

WEYL-EINSTEIN STRUCTURES ON K-CONTACT MANIFOLDS

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ABSTRACT. We show that a compact K-contact manifold (M, g, ξ) has a closed Weyl-Einstein connection compatible with the conformal structure $[g]$ if and only if it is Sasaki-Einstein.

1. INTRODUCTION

K -contact structures — see the definition in Section 3 — can be viewed as the odd-dimensional counterparts of *almost Kähler* structure, in the same way as *Sasakian* structures are the odd-dimensional counterparts of *Kähler structures*. It has been shown in [4], cf. also [1], that *compact* Einstein K -contact structures are actually Sasakian, hence Sasaki-Einstein. In this note, we consider the more general situation of a compact K -contact manifold (M, g, ξ) carrying in addition a *Weyl-Einstein* connection D compatible with the conformal class $[g]$, already considered by a number of authors, in particular in [9] and [12]. We show — Theorem 3.2 and Corollary 3.1 below — that g is then Einstein and D is the Levi-Civita connection of an Einstein metric g_0 in the conformal class $[g]$, which is actually equal g up to scaling, except if $(M, [g])$ is the flat conformal sphere. In all cases, the K -contact structure is Sasaki-Einstein.

2. CONFORMAL KILLING VECTOR FIELDS

Let (M, c) be a (positive definite) conformal manifold of dimension n . A vector field ξ on M is called *conformal Killing* with respect to c if it preserves c , meaning that for any metric g in c , the trace-free part $(\mathcal{L}_\xi g)_0$ of the Lie derivative $\mathcal{L}_\xi g$ of g along ξ is identically zero, hence that $\mathcal{L}_\xi g = f g$, for some function f , depending on ξ and g , and it is then easily checked that $f = -\frac{2\delta^g \eta_g}{n}$, where η_g denotes the 1-form dual to ξ and $\delta^g \eta_g$ the co-differential of η_g with respect to g . In particular, a conformal Killing vector field ξ on M is Killing with respect to some metric g in c if and only if $\delta^g \eta_g = 0$. In this section, we present a number of facts concerning conformal Killing vector fields for further use in this note.

Proposition 2.1. *Let (M, g) be a connected compact oriented Riemannian manifold of dimension n , $n \geq 2$, carrying a non-trivial parallel vector field T . Let ξ be any conformal Killing vector field on M with respect to the conformal class $[g]$ of g . Then, ξ is Killing with respect to g ; moreover, it commutes with T and the inner product $a := g(\xi, T)$ is constant.*

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Proof. Denote by $\eta = \xi^\flat$ the 1-form dual to ξ and by $\delta\eta$ the co-differential of η with respect to g ; then, ξ is Killing if and only if $\delta\eta = 0$. Denote by ∇ the Levi-Civita connection of g and by \mathcal{L}_T the Lie derivative along T ; then, $\nabla_T\xi = [T, \xi] = \mathcal{L}_T\xi$ is conformal Killing, and we have:

$$(2.1) \quad \delta(\nabla_T\eta) = \delta(\mathcal{L}_T\eta) = \mathcal{L}_T(\delta\eta) = T(\delta\eta).$$

Since T is non-trivial, we may assume $|T| \equiv 1$. Denote $a = g(\xi, T) = \eta(T)$. Since ξ is conformal Killing, $\nabla\xi = A - \frac{\delta\eta}{n}\text{Id}$, where A is skew-symmetric and Id denotes the identity; for any vector field X we then have: $da(X) = g(\nabla_X\xi, T) = -g(\nabla_T\xi, X) - \frac{2\delta\eta}{n}g(X, T)$. We thus get:

$$(2.2) \quad \nabla_T\eta = -da - \frac{2\delta\eta}{n}\theta,$$

where $\theta = T^\flat$ denotes the 1-form dual to T . By evaluating both members of (2.2) on T , we get:

$$(2.3) \quad \delta\eta = -n\,da(T),$$

whereas, by considering their co-differential and by using (2.1), we get:

$$(2.4) \quad \Delta a = -\frac{(n-2)}{n}T(\delta\eta),$$

where $\Delta a = \delta da$ denotes the Laplacian of a . Denote by v_g the volume form determined by g and the chosen orientation; from (2.3) and (2.4), we then infer:

$$\int_M a\,\Delta a\,v_g = -\frac{(n-2)}{n}\int_M aT(\delta\eta)\,v_g = \frac{(n-2)}{n}\int_M da(T)\,\delta\eta\,v_g = -\frac{(n-2)}{n^2}\int_M (\delta\eta)^2\,v_g,$$

hence

$$(2.5) \quad \int_M |da|^2\,v_g = \int_M a\,\Delta a\,v_g = -\frac{(n-2)}{n^2}\int_M (\delta\eta)^2\,v_g.$$

