# WEYL-EINSTEIN STRUCTURES ON K-CONTACT MANIFOLDS

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Abstract. We show that a compact K-contact manifold $(M, g, \xi)$ has a closed Weyl-Einstein connection compatible with the conformal structure $[g]$ if and only if it is Sasaki-Einstein.

## 1. Introduction

*K-contact* structures — see the definition in Section 3 — can be viewed as the odd-dimensional counterparts of *almost Kähler* structure, in the same way as *Sasakian* structures are the odd-dimensional counterparts of *Kähler structures*. It has been shown in [4], cf. also [1], that *compact* Einstein $K$-contact structures are actually Sasakian, hence Sasaki-Einstein. In this note, we consider the more general situation of a compact $K$-contact manifold $(M, g, \xi)$ carrying in addition a *Weyl-Einstein* connection $D$ compatible with the conformal class $[g]$, already considered by a number of authors, in particular in [9] and [12]. We show — Theorem 3.2 and Corollary 3.1 below — that $g$ is then Einstein and $D$ is the Levi-Civita connection of an Einstein metric $g_0$ in the conformal class $[g]$, which is actually equal $g$ up to scaling, except if $(M, [g])$ is the flat conformal sphere. In all cases, the $K$-contact structure is Sasaki-Einstein.

## 2. Conformal Killing vector fields

Let $(M, c)$ be a (positive definite) conformal manifold of dimension $n$. A vector field $\xi$ on $M$ is called *conformal Killing* with respect to $c$ if it preserves $c$, meaning that for any metric $g$ in $c$, the trace-free part $(\mathcal{L}_\xi g)_0$ of the Lie derivative $\mathcal{L}_\xi g$ of $g$ along $\xi$ is identically zero, hence that $\mathcal{L}_\xi g = f\, g$, for some function $f$, depending on $\xi$ and $g$, and it is then easily checked that $f = -\frac{2\delta^g \eta_g}{n}$, where $\eta_g$ denotes the 1-form dual to $\xi$ and $\delta^g \eta_g$ the co-differential of $\eta_g$ with respect to $g$. In particular, a conformal Killing vector field $\xi$ on $M$ is Killing with respect to some metric $g$ in $c$ if and only if $\delta^g \eta_g = 0$. In this section, we present a number of facts concerning conformal Killing vector fields for further use in this note.

**Proposition 2.1.** *Let $(M, g)$ be a connected compact oriented Riemannian manifold of dimension $n$, $n \geq 2$, carrying a non-trivial parallel vector field $T$. Let $\xi$ be any conformal Killing vector field on $M$ with respect to the conformal class $[g]$ of $g$. Then, $\xi$ is Killing with respect to $g$; moreover, it commutes with $T$ and the inner product $a := g(\xi, T)$ is constant.*

*Proof.* Denote by $\eta = \xi^\flat$ the 1-form dual to $\xi$ and by $\delta\eta$ the co-differential of $\eta$ with respect to $g$; then, $\xi$ is Killing if and only if $\delta\eta = 0$. Denote by $\nabla$ the Levi-Civita connection of $g$

and by $\mathcal{L}_T$ the Lie derivative along $T$; then, $\nabla_T\xi = [T,\xi] = \mathcal{L}_T\xi$ is conformal Killing, and we have:

$$\delta\big(\nabla_T\eta\big) = \delta(\mathcal{L}_T\eta) = \mathcal{L}_T(\delta\eta) = T(\delta\eta). \tag{2.1}$$

Since $T$ is non-trivial, we may assume $|T| \equiv 1$. Denote $a = g(\xi,T) = \eta(T)$. Since $\xi$ is conformal Killing, $\nabla\xi = A - \frac{\delta\eta}{n}\,\mathrm{Id}$, where $A$ is skew-symmetric and Id denotes the identity; for any vector field $X$ we then have: $\mathrm{d}a(X) = g(\nabla_X\xi,T) = -g(\nabla_T\xi,X) - \frac{2\delta\eta}{n}\,g(X,T)$. We thus get:

$$\nabla_T\eta = -\mathrm{d}a - \frac{2\delta\eta}{n}\,\theta, \tag{2.2}$$

where $\theta = T^\flat$ denotes the 1-form dual to $T$. By evaluating both members of (2.2) on $T$, we get:

