# KILLING AND CONFORMAL KILLING TENSORS

KONSTANTIN HEIL, ANDREI MOROIANU, UWE SEMMELMANN

Abstract. We introduce an appropriate formalism in order to study conformal Killing (symmetric) tensors on Riemannian manifolds. We reprove in a simple way some known results in the field and obtain several new results, like the classification of conformal Killing 2-tensors on Riemannian products of compact manifolds, Weitzenböck formulas leading to non-existence results, and construct various examples of manifolds with conformal Killing tensors.

2010 *Mathematics Subject Classification*: Primary: 53C25, 53C27, 53C40
*Keywords*: Killing tensors, conformal Killing tensors, Stäckel tensors.

## 1. Introduction

Killing $p$-tensors are symmetric $p$-tensors with vanishing symmetrized covariant derivative. This is a natural generalization of the Killing vector field equation for $p = 1$. Since many years Killing tensors, and more generally conformal Killing tensors, were intensively studied in the physics literature, e.g. in [25] and [30]. The original motivation came from the fact that symmetric Killing tensors define (polynomial) first integrals of the equations of motion, i.e. functions which are constant on geodesics. Conformal Killing tensors still define first integrals for null geodesics. Killing 2-tensors also appeared in the analysis of the stability of generalized black holes in $D = 11$ supergravity, e.g. in [11] and [22]. It turns out that trace-free Killing 2-tensors (also called Stäckel tensors) precisely correspond to the limiting case of a lower bound for the spectrum of the Lichnerowicz Laplacian on symmetric 2-tensors. More recently, Killing and conformal Killing tensors appeared in several other areas of mathematics, e.g. in connection with geometric inverse problems, integrable systems and Einstein-Weyl geometry, cf. [5], [8], [9], [13], [15], [23], [26].

Any parallel tensor is in particular a Killing tensor. The simplest non-parallel examples of Killing tensors can be constructed as symmetric products of Killing vector fields. For the standard sphere $\mathbb{S}^n$ there is a direct correspondence between Killing tensors and algebraic curvature tensors on $\mathbb{R}^{n+1}$. Other interesting examples are obtained as Ricci tensors of certain Riemannian manifolds, e.g. of natural reductive spaces.

*Date*: April 24, 2016.

The defining equation of trace-free conformal Killing tensors has the important property to be of finite type (or strongly elliptic). This leads to an explicit upper bound of the dimension of the space of conformal Killing tensors. In this respect, conformal Killing tensors are very similar to so-called conformal Killing forms, which were studied by the authors in several articles, e.g. [10], [20] and [27]. Moreover there is an explicit construction of Killing tensors starting from Killing forms, cf. Section 4.3 below.

The existing literature on symmetric Killing tensors is huge, especially coming from theoretical physics. One of the main obstacles in reading it is the old-fashioned formalism used in most articles in the subject.

In this article we introduce conformal Killing tensors in a modern, coordinate-free formalism. We use this formalism in order to reprove in a simpler way some known results, like Theorem 8.1 saying that the nodal set of a conformal Killing tensor has at least codimension 2, or Proposition 6.6 showing the non-existence of trace-free conformal Killing tensors on compact manifolds of negative sectional curvature. In addition we give a unified treatment of some subclasses of conformal Killing tensors, e.g. special conformal Killing tensors.

We obtain several new results, like the classification of Stäckel 2-tensors with at most two eigenvalues (which is also implicitly contained in the work of W. Jelonek, cf. [15]–[17]), or the description of conformal Killing 2-tensors on Riemannian products of compact manifolds (which turn out to be determined by Killing 2-tensors and Killing vector fields on the factors, cf. Theorem 5.1). We also prove a general Weitzenböck formula (Proposition 6.1) leading to non-existence results on certain compact manifolds.

Acknowledgments. This work was supported by the Procope Project No. 32977YJ. We are grateful to Mikko Salo who discovered an error in the proof of Proposition 6.6, and to Gregor Weingart who helped us to correct this error. We also thank the anonymous referee for having pointed out an error in the previous version of Theorem 5.1 and for several useful suggestions.