This readily implies that $da = 0$ and, either by (2.5) if $n > 2$ or by (2.3) if $n = 2$, that $\delta\eta = 0$, i.e. that ξ is Killing. Finally, by (2.2) we infer that $\nabla_T\xi = [T, \xi] = 0$. \square

Remark 2.1. Proposition 2.1 can be viewed as a particular case of a more general statement (Theorem 2.1 in [13]) concerning conformal Killing forms on Riemannian products.

The following well-known Proposition 2.2 was first established by T. Nagano in [14] and T. Nagano–K. Yano in [15] in the more general setting of complete Einstein manifolds. The sketch of proof given here for the convenience of the reader follows M. Obata's treatment in [16], cf. also [17] for a more general discussion.

Proposition 2.2. *Assume that (M^n, g) is a compact oriented Einstein manifold carrying a conformal Killing vector field which is not Killing. Then (M, g) is, up to constant rescaling, isometric to the round sphere \mathbb{S}^n .*

Proof. We first recall the following lemma, due to A. Lichnerowicz [11, §85], cf. also Theorems 3 and 4 in [16].

Lemma 2.1. *Let (M, g) be a connected compact Einstein manifold of dimension $n \geq 2$ of positive scalar curvature Scal (recall that Scal is automatically constant for $n \geq 3$ and constant by convention for $n = 2$). Denote by λ_1 the smallest positive eigenvalue of the Riemannian Laplacian acting on functions. Then,*

$$(2.6) \quad \lambda_1 \geq \frac{\text{Scal}}{(n-1)},$$

with equality if and only if $\text{grad}_g f$, the gradient of f with respect to g , is a conformal Killing vector field for each function f in the eigenspace of λ_1 .

Proof. As before denote by ∇ the Levi-Civita connection of the metric g and denote by Ric the Ricci tensor of g . For any 1-form η on M , denote by $\xi := \eta^\sharp$ the vector field dual to η with respect to g . The covariant derivative $\nabla\eta$ of η then splits as follows:

$$(2.7) \quad \nabla\eta = \frac{1}{2}(\mathcal{L}_\xi g)_0 + \frac{1}{2}d\eta - \frac{\delta\eta}{n}g,$$

where $(\mathcal{L}_\xi g)_0$ denotes the trace-free part of $\mathcal{L}_\xi g$. By using (2.7) the *Bochner identity*

$$(2.8) \quad \Delta\eta = \delta\nabla\eta + \text{Ric}(\xi)$$

can be rewritten as

$$(2.9) \quad \text{Ric}(\xi) = -\frac{1}{2}\delta(\mathcal{L}_\xi g)_0 + \frac{(n-1)}{n}d\delta\eta + \frac{1}{2}\delta d\eta.$$

Let λ be any positive eigenvalue of Δ and f any non-zero element of the corresponding eigenspace, so that $\Delta f = \lambda f$. By choosing $\eta := df$, so that $\xi = \text{grad}_g f$, and substituting $\text{Ric} = \frac{\text{Scal}}{n}g$ in (2.9), we get

$$(2.10) \quad \lambda df = \Delta df = \frac{\text{Scal}}{(n-1)}df + \frac{n}{2(n-1)}\delta(\mathcal{L}_\xi g)_0.$$

By contracting with df and integrating over M , we obtain

$$(2.11) \quad \left(\lambda - \frac{\text{Scal}}{(n-1)}\right) \int_M |df|^2 v_g = \frac{n}{4(n-1)} \int_M |(\mathcal{L}_\xi g)_0|^2 v_g \geq 0,$$

so that $\lambda \geq \frac{\text{Scal}}{(n-1)}$, with equality if and only if $(\mathcal{L}_\xi g)_0 = 0$, hence if and only if $\xi = \text{grad}_g f$ is conformal Killing. \square