$$\delta\eta = -n\,\mathrm{d}a(T), \tag{2.3}$$

whereas, by considering their co-differential and by using (2.1), we get:

$$\Delta a = -\frac{(n-2)}{n}\,T(\delta\eta), \tag{2.4}$$

where $\Delta a = \delta\mathrm{d}a$ denotes the Laplacian of $a$. Denote by $v_g$ the volume form determined by $g$ and the chosen orientation; from (2.3) and (2.4), we then infer:

$$\int_M a\,\Delta a\,v_g = -\frac{(n-2)}{n}\int_M aT(\delta\eta)\,v_g = \frac{(n-2)}{n}\int_M \mathrm{d}a(T)\,\delta\eta\,v_g = -\frac{(n-2)}{n^2}\int_M (\delta\eta)^2\,v_g,$$

hence

$$\int_M |\mathrm{d}a|^2\,v_g = \int_M a\,\Delta a\,v_g = -\frac{(n-2)}{n^2}\int_M (\delta\eta)^2\,v_g. \tag{2.5}$$

This readily implies that $\mathrm{d}a = 0$ and, either by (2.5) if $n > 2$ or by (2.3) if $n = 2$, that $\delta\eta = 0$, i.e. that $\xi$ is Killing. Finally, by (2.2) we infer that $\nabla_T\xi = [T,\xi] = 0$. ☐

**Remark 2.1.** Proposition 2.1 can be viewed as a particular case of a more general statement (Theorem 2.1 in [13]) concerning conformal Killing forms on Riemannian products.

The following well-known Proposition 2.2 was first established by T. Nagano in [14] and T. Nagano–K. Yano in [15] in the more general setting of complete Einstein manifolds. The sketch of proof given here for the convenience of the reader follows M. Obata's treatment in [16], cf. also [17] for a more general discussion.

**Proposition 2.2.** *Assume that $(M^n,g)$ is a compact oriented Einstein manifold carrying a conformal Killing vector field which is not Killing. Then $(M,g)$ is, up to constant rescaling, isometric to the round sphere $\mathbb{S}^n$.*

*Proof.* We first recall the following lemma, due to A. Lichnerowicz [11, §85], cf. also Theorems 3 and 4 in [16].

**Lemma 2.1.** *Let $(M, g)$ be a connected compact Einstein manifold of dimension $n \geq 2$ of positive scalar curvature* Scal *(recall that* Scal *is automatically constant for $n \geq 3$ and constant by convention for $n = 2$). Denote by $\lambda_1$ the smallest positive eigenvalue of the Riemannian Laplacian acting on functions. Then,*

$$\lambda_1 \geq \frac{\mathrm{Scal}}{(n-1)}, \tag{2.6}$$

*with equality if and only if* $\mathrm{grad}_g f$*, the gradient of $f$ with respect to $g$, is a conformal Killing vector field for each function $f$ in the eigenspace of $\lambda_1$.*

*Proof.* As before denote by $\nabla$ the Levi-Civita connection of the metric $g$ and denote by Ric the Ricci tensor of $g$. For any 1-form $\eta$ on $M$, denote by $\xi := \eta^\sharp$ the vector field dual to $\eta$ with respect to $g$. The covariant derivative $\nabla\eta$ of $\eta$ then splits as follows:

$$\nabla\eta = \frac{1}{2}\left(\mathcal{L}_\xi g\right)_0 + \frac{1}{2}\mathrm{d}\eta - \frac{\delta\eta}{n}\, g, \tag{2.7}$$

where $\left(\mathcal{L}_\xi g\right)_0$ denotes the trace-free part of $\mathcal{L}_\xi g$. By using (2.7) the *Bochner identity*

$$\Delta\eta = \delta\nabla\eta + \mathrm{Ric}(\xi) \tag{2.8}$$

can be rewritten as

$$\mathrm{Ric}(\xi) = -\frac{1}{2}\delta\left(\mathcal{L}_\xi g\right)_0 + \frac{(n-1)}{n}\,\mathrm{d}\delta\eta + \frac{1}{2}\delta\mathrm{d}\eta. \tag{2.9}$$

