm{d}x, \tag{12}$$

while in the local coordinates on Σ one has

$$|\nabla^\Sigma u|^2 = \sum_{j,k=1}^{n-1} g^{jk} \Big\langle \partial_j u + \frac{1}{2} \beta(\nu) \beta(\partial_j \nu) u, \partial_k u + \frac{1}{2} \beta(\nu) \beta(\partial_k \nu) u \Big\rangle_{\mathbb{C}^K}, \tag{13}$$

where $(g^{jk}) := (g_{jk})^{-1}$ with (g_{jk}) being the Riemannian metric on Σ induced by the embedding into $\mathbb{R}^n$.

For the subsequent analysis we introduce the Hilbert space

$$\mathcal{H} := \big\{ f \in L^2(\Sigma, \mathbb{C}^N) : f = \mathcal{B} f \big\}, \quad \|f\|^2_{\mathcal{H}} := \int_\Sigma |f|^2 \, \mathrm{d}s, \tag{14}$$

with $\mathcal{B}$ given in (3), and the self-adjoint operator L in $\mathcal{H}$ given by its sesquilinear form as follows:

$$L[f,f] = \int_\Sigma \Big[|\nabla f|^2 + \Big(H_2 - \frac{H_1^2}{4} \Big) |f|^2 \Big] \, \mathrm{d}s, \quad \mathcal{Q}(L) = H^1(\Sigma, \mathbb{C}^N) \cap \mathcal{H},$$

with $\mathcal{Q}(L)$ being the form domain (see Section 3). Remark that the operator L for $n = 3$ already appeared (without any further interpretation) in [2], and will arise naturally in the asymptotic spectral analysis of the Dirac operators A_m and $B_{m,M}$. Its importance is explained in the following assertion:

Lemma 2.4. *The operator L is unitarily equivalent to $\not{D}^2$.*

Before going through the proof of Lemma 2.4, we would like to emphasize that the result can be of its own interest as it allows one to investigate the eigenvalues of the intrinsic Dirac operator $\not{D}$ on Σ via those of a unitarily equivalent operator defined thanks to a matrix framework and local expressions but without any additional algebraic construction.

Proof. The proof is by direct computation, by constructing an explicit isomorphism between $L^2(\Sigma, \mathbb{C}^{N/2})$ and $\mathcal{H}$ and then by establishing a link with the extrinsically defined Dirac operator D^Σ using the Schrödinger-Lichnerowicz formula. Following the standard rules, see e.g. [8, Chapter 15] or [26, Appendix E], for $n \in \mathbb{N}$ we define $2^{[\frac{n}{2}]} \times 2^{[\frac{n}{2}]}$ Dirac matrices $\gamma_j(n)$ with $j \in \{1, \dots, n\}$ using the following iterative procedure:

- For $n = 1$, set $\gamma_1(1) := (1)$.
- For $n = 2$, set $\gamma_1(2) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\gamma_2(2) := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$.
- For $n = 2m + 1$ with $m \in \mathbb{N}$:

$$\begin{aligned} \gamma_j(2m+1) &:= \gamma_j(2m), \quad j = 1, \dots, 2m, \\ \gamma_{2m+1}(2m+1) &:= \pm \mathrm{i}^m \gamma_1(2m) \cdot \ldots \cdot \gamma_{2m}(2m) = \pm \begin{pmatrix} -I_{2^{m-1}} & 0 \\ 0 & I_{2^{m-1}} \end{pmatrix}, \end{aligned} \tag{15}$$

- For $n = 2m + 2$ with $m \in \mathbb{N}$:

$$\begin{aligned} \gamma_j(2m+2) &:= \begin{pmatrix} 0 & \gamma_j(2m+1) \\ \gamma_j(2m+1) & 0 \end{pmatrix}, \quad j = 1, \dots, 2m+1, \\ \gamma_{2m+2}(2m+2) &:= \begin{pmatrix} 0 & -\mathrm{i} I_{2^m} \\ \mathrm{i} I_{2^m} & 0 \end{pmatrix}. \end{aligned}$$

One easily checks that at a fixed $n \in \mathbb{N}$ the matrices $\gamma_j(n)$ are Hermitian and anticommute, the square of each of them is the identity matrix. Furthermore, if $\big(\gamma_j'(n)\big)$ is another set of matrices with these properties and of the same size, then there exists a unitary matrix C and a suitable choice of $\pm$ in (15) such that the equalities $\gamma_j'(n)C = \gamma_j(n)C$ hold for all j, see e.g. [8, Prop. 15.16]. Therefore, without loss of generality one may assume that the matrices α_j in the expression (3) of $\mathcal{B}$ and the matrices β_j used in the definition of D^Σ are chosen in the form

$$\alpha_j = \gamma_j(n+1), \quad j = 1, \dots, n+1, \qquad \beta_j = \gamma_j(n), \quad j = 1, \dots, n. \tag{16}$$

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $q \in \{n, n+1\}$ we define a matrix $\Gamma_q(x)$ by

$$\Gamma_q(x) = \sum_{j=1}^{n} x_j \gamma_j(q),$$

then one has the relations

$$\begin{gathered}\Gamma_n(x)\Gamma_n(y) + \Gamma_n(y)\Gamma_n(x) = 2\langle x, y\rangle_{\mathbb{R}^n} I, \quad x, y \in \mathbb{R}^n, \\ \Gamma(x) = \Gamma_{n+1}(x), \quad \beta(x) = \Gamma_n(x).\end{gathered} \tag{17}$$

Now we consider separately the cases of odd and even dimensions, as the constructions are rather different for the two cases.

Case 1: n is odd. Let $n = 2m+1$ with $m \in \mathbb{N}$. Represent $f \in \mathcal{H}$ as $f = (f_-, f_+)$ with $f_\pm \in L^2(\Sigma, \mathbb{C}^{N/2})$, then, under the convention (16), the condition $f = \mathcal{B}f$ takes the form

$$\begin{pmatrix} f_- \\ f_+ \end{pmatrix} = -\mathrm{i} \begin{pmatrix} 0 & -\mathrm{i} I_{2^m} \\ \mathrm{i} I_{2^m} & 0 \end{pmatrix} \begin{pmatrix} 0 & \Gamma_n(\nu) \\ \Gamma_n(\nu) & 0 \end{pmatrix} \begin{pmatrix} f_- \\ f_+ \end{pmatrix},$$

which holds if and only if $f_\pm = \pm \Gamma_n(\nu) f_\pm$. Therefore, the map

$$U : L^2(\Sigma, \mathbb{C}^{N/2}) \to \mathcal{H}, \quad (Uf)(s) = \frac{1}{2} \begin{pmatrix} \big(1 - \Gamma_n(\nu)\big) f \\ \big(1 + \Gamma_n(\nu)\big) f \end{pmatrix}$$

defines a unitary operator, and $Uf \in H^1(\Sigma, \mathbb{C}^N)$ iff $f \in H^1(\Sigma, \mathbb{C}^{N/2})$. As H_j are scalar functions, one has

$$\Big(H_2 - \frac{H_1^2}{4}\Big) |Uf|^2_{\mathbb{C}^N} = \Big(H_2 - \frac{H_1^2}{4}\Big) |f|^2_{\mathbb{C}^{N/2}}. \tag{18}$$

In order to compute $\big|\nabla(Uf)\big|^2$ we use local coordinates on Σ. One has

$$\begin{aligned}\big|\nabla(Uf)\big|^2 &= \frac{1}{4} \sum_{j,k=1}^{n-1} g^{j,k} \Big[\Big\langle \partial_j\Big(\big(1 - \Gamma_n(\nu)\big) f\Big), \partial_k\Big(\big(1 - \Gamma_n(\nu)\big) f\Big) \Big\rangle_{\mathbb{C}^{N/2}} \\ &\qquad + \Big\langle \partial_j\Big(\big(1 + \Gamma_n(\nu)\big) f\Big), \partial_k\Big(\big(1 + \Gamma_n(\nu)\big) f\Big) \Big\rangle_{\mathbb{C}^{N/2}} \Big] \\ &= \frac{1}{2} \sum_{j,k=1}^{n-1} g^{j,k} \Big[\langle \partial_j f, \partial_k f\rangle_{\mathbb{C}^{N/2}} + \Big\langle \partial_j\big(\Gamma_n(\nu) f\big), \partial_k\big(\Gamma_n(\nu) f\big) \Big\rangle_{\mathbb{C}^{N/2}} \Big].\end{aligned}$$

We have then

$$\Big\langle \partial_j\big(\Gamma_n(\nu)f\big), \partial_k\big(\Gamma_n(\nu)f\big)\Big\rangle_{\mathbb{C}^{N/2}}$$
$$= \Big\langle \Gamma_n(\nu)\partial_j f + \Gamma_n(\partial_j\nu)f, \Gamma_n(\nu)\partial_k f + \Gamma_n(\partial_k\nu)f\Big\rangle_{\mathbb{C}^{N/2}}$$
$$= \Big\langle \partial_j f + \Gamma_n(\nu)\Gamma_n(\partial_j\nu)f, \partial_k f + \Gamma_n(\nu)\Gamma_n(\partial_k\nu)f\Big\rangle_{\mathbb{C}^{N/2}},$$

and it follows that

$$\begin{aligned}\big|\nabla(Uf)\big|^2 &= \sum_{j,k=1}^{n-1} g^{j,k}\Big\langle \partial_j f + \frac{1}{2}\Gamma_n(\nu)\Gamma_n(\partial_j\nu)f, \partial_k f + \frac{1}{2}\Gamma_n(\nu)\Gamma_n(\partial_k\nu)f\Big\rangle_{\mathbb{C}^{N/2}}\\ &\quad + \frac{1}{4}\sum_{j,k=1}^{n-1} g^{j,k}\Big\langle \Gamma_n(\partial_k\nu)\Gamma_n(\partial_j\nu)f, f\Big\rangle_{\mathbb{C}^{N/2}}\\ &= |\nabla^\Sigma f|^2 + \frac{1}{4}\langle f, Vf\rangle_{\mathbb{C}^{N/2}}, \qquad V := \sum_{j,k=1}^{n-1} g^{j,k}\Gamma_n(\partial_k\nu)\Gamma_n(\partial_j\nu).\end{aligned}$$

Using the symmetry of $(g^{j,k})$ and the commutation relation (17) we compute

$$\begin{aligned}V &= \frac{1}{2}\sum_{j,k=1}^{n-1} g^{j,k}\Big(\Gamma_n(\partial_j\nu)\Gamma_n(\partial_k\nu) + \Gamma_n(\partial_k\nu)\Gamma_n(\partial_j\nu)\Big)\\ &= \sum_{j,k=1}^{n-1} g^{j,k}\langle\partial_j\nu, \partial_k\nu\rangle\, I = |\nabla\nu|^2 I = |W|^2 I = (H_1^2 - 2H_2)I.\end{aligned}$$

By combining with (18) we arrive at

$$L[Uf, Uf] = \int_\Sigma \Big(|\nabla^\Sigma f|^2 + \frac{H_2|f|^2}{2}\Big)\,\mathrm{d}s.$$

Due to the Schrödinger-Lichnerowicz formula (12) we conclude that $L = U^*(D^\Sigma)^2 U$, while $(D^\Sigma)^2$ is unitarily equivalent to $\not{D}^2$ as n is odd. This proves the claim for odd dimensions.

Case 2: n is even. Let $n = 2m$ with $m \in \mathbb{N}$. As for the previous case, we try to find a block representation for the condition $f = \mathcal{B}f$, which now takes the form

$$\Big(I_{2^m} + \mathrm{i}\gamma_{2m+1}(2m+1)\sum_{j=1}^{2m}\gamma_j(2m+1)\,\nu_j\Big)f = 0. \tag{19}$$

We first remark that for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we have the block representation

$$\sum_{j=1}^{2m}\gamma_j(2m+1)x_j \equiv \sum_{j=1}^{2m}\gamma_j(2m)x_j \equiv \Gamma_n(\nu) = \begin{pmatrix} 0 & \lambda(x)\\ \lambda(x)^* & 0\end{pmatrix}, \tag{20}$$
$$\lambda(x) := \sum_{j=1}^{2m-1}\gamma_j(2m-1)\,x_j - \mathrm{i}x_{2m}I_{2^{m-1}}.$$

Represent $f = (\psi_-, \psi_+)$ with $\psi_\pm \in L^2(\Sigma, \mathbb{C}^{N/2})$, then we rewrite the condition (19) in the block form

$$\left[\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \pm \mathrm{i} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & \lambda(\nu) \\ \lambda(\nu)^* & 0 \end{pmatrix}\right] \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $I := I_{2^{m-1}}$. Using $\lambda(\nu)\lambda(\nu)^* = \lambda(\nu)^*\lambda(\nu) = I$ we see that the condition $f = \mathcal{B}f$ can be rewritten as $\psi_- = \pm\mathrm{i}\lambda(\nu)\,\psi_+$. Hence, the map

$$U : L^2(\Sigma, \mathbb{C}^{N/2}) \to \mathcal{H}, \quad U\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm\mathrm{i}\lambda(\nu)\psi \\ \psi \end{pmatrix}$$

defines a unitary operator, and at each point of Σ there holds

$$\begin{aligned} \big|\nabla(U\psi)\big|^2 = \sum_{j,k=1}^{n-1} g^{j,k} \bigg(&\frac{1}{2} \Big\langle \mathrm{i}\lambda(\nu)\partial_j\psi + \mathrm{i}\lambda(\partial_j\nu)\psi, \mathrm{i}\lambda(\nu)\partial_k\psi + \mathrm{i}\lambda(\partial_k\nu)\psi \Big\rangle_{\mathbb{C}^{N/2}} \\ &+ \frac{1}{2} \langle \partial_j\psi, \partial_k\psi \rangle_{\mathbb{C}^{N/2}} \bigg). \end{aligned} \tag{21}$$

We then transform

$$\begin{aligned} &\frac{1}{2} \Big\langle \mathrm{i}\lambda(\nu)\partial_j\psi + \mathrm{i}\lambda(\partial_j\nu)\psi, \mathrm{i}\lambda(\nu)\partial_k\psi + \mathrm{i}\lambda(\partial_k\nu)\psi \Big\rangle_{\mathbb{C}^{N/2}} + \frac{1}{2}\langle \partial_j\psi, \partial_k\psi \rangle_{\mathbb{C}^{N/2}} \\ &\quad = \frac{1}{2} \Big\langle \partial_j\psi + \lambda(\nu)^*\lambda(\partial_j\nu)\psi, \partial_k\psi + \lambda(\nu)^*\lambda(\partial_k\nu)\psi \Big\rangle_{\mathbb{C}^{N/2}} + \frac{1}{2}\langle \partial_j\psi, \partial_k\psi \rangle_{\mathbb{C}^{N/2}} \\ &\quad = \Big\langle \partial_j\psi + \frac{1}{2}\lambda(\nu)^*\lambda(\partial_j\nu)\psi, \partial_k\psi + \frac{1}{2}\lambda(\nu)^*\lambda(\partial_k\nu)\psi \Big\rangle_{\mathbb{C}^{N/2}} \\ &\qquad + \frac{1}{4} \Big\langle \lambda(\nu)^*\lambda(\partial_j\nu)\psi, \lambda(\nu)^*\lambda(\partial_k\nu)\psi \Big\rangle_{\mathbb{C}^{N/2}} \\ &\quad = \Big\langle \partial_j\psi + \frac{1}{2}\lambda(\nu)^*\lambda(\partial_j\nu)\psi, \partial_k\psi + \frac{1}{2}\lambda(\nu)^*\lambda(\partial_k\nu)\psi \Big\rangle_{\mathbb{C}^{N/2}} \\ &\qquad + \frac{1}{4} \Big\langle \psi, \lambda(\partial_j\nu)^*\lambda(\partial_k\nu)\psi \Big\rangle_{\mathbb{C}^{N/2}}. \end{aligned}$$

The substitution into (21) gives

$$\begin{aligned} \big|\nabla(U\psi)\big|^2 = \sum_{j,k=1}^{n-1} g^{j,k} &\Big\langle \partial_j\psi + \frac{1}{2}\lambda(\nu)^*\lambda(\partial_j\nu)\psi, \partial_k\psi + \frac{1}{2}\lambda(\nu)^*\lambda(\partial_k\nu)\psi \Big\rangle_{\mathbb{C}^{N/2}} \\ &+ \frac{1}{4}\langle \psi, V\psi \rangle_{\mathbb{C}^{N/2}}, \qquad V := \sum_{j,k=1}^{2m-1} g^{j,k} \lambda(\partial_j\nu)^*\lambda(\partial_k\nu). \end{aligned}$$

In order to compute V we introduce

$$\widetilde{V} := \sum_{j,k=1}^{2m-1} g^{j,k} \lambda(\partial_j\nu)\lambda(\partial_k\nu)^*,$$

then

$$\begin{pmatrix} \widetilde{V} & 0 \\ 0 & V \end{pmatrix} = \sum_{j,k=1}^{2m-1} g^{j,k} \begin{pmatrix} \lambda(\partial_j\nu)\lambda(\partial_k\nu)^* & 0 \\ 0 & \lambda(\partial_j\nu)^*\lambda(\partial_k\nu) \end{pmatrix}$$

$$
\begin{aligned}
&= \sum_{j,k=1}^{2m-1} g^{j,k} \begin{pmatrix} 0 & \lambda(\partial_j\nu) \\ \lambda(\partial_j\nu)^* & 0 \end{pmatrix} \begin{pmatrix} 0 & \lambda(\partial_k\nu) \\ \lambda(\partial_k\nu)^* & 0 \end{pmatrix} \\
&= \sum_{j,k=1}^{2m-1} g^{j,k} \Gamma_n(\partial_j\nu)\Gamma_n(\partial_k\nu) \\
&= \frac{1}{2}\sum_{j,k=1}^{2m-1} g^{j,k} \Big(\Gamma_n(\partial_j\nu)\Gamma_n(\partial_k\nu) + \Gamma_n(\partial_k\nu)\Gamma_n(\partial_j\nu) \Big) \\
&= \sum_{j,k=1}^{2m-1} g^{j,k} \langle \partial_j\nu, \partial_k\nu\rangle\, I = |\nabla\nu|^2 I = |W|^2 I = (H_1^2 - 2H_2) I.
\end{aligned}
$$

In addition, as the functions H_j are scalar, we have

$$
\Big\langle U\psi, \Big(H_2 - \frac{H_1^2}{4}\Big) U\psi \Big\rangle_{\mathcal{H}} = \Big\langle \psi, \Big(H_2 - \frac{H_1^2}{4}\Big)\psi \Big\rangle_{L^2(\Sigma,\mathbb{C}^{N/2})},
$$

and then

$$
\begin{aligned}
&L[U\psi, U\psi] \\
&\quad = \int_\Sigma \sum_{j,k=1}^{n-1} g^{j,k} \Big\langle \partial_j\psi + \frac{1}{2}\lambda(\nu)^*\lambda(\partial_j\nu)\psi, \partial_k\psi + \frac{1}{2}\lambda(\nu)^*\lambda(\partial_k\nu)\psi \Big\rangle_{\mathbb{C}^{N/2}} \mathrm{d}s \\
&\qquad\qquad + \frac{1}{2}\langle\psi, H_2\psi\rangle_{L^2(\Sigma,\mathbb{C}^{N/2})}.
\end{aligned}
$$

Now consider the unitary transform $U_0 : L^2(\Sigma,\mathbb{C}^{N/2}) \to L^2(\Sigma,\mathbb{C}^{N/2})$ given by $U_0\psi = \lambda(\nu)^*\psi$, then a simple computation shows that

$$
\begin{aligned}
&L[UU_0\psi, UU_0\psi] \\
&\quad = \int_\Sigma \sum_{j,k=1}^{n-1} g^{j,k} \Big\langle \partial_j\psi + \frac{1}{2}\lambda(\nu)\lambda(\partial_j\nu)^*\psi, \partial_k\psi + \frac{1}{2}\lambda(\nu)\lambda(\partial_k\nu)^*\psi \Big\rangle_{\mathbb{C}^{N/2}} \mathrm{d}s \\
&\qquad\qquad + \frac{1}{2}\langle\psi, H_2\psi\rangle_{L^2(\Sigma,\mathbb{C}^{N/2})}.
\end{aligned}
$$

Using (20), for $\psi_\pm \in H^1(\Sigma,\mathbb{C}^{N/2})$ and $\psi := (\psi_-, \psi_+) \in H^1(\Sigma,\mathbb{C}^N)$ one has

$$
\begin{aligned}
&L[UU_0\psi_-, UU_0\psi_-] + L[U\psi_+, U\psi_+] \\
&\quad = \int_\Sigma \sum_{j,k=1}^{n-1} g^{j,k} \big\langle \partial_j\psi + \frac{1}{2}\Gamma_n(\nu)\Gamma_n(\partial_j\nu)\psi, \partial_k\psi + \frac{1}{2}\Gamma_n(\nu)\Gamma_n(\partial_k\nu)\psi \big\rangle_{\mathbb{C}^N} \mathrm{d}s \\
&\qquad\qquad + \frac{1}{2}\,\langle\psi, H_2\psi\rangle_{L^2(\Sigma,\mathbb{C}^N)}.
\end{aligned}
$$

By comparing with the Schrödinger-Lichnerowicz formula (12)–(13) we see that the operator $(U_0^* U^* L U U_0) \oplus (U^* L U)$ is unitarily equivalent to $(D^\Sigma)^2$. As $(D^\Sigma)^2$ is now unitarily equivalent to $\not{D}^2 \oplus \not{D}^2$ (because n is even), it follows that L is unitarily equivalent to $\not{D}^2$. □

Remark 2.5. One can also consider

$$\mathcal{H}' := \big\{ f \in L^2(\Sigma, \mathbb{C}^N) : f = -\mathcal{B} f \big\}, \quad \|f\|^2_{\mathcal{H}'} := \int_\Sigma |f|^2 \,\mathrm{d}s, \tag{22}$$

and the self-adjoint operator L' in $\mathcal{H}'$ given by its sesquilinear form

$$L'[f,f] = \int_\Sigma \Big[|\nabla f|^2 + \Big(H_2 - \frac{H_1^2}{4} \Big) |f|^2 \Big] \,\mathrm{d}s, \quad \mathcal{Q}(L') = H^1(\Sigma, \mathbb{C}^N) \cap \mathcal{H}',$$

then the preceding proof can be easily adapted to show that L' is also unitarily equivalent to $\not{D}^2$.