The proof of Proposition 2.2 goes as follows. First observe that we may assume $\text{Scal} > 0$, as any conformal Killing vector field is zero if $\text{Scal} < 0$ or parallel, hence Killing, if $\text{Scal} = 0$. Let ξ be any conformal Killing vector field on M , with dual 1-form η . From (2.9), we get:

$$(2.12) \quad \text{Ric}(\xi) = \frac{\text{Scal}}{n}\eta = \frac{(n-1)}{n}d\delta\eta + \frac{1}{2}\delta d\eta,$$

hence

$$(2.13) \quad \Delta(\delta\eta) = \frac{\text{Scal}}{(n-1)}\delta\eta.$$

From Lemma 3.21, we then infer that $\text{grad}_g(\delta\eta)$ is conformal Killing. By Theorem 5 in [17], this implies that $\delta\eta$ is constant, hence identically zero, unless (M, g) is isometric to the standard sphere \mathbb{S}^n . If $(M, g) \neq \mathbb{S}^n$, we then have $\delta\eta = 0$, meaning that ξ is Killing. \square

3. WEYL-EINSTEIN CONNECTIONS ON K-CONTACT MANIFOLDS

Definition 3.1. A *K-contact manifold* is an oriented Riemannian manifold (M, g) of odd dimension $n = 2m + 1$, endowed with a unit Killing vector field ξ whose covariant derivative $\varphi := \nabla\xi$ satisfies

$$(3.1) \quad \varphi^2 = -\text{Id} + \eta \otimes \xi,$$

where η is the metric dual 1-form of ξ .

Since ξ is Killing, we have $d\eta(X, Y) = 2g(\varphi(X), Y)$ for all vector fields X and Y . The kernel of the 2-form $d\eta$, equal to that of φ , is then spanned by ξ :

$$(3.2) \quad \ker(d\eta) = \ker(\varphi) = \mathbb{R}\xi.$$

It follows that the restriction of $d\eta$ to $\mathcal{D} := \ker(\eta)$ is non-degenerate, hence that \mathcal{D} is a contact distribution on M . Moreover, since $\eta(\xi) = 1$ and $\xi \lrcorner d\eta = 0$, ξ is the Reeb vector field of the contact 1-form η .

Denote by R the Riemannian curvature tensor defined by $R_{X,Y} := \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$. From (3.1) we easily infer:

Lemma 3.1. *For any K-contact structure, we have:*

$$(3.3) \quad R_{\xi,X}\xi = X - g(\xi, X)\xi,$$

for any vector field X .

Proof. We first recall the general *Kostant formula*:

$$(3.4) \quad \nabla_X(\nabla\xi) = R_{\xi,X},$$

for any vector field X and any Killing vector field ξ , on any Riemannian manifold, cf. [10]. In the current situation, we thus have

$$(3.5) \quad \nabla_X\varphi = R_{\xi,X},$$

for any vector field X . Since ξ is of norm 1, we infer: $R_{\xi,X}\xi = \nabla_X(\nabla\xi) - \nabla_{\nabla_X\xi}\xi = -\nabla_{\nabla_X\xi}\xi = -\varphi^2(X) = X - g(\xi, X)\xi$. \square

Remark 3.1. A *K-contact structure* (g, ξ) is called a *Sasaki structure* if

$$(3.6) \quad (\nabla_X\varphi)(Y) = \eta(Y)X - g(X, Y)\xi,$$

for any vector fields X, Y , or, equivalently in view of (3.5), if

$$(3.7) \quad R_{\xi,X} = \xi \wedge X,$$

(where the curvature R is viewed as a map from $\Lambda^2 TM$ to itself).

Lemma 3.2 (cf. [3]). *Viewed as endomorphism of the tangent bundle via the metric g , the Ricci tensor of any K-contact manifold satisfies*

$$(3.8) \quad \text{Ric}(\xi) = 2m \xi.$$

Proof. From (3.5) we get:

$$(3.9) \quad \nabla_{\xi}\varphi = 0,$$

and

$$(3.10) \quad \delta\varphi = \text{Ric}(\xi)$$

— here $\delta\varphi$ denotes the co-differential of the endomorphism φ and Ric is regarded as a field of endomorphisms of TM — whereas, from (3.1) we readily infer

$$(3.11) \quad \nabla_X\varphi \circ \varphi + \varphi \circ \nabla_X\varphi = \frac{1}{2}X \lrcorner d\eta \otimes \xi + \eta \otimes \varphi(X),$$