Let $\lambda$ be any positive eigenvalue of $\Delta$ and $f$ any non-zero element of the corresponding eigenspace, so that $\Delta f = \lambda\, f$. By choosing $\eta := \mathrm{d}f$, so that $\xi = \mathrm{grad}_g f$, and substituting $\mathrm{Ric} = \frac{\mathrm{Scal}}{n}\, g$ in (2.9), we get

$$\lambda\,\mathrm{d}f = \Delta\mathrm{d}f = \frac{\mathrm{Scal}}{(n-1)}\,\mathrm{d}f + \frac{n}{2(n-1)}\,\delta\left(\mathcal{L}_\xi g\right)_0. \tag{2.10}$$

By contracting with $\mathrm{d}f$ and integrating over $M$, we obtain

$$\left(\lambda - \frac{\mathrm{Scal}}{(n-1)}\right)\int_M |\mathrm{d}f|^2\, v_g = \frac{n}{4(n-1)}\int_M |\left(\mathcal{L}_\xi g\right)_0|^2\, v_g \geq 0, \tag{2.11}$$

so that $\lambda \geq \frac{\mathrm{Scal}}{(n-1)}$, with equality if and only if $\left(\mathcal{L}_\xi g\right)_0 = 0$, hence if and only if $\xi = \mathrm{grad}_g f$ is conformal Killing. ☐

The proof of Proposition 2.2 goes as follows. First observe that we may assume $\mathrm{Scal} > 0$, as any conformal Killing vector field is zero if $\mathrm{Scal} < 0$ or parallel, hence Killing, if $\mathrm{Scal} = 0$. Let $\xi$ be any conformal Killing vector field on $M$, with dual 1-form $\eta$. From (2.9), we get:

$$\mathrm{Ric}(\xi) = \frac{\mathrm{Scal}}{n}\,\eta = \frac{(n-1)}{n}\,\mathrm{d}\delta\eta + \frac{1}{2}\delta\mathrm{d}\eta, \tag{2.12}$$

hence

$$\Delta(\delta\eta) = \frac{\mathrm{Scal}}{(n-1)}\,\delta\eta. \tag{2.13}$$

From Lemma 2.1, we then infer that $\mathrm{grad}_g(\delta\eta)$ is conformal Killing. By Theorem 5 in [17], this implies that $\delta\eta$ is constant, hence identically zero, unless $(M,g)$ is isometric to the standard sphere $\mathbb{S}^n$. If $(M,g)\neq\mathbb{S}^n$, we then have $\delta\eta=0$, meaning that $\xi$ is Killing. ☐

## 3. Weyl-Einstein connections on K-contact manifolds

**Definition 3.1.** *A K-contact manifold is an oriented Riemannian manifold $(M,g)$ of odd dimension $n=2m+1$, endowed with a unit Killing vector field $\xi$ whose covariant derivative $\varphi:=\nabla\xi$ satisfies*

$$\varphi^2=-\mathrm{Id}+\eta\otimes\xi, \tag{3.1}$$

*where $\eta$ is the metric dual 1-form of $\xi$.*

Since $\xi$ is Killing, we have $\mathrm{d}\eta(X,Y)=2g(\varphi(X),Y)$ for all vector fields $X$ and $Y$. The kernel of the 2-form $\mathrm{d}\eta$, equal to that of $\varphi$, is then spanned by $\xi$:

$$\ker(\mathrm{d}\eta)=\ker(\varphi)=\mathbb{R}\xi. \tag{3.2}$$

It follows that the restriction of $\mathrm{d}\eta$ to $\mathcal{D}:=\ker(\eta)$ is non-degenerate, hence that $\mathcal{D}$ is a contact distribution on $M$. Moreover, since $\eta(\xi)=1$ and $\xi\lrcorner\mathrm{d}\eta=0$, $\xi$ is the Reeb vector field of the contact 1-form $\eta$.

Denote by R the Riemannian curvature tensor defined by $\mathrm{R}_{X,Y}:=\nabla_{[X,Y]}-[\nabla_X,\nabla_Y]$. From (3.1) we easily infer:

**Lemma 3.1.** *For any K-contact structure, we have:*

$$\mathrm{R}_{\xi,X}\xi=X-g(\xi,X)\xi, \tag{3.3}$$

*for any vector field $X$.*

*Proof.* We first recall the general *Kostant formula*:

$$\nabla_X(\nabla\xi)=\mathrm{R}_{\xi,X}, \tag{3.4}$$

for any vector field $X$ and any Killing vector field $\xi$, on any Riemannian manifold, cf. [10]. In the current situation, we thus have

$$\nabla_X\varphi=\mathrm{R}_{\xi,X}, \tag{3.5}$$

for any vector field $X$. Since $\xi$ is of norm 1, we infer: $\mathrm{R}_{\xi,X}\xi=\nabla_X(\nabla_\xi\xi)-\nabla_{\nabla_X\xi}\xi=-\nabla_{\nabla_X\xi}\xi=-\varphi^2(X)=X-g(\xi,X)\,\xi$. ☐