3 Preliminary constructions

As explained in the end of Subsection 1.1, our proofs of the main theorems rely on the investigation of reduced spectral problems in tubular neighborhoods of the hypersurface Σ. These reductions are performed by studying operators only acting in the normal variable, which are introduced and investigated in Subsection 3.1. The same operators play a role in the construction of an extension operator while proving Theorem 1.2. In Subsection 3.2 we gather various estimates for sesquilinear forms in tubular neighborhoods of the hypersurface Σ. These sesquilinear forms appear further on while proving the main results on the eigenvalue asymptotics.

3.1 One-dimensional model operators

Lemma 3.1. *Let $\delta > 0$ be fixed. For $\alpha > 0$, let S be the self-adjoint operator in $L^2(0,\delta)$ with*

$$S[f,f] = \int_0^\delta |f'|^2 \,\mathrm{d}t - \alpha \big| f(0) \big|^2, \quad \mathcal{Q}(S) = \big\{ f \in H^1(0,\delta) : f(\delta) = 0 \big\},$$

then for $\alpha \to +\infty$ one has $E_1(S) = -\alpha^2 + \mathcal{O}(e^{-\delta\alpha})$, and the associated eigenfunction ψ with $\|\psi\|_{L^2(0,\delta)} = 1$ satisfies $\big|\psi(0)\big|^2 = 2\alpha + \mathcal{O}(1)$.

Proof. One easily see that the operator S acts as $f \to -f''$ defined of the functions $f \in H^2(0,\delta)$ with $f'(0) + \alpha f(0) = f(\delta) = 0$. Let us estimate its first eigenvalue as $\alpha \to +\infty$. Look for negative eigenvalues $E = -k^2$ with $k > 0$, then using the boundary condition at δ we see that the associated normalized eigenfunction ψ is of the form $\psi(t) = c \sinh\big(k(\delta - t)\big)$ with $c \neq 0$ being a normalizing constant. The boundary condition at 0 gives $0 = \psi'(0) + \alpha\psi(0) = -k\cosh(k\delta) + \alpha \sinh(k\delta)$, i.e.

$$F(k\delta) = \alpha\delta, \quad F(x) := x \coth x. \tag{23}$$

One easily sees that $F : (0, +\infty) \to (1, +\infty)$ is strictly increasing and bijective, and for $\alpha\delta > 1$ the equation (23) admits a unique solution k, and then $k\delta \to +\infty$ for $\alpha \to +\infty$. Now rewrite (23) as $k = \alpha \tanh(k\delta)$. Due to $k\delta \to +\infty$ we have $\frac{3}{4} \le \tanh(k\delta) \le 1$ implying $3\alpha/4 \le k \le \alpha$. Then using the equation again we have $\alpha \tanh\big(\frac{3}{4}\alpha\delta\big) \le k \le \alpha$, while $\tanh\big(\frac{3}{4}\alpha\delta\big) = 1 + \mathcal{O}(e^{-3\delta\alpha/2})$. Therefore, with some $c_1 > 0$ one has $E_1(S) = -k^2 = -\alpha^2\big(1 + \mathcal{O}(e^{-3\delta\alpha/2})\big) \le -\alpha^2 + c_1 e^{-\delta\alpha}$ as $\alpha \to +\infty$.

In order to find the value of the normalizing constant c we use

$$1 = \|\psi\|^2_{L^2(0,\delta)} = |c|^2 \int_0^\delta \sinh^2\big(k(\delta-t)\big)\,\mathrm{d}t = |c|^2\Big(\frac{1}{4k}\sinh(2k\delta) - \frac{\delta}{2}\Big),$$

then

$$\big|\psi(0)\big|^2 = \big(\sinh^2(k\delta)\big)\Big(\frac{\sinh(2k\delta)}{4k} - \frac{\delta}{2}\Big)^{-1} = 2k + \mathcal{O}(1) = 2\alpha + \mathcal{O}(1). \qquad \square$$

Lemma 3.2. *Let $\delta > 0$ and $\beta \geq 0$ be fixed. For $\alpha > 0$, let S' be the self-adjoint operator in $L^2(0,\delta)$ given by*

$$S'[f,f] = \int_0^\delta |f'|^2\,\mathrm{d}t - \alpha\big|f(0)\big|^2 - \beta\big|f(\delta)\big|^2, \quad \mathcal{Q}(S') = H^1(0,\delta),$$

then for $\alpha \to +\infty$ one has $E_1(S') = -\alpha^2 + \mathcal{O}(e^{-\delta\alpha})$. Furthermore, there exist $b_\pm > 0$ and $b > 0$ such that

$$b^- j^2 - b \leq E_j(S') \leq b^+ j^2 \text{ for all } j \geq 2 \text{ and } \alpha \in \mathbb{R}. \tag{24}$$

Proof. The operator S' clearly acts as $f \mapsto -f''$ on the functions $f \in H^2(0,\delta)$ with $f'(0) + \alpha f(0) = f'(\delta) - \beta f(\delta) = 0$. To estimate $E_1(S')$ we remark that a value $E = -k^2$ with $k > 0$ is an eigenvalue of S' iff one can find $(C_1, C_2) \in \mathbb{C}^2 \setminus \big\{(0,0)\big\}$ such that the function $f : t \mapsto C_1 e^{kt} + C_2 e^{-kt}$ belongs to its domain. The boundary conditions give

$$\begin{aligned} 0 &= f'(0) + \alpha f(0) = (\alpha + k)C_1 + (\alpha - k)C_2, \\ 0 &= f'(\delta) - \beta f(\delta) = (k - \beta)e^{k\delta}C_1 - (k + \beta)e^{-k\delta}C_2, \end{aligned}$$

and one has a non-zero solution iff the determinant of the system vanishes, i.e. iff k solves $(k+\alpha)(k+\beta)e^{-k\delta} = (k-\alpha)(k-\beta)e^{k\delta}$, which we rewrite as

$$g(k) = h(k), \quad g(k) := \frac{k+\alpha}{k-\alpha}, \quad h(k) := \frac{k-\beta}{k+\beta}\, e^{2k\delta}. \tag{25}$$

Both g and h are continuous, and g is strictly decreasing on $(\alpha, +\infty)$ with $g(\alpha^+) = +\infty$ and $g(+\infty)$=1, while h is strictly increasing on $(\alpha, +\infty)$ being the product of two strictly increasing positive functions (we assume without loss of generality that $\alpha > \beta$), and $h(\alpha^+) = e^{2\alpha\delta}(\alpha - \beta)/(\alpha + \beta) < +\infty$ and $h(+\infty) = +\infty$. Therefore, there exists a unique solution k of (25) with $k \in (\alpha, +\infty)$. To obtain the required estimate we use again the monotonicity of h on $(\alpha, +\infty)$:

$$\frac{k+\alpha}{k-\alpha} = g(k) = h(k) > h(\alpha^+) = \frac{\alpha-\beta}{\alpha+\beta}\, e^{2\alpha\delta}.$$

We bound the last term from below very roughly by $e^{3\alpha\delta/2}$ then

$$\frac{k+\alpha}{k-\alpha} \geq e^{3\alpha\delta/2}, \quad k \leq \alpha\,\frac{1 + e^{-3\alpha\delta/2}}{1 - e^{-3\alpha\delta/2}} = \alpha\big(1 + \mathcal{O}(e^{-3\alpha\delta/2})\big).$$

By combining with $k > \alpha$ we arrive at the sought estimate

$$E_1(S') = -k^2 = -\alpha^2\big(1 + \mathcal{O}(e^{-3\alpha\delta/2})\big) = -\alpha^2 + \alpha^2\mathcal{O}(e^{-3\alpha\delta/2}) = -\alpha^2 + \mathcal{O}(e^{-\alpha\delta}).$$

To estimate $E_j(S')$ with $j \geq 2$ we consider the operator S'_D acting in $L^2(0,\delta)$ as $f \mapsto -f''$ on the functions $f \in H^2(0,\delta)$ with $f(0) = f'(\delta) - \beta f(\delta) = 0$. Remark that S'_D is independent of α, and by the min-max principle for any $\alpha \in \mathbb{R}$ one has $E_{j-1}(S'_D) \leq E_j(S') \leq E_j(S'_D)$: the upper bound follows from the fact that the sesquilinear form of S'_D is a restriction of that for S', while the lower bound follows from the fact that S'_D is a rank-one perturbation of S'. As the eigenvalues of S'_D satisfy the Weyl asymptotics $E_j(S'_D) \sim \pi^2 j^2/\delta^2$ as $j \to +\infty$, one arrives at the sought inequalities (24). □

3.2 Tubular coordinates

Recall that the shape operator W and curvatures of Σ were defined in Subsection 2.1. In what follows we will actively use tubular coordinates on both sides of Σ. In this section,

$$\text{let } \Omega_* \text{ be either } \Omega \text{ or } \Omega^c,$$

and let ν_* be the unit normal on Σ pointing to the exterior of Ω_*, i.e.

$$\nu_* := \nu,\ W_* := W \text{ for } \Omega_* = \Omega, \quad \nu_* := -\nu,\ W_* := -W \text{ for } \Omega_* = \Omega^c.$$

The principal curvatures and the (higher) mean curvatures of Σ with respect to ν_* will be denoted by h^*_j and H^*_k respectively, i.e.

$$h^*_j := h_j \text{ and } H^*_k = H_k \text{ for } \Omega_* = \Omega,$$
$$h^*_j := -h_j \text{ and } H^*_k = (-1)^k H_k \text{ for } \Omega_* = \Omega^c.$$

For small $\delta > 0$ denote

$$\Pi_\delta := \Sigma \times (0,\delta), \quad \Omega^\delta_* = \big\{x \in \Omega_* : \operatorname{dist}(x,\Sigma) < \delta\big\}$$

It is a well known result in differential geometry that there exists a small $\delta_0 > 0$ such that for sufficiently small $\delta > 0$ the map

$$\Phi_* : \Pi_\delta \to \Omega^\delta_*, \quad (s,t) \mapsto s - t\nu_*(s),$$

is a diffeomorphism, and $\operatorname{dist}\big(\Phi_*(s,t), \partial U\big) = t$ for $(s,t) \in \Pi_\delta$. Consider the associated unitary map

$$\Theta_\delta : L^2(\Omega^\delta_*) \to L^2(\Pi_\delta), \quad u \mapsto \sqrt{\det(\Phi'_*)}\, u \circ \Phi_*$$

We will use several times the following computations:

Lemma 3.3. *For $\gamma \in \mathbb{R}$ denote*

$$J_\gamma(u) \equiv J(u) := \int_{\Omega^\delta_*} |\nabla u|^2 \,\mathrm{d}x + \int_\Sigma \Big(\gamma + \frac{H^*_1}{2}\Big)|u|^2 \,\mathrm{d}s, \quad u \in H^1(\Omega^\delta_*).$$

There exist $\delta_0 > 0$ and $c > 0$ such that for any $\gamma \in \mathbb{R}$ and $\delta \in (0,\delta_0)$ the following assertions hold true with $v := \Theta_\delta u$:

(a) for any $u \in H^1(\Omega^\delta_*)$ with $u = 0$ on $\partial\Omega^\delta_* \setminus \Sigma$ one has

$$J(u) \leq \int_{\Pi_\delta} \Big[(1+c\delta)|\nabla_s v|^2 + |\partial_t v|^2 + \Big(H^*_2 - \frac{(H^*_1)^2}{4} + c\delta\Big)|v|^2\Big] \,\mathrm{d}s\,\mathrm{d}t + \gamma \int_\Sigma \big|v(s,0)\big|^2 \,\mathrm{d}s,$$

(b) for any $u \in H^1(\Omega_*^\delta)$ one has

$$J(u) \geq \int_{\Pi_\delta} \Big[(1-c\delta)|\nabla_s v|^2 + |\partial_t v|^2 + \Big(H_2^* - \frac{(H_1^*)^2}{4} - c\delta\Big)|v|^2\Big] \,\mathrm{d}s\,\mathrm{d}t$$
$$+ \gamma \int_{\partial U} \big|v(s,0)\big|^2 \,\mathrm{d}s - c\int_\Sigma \big|v(s,\delta)\big|^2 \,\mathrm{d}s,$$

where ∇_s is the gradient on Σ, i.e. with respect to the coordinates $s \in \Sigma$.

Proof. The metric G on Π_δ induced by the map Φ_* is given by $G = g\circ(1-tW_*)+\mathrm{d}t^2$, with g being the metric on Σ induced by the embedding in $\mathbb{R}^n$, and the volume form is $\det G\,\mathrm{d}s\,\mathrm{d}t = \varphi\,\mathrm{d}s\,\mathrm{d}t$ with $\mathrm{d}s$ being the volume form on Σ and the weight

$$\varphi(s,t) = \prod_{j=1}^{n-1} \big(1 - th_j^*(s)\big) = 1 + \sum_{j\geq 1} (-t)^j H_j^*(s). \tag{26}$$

Denote $w := u\circ\Phi_*$, then the standard change of variables gives, for any $u \in H^1(\Omega_*^\delta)$,

$$J(u) = \int_{\Pi_\delta} |\nabla w|^2 \varphi \,\mathrm{d}s\,\mathrm{d}t + \int_\Sigma \Big(\gamma + \frac{H_1^*}{2}\Big) \big|w(s,0)\big|^2 \,\mathrm{d}s,$$

and we remark that the condition $u = 0$ on $\partial\Omega_*^\delta \setminus \Sigma$ is equivalent to $w(\cdot,\delta) = 0$. Due to the above representation of the metric G, for a suitable fixed $c_0 > 0$ one can estimate, uniformly in u,

$$(1-c_0\delta)|\nabla_s w|^2 + |\partial_t w|^2 \leq |\nabla w|^2 \leq (1+c_0\delta)|\nabla_s w|^2 + |\partial_t w|^2,$$

with ∇_s being the gradient on Σ (i.e. with respect to the variable s), which gives

$$\int_{\Pi_\delta} \big((1-c_0\delta)|\nabla_s w|^2 + |\partial_t w|^2\big)\varphi \,\mathrm{d}s\,\mathrm{d}t + \int_\Sigma \Big(\gamma + \frac{H_1^*}{2}\Big) \big|w(s,0)\big|^2 \,\mathrm{d}s$$
$$\leq J(u) \leq \int_{\Pi_\delta} \big((1+c_0\delta)|\nabla_s w|^2 + |\partial_t w|^2\big)\varphi \,\mathrm{d}s\,\mathrm{d}t + \int_\Sigma \Big(\gamma + \frac{H_1^*}{2}\Big) \big|w(s,0)\big|^2 \,\mathrm{d}s. \tag{27}$$

Recall that $w = \varphi^{-\frac{1}{2}}v$, and that $\varphi = 1$ on Σ. Hence,

$$\Big(\gamma + \frac{H_1^*}{2}\Big) \big|w(s,0)\big|^2 = \Big(\gamma + \frac{H_1^*}{2}\Big) \big|v(s,0)\big|^2,$$

which allows to transform the last summand in (27). In addition,

$$|\nabla_s w|^2 \varphi = \Big|\nabla_s v - \frac{1}{2\varphi} v \nabla_s \varphi\Big|^2 = |\nabla_s v|^2 + \frac{|v|^2}{4\varphi^2}|\nabla_s\varphi|^2 - \frac{1}{\varphi}\Re\big(\langle\nabla_s v, v\nabla_s\varphi\rangle\big).$$

The Cauchy-Schwarz inequality gives $\big|\Re\langle\nabla_s v, v\nabla_s\varphi\rangle\big| \leq \delta|\nabla_s v|^2 + |v|^2|\nabla_s\varphi|^2/\delta$, and in view of the expression (26) for φ one has $|\nabla_s\varphi|^2 \leq c_1\delta^2$ for some $c_1 > 0$ and all $t \in (0,\delta)$. Therefore, for a suitable $c_2 > 0$ one estimates, uniformly in u,

$$(1-c_2\delta)|\nabla_s v|^2 - c_2\delta|v|^2 \leq (1\pm c_0\delta)|\nabla_s w|^2\varphi \leq (1+c_2\delta)|\nabla_s v|^2 + c_2\delta|v|^2.$$

We represent now

$$|\partial_t w|^2\varphi = \Big|\partial_t v - \frac{1}{2\varphi} v\,\partial_t\varphi\Big|^2 = |\partial_t v|^2 - \frac{\partial_t\varphi}{2\varphi}\partial_t\big(|v|^2\big) + \frac{(\partial_t\varphi)^2}{4\varphi^2}|v|^2$$

and performing an integration by parts with respect to t in the middle term we have

$$\int_{\Pi_\delta} |\partial_t w|^2 \varphi \,\mathrm{d}s\,\mathrm{d}t = \int_{\Pi_\delta} \Big(|\partial_t v|^2 + \Big(\partial_t\Big(\frac{\partial_t \varphi}{2\varphi}\Big) + \frac{(\partial_t\varphi)^2}{4\varphi^2}\Big)|v|^2\Big) \,\mathrm{d}s\,\mathrm{d}t$$
$$- \int_\Sigma \frac{H_1^*}{2} \big|v(s,0)\big|^2 \,\mathrm{d}s - \int_\Sigma \frac{(\partial_t\varphi)(s,\delta)}{2\varphi(s,\delta)} \big|v(s,\delta)\big|^2 \,\mathrm{d}s,$$

while the last summand vanishes for $v(\cdot,\delta)=0$, i.e. for $u=0$ on $\partial\Omega_*^\delta \setminus \Sigma$. Putting the above estimates together we obtain

$$J(u) \le \int_{\Pi_\delta} \Big((1+c_2\delta)|\nabla_s v|^2 + |\partial_t v|^2 + \Big(\frac{\partial_t^2\varphi}{2\varphi} - \frac{(\partial_t\varphi)^2}{4\varphi^2} + c_2\delta\Big)|v|^2 \,\mathrm{d}s\,\mathrm{d}t$$
$$+ \gamma \int_\Sigma \big|v(s,\delta)\big|^2 \,\mathrm{d}s, \quad u \in H^1(\Omega_*^\delta), \quad u = 0 \text{ on } \partial\Omega_*^\delta \setminus \Sigma,$$
$$J(u) \ge \int_{\Pi_\delta} \Big((1-c_2\delta)|\nabla_s v|^2 + |\partial_t v|^2 + \Big(\frac{\partial_t^2\varphi}{2\varphi} - \frac{(\partial_t\varphi)^2}{4\varphi^2} - c_2\delta\Big)|v|^2 \,\mathrm{d}s\,\mathrm{d}t$$
$$+ \gamma \int_\Sigma \big|v(s,0)\big|^2 \,\mathrm{d}s - \int_\Sigma \frac{(\partial_t\varphi)(s,\delta)}{2\varphi(s,\delta)} \big|v(s,\delta)\big|^2 \,\mathrm{d}s, \quad u \in H^1(\Omega_*^\delta).$$

It remains to estimate, with a suitable $c_3 > 0$,

$$\Big\| \frac{(\partial_t\varphi)(\cdot,\delta)}{2\varphi(\cdot,\delta)} \Big\|_{L^\infty(\Sigma)} \le c_3, \quad \Big\| \frac{\partial_t^2\varphi}{2\varphi} - \frac{(\partial_t\varphi)^2}{4\varphi^2} - \Big(H_2^* - \frac{(H_1^*)^2}{4}\Big) \Big\|_{L^\infty(\Sigma)} \le c_3\delta$$

and to choose $c := \max\{c_2, c_3\}$. □

4 Proof of Theorem 1.1

We are going to show that $E_j(A_m^2) \to E_j(D\!\!\!/^2)$ for each $j \in \mathbb{N}$ as $m \to -\infty$. Due to Lemma 2.4 for each $j \in \mathbb{N}$ there holds $E_j(D\!\!\!/^2) = E_j(L)$, hence, it is sufficient to prove that

$$E_j(L) = \lim_{m\to-\infty} E_j(A_m^2) \text{ for each } j \in \mathbb{N}. \tag{28}$$

The proof of (28) decomposes into three steps and relies on a dimension reduction argument.