## 2. Preliminaries

Let $(V, g)$ be a Euclidean vector space of dimension $n$. We denote with $\mathrm{Sym}^p V \subset V^{\otimes p}$ the $p$-fold symmetric tensor product of $V$. Elements of $\mathrm{Sym}^p V$ are symmetrized tensor products

$$v_1 \cdot \ldots \cdot v_p := \sum_{\sigma \in S_p} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(p)} ,$$

where $v_1, \ldots, v_p$ are vectors in $V$. In particular we have $v \cdot u = v \otimes u + u \otimes v$ for $u, v \in V$. Using the metric $g$, one can identify $V$ with $V^*$. Under this identification, $g \in \mathrm{Sym}^2 V^* \simeq \mathrm{Sym}^2 V$ can be written as $g = \frac{1}{2} \sum e_i \cdot e_i$, for any orthonormal basis $\{e_i\}$.

The direct sum $\mathrm{Sym} V := \bigoplus_{p \geq 0} \mathrm{Sym}^p V$ is endowed with a commutative product making $\mathrm{Sym} V$ into a $\mathbb{Z}$-graded commutative algebra. The scalar product $g$ induces a scalar product, also denoted by $g$, on $\mathrm{Sym}^p V$ defined by

$$g(v_1 \cdot \ldots \cdot v_p, w_1 \cdot \ldots \cdot w_p) \;=\; \sum_{\sigma \in S_p} g(v_1, w_{\sigma(1)}) \cdot \ldots \cdot g(v_p, w_{\sigma(p)}) \;.$$

With respect to this scalar product, every element $K$ of $\mathrm{Sym}^p V$ can be identified with a symmetric $p$-linear map (i.e. a polynomial of degree $p$) on $V$ by the formula

$$K(v_1, \ldots, v_p) = g(K, v_1 \cdot \ldots \cdot v_p) \;.$$

For every $v \in V$, the metric adjoint of the linear map $v\cdot : \mathrm{Sym}^p V \to \mathrm{Sym}^{p+1} V,\ K \mapsto v \cdot K$ is the contraction $v \lrcorner : \mathrm{Sym}^{p+1} V \to \mathrm{Sym}^p V,\ K \mapsto v \lrcorner\, K$, defined by $(v \lrcorner\, K)(v_1, \ldots, v_{p-1}) = K(v, v_1, \ldots, v_{p-1})$. In particular we have $v \lrcorner\, u^p = p g(v, u) u^{p-1}, \forall\ v, u \in V$.

We introduce the linear map $\deg : \mathrm{Sym} V \to \mathrm{Sym} V$, defined by $\deg(K) = p\, K$ for $K \in \mathrm{Sym}^p V$. Then we have $\sum e_i \cdot e_i \lrcorner\, K = \deg(K)$, where $\{e_i\}$ as usual denotes an orthonormal frame. Note that if $K \in \mathrm{Sym}^p T$ is considered as a polynomial of degree $p$ then $v \lrcorner K$ corresponds to the directional derivative $\partial_v K$ and the last formula is nothing else than the well-known Euler theorem on homogeneous functions.

Contraction and multiplication with the metric $g$ defines two additional linear maps:

$$\Lambda : \mathrm{Sym}^p V \to \mathrm{Sym}^{p-2} V, \quad K \mapsto \sum e_i \lrcorner\, e_i \lrcorner\, K$$

and

$$\mathrm{L} : \mathrm{Sym}^{p-2} V \to \mathrm{Sym}^p V, \quad K \mapsto \sum e_i \cdot e_i \cdot K \;,$$

which are adjoint to each other. Note that $\mathrm{L}(1) = 2g$ and $\Lambda K = \mathrm{tr}(K)$ for every $K \in \mathrm{Sym}^2 V$. It is straightforward to check the following algebraic commutator relations