hence

$$(3.12) \quad (\nabla_X\varphi)(\xi) = R_{\xi,X}\xi = X - \eta(X)\xi,$$

for any vector field X , from which we get

$$(3.13) \quad \text{Ric}(\xi, \xi) = n - 1 = 2m.$$

In view of (3.13) and (3.2), to prove Lemma 3.2 it is sufficient to check that $\varphi(\text{Ric}(\xi)) = 0$, or else, by (3.10), that $\varphi(\delta\varphi) = 0$. In view of (3.9), we have

$$(3.14) \quad \delta\varphi = - \sum_{i=1}^{2m} (\nabla_{e_i}\varphi)(e_i),$$

for any auxiliary (local) orthonormal frame of \mathcal{D} ; from (3.11) we thus get

$$(3.15) \quad \varphi(\delta\varphi) = \sum_{i=1}^{2m} (\nabla_{e_i}\varphi)(\varphi(e_i)).$$

Since φ is associated to the *closed* 2-form $d\eta$, for any vector field X we have:

$$g\left(\sum_{i=1}^{2m} (\nabla_{e_i}\varphi)(\varphi(e_i)), X\right) = -\frac{1}{2} \sum_{i=1}^{2m} g((\nabla_X\varphi)(e_i), \varphi(e_i)) = -g(\nabla_X\varphi, \varphi),$$

which is equal to zero since the norm of φ is constant. □

In the following statement, we denote by (\mathbb{S}^{2m+1}, c_0) the $(2m + 1)$ -dimensional sphere, equipped with the standard flat conformal structure c_0 .

Proposition 3.1. *Let (g, ξ) be any K-contact structure on (\mathbb{S}^{2m+1}, c_0) , such that g belongs to the conformal class c_0 . Then, g is of constant sectional curvature 1 and the K-contact structure is Sasaki-Einstein.*

Proof. Since c_0 is flat, the curvature R of g is of the form

$$(3.16) \quad R_{X,Y} = S(X) \wedge Y + X \wedge S(Y),$$

where, in general, for any n -dimensional Riemannian manifold (M, g) , the *normalized Ricci tensor* (or Schouten tensor) S is defined by

$$(3.17) \quad S = \frac{1}{(n-2)} \left(\text{Ric} - \frac{\text{Scal}}{2(n-1)} \text{Id} \right).$$

It then follows from (3.3), (3.8), and (3.16) that

$$(3.18) \quad S(X) = \frac{1}{(n-2)} \left[\left(\frac{\text{Scal}}{2(n-1)} - 1 \right) X + \left(n - \frac{\text{Scal}}{(n-1)} \right) g(\xi, X) \xi \right]$$

with $n = 2m + 1$ (as in (3.8), in (3.18) and in the sequel of the proof, Ric and S are regarded as endomorphisms of the tangent bundle via the metric g). In terms of the normalized Ricci tensor S , the contracted Bianchi identity $\delta \text{Ric} + \frac{d \text{Scal}}{2} = 0$, reads

$$(3.19) \quad \delta S + \frac{d \text{Scal}}{2(n-1)} = 0.$$

By using (3.19), we readily infer from (3.18) that Scal is constant, so that

$$(3.20) \quad (\nabla_X S)(Y) = \kappa (g(\nabla_X \xi, Y) \xi + g(\xi, Y) \nabla_X \xi),$$

for any vector fields X, Y , by setting:

$$(3.21) \quad \kappa := \frac{1}{(n-2)} \left(n - \frac{\text{Scal}}{(n-1)} \right).$$

Since the conformal structure is flat, the general Bianchi identity (cf. e.g. [6])

$$(3.22) \quad \delta W_Z(X, Y) = (n-3) g(Z, (\nabla_X S)(Y) - (\nabla_Y S)(X)),$$

where W denotes the Weyl tensor of g , implies that $(\nabla_X S)(Y)$ is *symmetric* in X, Y , while, by (3.20), $g((\nabla_X S)Y, \xi) = \kappa g(\nabla_X \xi, Y)$, which is anti-symmetric, as ξ is Killing; we thus get $\kappa = 0$, hence $\text{Scal} = n(n-1)$. By (3.18), this implies: $S = \frac{1}{2} \text{Id}$, whence by (3.17) $\text{Ric} = (n-1)g$, meaning that g is Einstein, hence of constant sectional curvature equal to 1, and the K -contact structure is then Sasaki-Einstein, cf. Remark 3.1. \square