**Remark 3.1.** A $K$-contact structure $(g,\xi)$ is called a *Sasaki structure* if

$$(\nabla_X\varphi)(Y)=\eta(Y)X-g(X,Y)\xi, \tag{3.6}$$

for any vector fields $X,Y$, or, equivalently in view of (3.5), if

$$\mathrm{R}_{\xi,X}=\xi\wedge X, \tag{3.7}$$

(where the curvature R is viewed as a map from $\Lambda^2\mathrm{T}M$ to itself).

**Lemma 3.2** (cf. [3]). *Viewed as endomorphism of the tangent bundle via the metric $g$, the Ricci tensor of any K-contact manifold satisfies*

$$\mathrm{Ric}(\xi) = 2m\,\xi. \tag{3.8}$$

*Proof.* From (3.5) we get:

$$\nabla_\xi \varphi = 0, \tag{3.9}$$

and

$$\delta\varphi = \mathrm{Ric}(\xi) \tag{3.10}$$

— here $\delta\varphi$ denotes the co-differential of the endomorphism $\varphi$ and Ric is regarded as a field of endomorphisms of $\mathrm{T}M$ — whereas, from (3.1) we readily infer

$$\nabla_X\varphi \circ \varphi + \varphi \circ \nabla_X\varphi = \frac{1}{2} X \lrcorner \mathrm{d}\eta \otimes \xi + \eta \otimes \varphi(X), \tag{3.11}$$

hence

$$(\nabla_X \varphi)(\xi) = \mathrm{R}_{\xi,X}\xi = X - \eta(X)\xi, \tag{3.12}$$

for any vector field $X$, from which we get

$$\mathrm{Ric}(\xi,\xi) = n - 1 = 2m. \tag{3.13}$$

In view of (3.13) and (3.2), to prove Lemma 3.2 it is sufficient to check that $\varphi\big(\mathrm{Ric}(\xi)\big) = 0$, or else, by (3.10), that $\varphi(\delta\varphi) = 0$. In view of (3.9), we have

$$\delta\varphi = -\sum_{i=1}^{2m} (\nabla_{e_i}\varphi)(e_i), \tag{3.14}$$

for any auxiliary (local) orthonormal frame of $\mathcal{D}$; from (3.11) we thus get

$$\varphi(\delta\varphi) = \sum_{i=1}^{2m} (\nabla_{e_i}\varphi)\big(\varphi(e_i)\big). \tag{3.15}$$

Since $\varphi$ is associated to the *closed* 2-form $\mathrm{d}\eta$, for any vector field $X$ we have:

$$g\big(\sum_{i=1}^{2m} (\nabla_{e_i}\varphi)\Big(\varphi(e_i)\big), X\Big) = -\frac{1}{2}\sum_{i=1}^{2m} g\big((\nabla_X\varphi)(e_i), \varphi(e_i)\big) = -g(\nabla_X\varphi, \varphi),$$

which is equal to zero since the norm of $\varphi$ is constant. □

In the following statement, we denote by $(\mathbb{S}^{2m+1}, c_0)$ the $(2m+1)$-dimensional sphere, equipped with the standard flat conformal structure $c_0$.

**Proposition 3.1.** *Let $(g,\xi)$ be any $K$-contact structure on $(\mathbb{S}^{2m+1}, c_0)$, such that $g$ belongs to the conformal class $c_0$. Then, $g$ has constant sectional curvature equal to* 1 *and the $K$-contact structure is then isomorphic to the standard Sasaki-Einstein structure.*

*Proof.* Since $c_0$ is flat, the curvature R of $g$ is of the form

$$\mathrm{R}_{X,Y} = \mathrm{S}(X) \wedge Y + X \wedge \mathrm{S}(Y), \tag{3.16}$$

where, in general, for any $n$-dimensional Riemannian manifold $(M, g)$, the *normalized Ricci tensor* (or Schouten tensor) S is defined by

$$\mathrm{S} = \frac{1}{(n-2)} \left( \mathrm{Ric} - \frac{\mathrm{Scal}}{2(n-1)} \mathrm{Id} \right). \tag{3.17}$$