In Subsection 4.1 we use the standard Dirichlet-Neumann bracketing to bound from above and from below the sesquilinear form of A_m^2. We reduce the spectral analysis of A_m^2 to the one of sesquilinear forms in tubular neighborhoods of the hypersurface Σ, and the sesquilinear forms are covered by the constructions of Subsection 3.2. Then, in Subsection 4.2, we obtain an upper bound by considering a well-chosen test function and by applying the min-max principle. The test function is constructed by taking advantage of the tensor structure of the sesquilinear form which bounds from above the sesquilinear form of A_m^2: it is obtained by taking the tensor product of the lowest mode of the operator in normal variable with an eigenfunction of the operator along the surface. The lower bound is more subtle and is handled in Subsection 4.3. One first decomposes the sesquilinear form which bounds from below the sesquilinear form of A_m^2 along the modes of a one-dimensional operator acting only in the normal variable, and then one concludes using estimates on the eigenvalues of the one-dimensional operator and the monotone convergence (Proposition 1.10).

Remark 4.1. Using the constructions mentioned in Remarks 2.2 and 2.5, the proof below can be easily modified in order to show that $E_j(L') = \lim_{m\to+\infty} E_j({A'_m}^2)$ for each $j \in \mathbb{N}$, which then gives Theorem 1.6.

4.1 Dirichlet-Neumann bracketing

For small $\delta > 0$ denote $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x,\Sigma) < \delta\}$ and $\Pi_\delta := \Sigma \times (0,\delta)$ and consider the diffeomorphisms $\Phi : \Pi_\delta \to \Omega_\delta$ given by $(s,t) \mapsto s - t\nu(s)$ together with the associated unitary maps $\Theta_\delta : L^2(\Omega_\delta,\mathbb{C}^N) \to L^2(\Pi_\delta,\mathbb{C}^N)$, $\Theta_\delta u = \sqrt{\det(\Phi')}\, u\circ\Phi$.

Consider the self-adjoint operator Z_m^+ in $L^2(\Omega_\delta,\mathbb{C}^N)$ given by

$$\begin{aligned} Z_m^+[u,u] &= \int_{\Omega_\delta} \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x + \int_\Sigma \Big(m + \frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s, \\ \mathcal{Q}(Z_m^+) &= \big\{u \in H^1(\Omega_\delta,\mathbb{C}^N) : u = \mathcal{B}u \text{ on } \Sigma, \quad u = 0 \text{ on } \partial\Omega_\delta\setminus\Sigma\big\}, \end{aligned} \tag{29}$$

the self-adjoint operator Z_m^- in $L^2(\Omega_\delta,\mathbb{C}^N)$ given by

$$\begin{aligned} Z_m^-[u,u] &= \int_{\Omega_\delta} \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x + \int_\Sigma \Big(m + \frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s, \\ \mathcal{Q}(Z_m^-) &= \big\{u \in H^1(\Omega_\delta,\mathbb{C}^N) : u = \mathcal{B}u \text{ on } \Sigma\big\}, \end{aligned} \tag{30}$$

and the self-adjoint operator Z'_m in $L^2(\Omega_\delta^c,\mathbb{C}^N)$ given by

$$Z'_m[u,u] = \int_{\Omega_\delta^c} \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x, \quad \mathcal{Q}(Z'_m) = H^1(\Omega_\delta^c,\mathbb{C}^N),$$

with $\Omega_\delta^c := \Omega \setminus \overline{\Omega_\delta}$. Due to the min-max principle for any $j \in \mathbb{N}$ and we have the eigenvalue inequality

$$E_j(Z_m^- \oplus Z'_m) \le E_j(A_m^2) \le E_j(Z_m^+).$$

(It is sufficient to apply Proposition 1.9: for the left inequality one takes $T = A_m^2$, $T' := Z_m^- \oplus Z'_m$, and $J : L^2(\Omega,\mathbb{C}^N) \mapsto (f^-, f') \in L^2(\Omega_\delta,\Omega_\delta^c)$ defined by $f^- := f|_{\Omega_\delta}$ and $f' := f|_{\Omega_\delta^c}$, while for the right inequality one takes $T := Z_m^+$, $T' := A_m^2$ and $J : L^2(\Omega_\delta) \to L^2(\Omega)$ the extension by zero.) Noting that $Z'_m \ge m^2$ we deduce that

$$E_j(Z_m^-) \le E_j(A_m^2) \le E_j(Z_m^+) \text{ for any } j \in \mathbb{N} \text{ with } E_j(Z_m^+) < m^2. \tag{31}$$

Using the change of coordinates of Lemma 3.3 to bound $Z_m^\pm[\Theta_\delta^* v, \Theta_\delta^* v]$ from above and below we then obtain

$$E_j(Z_m^+) \le E_j(Y_m^+), \quad E_j(Z_m^-) \ge E_j(Y_m^-) \text{ for any } j \in \mathbb{N}$$

with $Y_m^\pm$ being the self-adjoint operators in $L^2(\Pi_\delta,\mathbb{C}^N)$ given by

$$\begin{aligned} Y_m^+[v,v] &= \int_{\Pi_\delta} \bigg[(1+c\delta)|\nabla_s v|^2 + |\partial_t v|^2 + \Big(m^2 + H_2 - \frac{H_1^2}{4} + c\delta\Big)|v|^2\bigg]\,\mathrm{d}s\,\mathrm{d}t \\ &\quad + m\int_\Sigma \big|v(s,0)\big|^2\,\mathrm{d}s, \\ \mathcal{Q}(Y_m^+) &= \big\{v \in H^1(\Pi_\delta,\mathbb{C}^N) : v(\cdot,0) = \mathcal{B}v(\cdot,0) \text{ and } v(\cdot,\delta) = 0\big\}, \end{aligned}$$

$$
\begin{aligned}
Y_m^-[v,v] &= \int_{\Pi_\delta}\Big[(1-c\delta)|\nabla_s v|^2 + |\partial_t v|^2 + \Big(m^2 + H_2 - \frac{H_1^2}{4} - c\delta\Big)|v|^2\Big]\,\mathrm{d}s\,\mathrm{d}t\\
&\quad + m\int_\Sigma \big|v(s,0)\big|^2\,\mathrm{d}s - c\int_\Sigma \big|v(s,\delta)\big|^2\,\mathrm{d}s,\\
\mathcal{Q}(Y_m^-) &= \big\{v\in H^1(\Pi_\delta,\mathbb{C}^N):\, v(\cdot,0)=\mathcal{B}v(\cdot,0)\big\},
\end{aligned}
$$

where c is independent of $\delta\in(0,\delta_0)$ and $m\in\mathbb{R}$ is arbitrary. Therefore, we arrive at the two-sided estimate

$$
E_j(Y_m^-)\le E_j(A_m^2)\le E_j(Y_m^+) \text{ for any } j\in\mathbb{N} \text{ with } E_j(Y_m^+)<m^2. \tag{32}
$$

4.2 Upper bound

To obtain an upper bound for the eigenvalues of Y_m^+ let us consider the self-adjoint operator S in $L^2(0,\delta)$ with

$$
S[f,f]=\int_0^\delta |f'|^2\,\mathrm{d}t + m\big|f(0)\big|^2,\quad \mathcal{Q}(S)=\big\{f\in H^1(0,\delta):\, f(\delta)=0\big\}
$$

and let ψ be an eigenfunction for the first eigenvalue normalized by $\|\psi\|^2_{L^2(0,\delta)}=1$. The analysis of Lemma 3.1 shows that for some $b>0$ one has $E_1(S)\le -m^2+be^{-\delta|m|}$ as $(-m)$ is large, and then $S[\psi,\psi]+m^2\le be^{-\delta|m|}$.

Let $c>0$ be the same as in the above expressions for $Y_m^\pm$. For small $a\in\mathbb{R}$, let L_a be the self-adjoint operator in $\mathcal{H}$ given by

$$
\begin{aligned}
L_a[g,g] &= \int_\Sigma \Big[(1+ca)|\nabla g|^2 + \Big(H_2-\frac{H_1^2}{4}+ca\Big)|g|^2\Big]\,\mathrm{d}s,\\
\mathcal{Q}(L_a) &= H^1(\Sigma,\mathbb{C}^N)\cap\mathcal{H}.
\end{aligned} \tag{33}
$$

Remark that for $a=0$ we recover exactly the operator L and that due to the min-max principle one has

$$
E_j(L)=\lim_{a\to 0}E_j(L_a) \text{ for each } j\in\mathbb{N}. \tag{34}
$$

Let $j\in\mathbb{N}$ be fixed and $g_1,\dots,g_j$ be linearly independent eigenfunctions of L_δ for the first j eigenvalues, then the subspace $G:=\operatorname{span}(g_1,\dots,g_j)$ is j-dimensional and $L_\delta[g,g]/\|g\|^2_{\mathcal{H}}\le E_j(L_\delta)$ for any $0\ne g\in G$. Consider the subspace

$$
V=\{v\in L^2(\Pi_\delta,\mathbb{C}^N):\, v(s,t)=g(s)\psi(t),\ g\in G\}\subset\mathcal{Q}(Y_m^+),
$$

then for $v\in V$ with $v(s,t)=g(s)\psi(t)$ and $g\in G$ one has $\|v\|^2_{L^2(\Pi_\delta,\mathbb{C}^N)}=\|g\|^2_{\mathcal{H}}$ and

$$
\begin{aligned}
Y_m^+[v,v] &= L_\delta[g,g]\|\psi\|^2_{L^2(0,\delta)} + \Big(S[\psi,\psi]+m^2\|\psi\|^2_{L^2(0,\delta)}\Big)\|g\|^2_{\mathcal{H}}\\
&\le L_\delta[g,g]+be^{-\delta|m|}\|g\|^2_{\mathcal{H}} \le \big(E_j(L_\delta)+be^{-\delta|m|}\big)\|g\|^2_{\mathcal{H}}\\
&\equiv \big(E_j(L_\delta)+be^{-\delta|m|}\big)\|v\|^2_{L^2(\Pi_\delta,\mathbb{C}^N)}.
\end{aligned}
$$

As $\dim V=\dim G=j$, it follows by the min-max principle that

$$
E_j(Y_m^+)\le \sup_{0\ne v\in V}\frac{Y_m^+[v,v]}{\|v\|^2_{L^2(\Pi_\delta,\mathbb{C}^N)}}\le E_j(L_\delta)+be^{-\delta|m|},
$$

hence, $\limsup_{m\to-\infty} E_j(Y_m^+) \le E_j(L_\delta)$. As $\delta > 0$ can be chosen arbitrarily small, the convergence (34) implies $\limsup_{m\to-\infty} E_j(Y_m^+) \le E_j(L)$, and then due to the upper bound (32) we arrive at

$$\limsup_{m\to-\infty} E_j(A_m^2) \le E_j(L). \tag{35}$$

4.3 Lower bound

Now let us pass to a lower bound for $E_j(Y_m^-)$. In the constructions below, the constant $c > 0$ is the same as in the expression for Y_m^-. Let S' be the self-adjoint operator in $L^2(0,\delta)$ with

$$S'[f,f] = \int_0^\delta |f'|^2 \,\mathrm{d}t + m\big|f(0)\big|^2 - c\big|f(\delta)\big|^2, \quad \mathcal{Q}(S') = H^1(0,\delta).$$

Let $\psi_k \in L^2(0,\delta)$ with $k \in \mathbb{N}$ be real-valued eigenfunctions of S' for the eigenvalues $E_k(S')$ forming an orthonormal basis in $L^2(0,\delta)$, which induces the unitary transforms $\Theta : L^2(0,\delta) \to \ell^2(\mathbb{N})$ given by $(\Theta f)_k = \langle \psi_k, f\rangle_{L^2(0,\delta)}$, $k \in \mathbb{N}$. Recall that due to the analysis of Lemma 3.2 we have, with some $b^\pm > 0$, $b > 0$ and $b_0 > 0$,

$$E_1(S') \ge -m^2 - be^{-\delta|m|} \text{ as } m \to -\infty, \tag{36}$$

$$b^- k^2 - b_0 \le E_k(S') \le b^+ k^2 \text{ for all } k \ge 2 \text{ and } m \in \mathbb{R}. \tag{37}$$

Let us give some more details on the subsequent constructions. Let Y_m be the self-adjoint operator whose sesquilinear form is given by the same expression as the one for Y_m^- but on the larger form domain $\mathcal{Q}(Y_m) = H^1(\Pi_\delta, \mathbb{C}^N)$. It follows easily that the new operator Y_m admits a separation of variables. Namely, for small $a \in \mathbb{R}$ we consider the self-adjoint operator Λ_a in $L^2(\Sigma, \mathbb{C}^N)$ given by

$$\Lambda_a[g,g] = \int_\Sigma \Big[(1+ca)|\nabla g|^2 + \Big(H_2 - \frac{H_1^2}{4} + ca\Big)|g|^2\Big]\,\mathrm{d}s, \quad \mathcal{Q}(\Lambda_a) = H^1(\Sigma, \mathbb{C}^N),$$

i.e. its sesquilinear form is given by the same expression as the one for L_a in (33) but without the restriction $g \in \mathcal{H}$. Now, if one identifies $L^2(\Pi_\delta, \mathbb{C}^N) = L^2(0,\delta) \otimes L^2(\Sigma, \mathbb{C}^N)$, then $Y_m = (S' + m^2) \otimes 1 + 1 \otimes \Lambda_{-\delta}$. Using the unitary transform

$$\Xi : L^2(\Pi_\delta) \to \ell^2(\mathbb{N}) \otimes L^2(\Sigma, \mathbb{C}^N),$$
$$\Xi v = (v_k), \quad v_k := \int_0^\delta \psi_k(t) v(t,\cdot)\,\mathrm{d}t \in L^2(\Sigma, \mathbb{C}^N),$$

and the spectral theorem we see that the operator $\widehat{Y}_m := \Xi Y_m \Xi^*$ is given by

$$\widehat{Y}_m\big[(v_k),(v_k)\big] = \sum_{k\in\mathbb{N}} \Big(\Lambda_{-\delta}[v_k, v_k] + \big(E_k(S') + m^2\big)\|v_k\|^2_{L^2(\Sigma,\mathbb{C}^N)}\Big),$$

while the form domain $\mathcal{Q}(\widehat{Y}_m)$ consists of all $(v_k) \in \ell^2(\mathbb{N}) \otimes L^2(\Sigma, \mathbb{C}^N)$ with $v_k \in H^1(\Sigma, \mathbb{C}^N)$ such that the right-hand side of the preceding expression is finite. Using the two-sided estimate (37) we can rewrite

$$\mathcal{Q}(\widehat{Y}_m) = \Big\{(v_k) \in \ell^2(\mathbb{N}) \otimes L^2(\Sigma, \mathbb{C}^N) : \ v_k \in H^1(\Sigma, \mathbb{C}^N) \text{ for each } k \in \mathbb{N} \\ \text{and } \sum_{k\in\mathbb{N}} \Big(\|v_k\|^2_{H^1(\Sigma,\mathbb{C}^N)} + k^2 \|v_k\|^2_{L^2(\Sigma,\mathbb{C}^N)} \Big) < \infty \Big\}. \quad (38)$$

As the sesquilinear form for Y_m^- is simply the restriction of that for Y_m on the functions v with $v(\cdot, 0) = \mathcal{B}v(\cdot, 0)$, for the operator $\widehat{Y}_m^- := \Xi Y_m^- \Xi^*$ we have

$$\mathcal{Q}(\widehat{Y}_m^-) = \big\{\widehat{v} = (v_k) \in \mathcal{Q}(\widehat{Y}_m) : (1 - \mathcal{B})(\Xi^* \widehat{v})(\cdot, 0) = 0\big\}. \quad (39)$$

Using the lower bounds (36) and (37) for $E_k(S')$, for all $\widehat{v} = (v_k) \in \mathcal{Q}(\widehat{Y}_m^-)$ we obtain the inequality $\widehat{Y}_m^-[\widehat{v}, \widehat{v}] \geq w_m(\widehat{v}, \widehat{v})$ with the sesquilinear form w_m defined on $\mathcal{D}(w_m) := \mathcal{Q}(\widehat{Y}_m^-)$ by

$$w_m(\widehat{v}, \widehat{v}) := \Lambda_{-\delta}[v_1, v_1] - b e^{-\delta|m|} \|v_1\|^2_{L^2(\Sigma,\mathbb{C}^N)} \\ + \sum_{k\geq 2} \Big(\Lambda_{-\delta}[v_k, v_k] + (b^- k^2 - b_0 + m^2) \|v_k\|^2_{L^2(\Sigma,\mathbb{C}^N)} \Big).$$

It follows from representation (38) that the form w_m is lower semibounded and from reprentation (39) that it is closed. Thus, it defines a self-adjoint operator W_m in $\ell^2(\mathbb{N}) \otimes L^2(\Sigma, \mathbb{C}^N)$ with compact resolvent. For any $j \in \mathbb{N}$ we have then

$$E_j(A_m^2) \geq E_j(Y_m^-) = E_j(\widehat{Y}_m^-) \geq E_j(W_m). \quad (40)$$

We are now in the classical situation for the monotone convergence (Proposition 1.10) to analyze the eigenvalues of W_m. Namely, consider the set

$$\mathcal{Q}_\infty := \Big\{\widehat{v} = (v_k) \in \bigcap_{m<0} \mathcal{Q}(W_m) \equiv \mathcal{Q}(\widehat{Y}_m^-) : \quad \sup_{m<0} W_m[\widehat{v}, \widehat{v}] < +\infty \Big\}. \quad (41)$$

It is easily seen that a vector $\widehat{v} = (v_k) \in \mathcal{Q}(\widehat{Y}_m^-)$ belongs to $\mathcal{Q}_\infty$ if and only if $v_k = 0$ for $k \geq 2$ and $0 = (1 - \mathcal{B})(\Xi^* \widehat{v})(\cdot, 0) \equiv \psi_1(0)(1 - \mathcal{B})v_1$, i.e. $v_1 \in \mathcal{H}$. This gives the equality

$$\mathcal{Q}_\infty = \big\{\widehat{v} = e_1 \otimes v_1 : \ v_1 \in H^1(\Sigma, \mathbb{C}^N) \cap \mathcal{H}\big\}, \quad e_1 = (1, 0, 0, \dots) \in \ell^2(\mathbb{N}).$$

For each $\widehat{v} \in \mathcal{Q}_\infty$ one has

$$\lim_{m\to-\infty} W_m[\widehat{v}, \widehat{v}] = \lim_{m\to-\infty} \Big(\Lambda_{-\delta}[v_1, v_1] - c_1 e^{-\delta|m|} \|v_1\|^2_{L^2(\Sigma,\mathbb{C}^N)} \Big) \\ = \Lambda_{-\delta}[v_1, v_1] \equiv L_{-\delta}[v_1, v_1];$$

we recall that L_a was defined in (33). Let W_∞ be the self-adjoint operator in the Hilbert space $\mathcal{H}_\infty := e_1 \otimes \mathcal{H}$ with $\mathcal{Q}(W_\infty) = \mathcal{Q}_\infty$ and $W_\infty[e_1 \otimes v_1, e_1 \otimes v_1] = L_{-\delta}[v_1, v_1]$, then the monotone convergence principle (Proposition 1.10) gives $\lim_{m\to-\infty} E_j(W_m) = E_j(W_\infty)$ for each $j \in \mathbb{N}$. On the other hand, the operator W_∞ is unitarily equivalent to $L_{-\delta}$, and by combining with (40) we have $\liminf_{m\to-\infty} E_j(A_m) \geq E_j(L_{-\delta})$. As δ can be arbitrarily small, the convergence (34) implies $\liminf_{m\to-\infty} E_j(A_m) \geq E_j(L)$. In combination with the upper bound (35) one arrives at the sought limit (28), which proves Theorem 1.1.