Definition 3.2. *A Weyl connection on a conformal manifold (M, c) is a torsion-free linear connection D which preserves the conformal class c .*

The latter condition means that for any metric g in the conformal class c , there exists a real 1-form, θ^g , called the *Lee form* of D with respect to g , such that $Dg = -2\theta^g \otimes g$, and D is then related to the Levi-Civita connection, ∇^g , of g by

$$(3.23) \quad D_X Y = \nabla_X^g Y + \theta^g(X)Y + \theta^g(Y)X - g(X, Y) (\theta^g)^\sharp,$$

cf. e.g. [5]. A Weyl connection D is said to be *closed* if it is locally the Levi-Civita connection of a (local) metric in c , *exact* if it is the Levi-Civita connection of a (globally defined) metric in c ; equivalently, D is closed, respectively exact, if its Lee form is closed, respectively exact, with respect to one, hence any, metric in c .

If M is compact, for any Weyl connection on (M, c) there exists a distinguished metric, say g_0 , in c , usually called the *Gauduchon metric* of D , unique up to scaling, whose Lee form θ^{g_0} is co-closed with respect to g_0 , [7]. If D is closed, θ^{g_0} is then g_0 -harmonic, identically zero if D is exact.

The *Ricci tensor*, Ric^D , of a Weyl connection D is the bilinear defined by $\text{Ric}(X, Y) = \text{trace}\{Z \mapsto R_{X,Z}^D Y\} = \sum_{i=1}^n g(R_{X,e_i}^D Y, e_i)$, for any metric g in c and any g -orthonormal basis $\{e_i\}_{i=1}^n$. The Ricci tensor Ric^D defined that way is symmetric if and only if D is closed.

A Weyl connection D is called *Weyl-Einstein* if the trace-free component of the symmetric part of Ric^D is identically zero. A closed Weyl-Einstein connection is locally the Levi-Civita connection of a (local) Einstein metric in c ; an exact Weyl-Einstein connection is the Levi-Civita connection of a (globally defined) Einstein metric.

We here recall the following well-known fact, first observed in [18], cf. also [8].

Theorem 3.1. *Let D be a Weyl-Einstein connection defined on a compact connected oriented conformal manifold (M, c) and denote by g_0 its Gauduchon metric. Then the vector field T on M dual to the Lee form θ^{g_0} is Killing with respect to g_0 . If D is closed, T is parallel with respect to g_0 , identically zero if and only if D is exact, and D is then the Levi-Civita connection of g_0 .*

The aim of this section is to prove the following:

Theorem 3.2. *Let (M, g, ξ) be a compact K -contact manifold of dimension $n = 2m + 1$, $m \geq 1$, carrying a closed Weyl-Einstein structure D compatible with the conformal class $c = [g]$. Then g is Einstein and D is the Levi-Civita connection of an Einstein metric g_0 in c , which is equal to g , up to scaling, except if (M, c) is the flat conformal sphere (\mathbb{S}^{2m+1}, c_0) ; in the latter case, the K -contact structure is isomorphic to the standard Sasaki-Einstein structure of \mathbb{S}^{2m+1} .*

Proof. In view of Proposition 3.1, we may assume that (M, c) is not isomorphic to the flat conformal sphere (\mathbb{S}^{2m+1}, c_0) . Let $g_0 := e^{2f}g$ denote the Gauduchon metric of D and let T denote the g_0 -dual of the Lee form of D with respect to g_0 . According to Theorem 3.1, T is ∇^{g_0} -parallel. We first show that $T \equiv 0$, i.e. that the closed Weyl-Einstein connection D is actually exact.

Assume, for a contradiction, that T is non-zero. By rescaling the Gauduchon metric g_0 if necessary, we may assume that $g_0(T, T) = 1$. Denote by η , resp. η_0 , the 1-form dual to ξ with respect to g , resp. g_0 . Both η and η_0 are contact 1-forms for the contact distribution \mathcal{D} , and, as already noticed, ξ is the Reeb vector field of η . According to Proposition 2.1, ξ , which is Killing with respect to g , hence conformal Killing with respect to g_0 , is actually Killing with respect to g_0 as well, commutes with T , and the inner product $a := g_0(\xi, T) = \eta_0(T)$ is constant; we then have: $\mathcal{L}_T \eta_0 = 0$, hence that $T \lrcorner d\eta_0 = \mathcal{L}_T \eta_0 - d(\eta_0(T)) = -da = 0$. Moreover, since $\eta_0(T) = a$ and $T \lrcorner d\eta_0 = 0$, a cannot be zero — otherwise, η_0 would not be a contact 1-form — and $\xi_0 := a^{-1}T$ is then the Reeb vector