It then follows from (3.3), (3.8), and (3.16) that

$$\mathrm{S}(X) = \frac{1}{(n-2)} \left[ \left( \frac{\mathrm{Scal}}{2(n-1)} - 1 \right) X + \left( n - \frac{\mathrm{Scal}}{(n-1)} \right) g(\xi, X)\, \xi \right] \tag{3.18}$$

with $n = 2m+1$ (as in (3.8), in (3.18) and in the sequel of the proof, Ric and S are regarded as endomorphisms of the tangent bundle via the metric $g$). In terms of the normalized Ricci tensor S, the contracted Bianchi identity $\delta\mathrm{Ric} + \frac{\mathrm{d\,Scal}}{2} = 0$, reads

$$\delta \mathrm{S} + \frac{\mathrm{d\,Scal}}{2(n-1)} = 0. \tag{3.19}$$

By using (3.19), we readily infer from (3.18) that Scal is constant, so that

$$(\nabla_X \mathrm{S})(Y) = \kappa \left( g(\nabla_X \xi, Y)\xi + g(\xi, Y)\, \nabla_X \xi \right), \tag{3.20}$$

for any vector fields $X, Y$, by setting:

$$\kappa := \frac{1}{(n-2)} \left( n - \frac{\mathrm{Scal}}{(n-1)} \right). \tag{3.21}$$

Since the conformal structure is flat, the general Bianchi identity (cf. e.g. [6])

$$\delta \mathrm{W}_Z(X, Y) = (n-3)\, g\big(Z, (\nabla_X \mathrm{S})(Y) - (\nabla_Y \mathrm{S})(X)\big), \tag{3.22}$$

where W denotes the Weyl tensor of $g$, implies that $(\nabla_X \mathrm{S})(Y)$ is *symmetric* in $X, Y$, while, by (3.20), $g\big((\nabla_X \mathrm{S})Y, \xi\big) = \kappa\, g(\nabla_X \xi, Y)$, which is anti-symmetric, as $\xi$ is Killing; we thus get $\kappa = 0$, hence by (3.21), $\mathrm{Scal} = n(n-1)$. By (3.18), this implies $\mathrm{S} = \frac{1}{2}\mathrm{Id}$, so (3.16) shows that $g$ is a metric of constant sectional curvature equal to 1.

Finally, (3.5) shows that $\nabla_X \varphi = \xi \wedge X$ for every tangent vector $X$, meaning that the K-contact structure is Sasaki-Einstein, and it is well known that the isometry group of $\mathbb{S}^{2m+1}$ acts transitively on the set of Sasaki-Einstein structures on the sphere. □

**Definition 3.2.** *A Weyl connection on a conformal manifold $(M, c)$ is a torsion-free linear connection $D$ which preserves the conformal class $c$.*

The latter condition means that for any metric $g$ in the conformal class $c$, there exists a real 1-form, $\theta^g$, called the *Lee form* of $D$ with respect to $g$, such that $Dg = -2\theta^g \otimes g$, and $D$ is then related to the Levi-Civita connection, $\nabla^g$, of $g$ by

$$D_X Y = \nabla^g_X Y + \theta^g(X) Y + \theta^g(Y) X - g(X, Y)\, (\theta^g)^{\sharp_g}, \tag{3.23}$$

cf. e.g. [5]. A Weyl connection $D$ is said to be *closed* if it is locally the Levi-Civita connection of a (local) metric in $c$, *exact* if it is the Levi-Civita connection of a (globally defined) metric in $c$; equivalently, $D$ is closed, respectively exact, if its Lee form is closed, respectively exact, with respect to one, hence any, metric in $c$.

If $M$ is compact, for any Weyl connection on $(M, c)$ there exists a distinguished metric, say $g_0$, in $c$, usually called the *Gauduchon metric* of $D$, unique up to scaling, whose Lee form $\theta^{g_0}$ is co-closed with respect to $g_0$, [7]. If $D$ is closed, $\theta^{g_0}$ is then $g_0$-harmonic, identically zero if $D$ is exact.