5 Proof of Theorem 1.2

The proof relies on the construction of an extension operator from $H^1(\Sigma)$ to $H^1(\Omega^c)$ suitably controled in the regime $M \to +\infty$. This is done in Subsection 5.1 using alternative expressions of the sesquilinear forms of A_m^2 and $B_{m,M}^2$. Afterwards, we prove Theorem 1.2 by obtaining separately the upper and lower bounds for all $j \in \mathbb{N}$:

$$\limsup_{M\to+\infty} E_j(B_{m,M}^2) \le E_j(A_m^2), \quad E_j(A_m^2) \le \liminf_{M\to+\infty} E_j(B_{m,M}^2).$$

In Subsection 5.2 we prove the upper bound by constructing an adequate test function, which is done by applying the above extension operator to the eigenfunctions of A_m^2. The lower bound is proved in Subsection 5.3 using a Neumann bracketing argument, which allows for a decoupling along Σ. Hence, one is reduced to the study of the direct sum of two operators in Ω and Ω^c. One shows first that the one in Ω^c produces eigenvalues diverging to $+\infty$, and only the operator in Ω is of relevance for the low-lying eigenvalues. The problem in Ω in the regime $M \to +\infty$ appears then to be covered by the monotone convergence (Proposition 1.10).

5.1 Preliminary estimates

We are going to prove that for each $m \in \mathbb{R}$ and $j \in \mathbb{N}$ one has $\lim_{M\to+\infty} E_j(B_{m,M}^2) = E_j(A_m^2)$. We recall that $\mathcal{Q}(B_{m,M}^2) \equiv \mathcal{D}(B_{m,M}) = H^1(\mathbb{R}^n, \mathbb{C}^N)$, and

$$\begin{aligned} B_{m,M}^2[u,u] &\equiv \langle B_{m,M}u, B_{m,M}u\rangle_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \\ &= \int_\Omega \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x + \int_{\Omega^c} \big(|\nabla u|^2 + M^2|u|^2\big)\,\mathrm{d}x \\ &\qquad + (M-m)\Big(\int_\Sigma |\mathcal{P}_- u|^2\,\mathrm{d}s - \int_\Sigma |\mathcal{P}_+ u|^2\,\mathrm{d}s\Big), \\ &= \int_\Omega \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x + \int_{\Omega^c} \big(|\nabla u|^2 + M^2|u|^2\big)\,\mathrm{d}x \\ &\qquad + 2(M-m)\int_\Sigma |\mathcal{P}_- u|^2\,\mathrm{d}s + (m-M)\int_\Sigma |u|^2\,\mathrm{d}s \end{aligned} \tag{42}$$

where $\mathcal{P}_\pm(s) := \dfrac{1 \pm \mathcal{B}(s)}{2}$ for $s \in \Sigma$, while

$$\mathcal{Q}(A_m^2) \equiv \mathcal{D}(A_m) = \big\{u \in H^1(\Omega, \mathbb{C}^N) : \ \mathcal{P}_- u = 0 \text{ on } \Sigma\big\},$$

$$A_m^2[u,u] \equiv \langle A_m u, A_m u\rangle_{L^2(\Omega,\mathbb{C}^N)} = \int_\Omega \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x + \int_\Sigma \Big(m + \frac{H_1}{2}\Big)\,|u|^2\,\mathrm{d}s.$$

Taking any $\varepsilon \in \mathbb{R}$ we rewrite the above expression for $B_{m,M}^2[u,u]$ as

$$\begin{aligned} B_{m,M}^2[u,u] &= \int_\Omega \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x \\ &\quad + \int_\Sigma \Big(m - \varepsilon + \frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s + 2(M-m)\int_\Sigma |\mathcal{P}_- u|^2\,\mathrm{d}s \\ &\qquad + \int_{\Omega^c} \big(|\nabla u|^2 + M^2|u|^2\big)\,\mathrm{d}x - \int_\Sigma \Big(M - \varepsilon + \frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s. \end{aligned} \tag{43}$$

Let us start with an additional estimate which will allow us to control the term in the last line of (43).

Lemma 5.1. *For $\gamma > 0$ let R_γ be the self-adjoint operator in $L^2(\Omega^c)$ given by*

$$R_\gamma[u,u] = \int_{\Omega^c} |\nabla u|^2 \,\mathrm{d}x - \int_\Sigma \Big(\gamma + \frac{H_1}{2}\Big)|u|^2 \,\mathrm{d}s, \quad \mathcal{Q}(R_\gamma) = H^1(\Omega^c), \tag{44}$$

then:

(a) For some fixed $C > 0$ and all large $\gamma > 0$ there exists a linear map $F_\gamma : H^1(\Sigma) \to H^1(\Omega^c)$ such that for all $f \in H^1(\Sigma)$ one has $F_\gamma f = f$ on Σ and

$$R_\gamma[F_\gamma f, F_\gamma f] + \gamma^2 \|F_\gamma f\|^2_{L^2(\Omega^c)} \le \frac{C}{\gamma}\|f\|^2_{H^1(\Sigma)}.$$

(b) For some $C_0 > 0$ there holds $E_1(R_\gamma) \ge -\gamma^2 - C_0$ for $\gamma \to +\infty$.

Proof. For a small $\delta > 0$ consider the sets $\Omega^c_\delta := \big\{x \in \Omega^c : \operatorname{dist}(x,\Sigma) < \delta\big\}$ and $\Pi_\delta := \Sigma \times (0,\delta)$ together with the the diffeomorphisms $\Phi^c : \Pi_\delta \to \Omega^c_\delta$ given by $\Phi^c(s,t) \mapsto s + t\nu(s)$ and the associated unitary maps $\Theta^c_\delta : L^2(\Omega^c_\delta) \to L^2(\Pi_\delta)$ with $\Theta^c_\delta u = \sqrt{\det\big((\Phi^c)'\big)}\ u \circ \Phi^c$.

Let us prove (a). Consider the self-adjoint operator S in $L^2(0,\delta)$ given by

$$S[f,f] = \int_0^\delta |f'|^2 \,\mathrm{d}t - \gamma\big|f(0)\big|^2, \quad \mathcal{Q}(S) = \big\{f \in H^1(0,\delta) : f(\delta) = 0\big\}$$

and let ψ be an eigenfunction for the first eigenvalue normalized by $\psi(0) = 1$. By Lemma 3.1, with some $b > 0$ one has $E_1(S) \le -\gamma^2 + b$ and $\|\psi\|^2_{L^2(0,\delta)} \le b/\gamma$ as γ is large. For $f \in H^1(\Sigma)$ define $v \in H^1(\Pi_\delta)$ by $v = f \otimes \psi$, i.e. $v(s,t) = f(s)\psi(t)$, and then set

$$(F_\gamma f)(x) := \begin{cases} (\Theta^c_\delta)^{-1} v & \text{in } \Omega^c_\delta, \\ 0 & \text{in } \Omega^c \setminus \Omega^c_\delta. \end{cases}$$

Due to $f \in H^1(\Sigma)$ and $\psi(\delta) = 0$ one has $F_\gamma f \in H^1(\Omega^c)$, and the equality $F_\gamma f|_\Sigma = v(\cdot,0) = f$ holds by construction. Furthermore, using the result and the notation of Lemma 3.3(a) we obtain, with some $a > 0$,

$$\begin{aligned} R_\gamma[F_\gamma f, F_\gamma f] + \gamma^2\|F_\gamma f\|^2 &= J_{-\gamma}(F_\gamma f) + \gamma^2 \|F_\gamma f\|^2 \\ &\le \int_{\Pi_\delta} \big(a|\nabla_s v|^2 + |\partial_t v|^2 + (\gamma^2 + a)|v|^2\big)\,\mathrm{d}s\,\mathrm{d}t - \gamma \int_\Sigma |F_\gamma f|^2 \,\mathrm{d}s \\ &= \Big(a \int_\Sigma |\nabla_s f|^2 \,\mathrm{d}s + \big(E_1(S) + \gamma^2 + a\big)\|f\|^2_{L^2(\Sigma)}\Big)\,\|\psi\|^2_{L^2(0,\delta)} \\ &\le \Big(a \int_\Sigma |\nabla_s f|^2 \,\mathrm{d}s + \big(be^{-\delta\gamma} + a\big)\|f\|^2_{L^2(\Sigma)}\Big)\,\frac{b}{\gamma} \le \frac{C}{\gamma}\|f\|^2_{H^1(\Sigma)} \end{aligned}$$

with $C := b\big(b+a\big)$. Hence, the assertion (a) is proved.

To prove (b) we remark first that due to the min-max principle one has the inequality $E_1(R_\gamma) \ge E_1(R^0_\gamma \oplus R'_\gamma)$ where R^0_γ is the operator in $L^2(\Omega^c_\delta)$ given by

$$R^0_\gamma[u,u] = \int_{\Omega^c_\delta} |\nabla u|^2 \,\mathrm{d}x - \int_\Sigma \Big(\gamma + \frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s, \quad \mathcal{Q}(R^0_\gamma) = H^1(\Omega^c_\delta),$$

and R'_γ is the self-adjoint operator in $L^2(\Omega'_\delta)$, with $\Omega'_\delta := \Omega^c \setminus \overline{\Omega^c_\delta}$, given by

$$R'_\gamma[u,u] = \int_{\Omega'_\delta} |\nabla u|^2 \,\mathrm{d}x, \quad \mathcal{Q}(R'_\gamma) = H^1(\Omega'_\delta).$$

Due to $R'_\gamma \geq 0$ one has $E_1(R_\gamma) \geq \min\{E_1(R^0_\gamma), 0\}$. By Lemma 3.3(b) one has $E_1(R^0_\gamma) \geq E_1(X_\gamma)$ with R being the self-adjoint operator in $L^2(\Pi_\delta)$ with

$$X_\gamma[v,v] = \int_{\Pi_\delta} \Big[a'|\nabla_s v|^2 + |\partial_t v|^2 - a'|v|^2\Big] \,\mathrm{d}s\,\mathrm{d}t - \gamma \int_\Sigma \big|v(s,0)\big|^2 \,\mathrm{d}s - a' \int_{\partial U} \big|v(s,\delta)\big|^2 \,\mathrm{d}s$$

and $\mathcal{Q}(X_\gamma) = H^1(\Omega^c_\delta)$, with some $a' > 0$. Let S' be the self-adjoint operator in $L^2(0,\delta)$ given by

$$S'[f,f] = \int_0^\delta |f'|^2 \,\mathrm{d}t - \gamma\big|f(0)\big|^2 - a'\big|f(\delta)\big|^2, \quad \mathcal{Q}(S') = H^1(0,\delta).$$

As $|\nabla_s v|^2 \geq 0$, due to Fubini's theorem one has $E_1(X_\gamma) \geq E_1(S') - a'$, and now it is sufficient to remark that by Lemma 3.2 one has $E_1(S') \geq -\gamma^2 - a_0$ with some $a_0 > 0$ as $\gamma \to +\infty$. ☐

5.2 Upper bound

Pick $m \in \mathbb{R}$ and $j \in \mathbb{N}$, and let $u_1, \dots, u_j$ be linearly independent eigenfunctions of A^2_m for the first j eigenvalues, then for any function $u \in V := \operatorname{span}(u_1, \dots, u_j)$ there holds $A^2_m[u,u] \leq E_j(A^2_m)\|u\|^2_{L^2(\Omega,\mathbb{C}^N)}$. Recall that the standard elliptic regularity argument implies one has $V \subset C^\infty(\overline{\Omega}, \mathbb{C}^N)$,see Proposition A.2(b) for details, and then

$$a := \sup\big\{\|u\|^2_{H^1(\Sigma,\mathbb{C}^N)} : u \in V \text{ with } \|u\|^2_{L^2(\Omega,\mathbb{C}^N)} = 1\big\} < \infty.$$

Using the linear map F_γ as in Lemma 5.1(a), for $u \in V$ define $\widetilde{u} \in H^1(\mathbb{R}^n, \mathbb{C}^N)$ by

$$\widetilde{u} = \begin{cases} u & \text{in } \Omega, \\ (F_M \otimes 1)(u|_\Sigma) & \text{in } \Omega^c. \end{cases}$$

with 1 understood as the identity operator in $\mathbb{C}^N$, then for any $u \in V$ we have

$$\int_{\Omega^c} \big(|\nabla\widetilde{u}|^2 + M^2|\widetilde{u}|^2\big) \,\mathrm{d}x - \int_\Sigma \Big(M + \frac{H_1}{2}\Big)|\widetilde{u}|^2 \,\mathrm{d}s$$
$$\equiv \Big((R_M + M^2) \otimes 1\Big)[\widetilde{u}, \widetilde{u}] \leq \frac{C}{M}\|u\|^2_{H^1(\Sigma,\mathbb{C}^N)} \leq \frac{Ca}{M}\|u\|^2_{L^2(\Omega,\mathbb{C}^N)}$$

with $C > 0$ independent of u. Noting that for $u \in V$ we have $\mathcal{P}_- u = 0$ on Σ and substituting the preceding upper bound into (43) with the choice $\varepsilon = 0$ we arrive at

$$B^2_{m,M}[\widetilde{u},\widetilde{u}] = A^2_m[u,u] + \big((R_M + M^2) \otimes 1\big)[\widetilde{u},\widetilde{u}] \leq \Big(E_j(A^2_m) + \frac{Ca}{M}\Big)\|u\|^2_{L^2(\Omega,\mathbb{C}^N)}.$$

For $u \in V$ there holds $\|\widetilde{u}\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \geq \|u\|^2_{L^2(\Omega,\mathbb{C}^N)}$, and $\widetilde{V} := \{\widetilde{u} : u \in V\}$ is therefore a j-dimensional subspace of $H^1(\mathbb{R}^n, \mathbb{C}^N) \equiv \mathcal{Q}(B^2_{m,M})$. The min-max principle gives

$$E_j(B^2_{m,M}) \le \sup_{0\ne v\in\widetilde V} \frac{B^2_{m,M}[v,v]}{\|v\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)}} = \sup_{0\ne u\in V} \frac{B^2_{m,M}[\widetilde u,\widetilde u]}{\|\widetilde u\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)}}$$
$$\le \sup_{0\ne u\in V} \frac{\Big(E_j(A^2_m)+\dfrac{Ca}{M}\Big)\|u\|^2_{L^2(\Omega,\mathbb{C}^N)}}{\|\widetilde u\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)}} \le E_j(A^2_m)+\frac{Ca}{M},$$

which implies $\limsup_{M\to+\infty} E_j(B^2_{m,M}) = E_j(A^2_m)$.

5.3 Lower bound

Now we use the representation (43) with an arbitrary fixed $\varepsilon>0$. By the min-max principle, for any $j\in\mathbb{N}$ one has

$$E_j(B^2_{m,M}) \ge E_j(K_{m,M,\varepsilon}\oplus K^c_{M,\varepsilon}) \tag{45}$$

where $K_{m,M,\varepsilon}$ is the self-adjoint operator in $L^2(\Omega,\mathbb{C}^N)$ with the form domain $\mathcal{Q}(K_{m,M,\varepsilon}) = H^1(\Omega,\mathbb{C}^N)$ and

$$K_{m,M,\varepsilon}[u,u] = \int_\Omega \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x + \int_\Sigma \Big(m-\varepsilon+\frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s + 2(M-m)\int_\Sigma |\mathcal{P}_-u|^2\,\mathrm{d}s,$$

and $K^c_{M,\varepsilon}$ is the self-adjoint operator in $L^2(\Omega^c,\mathbb{C}^N)$ with the form domain $\mathcal{Q}(K^c_{M,\varepsilon}) = H^1(\Omega^c,\mathbb{C}^N)$ and

$$K^c_{M,\varepsilon}[u,u] = \int_{\Omega^c} \big(|\nabla u|^2 + M^2|u|^2\big)\,\mathrm{d}x - \int_\Sigma \Big(M-\varepsilon+\frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s.$$

Using the operator R_γ from Lemma 5.1 one easily sees that $K^c_{M,\varepsilon} = (R_{M-\varepsilon}\otimes 1) + M^2$ with 1 being the identity in $\mathbb{C}^N$, and then, using Lemma 5.1(b), $E_1(K^c_{M,\varepsilon}) = E_1(R_{M-\varepsilon})+M^2 \ge \varepsilon M$ as M is large. Due to the upper bound proved in the preceding subsection we know already that for each fixed $j\in\mathbb{N}$ there holds $E_j(B^2_{m,M}) = \mathcal{O}(1)$ for large M, hence, Eq. (45) implies

$$E_j(B^2_{m,M}) \ge \min\big\{E_j(K_{m,M,\varepsilon}), E_1(K^c_{M,\varepsilon})\big\} = E_j(K_{m,M,\varepsilon}) \text{ as } M\to+\infty.$$

As the operators $K_{m,M,\varepsilon}$ are increasing with respect to M, one uses the monotone convergence (Proposition 1.10) for each $j\in\mathbb{N}$ to obtain $\lim_{M\to+\infty} E_j(K_{m,M,\varepsilon}) = E_j(C_{m,\varepsilon})$, where $C_{m,\varepsilon}$ is the self-adjoint operator in $L^2(\Omega,\mathbb{C}^N)$ given by

$$C_{m,\varepsilon}[u,u] = \int_\Omega \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x + \int_\Sigma \Big(m-\varepsilon+\frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s,$$
$$\mathcal{Q}(C_{m,\varepsilon}) = \big\{u\in H^1(\Omega,\mathbb{C}^N):\ \mathcal{P}_-u=0 \text{ on } \Sigma\big\} \equiv \mathcal{Q}(A^2_m).$$

This shows that $\liminf_{M\to+\infty} E_j(B^2_{m,M}) \ge E_j(C_{m,\varepsilon})$. As $\varepsilon>0$ is arbitrary and we have the obvious limit $\lim_{\varepsilon\to0} E_j(C_{m,\varepsilon}) = E_j(C_{m,0}) \equiv E_j(A^2_m)$, we arrive at the sought lower bound $\liminf_{M\to+\infty} E_j(B^2_{m,M}) \ge E_j(A^2_m)$, which finishes the proof.

6 Proof of Theorem 1.3

We are going to show that for each $j \in \mathbb{N}$ the eigenvalues $E_j(B^2_{m,M})$ converge to $E_j(\not{D}^2)$ as $m \to -\infty$ and $M \to +\infty$ with $m/M \to 0$. Due to Lemma 2.4 for each $j \in \mathbb{N}$ there holds $E_j(\not{D}^2) = E_j(L)$, hence, it is sufficient to prove that $E_j(B^2_{m,M})$ converges to $E_j(L)$ in the same asymptotic regime.

The proof is by combining in a new way several components used for Theorem 1.1 and 1.2. For the upper bound one extends the eigenfunctions of the operator L on both sides of Σ by taking tensor products with the first eigenfunctions of the model one-dimensional operators in the normal direction. These extensions are then used as test functions in the min-max principle. For the lower bound we again decouple the two sides of Σ and eliminate the contribution in Ω^c by acting as in the proof of Theorem 1.2. The analysis of the part in Ω is then quite similar to the constructions in the proof of Theorem 1.1: one is first reduced to the analysis in a thin tubular neighborhood of Ω, and then one applies a unitary transform (expansion in the eigenfunctions of the operator in the normal direction) to obtain a monotone family of operators.

6.1 Upper bound

Let us recall the important technical ingredients. For small $\delta > 0$ consider the sets $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \Sigma) < \delta\}$ and $\Pi_\delta := \Sigma \times (0, \delta)$ as well as the diffeomorphisms $\Phi : \Pi_\delta \to \Omega_\delta$ given by $\Phi(s,t) = s - t\nu(s)$ and the associated unitary maps $\Theta_\delta : L^2(\Omega_\delta, \mathbb{C}^N) \to L^2(\Pi_\delta, \mathbb{C}^N)$ with $\Theta_\delta u = \sqrt{\det(\Phi')}\; u \circ \Phi$.