field of η_0 . Since $\eta_0 = e^{2f}\eta$, the Reeb vector fields ξ_0 and ξ are related by

$$(3.24) \quad \xi_0 = e^{-2f}\xi + Z_f,$$

where Z_f is the section of \mathcal{D} defined by

$$(3.25) \quad (Z_f \lrcorner d\eta)|_{\mathcal{D}} = 2e^{-2f}df|_{\mathcal{D}}.$$

Since M is compact, f has critical points and for each of them, say x , it follows from (3.25) that $Z_f(x) = 0$, hence $\xi_0(x) = a^{-1}T(x) = e^{-2f(x)}\xi(x)$. Since, $g_0(T(x), T(x)) = 1$ and $g_0(\xi(x), T(x)) = a$ for any x , we infer that $e^{2f(x)} = a^2$ for *any* critical point x of f , in particular for points where f takes its minimal or its maximal value. It follows that f is *constant*, with $e^{2f} \equiv a^2$, that $g_0 = a^2g$ and $\xi = aT$. In particular, η and $\eta_0 = a^2\eta$ are parallel, with respect to g and g_0 , hence closed. This contradicts the fact that they are contact 1-forms.

In view of the above, T must be identically zero. This means that D is the Levi-Civita connection of the Gauduchon metric g_0 , which is thus Einstein. Since ξ with respect to g , hence conformal Killing with respect to $g_0 = e^{2f}g$, and (M, c) is not isomorphic to the flat conformal sphere (\mathbb{S}^{2m+1}, c_0) , it follows from Proposition 2.2 that ξ is Killing with respect to g_0 as well. We thus have $df(\xi) = 0$, hence

$$(3.26) \quad g(\xi, \text{grad}_g f) = 0.$$

Let λ denote the Einstein constant of (M, g_0) , so that $\text{Ric}^0 = \lambda g_0 = e^{2f}\lambda g$. The classical formula relating the Ricci tensors Ric and Ric^0 of g and g_0 reads (cf. [2], p. 59):

$$(3.27) \quad \text{Ric}^0 = \text{Ric} - (2m-1)(\nabla^g df - df \otimes df) + (\Delta^g f - (2m-1)|df|_g^2)g.$$

Contracting (3.27) with ξ and using Proposition 3.2 we get

$$\lambda e^{2f}\eta = 2m\eta - (2m-1)\nabla_\xi^g df + (\Delta^g f - (2m-1)|df|_g^2)\eta.$$

Taking the metric duals with respect to g this equation reads

$$(3.28) \quad \nabla_\xi^g(\text{grad}_g f) = h\xi, \quad \text{with} \quad h := \frac{1}{2m-1}(\Delta^g f - (2m-1)|df|_g^2 + 2m - \lambda e^{2f}).$$

On the other hand, we have $0 = d\mathcal{L}_\xi f = \mathcal{L}_\xi df$, thus $\mathcal{L}_\xi(\text{grad}_g f) = 0$ and therefore

$$\nabla_\xi^g(\text{grad}_g f) = \nabla_{\text{grad}_g f}^g \xi = \varphi(\text{grad}_g f).$$

Since the image of φ is orthogonal to ξ , (3.28) implies that $\varphi(\text{grad}_g f) = 0$, thus by (3.2), $\text{grad}_g f$ is proportional to ξ . From (3.26) we thus get $\text{grad}_g f = 0$, so f is constant and D is the Levi-Civita connection of g , and hence g is Einstein. \square

As a direct corollary of Theorem 3.2 above together with Theorem 1.1 in [1] (see also [4]), we obtain the following result:

Corollary 3.1. *If (M^{2m+1}, g, ξ) is a compact K-contact manifold carrying a closed Weyl-Einstein structure compatible with g , then M is Sasaki-Einstein.*

Remark 3.2. In [12] it is claimed that if (M^{2m+1}, g, ξ) is a compact K -contact manifold carrying a compatible closed Weyl-Einstein structure, then M is Sasakian if and only if it is η -Einstein. Our above result show that the hypotheses in [12] already imply both conditions.

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