The *Ricci tensor*, $\mathrm{Ric}^D$, of a Weyl connection $D$ is the bilinear defined by $\mathrm{Ric}(X, Y) = \mathrm{trace}\{Z \mapsto \mathrm{R}^D_{X,Z}Y\} = \sum_{i=1}^n g(\mathrm{R}^D_{X,e_i}Y, e_i)$, for any metric $g$ in $c$ and any $g$-orthonormal basis $\{e_i\}_{i=1}^n$. The Ricci tensor $\mathrm{Ric}^D$ defined that way is symmetric if and only if $D$ is closed.

A Weyl connection $D$ is called *Weyl-Einstein* if the trace-free component of the symmetric part of $\mathrm{Ric}^D$ is identically zero. A closed Weyl-Einstein connection is locally the Levi-Civita connection of a (local) Einstein metric in $c$; an exact Weyl-Einstein connection is the Levi-Civita connection of a (globally defined) Einstein metric.

We here recall the following well-known fact, first observed in [18], cf. also [8].

**Theorem 3.1.** *Let $D$ be a Weyl-Einstein connection defined on a compact connected oriented conformal manifold $(M, c)$ and denote by $g_0$ its Gauduchon metric. Then the vector field $T$ on $M$ dual to the Lee form $\theta^{g_0}$ is Killing with respect to $g_0$. If $D$ is closed, $T$ is parallel with respect to $g_0$, identically zero if and only if $D$ is exact, and $D$ is then the Levi-Civita connection of $g_0$.*

The aim of this section is to prove the following:

**Theorem 3.2.** *Let $(M, g, \xi)$ be a compact K-contact manifold of dimension $n = 2m + 1$, $m \geq 1$, carrying a closed Weyl-Einstein structure $D$ compatible with the conformal class $c = [g]$. Then $g$ is Einstein and $D$ is the Levi-Civita connection of an Einstein metric $g_0$ in $c$, which is equal to $g$, up to scaling, except if $(M, c)$ is the flat conformal sphere $(\mathbb{S}^{2m+1}, c_0)$; in the latter case, the $K$-contact structure is isomorphic to the standard Sasaki-Einstein structure of $\mathbb{S}^{2m+1}$.*

*Proof.* In view of Proposition 3.1, we may assume that $(M, c)$ is not isomorphic to the flat conformal sphere $(\mathbb{S}^{2m+1}, c_0)$. Let $g_0 := e^{2f} g$ denote the Gauduchon metric of $D$ and let $T$ denote the $g_0$-dual of the Lee form of $D$ with respect to $g_0$. According to Theorem 3.1, $T$ is $\nabla^{g_0}$-parallel. We first show that $T \equiv 0$, i.e. that the closed Weyl-Einstein connection $D$ is actually exact.

Assume, for a contradiction, that $T$ is non-zero. By rescaling the Gauduchon metric $g_0$ if necessary, we may assume that $g_0(T, T) = 1$. Denote by $\eta$, resp. $\eta_0$, the 1-form dual to $\xi$ with respect to $g$, resp. $g_0$. Both $\eta$ and $\eta_0$ are contact 1-forms for the contact distribution $\mathcal{D}$, and, as already noticed, $\xi$ is the Reeb vector field of $\eta$. According to Proposition 2.1, $\xi$, which is Killing with respect to $g$, hence conformal Killing with respect to $g_0$, is actually Killing with respect to $g_0$ as well, commutes with $T$, and the inner product $a := g_0(\xi, T) = \eta_0(T)$ is constant; we then have: $\mathcal{L}_T \eta_0 = 0$, hence that $T \lrcorner \mathrm{d}\eta_0 = \mathcal{L}_T \eta_0 - \mathrm{d}\big(\eta_0(T)\big) = -\mathrm{d}a = 0$. Moreover, since $\eta_0(T) = a$ and $T \lrcorner \mathrm{d}\eta_0 = 0$, $a$ cannot be zero

— otherwise, $\eta_0$ would not be a contact 1-form — and $\xi_0 := a^{-1} T$ is then the Reeb vector field of $\eta_0$. Since $\eta_0 = e^{2f}\eta$, the Reeb vector fields $\xi_0$ and $\xi$ are related by

$$\xi_0 = e^{-2f}\,\xi + Z_f, \tag{3.24}$$

where $Z_f$ is the section of $\mathcal{D}$ defined by

$$(Z_f \lrcorner \mathrm{d}\eta)_{|\mathcal{D}} = 2e^{-2f}\,\mathrm{d}f_{|\mathcal{D}}. \tag{3.25}$$