Consider the self-adjoint operator S in $L^2(0, \delta)$ with

$$S[f,f] = \int_0^\delta |f'|^2 \,\mathrm{d}t + m\big|f(0)\big|^2, \quad \mathcal{Q}(S) = \big\{f \in H^1(0,\delta) : f(\delta) = 0\big\}$$

and let ψ be an eigenfunction for the first eigenvalue normalized by $\|\psi\|^2_{L^2(0,\delta)} = 1$. By Lemma 3.1 with some $b > 0$ one has

$$E_1(S) \le -m^2 + b e^{-\delta|m|}, \quad \big|\psi(0)\big|^2 \le b|m|, \text{ as } (-m) \text{ is large.}$$

Also recall that due to Lemma 5.1(a) one can find $c > 0$ such that for $\delta \in (0, \delta_0)$ and $u \in H^1(\Omega_\delta)$ with $u = 0$ on $\partial\Omega_\delta \setminus \Sigma$ there holds, with $w := \Theta_\delta u$,

$$\begin{aligned} \int_{\Omega_\delta} |\nabla u|^2 \,\mathrm{d}x &+ \int_{\partial U} \Big(m + \frac{H_1}{2}\Big)|u|^2 \,\mathrm{d}s \\ &\le \int_{\Pi_\delta} \bigg[(1 + c\delta)|\nabla_s w|^2 + |\partial_t w|^2 + \Big(H_2 - \frac{H_1^2}{4} + c\delta\Big)|w|^2\bigg] \,\mathrm{d}s\,\mathrm{d}t \\ &\qquad + m \int_\Sigma \big|w(s,0)\big|^2 \,\mathrm{d}s. \end{aligned} \tag{46}$$

We will use the representation (43) with $\varepsilon = 0$, i.e.

$$\begin{aligned} B^2_{m,M}[u,u] &= \int_\Omega \big(|\nabla u|^2 + m^2|u|^2\big) \,\mathrm{d}x + \int_\Sigma \Big(m + \frac{H_1}{2}\Big)|u|^2 \,\mathrm{d}s + 2(M - m) \int_\Sigma |\mathcal{P}_- u|^2 \,\mathrm{d}s \\ &+ \int_{\Omega^c} \big(|\nabla u|^2 + M^2|u|^2\big) \,\mathrm{d}x - \int_\Sigma \Big(M + \frac{H_1}{2}\Big)|u|^2 \,\mathrm{d}s, \quad u \in H^1(\mathbb{R}^n, \mathbb{C}^N). \end{aligned} \tag{47}$$

For small $a \in \mathbb{R}$ consider the operator L_a in $\mathcal{H}$ given by

$$\begin{aligned} L_a[g,g] &= \int_\Sigma \Big[(1+ca)|\nabla g|^2 + \Big(H_2 - \frac{H_1^2}{4} + ca\Big)|g|^2\Big]\,\mathrm{d}s, \\ \mathcal{Q}(L_a) &= H^1(\Sigma,\mathbb{C}^N)\cap\mathcal{H}. \end{aligned} \tag{48}$$

Finally, by Lemma 5.1 for large $M>0$ there exists $C>0$ and a linear extension map $F_M : H^1(\Sigma,\mathbb{C}^N) \to H^1(\Omega^c,\mathbb{C}^N)$ with $(F_M f)|_\Sigma = f$ and

$$\int_{\Omega^c} \big(|\nabla F_M f|^2 + M^2|F_M f|^2\big)\,\mathrm{d}x - \int_\Sigma \Big(M + \frac{H_1}{2}\Big)|F_M f|^2\,\mathrm{d}s \le \frac{C}{M}\,\|f\|^2_{H^1(\Sigma,\mathbb{C}^N)}.$$

for all $f \in H^1(\Sigma,\mathbb{C}^N)$.

Let $j\in\mathbb{N}$ and $v_1,\dots,v_j$ be linearly independent eigenfunctions of L_δ for the first j eigenvalues, then for $v \in V := \operatorname{span}(v_1,\dots,v_j)$ one has $L_\delta[v,v] \le E_j(L_\delta)\,\|v\|^2_{\mathcal{H}} \equiv E_j(L_\delta)\,\|v\|^2_{L^2(\Sigma,\mathbb{C}^N)}$. Denote

$$a_0 := \sup\big\{\|v\|^2_{H^1(\Sigma,\mathbb{C}^N)} : v\in V \text{ with } \|v\|^2_{\mathcal{H}} = 1\big\} < \infty.$$

For $v\in V$ construct $u \in H^1(\mathbb{R}^n,\mathbb{C}^N)$ as follows:

$$u = \begin{cases} \Theta_\delta^{-1}(v\otimes\psi) & \text{in } \Omega_\delta, \\ \psi(0)F_M v & \text{in } \Omega^c, \\ 0 & \text{in } \Omega\setminus\Omega_\delta. \end{cases}$$

By construction one has

$$\|u\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \ge \|u\|^2_{L^2(\Omega_\delta,\mathbb{C}^N)} = \|v\|^2_{L^2(\Sigma,\mathbb{C}^N)}\|\psi\|^2_{L^2(0,\delta)} = \|v\|^2_{L^2(\Sigma,\mathbb{C}^N)} \equiv \|v\|^2_{\mathcal{H}},$$

hence, the subspace $U := \{u : v\in V\} \subset H^1(\mathbb{R}^n,\mathbb{C}^N)$ is j-dimensional. By the above properties of F_M and ψ one has

$$\begin{aligned} &\int_{\Omega^c} \big(|\nabla u|^2 + M^2|u|^2\big)\,\mathrm{d}x - \int_\Sigma \Big(M+\frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s \\ &\quad = \big|\psi(0)\big|^2\bigg(\int_{\Omega^c} \big(|\nabla F_M v|^2 + M^2|F_M v|^2\big)\,\mathrm{d}x - \int_\Sigma \Big(M+\frac{H_1}{2}\Big)|F_M v|^2\,\mathrm{d}s\bigg) \\ &\qquad \le \big|\psi(0)\big|^2\,\frac{C}{M}\,\|v\|^2_{H^1(\Sigma,\mathbb{C}^N)} \le b|m|\,\frac{C}{M}\,a_0\|v\|^2_{L^2(\Sigma,\mathbb{C}^N)} \equiv a_0 bC\frac{|m|}{M}\,\|v\|^2_{\mathcal{H}}, \end{aligned}$$

and due to (46) there holds

$$\int_\Omega \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x + \int_\Sigma \Big(m+\frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s + 2(M-m)\int_\Sigma |\mathcal{P}_- u|^2\,\mathrm{d}s$$

$$
\begin{aligned}
&\equiv \int_{\Omega_\delta} \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x + \int_\Sigma \Big(m + \frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s\\
&\le \int_0^\delta \int_\Sigma \bigg[(1+c\delta)\big|\nabla_s(v\otimes\psi)\big|^2 + \big|\partial_t(v\otimes\psi)\big|^2\\
&\qquad + \Big(m^2 + H_2 - \frac{H_1^2}{4} + c\delta\Big)\big|(v\otimes\psi)\big|^2\bigg]\,\mathrm{d}s\,\mathrm{d}t + m\int_\Sigma \big|(v\otimes\psi)(s,0)\big|^2\,\mathrm{d}s\\
&= \bigg(\int_\Sigma \Big[(1+c\delta)|\nabla v|^2 + \Big(H_2 - \frac{H_1^2}{4} + c\delta\Big)|v|^2\Big]\,\mathrm{d}s\bigg)\,\|\psi\|^2_{L^2(0,\delta)}\\
&\qquad + \bigg(\int_0^\delta |\psi'|^2\,\mathrm{d}t + m\big|\psi(0)\big|^2 + m^2\|\psi\|^2_{L^2(0,\delta)}\bigg)\,\|v\|^2_{L^2(\Sigma,\mathbb{C}^N)}\\
&= L_\delta[v,v] + \big(E_1(S)+m^2\big)\|v\|^2_{\mathcal{H}} \le \big(E_j(L_\delta) + be^{-\delta|m|}\big)\|v\|^2_{\mathcal{H}}.
\end{aligned}
$$

Inserting the preceding inequalities into the expression (47) for $B^2_{m,M}$ one sees that for all $u \in U$ there holds

$$
B^2_{m,M}[u,u] \le \Big(E_j(L_\delta) + be^{-\delta|m|} + a_0 bC\frac{|m|}{M}\Big)\|v\|^2_{\mathcal{H}}, \quad \|u\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \ge \|v\|^2_{\mathcal{H}},
$$

and the min-max principle gives

$$
E_j(B^2_{m,M}) \le \max_{0\ne u\in U} \frac{B^2_{m,M}[u,u]}{\|u\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)}} = \max_{0\ne v\in V} \frac{B^2_{m,M}[u,u]}{\|u\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)}} \le E_j(L_\delta) + be^{-\delta|m|} + a_0 bC\frac{|m|}{M}.
$$

Therefore, one has $\limsup_{m\to-\infty,\, m/M\to 0} E_j(B^2_{m,M}) \le E_j(L_\delta)$. As δ can be chosen arbitrarily small and $\lim_{\delta\to 0} E_j(L_\delta) = E_j(L_0) \equiv E_j(L)$ one arrives at

$$
\limsup_{m\to-\infty,\, m/M\to 0} E_j(B^2_{m,M}) \le E_j(L). \tag{49}
$$

6.2 Lower bound

Now we will use the representation (43) with $\varepsilon = \varepsilon_0/|m|$ and an arbitrary but fixed $\varepsilon_0 > 0$, i.e.

$$
\begin{aligned}
&B^2_{m,M}[u,u]\\
&= \int_\Omega \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x + \int_\Sigma \Big(m - \frac{\varepsilon_0}{|m|} - \frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s + 2(M-m)\int_\Sigma |\mathcal{P}_- u|^2\,\mathrm{d}s\\
&\quad + \int_{\Omega^c} \big(|\nabla u|^2 + M^2|u|^2\big)\,\mathrm{d}x - \int_\Sigma \Big(M - \frac{\varepsilon_0}{|m|} + \frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s, \quad u \in H^1(\mathbb{R}^n,\mathbb{C}^N).
\end{aligned}
$$

Due to the min-max principle for any $j \in \mathbb{N}$ one has

$$
E_j(B^2_{m,M}) \ge E_j(K_{m,M} \oplus K^c_{m,M}), \tag{50}
$$

where $K_{m,M}$ is the self-adjoint operator in $L^2(\Omega,\mathbb{C}^N)$ with the form domain given by $\mathcal{Q}(K_{m,M}) = H^1(\Omega,\mathbb{C}^N)$ and

$$
K_{m,M}[u,u] = \int_\Omega \big(|\nabla u|^2 + m^2|u|^2\big)\,\mathrm{d}x
$$

$$+\int_\Sigma \Big(m-\frac{\varepsilon_0}{|m|}-\frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s+2(M-m)\int_\Sigma |\mathcal{P}_-u|^2\,\mathrm{d}s$$

and $K^c_{m,M}$ is the self-adjoint operator in $L^2(\Omega^c,\mathbb{C}^N)$ with $\mathcal{Q}(K^c_{m,M})=H^1(\Omega^c,\mathbb{C}^N)$ and

$$K^c_{m,M}[u,u]=\int_{\Omega^c}\big(|\nabla u|^2+M^2|u|^2\big)\,\mathrm{d}x-\int_\Sigma\Big(M-\frac{\varepsilon_0}{|m|}+\frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s.$$

Using the operator R_γ from Lemma 5.1 we see that in the asymptotic regime under consideration we have, with some $C_0>0$,

$$\begin{aligned}E_1(K^c_{m,M})=E_1(R_{M-\varepsilon_0/m})+M^2\ge M^2-\Big(M-\frac{\varepsilon_0}{|m|}\Big)^2-C_0\\ =2\varepsilon_0\frac{M}{|m|}-\frac{\varepsilon_0^2}{m^2}-C_0\to+\infty.\end{aligned}$$

As we have already the upper bound $E_j(B^2_{m,M})=\mathcal{O}(1)$, it follows from (50) that $E_j(B^2_{m,M})\ge E_j(K_{m,M})$. One can assume in addition that $M\ge 0$ and $m\le 0$, then $2(M-m)\ge -2m\ge 2|m|$, which implies

$$E_j(B^2_{m,M})\ge E_j(K_m),\tag{51}$$

with K_m being the self-adjoint operator in $L^2(\Omega,\mathbb{C}^N)$ with $\mathcal{Q}(K_m)=H^1(\Omega,\mathbb{C}^N)$ and

$$K_m[u,u]=\int_\Omega\big(|\nabla u|^2+m^2|u|^2\big)\,\mathrm{d}x+\int_\Sigma\Big(m-\frac{\varepsilon_0}{|m|}-\frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s+2|m|\int_\Sigma|\mathcal{P}_-u|^2\,\mathrm{d}s.$$

In order to obtain a lower bound for the eigenvalues of K_m we take a small $\delta>0$ and consider the domains $\Omega_\delta=\big\{x\in\Omega:\mathrm{dist}(x,\delta)\big\}$ and $\Omega^c_\delta:=\Omega\setminus\overline{\Omega_\delta}$, then due to the min-max principle one has

$$E_j(K_m)\ge E_j(K'_m\oplus K''_m),\tag{52}$$

where K'_m is the self-adjoint operator in $L^2(\Omega_\delta,\mathbb{C}^N)$ with the form domain $\mathcal{Q}(K'_m)=H^1(\Omega_\delta,\mathbb{C}^N)$ and

$$K'_m[u,u]=\int_{\Omega_\delta}\big(|\nabla u|^2+m^2|u|^2\big)\,\mathrm{d}x+\int_\Sigma\Big(m-\frac{\varepsilon_0}{|m|}-\frac{H_1}{2}\Big)|u|^2\,\mathrm{d}s+|m|\int_\Sigma|\mathcal{P}_-u|^2\,\mathrm{d}s,$$

while K''_m is the self-adjoint operator in $L^2(\Omega^c_\delta,\mathbb{C}^N)$ with

$$\mathcal{Q}(K''_m)=H^1(\Omega^c_\delta,\mathbb{C}^N),\quad K''_m[u,u]=\int_{\Omega^c_\delta}\big(|\nabla u|^2+m^2|u|^2\big)\,\mathrm{d}x,$$

and $E_1(K''_m)\ge m^2\to+\infty$. By combining (51) and (52) one sees that $E_j(B^2_{m,M})\ge E_j(K'_m\oplus K''_m)$. As we already have proved the upper bound $E_j(B^2_{m,M})=\mathcal{O}(1)$, it follows that

$$E_j(B^2_{m,M})\ge E_j(K'_m).\tag{53}$$

Using now the diffeomorphism

$$\Phi:\Pi_\delta\to\Omega_\delta,\quad \Pi_\delta:=\Sigma\times(0,\delta),\quad \Phi(s,t)\mapsto s-t\nu(s),$$

and the unitary maps $\Theta_\delta : L^2(\Omega_\delta, \mathbb{C}^N) \to L^2(\Pi_\delta, \mathbb{C}^N)$, $\Theta_\delta u = \sqrt{\det(\Phi')}\, u \circ \Phi$, with the help of Lemma 3.3(b) one obtains $E_j(K'_m) = E_j(\Theta_\delta^* K'_m \Theta_\delta) \geq E_j(K^0_m)$ with K^0_m being the self-adjoint operator in $L^2(\Pi_\delta, \mathbb{C}^N)$ given by

$$\begin{aligned} K^0_m[v,v] &= \int_{\Pi_\delta} \Big[(1-c\delta)|\nabla_s v|^2 + |\partial_t v|^2 + \Big(H_2 - \frac{H_1^2}{4} - c\delta \Big) |v|^2 \Big] \,\mathrm{d}s\,\mathrm{d}t \\ &+ \Big(m - \frac{\varepsilon_0}{|m|} \Big) \int_\Sigma \big| v(s,0) \big|^2 \,\mathrm{d}s - c \int_\Sigma \big| v(s,\delta) \big|^2 \,\mathrm{d}s + |m| \int_\Sigma \big| \mathcal{P}_- v(s,0) \big|^2 \,\mathrm{d}s \end{aligned} \tag{54}$$

on the form domain $\mathcal{Q}(K^0_m) = H^1(\Pi_\delta, \mathbb{C}^N)$, where $c > 0$ is chosen independent of δ and v. With this choice of c, let S' be the self-adjoint operator in $L^2(0,\delta)$ with

$$S'[f,f] = \int_0^\delta |f'|^2 \,\mathrm{d}t + \Big(m - \frac{\varepsilon_0}{|m|} \Big) \big| f(0) \big|^2 - c \big| f(\delta) \big|^2, \quad \mathcal{Q}(S') = H^1(0,\delta).$$

and $\psi_k \in L^2(0,\delta)$ with $k \in \mathbb{N}$ be its eigenfunctions for the eigenvalues $E_k(S')$ forming an orthonormal basis in $L^2(0,\delta)$. Due to Lemma 3.2 we have, with some $b^\pm > 0$, $b > 0$ and $b_0 > 0$,

$$E_1(S') \geq -\Big(|m| + \frac{\varepsilon_0}{|m|} \Big)^2 - b e^{-\delta|m|} \geq -m^2 - 3\varepsilon_0 \text{ as } m \to -\infty, \tag{55}$$

$$b^- k^2 - b_0 \leq E_k(S') \leq b^+ k^2 \text{ for all } k \geq 2 \text{ and } m \in \mathbb{R}. \tag{56}$$

For small $a \in \mathbb{R}$, in addition to the operator L_a in $\mathcal{H}$ defined in (48) we consider the self-adjoint operator Λ_a in $L^2(\Sigma, \mathbb{C}^N)$ given by

$$\Lambda_a[g,g] = \int_\Sigma \Big[(1+ca)|\nabla g|^2 + \Big(H_2 - \frac{H_1^2}{4} + ca \Big) |g|^2 \Big] \,\mathrm{d}s, \quad \mathcal{Q}(\Lambda_a) = H^1(\Sigma, \mathbb{C}^N).$$

Let K^1_m be the self-adjoint operator in $L^2(\Pi_\delta)$ having the same form domain as K^0_m and with the sesquilinear form obtained from the one of K^0_m by omitting the last summand in (54), then K^1_m admits a separation of variables: using the identification $L^2(\Pi_\delta) \simeq L^2(0,\delta) \otimes L^2(\Sigma, \mathbb{C}^N)$ one has $K^1_m = S' \otimes 1 + 1 \otimes \Lambda_{-\delta}$. Using the unitary transform

$$\Theta : L^2(0,\delta) \to \ell^2(\mathbb{N}), \quad (\Theta f)_k = \langle \psi_k, f \rangle_{L^2(0,\delta)}, \quad k \in \mathbb{N},$$

the identification $L^2(\Pi_\delta) \simeq L^2(0,\delta) \otimes L^2(\Sigma, \mathbb{C}^N)$ and another unitary transform

$$\begin{aligned} \Xi := \Theta \otimes 1 : L^2(\Pi_\delta) \to \ell^2(\mathbb{N}) \otimes L^2(\Sigma, \mathbb{C}^N), \\ \Xi v = (v_k) =: \widehat{v}, \quad v_k := \int_0^\delta \psi_k(t)\, v(t,\cdot) \,\mathrm{d}t \in L^2(\Sigma, \mathbb{C}^N), \end{aligned}$$

for the self-adjoint operator $\widehat{K}^1_m := \Xi K^1_m \Xi^*$ in $\ell^2(\mathbb{N}) \otimes L^2(\Sigma, \mathbb{C}^N)$ one has

$$\widehat{K}^1_m[\widehat{v}, \widehat{v}] = \sum_{k \in \mathbb{N}} \Big(\Lambda_{-\delta}[v_k, v_k] + \big(E_k(S') + m^2 \big) \|v_k\|^2_{L^2(\Sigma, \mathbb{C}^N)},$$

while $\mathcal{Q}(K^1_m)$ consists of all $\widehat{v} \in \ell^2(\mathbb{N}) \otimes L^2(\Sigma, \mathbb{C}^N)$ with $v_k \in H^1(\Sigma, \mathbb{C}^N)$ such that the right-hand side of the preceding expression is finite. Using the two-sided estimate (56) one can rewrite

$$\mathcal{Q}(\widehat{K}^1_m) = \Big\{\widehat{v} = (v_k) \in \ell^2(\mathbb{N}) \otimes L^2(\Sigma, \mathbb{C}^N) : \ v_k \in H^1(\Sigma, \mathbb{C}^N) \text{ for each } k \in \mathbb{N} \\ \text{and } \sum_{k\in\mathbb{N}} \Big(\|v_k\|^2_{H^1(\Sigma,\mathbb{C}^N)} + k^2 \|v_k\|^2_{H^1(\Sigma,\mathbb{C}^N)} \Big) < \infty \Big\}. \quad (57)$$

For the operator $\widehat{K}^0_m := \Xi K^0_m \Xi^*$ one has the same form domain and

$$\widehat{K}^0_m[\widehat{v}, \widehat{v}] = \sum_{k\in\mathbb{N}} \Big(\Lambda_{-\delta}[v_k, v_k] + \big(E_k(S') + m^2\big) \|v_k\|^2_{L^2(\Sigma,\mathbb{C}^N)} \Big) + |m| \int_\Sigma \big|\mathcal{P}_- \Xi^* \widehat{v}(\cdot, 0)\big|^2 \, \mathrm{d}s.$$

Using the lower bounds (55) and (56) for $E_k(S')$, for any $\widehat{v} \in \mathcal{Q}(\widehat{K}^0_m)$ we obtain the inequality $\widehat{K}^0_m[\widehat{v}, \widehat{v}] \geq w_m(\widehat{v}, \widehat{v})$ with the sesquilinear form w_m in $\ell^2(\mathbb{N}) \otimes L^2(\Sigma, \mathbb{C}^N)$ defined on $\mathcal{D}(w_m) := \mathcal{Q}(\widehat{K}^0_m)$ by

$$w_m(\widehat{v}, \widehat{v}) := \Lambda_{-\delta}[v_1, v_1] - 3\varepsilon_0 \|v_1\|^2_{L^2(\Sigma,\mathbb{C}^N)} \\ + \sum_{k\geq 2} \Big(\Lambda_{-\delta}[v_k, v_k] + (b^- k^2 - b_0 + m^2) \|v_k\|^2_{L^2(\Sigma,\mathbb{C}^N)} \Big) + |m| \int_\Sigma \big|\mathcal{P}_- \Xi^* \widehat{v}(\cdot, 0)\big|^2 \, \mathrm{d}s.$$

Using the above representation (57) one sees that the form w_m is lower semibounded and closed, hence it generates a self-adjoint operator W_m in $\ell^2(\mathbb{N}) \otimes L^2(\Sigma, \mathbb{C}^N)$ with compact resolvent, and then $E_j(\hat{K}^0_m) \geq E_j(W_m)$ for all $j \in \mathbb{N}$. By summarizing all the preceding constructions, for any $j \in \mathbb{N}$ in the asymptotic regime under consideration one has

$$E_j(B^2_{m,M}) \geq E_j(W_m). \quad (58)$$

For the analysis of the eigenvalues of W_m as $m \to -\infty$ we are now in the classical situation for the monotone convergence (Proposition 1.10), as W_m are increasing with respect to $|m|$. Namely, consider the set

$$\mathcal{Q}_\infty := \Big\{ \widehat{v} = (v_k) \in \bigcap_{m<0} \mathcal{Q}(W_m) \equiv \mathcal{Q}(\widehat{K}^0_m), \quad \sup_{m<0} W_m[\widehat{v}, \widehat{v}] < +\infty \Big\},$$

then a vector $\widehat{v} = (v_k) \in \mathcal{Q}(\widehat{K}^0_m)$ belongs to $\mathcal{Q}_\infty$ iff the following two conditions are satisfied: (i) $v_k = 0$ for all $k \geq 2$ and (ii) $\mathcal{P}_- \Xi^* \widehat{v}(\cdot, 0) = 0$. The condition (i) gives $v = e_1 \otimes v_1$ with $e_1 = (1, 0, 0, \dots) \in \ell^2(\mathbb{N})$, and then the condition (ii) reduces to $\mathcal{P}_- v_1 = 0$, i.e. $v_1 \in \mathcal{H}$. Therefore, there holds $\mathcal{Q}_\infty = \big\{ e_1 \otimes v_1 : \, v_1 \in H^1(\Sigma, \mathbb{C}^N) \cap \mathcal{H} \big\}$. Moreover, for any $e_1 \otimes v_1 \in \mathcal{Q}_\infty$ one has

$$\lim_{m\to-\infty} W_m[e_1 \otimes v_1, e_1 \otimes v_1] = L_{-\delta}[v_1, v_1] - 3\varepsilon_0 \|v_1\|^2_{\mathcal{H}},$$

while we recall that $L_{-\delta}$ is defined as in (48). Therefore, if one denotes by W_∞ the self-adjoint operator in $e_1 \otimes \mathcal{H}$ given by

$$W_\infty[e_1 \otimes v_1, e_1 \otimes v_1] = L_{-\delta}[v_1, v_1] - 3\varepsilon_0 \|v_1\|^2_{\mathcal{H}},$$

then it follows by the monotone convergence (Proposition 1.10) that for each $j \in \mathbb{N}$ there holds $\lim_{m\to-\infty} E_j(W_m) = E_j(W_\infty) \equiv E_j(L_{-\delta}) - 3\varepsilon_0$. By (58) one has $\liminf_{M\to+\infty, m\to-\infty, m/M\to 0} \geq E_j(L_{-\delta}) - 3\varepsilon_0$. As both δ and ε_0 can be chosen arbitrarily small and we have the convergence $\lim_{a\to 0} E_j(L_a) = E_j(L)$, we arrive at the inequality $\liminf_{M\to+\infty, m\to-\infty, m/M\to 0} E_j(B^2_{m,M}) \geq E_j(L)$. By combining it with the upper bound (49) we arrive at the result of Theorem 1.3.

A Basic properties of Euclidean Dirac operators

In the present section we would like to recall some key facts on the Euclidean Dirac operators $B_{m,M}$ and A_m. All the properties are very standard and well known for $n \in \{2,3\}$, but we were not able to find a suitable single reference covering arbitrary dimensions.

The analysis of $B_{m,M}$ will be provided in a slightly more general setting, with the hope that the constructions can be of use for other works. Denote

$$B_M := B_{M,M}.$$

Proposition A.1. *Let $V \in L^\infty(\mathbb{R}^n)$ be real-valued with compact support. Consider the linear operator $C := B_M + V\,\alpha_{n+1}$ in $L^2(\mathbb{R}^n,\mathbb{C}^N)$ with domain $\mathcal{D}(C) = H^1(\mathbb{R}^n,\mathbb{C}^N)$, then:*

(a) The operator C is self-adjoint, and it is essentially self-adjoint on $C_c^\infty(\mathbb{R}^n,\mathbb{C}^N)$.

(b) The essential spectrum of C is $\big(-\infty,-|M|\,\big] \cup \big[\,|M|,+\infty\big)$, and there are at most finitely many discrete eigenvalues in $\big(-|M|,|M|\big)$.

(c) Assume in addition that $n \notin 4\mathbb{Z}$. If E is an eigenvalue of C, then $(-E)$ is also an eigenvalue of C of the same multiplicity.

In particular, the three assertions hold for $C := B_{m,M} \equiv B_M + (m-M)1_\Omega\alpha_{n+1}$.

Proof. (a) As both C and B_0 are symmetric and only differ by the bounded symmetric operator $(M+V)\alpha_{n+1}$, the (essential) self-adjointness of C is equivalent to that of B_0. First, it is directly seen that B_0 is symmetric. Let S be the restriction of B_0 to $C_c^\infty(\mathbb{R}^n,\mathbb{C}^N)$, then S is a densely defined symmetric operator, and by applying the definitions one sees that its adjoint S^* is given by

$$S^*u = D_0 u, \quad \mathcal{D}(S^*) = \big\{u \in L^2(\mathbb{R}^n,\mathbb{C}^N) : \, D_0 u \in L^2(\mathbb{R}^n,\mathbb{C}^N)\big\}$$

with D_0 acting in the sense of distributions. In order to complete the proof we simply need to show that $S^* = B_0$. As both operators are given by the same differential expression D_0 and S^* is clearly an extension of B_0, one only needs to check the inclusion $\mathcal{D}(S^*) \subset \mathcal{D}(B_0)$. Let a function u belong to $\mathcal{D}(S^*)$, i.e.

$$u \in L^2(\mathbb{R}^n,\mathbb{C}^N) \text{ and } -\mathrm{i}\sum_{j=1}^n \alpha_j\partial_j u \in L^2(\mathbb{R}^n,\mathbb{C}^N).$$

We need to show that this implies $u \in H^1(\mathbb{R}^n,\mathbb{C}^N)$. We remark first that the Fourier transform $\widehat{u}$ of u satisfies $\widehat{u} \in L^2(\mathbb{R}^n,\mathbb{C}^N)$ and $\Gamma(\xi)\widehat{u} \in L^2(\mathbb{R}^n,\mathbb{C}^N)$. The matrices $\Gamma(\xi)$ are Hermitian, and, by construction, $\Gamma(\xi)^2 = |\xi|^2 I_N$. In particular, one can represent $\Gamma(\xi) = |\xi| U(\xi)$ with the unitary matrices $U(\xi) := \Gamma(\xi/|\xi|)$. As the pointwise multiplication by $U(\cdot)$ is an isomorphism of $L^2(\mathbb{R}^n,\mathbb{C}^N)$, it follows that the condition $\Gamma(\xi)\widehat{u} \in L^2(\mathbb{R}^n,\mathbb{C}^N)$ yields $|\xi|\widehat{u} \in L^2(\mathbb{R}^n,\mathbb{C}^N)$. Due to $|\xi_j| \le |\xi|$ we obtain $\xi_j\widehat{u} \in L^2(\mathbb{R}^n,\mathbb{C}^N)$ for each $j = 1,\dots n$. By applying the inverse Fourier transform this implies $\partial_j u \in L^2(\mathbb{R}^n,\mathbb{C}^N)$ for each $j = 1,\dots n$, and then $u \in H^1(\mathbb{R}^n,\mathbb{C}^N)$.

(b) Let us start by showing the inclusion

$$\big(-\infty,-|M|\big] \cup \big[|M|,+\infty\big) \subset \operatorname{spec}_{\mathrm{ess}} C. \tag{59}$$

Denote $e_1 := (1, 0, \dots, 0) \in \mathbb{R}^n$. For $\rho > 0$, denote during the proof

$$b_\rho := \{x \in \mathbb{R}^n : |x| < \rho\}.$$

Let $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi = 1$ in b_1 and $\chi = 0$ outside b_2, and $E \in \mathbb{R}$ with $|E| > |M|$. Let us choose $\eta \in \mathbb{C}^N$ with

$$\varphi := (\sqrt{E^2 - M^2}\,\alpha_1 + M\alpha_{n+1} + E\, I_N)\eta \neq 0,$$

which is possible as the matrix $\sqrt{E^2 - M^2}\,\alpha_1 + M\alpha_{n+1} + E\, I_N$ is non-zero (otherwise α_1 could not anticommute with α_{n+1}). For large $k \in \mathbb{N}$ consider the following functions $u_k \in H^1(\mathbb{R}^n, \mathbb{C}^N)$:

$$\begin{aligned} u_k(x) &= \chi\Big(\frac{x}{k} - ke_1\Big) e^{\mathrm{i}\sqrt{E^2-M^2}\,x_1}\varphi \\ &\equiv \chi\Big(\frac{x}{k} - ke_1\Big) e^{\mathrm{i}\sqrt{E^2-M^2}\,x_1}(\sqrt{E^2 - M^2}\,\alpha_1 + M\alpha_{n+1} + E\, I_N)\eta, \end{aligned}$$

then one easily checks that $\|u_k\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)} = ck^n > 0$ with $c = \|\chi\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)} > 0$ independent of k. On the other hand, for large k one has $V = 0$ on the support of u_k, hence,

$$\begin{aligned} (Cu_k)(x) &= -\mathrm{i}\sum_{j=1}^n \alpha_j \partial_j u_k(x) + M\alpha_{n+1} u_k(x) \\ &= -\frac{\mathrm{i}}{k}\sum_{j=1}^n \alpha_j \partial_j \chi\Big(\frac{x}{k} - ke_1\Big) e^{\mathrm{i}\sqrt{E^2-M^2}\,x_1}\varphi \\ &\quad + \chi\Big(\frac{x}{k} - ke_1\Big)\Big(\sqrt{E^2 - M^2}\,\alpha_1 + M\alpha_{n+1}\Big) e^{\mathrm{i}\sqrt{E^2-M^2}\,x_1}\varphi, \end{aligned}$$

and then $(C - E)u_k = v_k + w_k$ with

$$\begin{aligned} v_k(x) &= -\frac{\mathrm{i}}{k}\sum_{j=1}^n \alpha_j \partial_j \chi\Big(\frac{x}{k} - ke_1\Big) e^{\mathrm{i}\sqrt{E^2-M^2}\,x_1}\varphi, \\ w_k(x) &= \chi\Big(\frac{x}{k} - ke_1\Big) e^{\mathrm{i}\sqrt{E^2-M^2}\,x_1}\Big(\sqrt{E^2 - M^2}\,\alpha_1 + M\alpha_{n+1} - E\, I_N\Big)\varphi \\ &\equiv \chi\Big(\frac{x}{k} - ke_1\Big) e^{\mathrm{i}\sqrt{E^2-M^2}\,x_1}\Big(\sqrt{E^2 - M^2}\,\alpha_1 + M\alpha_{n+1} - E\, I_N\Big) \\ &\qquad \times \Big(\sqrt{E^2 - M^2}\,\alpha_1 + M\alpha_{n+1} + E\Big)\eta = 0. \end{aligned}$$

One estimates easily $\|v_k\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)} = \mathcal{O}(k^{n-2})$ for $k \to +\infty$, and this shows that $\|(C-E)u_k\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)}/\|u_k\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)} = \mathcal{O}(1/k) \to 0$ and yields $E \in \operatorname{spec} C$. It follows that $\big(-\infty, -|M|\big) \cup \big(|M|, +\infty\big) \subset \operatorname{spec} C$, and one can take the closure on the left-hand side as $\operatorname{spec} C$ is a closed set. Furthermore, as the set $\big(-\infty, -|M|\big] \cup \big[|M|, +\infty\big)$ has no isolated points, it is included into the essential spectrum of C. Hence, the claim (59) is proved.

Now it remains to check that C has no essential spectrum in $\big(-|M|, |M|\big)$ and that it has at most finitely many discrete eigenvalues, which will be done by an iterated application of the min-max principle. For $E \in \mathbb{R}$ and a self-adjoint semibounded from below operator T we will denote

$$\mathcal{N}(E, T) := \#\big\{j \in \mathbb{N} : E_j(T) < E\big\}.$$

In other words, if $\operatorname{spec}_{\mathrm{ess}} T \cap (-\infty, E) \neq \emptyset$, then $\mathcal{N}(E,T) = +\infty$, otherwise $\mathcal{N}(E,T)$ is the number of eigenvalues of T in $(-\infty, E)$, where each eigenvalue counted according to its multiplicity. In these terms, we simply need to show that $\mathcal{N}(M^2, C^2) < +\infty$.

Choose $r > 0$ large such that the support of V is contained in b_r, and then pick any $R > r$ and real-valued functions $\chi_1, \chi_2 \in C^\infty(\mathbb{R}^2)$ with

$$\chi_1^2 + \chi_2^2 = 1, \quad \chi_1 = 1 \text{ in } b_r, \quad \chi_2 = 1 \text{ in } \mathbb{R}^2 \setminus b_R.$$

Let $u \in \mathcal{D}(C)$, then for each $k \in \{1,2\}$ one also has $\chi_k u \in \mathcal{D}(C)$ and $C(\chi_k u) = \chi_k C u - \mathrm{i}\Gamma(\nabla\chi_k)u$, and

$$\big\|C(\chi_k u)\big\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)} = \int_{\mathbb{R}^n} \Big(\chi_k^2 |Cu|^2 + \big|\mathrm{i}\Gamma(\nabla\chi_k)u\big|^2\Big)\,\mathrm{d}x + \mathcal{J}_k,$$

$$\begin{aligned}\mathcal{J}_k &= 2\Re \int_{\mathbb{R}^n} \big\langle \chi_k C u, -\mathrm{i}\Gamma(\nabla\chi_k)u\big\rangle_{\mathbb{C}^n}\,\mathrm{d}x\\ &= \Re \int_{\mathbb{R}^n} \big\langle Cu, -\mathrm{i}\Gamma(2\chi_k\nabla\chi_k)u\big\rangle_{\mathbb{C}^n}\,\mathrm{d}x = \Re \int_{\mathbb{R}^n} \big\langle Cu, -\mathrm{i}\Gamma\big(\nabla(\chi_k^2)\big)u\big\rangle_{\mathbb{C}^n}\,\mathrm{d}x.\end{aligned}$$

From $\chi_1^2 + \chi_2^2 = 1$ we infer $\nabla(\chi_1^2 + \chi_2^2) = 0$ and then $\mathcal{J}_1 + \mathcal{J}_2 = 0$. Therefore,

$$\begin{aligned}&\big\|C(\chi_1 u)\big\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)} + \big\|C(\chi_2 u)\big\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)}\\ &\qquad = \int_{\mathbb{R}^n} (\chi_1^2 + \chi_2^2)|Cu|^2\,\mathrm{d}x + \int_{\mathbb{R}^2} \big(|\nabla\chi_1|^2 + |\nabla\chi_2|^2)|u|^2\big)\,\mathrm{d}x\\ &\qquad\qquad = \int_{\mathbb{R}^n} |Cu|^2\,\mathrm{d}x + \int_{\mathbb{R}^n} W|u|^2\,\mathrm{d}x, \quad W := |\nabla\chi_1|^2 + |\nabla\chi_2|^2, \end{aligned} \tag{60}$$

Recall that W is supported in $\overline{b_R \setminus b_r}$, while the support of $\chi_2 u$ does not intersect the support of V, which gives

$$\big\|C(\chi_2 u)\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)} = \big\|B_M(\chi_2 u)\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)} = \int_{\mathbb{R}^n} \big(|\nabla(\chi_2 u)|^2 + M^2|\chi_2 u|^2\big)\,\mathrm{d}x$$

(for $u \in C_c^\infty(\mathbb{R}^n, \mathbb{C}^N)$ this is a simple integration by parts, and it is then extended by density to the whole of $\mathcal{D}(C)$ as $C_c^\infty(\mathbb{R}^n,\mathbb{C}^N)$ is a domain of essential self-adjointness as shown above). This allows one to rewrite (60) as

$$\begin{aligned}\big\|Cu\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)} &= \int_{b_R} \big(\big|C(\chi_1 u)\big|^2 - W|\chi_1 u|^2\big)\,\mathrm{d}x\\ &\qquad + \int_{\mathbb{R}^n} \big(|\nabla(\chi_2 u)|^2 + M^2|\chi_2 u|^2 - W|\chi_2 u|^2\big)\,\mathrm{d}x.\end{aligned} \tag{61}$$

For $v \in L^2(b_R, \mathbb{C}^N)$ let us denote by v' its extension by zero to the whole of $\mathbb{R}^n$ and consider the following sesqulinear form s_1 in $L^2(b_R, \mathbb{C}^N)$:

$$s_1(v,v) = \int_{b_R} |Cv'|^2\,\mathrm{d}x, \quad \mathcal{D}(s_1) = H_0^1(b_R, \mathbb{C}^N).$$

The form s_1 is clearly non-negative and densely defined. Let us show that it is also closed. Let $(v_k) \subset \mathcal{D}(S_1)$ and $v \in L^2(b_R, \mathbb{C}^N)$ with $\|v_k - v\|_{L^2(b_R,\mathbb{C}^N)} \to 0$ for $k \to \infty$

and $s_1(v_k - v_l, v_k - v_l) \equiv \|C(v'_k - v'_l)\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \to 0$ as $k, l \to +\infty$. Then one has $\|v'_k - v'\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \to 0$ for $k \to \infty$, and the closedness of the operator C implies that $v' \in \mathcal{D}(C)$ with $Cv' = \lim_{k\to\infty} Cv'_k$. Hence, the function v is such that its extension by zero belongs to $H^1(\mathbb{R}^n, \mathbb{C}^N)$, and this shows $v \in H^1_0(b_R, \mathbb{C}^N) \equiv \mathcal{D}(s_1)$ and then $s_1(v_k - v, v_k - v) \equiv \|C(v'_k - v')\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \to 0$. The semiboundedness and closedness of s_1 imply the existence of a self-adjoint operator S_1 in $L^2(b_R, \mathbb{C}^N)$ with $S_1[v, v] = s_1(v, v)$ for $v \in \mathcal{Q}(S_1) = \mathcal{D}(s_1)$, and S_1 has compact resolvent as $\mathcal{Q}(S_1)$ is compactly embedded into $L^2(b_R, \mathbb{C}^N)$.