Since $M$ is compact, $f$ has critical points and for each of them, say $x$, it follows from (3.25) that $Z_f(x) = 0$, hence $\xi_0(x) = a^{-1} T(x) = e^{-2f(x)}\,\xi(x)$. Since, $g_0(T(x), T(x)) = 1$ and $g_0(\xi(x), T(x)) = a$ for any $x$, we infer that $e^{2f(x)} = a^2$ for *any* critical point $x$ of $f$, in particular for points where $f$ takes its minimal or its maximal value. It follows that $f$ is *constant*, with $e^{2f} \equiv a^2$, that $g_0 = a^2\,g$ and $\xi = a\,T$. In particular, $\eta$ and $\eta_0 = a^2\,\eta$ are parallel, with respect to $g$ and $g_0$, hence closed. This contradicts the fact that they are contact 1-forms.

In view of the above, $T$ must be identically zero. This means that $D$ is the Levi-Civita connection of the Gauduchon metric $g_0$, which is thus Einstein. Since $\xi$ with respect to $g$, hence conformal Killing with respect to $g_0 = e^{2f}g$, and $(M, c)$ is not isomorphic to the flat conformal sphere $(\mathbb{S}^{2m+1}, c_0)$, it follows from Proposition 2.2 that $\xi$ is Killing with respect to $g_0$ as well. We thus have $\mathrm{d}f(\xi) = 0$, hence

$$g(\xi, \mathrm{grad}_g f) = 0. \tag{3.26}$$

Let $\lambda$ denote the Einstein constant of $(M, g_0)$, so that $\mathrm{Ric}^0 = \lambda g_0 = e^{2f}\lambda g$. The classical formula relating the Ricci tensors Ric and $\mathrm{Ric}^0$ of $g$ and $g_0$ reads (cf. [2], p. 59):

$$\mathrm{Ric}^0 = \mathrm{Ric} - (2m-1)(\nabla^g \mathrm{d}f - \mathrm{d}f \otimes \mathrm{d}f) + (\Delta^g f - (2m-1)|\mathrm{d}f|_g^2)g. \tag{3.27}$$

Contracting (3.27) with $\xi$ and using Proposition 3.2 we get

$$\lambda e^{2f}\eta = 2m\eta - (2m-1)\nabla^g_\xi \mathrm{d}f + (\Delta^g f - (2m-1)|\mathrm{d}f|_g^2)\eta.$$

Taking the metric duals with respect to $g$ this equation reads

$$\nabla^g_\xi(\mathrm{grad}_g f) = h\xi, \qquad \text{with} \qquad h := \frac{1}{2m-1}\left(\Delta^g f - (2m-1)|\mathrm{d}f|_g^2 + 2m - \lambda e^{2f}\right). \tag{3.28}$$

On the other hand, we have $0 = \mathrm{d}\mathcal{L}_\xi f = \mathcal{L}_\xi \mathrm{d}f$, thus $\mathcal{L}_\xi(\mathrm{grad}_g f) = 0$ and therefore

$$\nabla^g_\xi(\mathrm{grad}_g f) = \nabla^g_{\mathrm{grad}_g f}\xi = \varphi(\mathrm{grad}_g f).$$

Since the image of $\varphi$ is orthogonal to $\xi$, (3.28) implies that $\varphi(\mathrm{grad}_g f) = 0$, thus by (3.2), $\mathrm{grad}_g f$ is proportional to $\xi$. From (3.26) we thus get $\mathrm{grad}_g f = 0$, so $f$ is constant and $D$ is the Levi-Civita connection of $g$, and hence $g$ is Einstein. □

As a direct corollary of Theorem 3.2 above together with Theorem 1.1 in [1] (see also [4]), we obtain the following result:

**Corollary 3.1.** *If $(M^{2m+1}, g, \xi)$ is a compact K-contact manifold carrying a closed Weyl-Einstein structure compatible with $g$, then $M$ is Sasaki-Einstein.*

**Remark 3.2.** *In* [12] *it is claimed that if* $(M^{2m+1}, g, \xi)$ *is a compact K-contact manifold carrying a compatible closed Weyl-Einstein structure, then* $M$ *is Sasakian if and only if it is* $\eta$*-Einstein. Our above result show that the hypotheses in* [12] *already imply both conditions.*

## References


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