Denote by S_0 the (scalar) Schrödinger operator $-\Delta - W$ in $L^2(\mathbb{R}^n)$, i.e.

$$S_0[u, u] = \int_{\mathbb{R}^n} \big(|\nabla u|^2 - W|u|^2\big)\, \mathrm{d}x, \quad \mathcal{Q}(S_0) = H^1(\mathbb{R}^n),$$

and consider the linear map $J : \mathcal{D}(C) \equiv \mathcal{Q}(C^2) \mapsto \mathcal{Q}(S_1) \times \mathcal{Q}(S_0 \otimes I_N)$ given by $Ju = \big((\chi_1 u)|_{b_R}, \chi_2 u\big)$. For any $u \in \mathcal{D}(C)$ one has then $\|Ju\|_{L^2(b_R,\mathbb{C}^N)\times L^2(\mathbb{R}^n,\mathbb{C}^N)} = \|u\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)}$, and the above representation (61) can be rewritten as

$$\begin{aligned} C^2[u, u] &= \|Cu\|^2_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \\ &= (S_1 - W)[(\chi_1 u)|_{b_R}, (\chi_1 u)|_{b_R}] + \big((S_0 + M^2) \otimes I_N\big)[\chi_2 u, \chi_2 u] \\ &\equiv \Big((S_1 - W) \oplus \big((S_0 + M^2) \otimes I_N\big)\Big)[Ju, Ju]. \end{aligned}$$

A standard application of the min-max principle (see e.g. Proposition 1.9) yields the inequality

$$\begin{aligned} \mathcal{N}(M^2, C^2) &\le \mathcal{N}\Big(M^2, (S_1 - W) \oplus \big((S_0 + M^2) \otimes I_N\big)\Big) \\ &\equiv \mathcal{N}(M^2, S_1 - W) + N\mathcal{N}(0, S_0). \end{aligned}$$

As $S_1 - W$ is semibounded from below with compact resolvent, one has $\mathcal{N}(M^2, S_1 - W) < \infty$, and in order to show $\mathcal{N}(M^2, C^2) < \infty$ it is sufficient to prove that $\mathcal{N}(0, S_0) < \infty$.

Recall that W vanishes outside b_R. Let T_R be the self-adjoint operator in $L^2(b_R)$ with

$$T_R[u, u] = \int_{b_R} \big(|\nabla u|^2 - W|u|^2\big)\, \mathrm{d}x, \quad \mathcal{Q}(T_R) = H^1(b_R),$$

denote by O_R the zero operator in $L^2(\mathbb{R}^2 \setminus b_R)$, and consider the linear map

$$J' : \mathcal{Q}(S_0) \to \mathcal{Q}(T_R) \times \mathcal{Q}(O_R), \quad J'u = (u|_{b_R}, u|_{\mathbb{R}^2\setminus b_R}).$$

One has then

$$S_0[u, u] \ge \int_{b_R} \big(|\nabla u|^2 - W|u|^2\big)\, \mathrm{d}x \equiv (T_R \oplus O_R)[J'u, J'u]$$

implying $\mathcal{N}(0, S_0) \le \mathcal{N}(0, T_R \oplus O_R) \equiv \mathcal{N}(0, T_R) + \mathcal{N}(0, O_R) \equiv \mathcal{N}(0, T_R) < \infty$, as T_R is semibounded from below with compact resolvent. This shows that $\mathcal{N}(M^2, C^2) < \infty$ and finishes the proof of(b).

(c) According to the general theory of Clifford algebras, see e.g. Theorem 15.19 in [8], for $n \notin 4\mathbb{Z}$ there exists an antilinear map $\theta : \mathbb{C}^N \to \mathbb{C}^N$ with $\theta^2 \in \{-1, +1\}$ and[2]

$$\theta\alpha_j = \alpha_j\theta \text{ for } j = 1, \dots, n, \quad \theta(\mathrm{i}\alpha_{n+1}) = (\mathrm{i}\alpha_{n+1})\theta. \tag{62}$$

The pointwise map Θ defined by $(\Theta u)(x) = \theta\big(u(x)\big)$ is clearly an isomorphism of $H^1(\mathbb{R}^n, \mathbb{C}^N)$. Let $u \in \ker(C - E)$, $E \in \mathbb{R}$, then $\Theta u \in \mathcal{D}(C)$ and

$$Eu = -\mathrm{i}\sum_{j=1}^{n} \alpha_j \frac{\partial u}{\partial x_j} + V\alpha_{n+1}u.$$

The last equality in (62) rewrites as $\theta\alpha_{n+1} = -\alpha_{n+1}\theta$. We compute then

$$\begin{aligned}-E\Theta u = \Theta(-Eu) &= \Theta\Big(\mathrm{i}\sum_{j=1}^{n} \alpha_j \frac{\partial u}{\partial x_j} - V\alpha_{n+1}u\Big)\\ &= -\mathrm{i}\sum_{j=1}^{n} \theta\alpha_j \frac{\partial u}{\partial x_j} - V\theta\alpha_{n+1}u = -\mathrm{i}\sum_{j=1}^{n} \alpha_j\theta \frac{\partial u}{\partial x_j} + V\alpha_{n+1}\theta u = C\Theta u,\end{aligned}$$

which shows that $\Theta u \in \ker(C + E)$. By construction one has $\Theta^2 \in \{1, -1\}$, hence, Θ is a bijection and the claim follows. □

Let us now discuss the basic properties of the operator A_m.

Proposition A.2. *The following assertions hold:*

(a) *The operator A_m is self-adjoint, and, in addition, it is essentially self-adjoint on $\mathcal{D}_\infty(A_m) := C^\infty(\overline{\Omega}, \mathbb{C}^N) \cap \mathcal{D}(A_m)$,*

(b) *The operator A_m has compact resolvent, and its eigenfunctions belong to $\mathcal{D}_\infty(A_m)$,*

(c) *Assume in addition that $n \notin 4\mathbb{Z}$, then the spectrum of A_m is symmetric with respect to zero, i.e.* $\dim\ker(C - E) = \dim\ker(C + E)$ *for any $E \in \mathbb{R}$.*

The above assertions also hold with A_m replaced by A'_m.

Proof. The assertions (a) and (b) are standard properties of elliptic Dirac operators. One can use e.g. [3, Ex. 4.20] or [4, Ex. 7.26], by noting that $\mathcal{B}$ is a boundary chirality operator, which shows that A_m belongs to the class of Dirac operators with local elliptic boundary conditions. Then [3, Thm. 4.11] implies that the restriction R of A_m on $\mathcal{D}_\infty(A_m)$ is essentially self-adjoint, while the boundary regularity theorem [3, Thm. 4.9] implies that $\overline{R} \subset A_m$. As A_m is clearly symmetric, this implies that A_m is self-adjoint and proves (a). The same boundary regularity theorem [3, Thm. 4.9] implies that all eigenfunctions are smooth up to the boundary, and the compactness of the resolvent of A_m follows then from the compactness of the embedding of $H^1(\Omega)$ into $L^2(\Omega)$. This proves (b).

[2]In the language of [8, Theorem 15.19], the matrices $\alpha_1, \dots, \alpha_n, \mathrm{i}\alpha_{n+1}$ form a basis of a minimal representation of the Clifford algebra of $\mathbb{R}^{1,n}$, while θ is a real (if $\theta^2 = 1$) or quaternionic (if $\theta^2 = -1$) charge conjugation, which is shown to exist for $n - 1 \notin 4\mathbb{Z} + 3$.

The proof of (c) is very similar to the proof of Proposition A.1(c) with an additional attention given to the operator domain. Namely, for $n \notin 4\mathbb{Z}$ there exist an antilinear map $\theta : \mathbb{C}^N \to \mathbb{C}^N$ with $\theta^2 \in \{-1, +1\}$ satisfying (62), and the pointwise map Θ defined by $(\Theta u)(x) = \theta\big(u(x)\big)$ is again an isomorphism of $H^1(\Omega, \mathbb{C}^N)$. Furthermore, if $v \in H^1(\Omega, \mathbb{C}^N)$ satisfies the boundary condition $v = \mathcal{B}v$ on Σ, then due to

$$\Theta\mathcal{B}(s) = \Theta\big(- \mathrm{i}\alpha_{n+1}\Gamma(s)\big) = -\mathrm{i}\alpha_{n+1}\Theta\Gamma(s) = -\mathrm{i}\alpha_{n+1}\Gamma(s)\Theta = \mathcal{B}\Theta(s)$$

one also has $\Theta v = \mathcal{B}\Theta v$ on Σ. Therefore, if $v \in \mathcal{D}(A_m)$, then also $\Theta v \in \mathcal{D}(A_m)$. Finally, let $v \in \ker(A_m - E)$, $E \in \mathbb{R}$, then the same computation as in Proposition A.1(c) gives $\Theta v \in \ker(A_m + E)$.

The preceding arguments apply to A'_m as well. First, $(-\mathcal{B})$ is still a boundary chirality operator in the sense of [3], which implies the assertions (a) and (b). The proof of (c) is the same by noting that $v = -\mathcal{B}v$ on Σ yields $\Theta v = -\mathcal{B}\Theta v$ on Σ. □

B Schrödinger-Lichnerowicz formula for Euclidean hypersurfaces

Let $\Sigma \subset \mathbb{R}^n$ be a smooth compact hypersurface with the outer unit normal field ν and endowed with the Riemannian metric induced by the embedding. Recall that the standard scalar product in $\mathbb{R}^n$ gives rise to the induced scalar product in $T\Sigma$, which we simply denote by $\langle\cdot,\cdot\rangle$ in this section. Denote by W the Weingarten operator, $WX = \nabla_X \nu$ for $X \in T\Sigma$, with ∇ being the gradient in $\mathbb{R}^n$. Recall that the Levi-Civita connection ∇' on Σ is given by the Gauss formula

$$\nabla'_X Y = \nabla_X Y + \langle WX, Y\rangle\nu, \quad X, Y \in T\Sigma.$$

We denote

$$H_1 := \operatorname{tr} W, \quad |W|^2 := \operatorname{tr}(W^2), \quad H_2 := \frac{H_1^2 - |W|^2}{2},$$

i.e. H_1 is the mean curvature and H_2 is the half of the scalar curvature of Σ.

Let $N \in \mathbb{N}$ and $\gamma_1, \dots, \gamma_n$ be $N \times N$ anticommuting Hermitian matrices satisfying $\gamma_j^2 = I$, with I being the $N \times N$ identity matrix, then the matrices

$$\gamma(x) := \sum_{j=1}^{n} x_j \gamma_j, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

satisfy the Clifford commutation relation $\gamma(x)\gamma(y) + \gamma(y)\gamma(x) = 2\langle x, y\rangle I$ for all $x, y \in \mathbb{R}^n$. Let us recall the definition of the associated extrinsically defined Dirac operator D^Σ on Σ following [13, Sec. 1–3].[3] The induced spin connection ∇^Σ on Σ is defined by

$$\nabla^\Sigma_X \psi = \nabla_X + \frac{1}{2}\gamma(\nu)\gamma(WX) : C^\infty(\Sigma, \mathbb{C}^N) \to C^\infty(\Sigma, \mathbb{C}^N), \quad X \in T\Sigma,$$

then D^Σ acts on functions $\psi \in C^\infty(\Sigma, \mathbb{C}^N)$ by

$$D^\Sigma\psi := -\gamma(\nu) \sum_{j=1}^{n-1} \gamma(e_j)\nabla^\Sigma_{e_j}\psi$$

[3]See footnote 1 on page 10.

with $(e_1,\dots,e_{n-1})$ being an orthonormal frame of $T\Sigma$. Recall that $\gamma(e_j)$ anticommute with $\gamma(\nu)$ and, furthermore,

$$\sum_{j=1}^{n-1}\gamma(e_j)\gamma(We_j)=H_1\,I \tag{63}$$

(which is seen by testing on an eigenbasis of W), and we may rewrite

$$D^\Sigma\psi=\frac{H_1}{2}\,\psi-\gamma(\nu)\sum_{j=1}^{n-1}\gamma(e_j)\nabla_{e_j}\psi,$$

Being viewed as an operator in $L^2(\Sigma,\mathbb{C}^N)$, the operator D^Σ is known to be essentially self-adjoint on $C^\infty(\Sigma,\mathbb{R}^N)$. We would like to provide a elementary direct proof, adapted to the Euclidean setting, of the eminent Schrödinger-Lichnerowicz formula

$$(D^\Sigma)^2=(\nabla^\Sigma)^*\nabla^\Sigma+\frac{H_2}{2}\,I, \tag{64}$$

where the first term on the right-hand side is the Bochner Laplacian associated with the above spin connection ∇^Σ, which is a self-adjoint operator in $L^2(\Sigma,\mathbb{C}^N)$. (We refer to the original papers [16, 21] and other sources [7, 10–12] for a more general setting.)

In what follows we use the standard identification of $T\Sigma$ and $T^*\Sigma$ with the help of the musical isomorphism. For $\psi\in C^\infty(\Sigma,\mathbb{C}^N)$ we have the decomposition

$$\nabla^\Sigma\psi=\sum_{j=1}^{n-1}e_j\otimes\nabla^\Sigma_{e_j}\psi=\sum_{j=1}^{n-1}e_j\otimes\Big(\nabla_{e_j}+\frac12\,\gamma(\nu)\gamma(We_j)\Big)\psi. \tag{65}$$

To compute the adjoint $(\nabla^\Sigma)^*:T^*\Sigma\otimes C^\infty(\Sigma,\mathbb{C}^N)\to C^\infty(\Sigma,\mathbb{C}^N)$, let $X\in T\Sigma\simeq T^*\Sigma$ and $\varphi,\psi\in C^\infty(\Sigma,\mathbb{C}^N)$ then

$$\begin{aligned}\big\langle(\nabla^\Sigma)^*(X\otimes\varphi),\psi\big\rangle_{L^2(\Sigma,\mathbb{C}^N)}&=\langle X\otimes\varphi,\nabla^\Sigma\psi\rangle_{T^*\Sigma\otimes L^2(\Sigma,\mathbb{C}^N)}\\&=\big\langle\varphi,\nabla_X\psi+\frac12\,\gamma(\nu)\gamma(WX)\psi\big\rangle_{L^2(\Sigma,\mathbb{C}^N)}\\&=\big\langle\varphi,\nabla_X\psi\big\rangle_{L^2(\Sigma,\mathbb{C}^N)}+\Big\langle\frac12\,\gamma(WX)\gamma(\nu)\varphi,\psi\Big\rangle_{L^2(\Sigma,\mathbb{C}^N)}.\end{aligned}$$

Using Leibniz rule and the divergence theorem we have

$$\begin{aligned}\langle\varphi,\nabla_X\psi\rangle_{L^2(\Sigma,\mathbb{C}^N)}&=\int_\Sigma X\langle\varphi,\psi\rangle_{\mathbb{C}^N}\,\mathrm{d}s-\langle\nabla_X\varphi,\psi\rangle_{L^2(\Sigma,\mathbb{C}^N)}\\&=-\big\langle(\operatorname{div}_\Sigma X)\varphi+\nabla_X\varphi,\psi\big\rangle_{L^2(\Sigma,\mathbb{C}^N)},\end{aligned}$$

where $\operatorname{div}_\Sigma$ is the divergence on Σ,

$$\operatorname{div}_\Sigma X=\sum_{j=1}^{n-1}\langle e_j,\nabla'_{e_j}X\rangle.$$

Therefore,

$$(\nabla^\Sigma)^*(X\otimes\varphi)=-(\operatorname{div}_\Sigma X)\,\varphi-\nabla_X\varphi+\frac12\,\gamma(WX)\gamma(\nu)\varphi.$$

By combining (65) with the last expression, for $\psi \in C^\infty(\Sigma, \mathbb{C}^N)$ one obtains

$$
\begin{aligned}
(\nabla^\Sigma)^* \nabla^\Sigma \psi &= \sum_{j=1}^{n-1} (\nabla^\Sigma)^* \Big[e_j \otimes \Big(\nabla_{e_j} + \frac{1}{2}\, \gamma(\nu)\gamma(We_j) \Big) \psi \Big] \\
&= - \sum_{j=1}^{n-1} (\operatorname{div}_\Sigma e_j) \Big(\nabla_{e_j} + \frac{1}{2}\, \gamma(\nu)\gamma(We_j) \Big) \psi \\
&\quad + \sum_{j=1}^{n-1} \Big\{ - \nabla_{e_j} \Big(\nabla_{e_j} + \frac{1}{2}\, \gamma(\nu)\gamma(We_j) \Big) \psi \\
&\quad + \frac{1}{2}\, \gamma(We_j)\gamma(\nu) \Big(\nabla_{e_j} + \frac{1}{2}\, \gamma(\nu)\gamma(We_j) \Big) \psi \Big\} =: S_1 + S_2.
\end{aligned}
\tag{66}
$$

To simplify S_1 we first use the Leibniz rule and the orthogonality of (e_j) to obtain

$$
\begin{aligned}
\operatorname{div}_\Sigma e_j &= \sum_{k=1}^{n-1} \langle e_k, \nabla'_{e_k} e_j \rangle = - \sum_{k=1}^{n-1} \langle \nabla'_{e_k} e_k, e_j \rangle, \\
S_1 &= \sum_{j,k=1}^{n-1} \langle \nabla'_{e_k} e_k, e_j \rangle \nabla_{e_j} \psi + \frac{1}{2} \sum_{j,k=1}^{n-1} \langle \nabla'_{e_k} e_k, e_j \rangle \gamma(\nu)\gamma(We_j)\psi \\
&= \sum_{k=1}^{n-1} \Big(\sum_{j=1}^{n-1} \langle \nabla'_{e_k} e_k, e_j \rangle \nabla_{e_j} \psi \Big) + \frac{1}{2} \sum_{k=1}^{n-1} \gamma(\nu)\gamma\Big(W \sum_{j=1}^{n-1} \langle \nabla'_{e_k} e_k, e_j \rangle e_j \Big) \\
&= \sum_{k=1}^{n-1} \nabla_{\nabla'_{e_k} e_k} \psi + \frac{1}{2} \sum_{k=1}^{n-1} \gamma(\nu)\gamma\Big(W \nabla'_{e_k} e_k \Big).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
S_2 = \sum_{j=1}^{n-1} \Big\{ & - \nabla_{e_j} \nabla_{e_j} \psi - \frac{1}{2}\, \gamma(We_j)\gamma(We_j)\psi - \frac{1}{2}\, \gamma(\nu)\gamma\big(\nabla_{e_j}(We_j)\big)\psi \\
& - \frac{1}{2}\, \gamma(\nu)\gamma(We_j)\nabla_{e_j}\psi + \frac{1}{2}\, \gamma(We_j)\gamma(\nu)\nabla_{e_j}\psi + \frac{1}{4}\, \gamma(We_j)\gamma(\nu)\gamma(\nu)\gamma(We_j)\psi \Big\} \\
= \sum_{j=1}^{n-1} \Big\{ & - \nabla_{e_j} \nabla_{e_j} \psi - \frac{1}{2}\, \gamma(\nu)\gamma\big(\nabla_{e_j}(We_j)\big)\psi - \gamma(\nu)\gamma(We_j)\nabla_{e_j}\psi \Big\} \psi - \frac{1}{4}\, |W|^2 \psi,
\end{aligned}
$$

and then

$$
\begin{aligned}
(\nabla^\Sigma)^* \nabla^\Sigma \psi = \sum_{j=1}^{n-1} \Big[& \nabla_{\nabla'_{e_j} e_j} \psi - \nabla_{e_j} \nabla_{e_j} \psi \\
& + \frac{1}{2} \gamma(\nu)\gamma\Big(W \nabla'_{e_j} e_j - \nabla_{e_j}(We_j) \Big)\psi - \gamma(\nu)\gamma(We_j)\nabla_{e_j}\psi \Big] - \frac{1}{4}\, |W|^2 \psi.
\end{aligned}
$$

Using $\nabla'_{e_j}(We_j) = \nabla_{e_j}(We_j) + |We_j|^2 \nu$ and Leibniz rule we have

$$
W \nabla'_{e_j} e_j - \nabla_{e_j}(We_j) = W \nabla'_{e_j} e_j - \nabla'_{e_j}(We_j) + |We_j|^2 \nu = -(\nabla'_{e_j} W) e_j + |We_j|^2 \nu
$$

implying $\gamma(\nu)\gamma\big(W\nabla'_{e_j}e_j-\nabla_{e_j}(We_j)\big)\psi=-\gamma(\nu)\gamma\big((\nabla'_{e_j}W)e_j\big)\psi+|We_j|^2\psi$, and then

$$(\nabla^\Sigma)^*\nabla^\Sigma\psi=\sum_{j=1}^{n-1}\Bigg[\nabla_{\nabla'_{e_j}e_j}\psi-\nabla_{e_j}\nabla_{e_j}\psi \\ -\frac{1}{2}\,\gamma(\nu)\,\gamma\big((\nabla'_{e_j}W)e_j\big)\psi-\gamma(\nu)\,\gamma(We_j)\nabla_{e_j}\psi\Bigg]+\frac{1}{4}\,|W|^2\psi. \tag{67}$$

On the other hand,

$$\begin{aligned}(D^\Sigma)^2\psi&=\Big(\frac{H_1}{2}-\gamma(\nu)\sum_{j=1}^{n-1}\gamma(e_j)\nabla_{e_j}\Big)\Big(\frac{H_1\,\psi}{2}-\gamma(\nu)\sum_{j=1}^{n-1}\gamma(e_j)\nabla_{e_j}\psi\Big)\\
&=\frac{H_1^2}{4}\,\psi-\frac{1}{2}\Big(\gamma(\nu)\sum_{j=1}^{n-1}\gamma(e_j)\nabla_{e_j}H_1\Big)\,\psi-\frac{H_1}{2}\,\gamma(\nu)\sum_{j=1}^{n-1}\gamma(e_j)\nabla_{e_j}\psi\\
&\quad-\frac{H_1}{2}\,\gamma(\nu)\sum_{j=1}^{n-1}\gamma(e_j)\nabla_{e_j}\psi+\gamma(\nu)\sum_{j,k=1}^{n-1}\gamma(e_j)\gamma(We_j)\gamma(e_k)\nabla_{e_k}\psi\\
&\quad+\gamma(\nu)\sum_{j,k=1}^{n-1}\gamma(e_j)\gamma(\nu)\gamma(\nabla_{e_j}e_k)\nabla_{e_k}\psi+\gamma(\nu)\sum_{j,k=1}^{n-1}\gamma(e_j)\gamma(\nu)\gamma(e_k)\nabla_{e_j}\nabla_{e_k}\psi.\end{aligned} \tag{68}$$

The sum of the third, forth and fifth terms is zero, in fact,

$$\begin{aligned}&-\frac{H_1}{2}\,\gamma(\nu)\sum_{j=1}^{n-1}\gamma(e_j)\nabla_{e_j}\psi-\frac{H_1}{2}\,\gamma(\nu)\sum_{j=1}^{n-1}\gamma(e_j)\nabla_{e_j}\psi\\
&\quad+\gamma(\nu)\sum_{j,k=1}^{n-1}\gamma(e_j)\gamma(We_j)\gamma(e_k)\nabla_{e_k}\psi\\
&=-H_1\,\gamma(\nu)\sum_{j=1}^{n-1}\gamma(e_k)\nabla_{e_k}\psi+\gamma(\nu)\sum_{j,k=1}^{n-1}\gamma(e_j)\gamma(We_j)\gamma(e_k)\nabla_{e_k}\psi\\
&=\gamma(\nu)\sum_{k=1}^{n-1}\Big(-H_1+\sum_{j=1}^{n-1}\gamma(e_j)\gamma(We_j)\Big)\gamma(e_k)\nabla_{e_k}\psi=0\end{aligned}$$

as the term in the parentheses vanishes due to (63). Therefore, Eq. (68) reads as

$$\begin{aligned}(D^\Sigma)^2\psi&=\frac{H_1^2}{4}\,\psi-\frac{1}{2}\,\gamma(\nu)\sum_{j=1}^{n-1}\gamma(e_j)(\nabla_{e_j}H_1)\,\psi-\sum_{j,k=1}^{n-1}\gamma(e_j)\gamma(\nabla_{e_j}e_k)\nabla_{e_k}\psi\\
&\quad-\sum_{j,k=1}^{n-1}\gamma(e_j)\gamma(e_k)\nabla_{e_j}\nabla_{e_k}\psi.\end{aligned} \tag{69}$$

We transform the last summand as follows:

$$\sum_{j,k=1}^{n-1}\gamma(e_j)\gamma(e_k)\nabla_{e_j}\nabla_{e_k}\psi=\frac{1}{2}\,\sum_{j,k=1}^{n-1}\Big(\gamma(e_j)\gamma(e_k)\nabla_{e_j}\nabla_{e_k}\psi+\gamma(e_k)\gamma(e_j)\nabla_{e_k}\nabla_{e_j}\psi\Big)$$

$$= \frac{1}{2}\sum_{j,k=1}^{n-1}\Big(\gamma(e_j)\gamma(e_k)+\gamma(e_k)\gamma(e_j)\Big)\nabla_{e_j}\nabla_{e_k}\psi$$

$$+\frac{1}{2}\sum_{j,k=1}^{n-1}\gamma(e_k)\gamma(e_j)\Big(\nabla_{e_k}\nabla_{e_j}-\nabla_{e_j}\nabla_{e_k}\Big)\psi$$

$$=\sum_{j=1}^{n-1}\nabla_{e_j}\nabla_{e_j}\psi+\frac{1}{2}\,J,$$

where

$$J:=\sum_{j,k=1}^{n-1}\gamma(e_j)\gamma(e_k)\big(\nabla_{e_j}\nabla_{e_k}-\nabla_{e_k}\nabla_{e_j}\big)\psi\equiv\sum_{j,k=1}^{n-1}\gamma(e_j)\gamma(e_k)\nabla_{[e_j,e_k]}\psi.$$

Representing $[e_j,e_k]=\sum_{l=1}^{n-1}\big\langle e_l,[e_j,e_k]\big\rangle e_k$ we have

$$J=\sum_{j,k,l=1}^{n-1}\gamma(e_j)\gamma(e_k)\Big[\langle e_l,\nabla'_{e_j}e_k\rangle-\langle e_l,\nabla'_{e_k}e_j\rangle\Big]\nabla_{e_l}\psi,$$

and using

$$\sum_{j=1}^{n-1}e_j\langle e_l,\nabla'_{e_k}e_j\rangle=-\sum_{j=1}^{n-1}e_j\langle\nabla'_{e_k}e_l,e_j\rangle=-\nabla'_{e_k}e_l$$

we rewrite

$$J=-\sum_{j,k=1}^{n-1}\gamma(e_j)\,\gamma(\nabla'_{e_j}e_k)\nabla_{e_k}\psi+\sum_{j,k=1}^{n-1}\gamma(\nabla'_{e_j}e_k)\,\gamma(e_j)\,\nabla_{e_k}\psi$$

$$=-2\sum_{j,k=1}^{n-1}\gamma(e_j)\,\gamma(\nabla'_{e_j}e_k)\nabla_{e_k}\psi$$

$$+\sum_{j,k=1}^{n-1}\Big(\gamma(e_j)\,\gamma(\nabla'_{e_j}e_k)+\gamma(\nabla'_{e_j}e_k)\,\gamma(e_j)\Big)\nabla_{e_k}\psi$$

$$=-2\sum_{j,k=1}^{n-1}\gamma(e_j)\,\gamma(\nabla'_{e_j}e_k)\nabla_{e_k}\psi+2\sum_{j,k=1}^{n-1}\langle e_j,\nabla'_{e_j}e_k\rangle\nabla_{e_k}\psi$$

$$=-2\sum_{j,k=1}^{n-1}\gamma(e_j)\,\gamma(\nabla'_{e_j}e_k)\nabla_{e_k}\psi-2\sum_{j,k=1}^{n-1}\langle\nabla'_{e_j}e_j,e_k\rangle\nabla_{e_k}\psi$$

$$=-2\sum_{j,k=1}^{n-1}\gamma(e_j)\,\gamma(\nabla'_{e_j}e_k)\nabla_{e_k}\psi-2\sum_{j=1}^{n-1}\nabla_{\nabla'_{e_j}e_j}\psi.$$

The substitution into (69) gives

$$(D^\Sigma)^2\psi=\frac{H_1^2}{4}\,\psi-\frac{1}{2}\,\gamma(\nu)\sum_{j=1}^{n-1}\gamma(e_j)(\nabla_{e_j}H_1)\,\psi-\sum_{j,k=1}^{n-1}\gamma(e_j)\gamma(\nabla_{e_j}e_k)\nabla_{e_k}\psi$$

$$-\sum_{j=1}^{n-1}\nabla_{e_j}\nabla_{e_j}\psi+\sum_{j,k=1}^{n-1}\gamma(e_j)\,\gamma(\nabla'_{e_j}e_k)\nabla_{e_k}\psi+\sum_{j=1}^{n-1}\nabla_{\nabla'_{e_j}e_j}\psi.$$

The sum of the third and the fifth terms simplifies as

$$
\begin{aligned}
-\sum_{j,k=1}^{n-1}\gamma(e_j)\gamma(\nabla_{e_j}e_k)\nabla_{e_k}\psi+\sum_{j,k=1}^{n-1}\gamma(e_j)\,\gamma(\nabla'_{e_j}e_k)\nabla_{e_k}\psi&\\
=\sum_{j,k=1}^{n-1}\gamma(e_j)\gamma(\nabla'_{e_j}e_k-\nabla_{e_j}e_k)\nabla_{e_k}\psi=\sum_{j,k=1}^{n-1}\gamma(e_j)\gamma\big(\langle We_j,e_k\rangle\nu\big)\nabla_{e_k}\psi&\\
=\sum_{j,k=1}^{n-1}\gamma\big(e_j\langle e_j,We_k\rangle\big)\gamma(\nu)\nabla_{e_k}\psi=\sum_{k=1}^{n-1}\gamma(We_k)\gamma(\nu)\nabla_{e_k}\psi,&
\end{aligned}
$$

hence,

$$
\begin{aligned}
(D^\Sigma)^2\psi=\frac{H_1^2}{4}\,\psi-\frac{1}{2}\,\gamma(\nu)\sum_{j=1}^{n-1}\gamma(e_j)(\nabla_{e_j}H_1)\,\psi&\\
+\sum_{j=1}^{n-1}\gamma(We_j)\gamma(\nu)\nabla_{e_j}\psi-\sum_{j=1}^{n-1}\nabla_{e_j}\nabla_{e_j}\psi+\sum_{j=1}^{n-1}\nabla_{\nabla'_{e_j}e_j}\psi.&
\end{aligned}
$$

By comparing the last expression with (67) we obtain

$$
\begin{aligned}
&D_\Sigma^2\psi-(\nabla^\Sigma)^*\nabla^\Sigma\psi\\
&\quad=\frac{H_1^2}{4}\,\psi-\frac{1}{4}|W|^2\,\psi-\frac{1}{2}\,\gamma(\nu)\sum_{j=1}^{n-1}\gamma(e_j)(\nabla_{e_j}H_1)\,\psi+\sum_{j=1}^{n-1}\gamma(We_j)\gamma(\nu)\nabla_{e_j}\psi\\
&\qquad+\frac{1}{2}\sum_{j=1}^{n-1}\gamma(\nu)\,\gamma\big((\nabla'_{e_j}W)e_j\big)\psi+\sum_{j=1}^{n-1}\gamma(\nu)\,\gamma(We_j)\nabla_{e_j}\psi.
\end{aligned}
$$

The sum of the fourth and sixth term on the right hand is zero, hence,

$$
\begin{aligned}
&(D^\Sigma)^2\psi-(\nabla^\Sigma)^*\nabla^\Sigma\psi\\
&\quad=\frac{H_2}{2}\,\psi-\frac{1}{2}\,\gamma(\nu)\sum_{j=1}^{n-1}\gamma(e_j)(\nabla_{e_j}H_1)\,\psi+\frac{1}{2}\sum_{j=1}^{n-1}\gamma(\nu)\,\gamma\big((\nabla'_{e_j}W)e_j\big)\psi\\
&\quad=\frac{H_2}{2}\,\psi+\frac{1}{2}\,\gamma(\nu)\gamma\Big(\sum_{j=1}^{n-1}(\nabla'_{e_j}W)e_j-\sum_{j=1}^{n-1}(\nabla_{e_j}H_1)e_j\Big)\psi.
\end{aligned}
$$

Therefore, to show the sought identity (64) it is sufficient to prove the equality

$$
\sum_{j=1}^{n-1}(\nabla'_{e_j}W)e_j=\sum_{j=1}^{n-1}(\nabla_{e_j}H_1)e_j. \tag{70}
$$

In order to check (70) let us remark that $\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z=0$ for any $X,Y,Z\in T\Sigma$. Using the definition of ∇' we have

$$
\begin{aligned}
0&=\nabla_X\big(\nabla'_Y Z-\langle WY,Z\rangle\nu\big)-\nabla_Y\big(\nabla'_X Z-\langle WX,Z\rangle\nu\big)-\nabla'_{[X,Y]}Z+\big\langle W[X,Y],Z\big\rangle\nu\\
&=\nabla'_X\big(\nabla'_Y Z-\langle WY,Z\rangle\nu\big)-\big\langle WX,\nabla'_Y Z-\langle WY,Z\rangle\nu\big\rangle-\nabla'_Y\big(\nabla'_X Z-\langle WX,Z\rangle\nu\big)
\end{aligned}
$$

$$+ \big\langle WY, \nabla'_X Z - \langle WX, Z\rangle \nu \big\rangle - \nabla'_{[X,Y]} Z + \big\langle W[X,Y], Z\big\rangle \nu.$$

Using $\nabla'_X \nu = \nabla_X \nu = WX$ we then arrive at

$$\begin{aligned}
0 &= \nabla'_X \nabla'_Y Z - \big\langle (\nabla'_X W)Y, Z\big\rangle \nu - \big\langle W(\nabla'_X Y), Z\big\rangle \nu - \langle WY, \nabla'_X Z\rangle \nu - \langle WY, Z\rangle WX \\
&\quad - \langle WX, \nabla'_Y Z\rangle \nu - \nabla'_Y \nabla'_X Z + \big\langle (\nabla'_Y W)X, Z\big\rangle \nu + \langle W\nabla'_Y X, Z\rangle \nu \\
&\quad + \langle WX, \nabla'_Y Z\rangle \nu + \langle WX, Z\rangle WY + \langle WY, \nabla'_X Z\rangle - \nabla'_{[X,Y]} Z + \big\langle W[X,Y], Z\big\rangle \nu \\
&= \nabla'_X \nabla'_Y Z - \nabla'_Y \nabla'_X Z - \nabla'_{[X,Y]} Z - \langle WY, Z\rangle WX + \langle WX, Z\rangle WY \\
&\quad + \big\langle (\nabla'_Y W)X, Z\big\rangle \nu - \big\langle (\nabla'_X W)Y, Z\big\rangle \nu \\
&\quad + \langle W\nabla'_Y X, Z\rangle \nu - \big\langle W\nabla'_X Y, Z\big\rangle \nu + \big\langle W[X,Y], Z\big\rangle \nu.
\end{aligned}$$

As $\nabla'_X Y - \nabla'_Y X = [X,Y]$, the sum of the three terms in the last line vanishes. Considering the normal components of the remaining equality we obtain $\big\langle (\nabla'_Y W)X, Z\big\rangle = \big\langle (\nabla'_X W)Y, Z\big\rangle$, and then $\big\langle (\nabla'_Y W)X, Z\big\rangle = \big\langle Y, (\nabla'_X W)Z\big\rangle$. Taking $Y = Z = e_k$ and summing over k we arrive at

$$\sum_{k=1}^{n-1} \big\langle (\nabla'_{e_k} W)X, e_k\big\rangle = \sum_{j=1}^{n-1} \big\langle e_k, (\nabla'_X W)e_k\big\rangle, \text{ i.e. } \sum_{k=1}^{n-1} \big\langle X, (\nabla'_{e_k} W)e_k\big\rangle = \nabla_X H_1.$$

Using the last equality for $X = e_j$ we obtain

$$\sum_{j=1}^{n-1} \sum_{k=1}^{n-1} \big\langle e_j, (\nabla'_{e_k} W)e_k\big\rangle e_j = \sum_{j=1}^{n-1} (\nabla_{e_j} H_1) e_j.$$

The left-hand side of the last equality simplifies to $\sum_{k=1}^{n-1} (\nabla'_{e_k} W)e_k$, which gives (70) and finishes the proof of (64).

C Dirac operator on a loop

Let $\Sigma \subset \mathbb{R}^2$ be a smooth loop of length $\ell > 0$. We give an explicit computation for the eigenvalues of the intrinsic Dirac operator $\not{D}$ on Σ. Consider first the associated extrinsically defined Dirac operator D^Σ using the notation introduced in Subsection 2.2. Denote by $\mathbb{T} := \mathbb{R}/(\ell\mathbb{Z})$ and let $\gamma : \mathbb{T} \to \mathbb{R}^2$ be an arc-length parametrization of Σ, which provides a global coordinate of Σ. To be definite, assume that $\gamma(s)$ runs through Σ in the anti-clockwise direction as s runs from 0 to ℓ, which amounts to the choice of an orientation. We take (e) with $e = \tau := \gamma'$ as an orthonormal frame tangent to Σ and denote by ν the outer unit normal. In addition, let us make an explicit choice of 2×2 matrices β_1 and β_2 satisfying the Clifford commutation relation: we choose them as the Pauli matrices,

$$\beta_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$

then, by setting $\mathcal{N} := \nu_1 + \mathrm{i}\nu_2$ and $\mathcal{T} := \tau_1 + \mathrm{i}\tau_2$,

$$\beta(\nu) = \begin{pmatrix} 0 & \overline{\mathcal{N}} \\ \mathcal{N} & 0 \end{pmatrix}, \quad \beta(e) = \begin{pmatrix} 0 & \overline{\mathcal{T}} \\ \mathcal{T} & 0 \end{pmatrix}$$

Using (11) we realize D^Σ as an operator in $L^2(\mathbb{T},\mathbb{C}^2)$: for $\psi\in C^\infty(\mathbb{T},\mathbb{C}^2)$ one has

$$D^\Sigma\psi=\Big(\frac{H_1}{2}-\beta(\nu)\sum_{j=1}^{n-1}\beta(e_j)\nabla_{e_j}\Big)\psi=\frac{\kappa}{2}\psi-\begin{pmatrix}0&\overline{\mathcal{N}}\\ \mathcal{N}&0\end{pmatrix}\begin{pmatrix}0&\overline{\mathcal{T}}\\ \mathcal{T}&0\end{pmatrix}\psi'.$$

where κ is the curvature of Σ. The above choice of orientation gives $\mathcal{T}=\mathrm{i}\,\mathcal{N}$ and

$$D^\Sigma\psi=\frac{\kappa}{2}\psi+\mathrm{i}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\psi'.$$

Consider the function $K:\mathbb{R}\to\mathbb{R}$ given by

$$K(s):=\frac{1}{2}\int_0^s\kappa(t)\,\mathrm{d}t.$$

Using the well-known identity

$$\int_0^\ell\kappa(t)\,\mathrm{d}t=2\pi$$

we conclude that $K(\cdot+\ell)=\pi+K$. Hence, using the unitary transform

$$U:L^2((0,\ell),\mathbb{C}^2)\phi\mapsto\begin{pmatrix}e^{\mathrm{i}K}&0\\0&e^{-\mathrm{i}K}\end{pmatrix}\phi\in L^2(\mathbb{T},\mathbb{C}^2)$$

we rewrite

$$U^{-1}D^\Sigma U\phi=\mathrm{i}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\phi',$$

and remark that $U\phi\in C^\infty(\mathbb{T},\mathbb{C}^2)$ if and only if ϕ extends to a function from $C^\infty(\mathbb{R},\mathbb{C}^2)$ with $\phi(\cdot+\ell)=-\phi$.

It follows that D^Σ is unitarily equivalent to $D\oplus(-D)$, where D is the operator $\phi\to-\mathrm{i}\phi'$ on $(0,\ell)$ with the antiperiodic boundary condition $\phi(\ell)=-\phi(0)$, and one easily shows that the eigenvalues of D are $(2r-1)\pi/\ell$, $r\in\mathbb{Z}$. In addition, as the dimension $n=2$ is even, the operator D^Σ is also unitarily equivalent to $\not{D}\oplus(-\not{D})$, which means that $\not{D}$ has the same eigenvalues as D.

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