$$[\,\Lambda, \mathrm{L}\,] \;=\; 2n\,\mathrm{id} \;+\; 4\deg, \quad [\,\deg, \mathrm{L}\,] = 2\,\mathrm{L}, \quad [\,\deg, \Lambda\,] = -\,2\,\Lambda \;, \tag{1}$$

and for every $v \in V$:

$$[\Lambda,\, v\,\cdot\,] \;=\; 2\,v\,\lrcorner\,, \quad [\,v\,\lrcorner\,,\, \mathrm{L}\,] \;=\; 2\,v\cdot\,, \quad [\,\Lambda,\, v\,\lrcorner\,] \;=\; 0 \;=\; [\,\mathrm{L},\, v\cdot\,] \;. \tag{2}$$

For $V = \mathbb{R}^n$, the standard $\mathrm{O}(n)$-representation induces a reducible $\mathrm{O}(n)$-representation on $\mathrm{Sym}^p V$. We denote by $\mathrm{Sym}^p_0 V := \ker(\Lambda : \mathrm{Sym}^p V \to \mathrm{Sym}^{p-2} V)$ the space of trace-free symmetric $p$-tensors.

It is well known that $\mathrm{Sym}^p_0 \mathbb{R}^n$ is an irreducible $\mathrm{O}(n)$-representation and we have the following decomposition into irreducible summands

$$\mathrm{Sym}^p V \;\cong\; \mathrm{Sym}^p_0 V \;\oplus\; \mathrm{Sym}^{p-2}_0 V \;\oplus\; \ldots \;,$$

where the last summand in the decomposition is $\mathbb{R}$ for $p$ even and $V$ for $p$ odd. The summands $\mathrm{Sym}^{p-2i}_0 V$ are embedded into $\mathrm{Sym}^p V$ via the map $\mathrm{L}^i$. Corresponding to the decomposition above any $K \in \mathrm{Sym}^p V$ can be decomposed as

$$K \;=\; K_0 \;+\; \mathrm{L} K_1 \;+\; \mathrm{L}^2 K_2 \;+\; \ldots$$

with $K_i \in \mathrm{Sym}^{p-2i}_0 V$, i.e. $\Lambda K_i = 0$. We will call this decomposition the *standard decomposition* of $K$. In the following, the subscript 0 always denotes the projection of an element from $\mathrm{Sym}^p V$ onto its component in $\mathrm{Sym}^p_0 V$. Note that for any $v \in V$ and $K \in \mathrm{Sym}^p_0 V$ we have the following projection formula

$$(v \cdot K)_0 \;=\; v \cdot K \;-\; \tfrac{1}{n+2(p-1)}\, \mathrm{L}\,(v \lrcorner K) \;. \tag{3}$$

Indeed, using the commutator relation (1) we have $\Lambda(\mathrm{L}\,(v \lrcorner K)) = (2n + 4(p-1))\,(v \lrcorner K)$, since $\Lambda$ commutes with $v \lrcorner$ and $\Lambda K = 0$. Moreover $\Lambda(v \cdot K) = 2\, v \lrcorner K$. Thus the right-hand side of (3) is in the kernel of $\Lambda$, i.e. it computes the projection $(v \cdot K)_0$.

Recall the classical decomposition into irreducible $\mathrm{O}(n)$ representations

$$V \otimes \mathrm{Sym}^p_0 V \;\cong\; \mathrm{Sym}^{p+1}_0 V \;\oplus\; \mathrm{Sym}^{p-1}_0 V \;\oplus\; \mathrm{Sym}^{p,1} V \;, \tag{4}$$

where $V = \mathbb{R}^n$ is the standard $\mathrm{O}(n)$-representation of highest weight $(1, 0, \ldots, 0)$, $\mathrm{Sym}^p_0 V$ is the irreducible representation of highest weight $(p, 0, \ldots, 0)$ and $\mathrm{Sym}^{p,1} V$ is the irreducible representation of highest weight $(p, 1, 0, \ldots, 0)$. We note that $\mathrm{Sym}^{p+1}_0 V$ is the so-called Cartan summand. Its highest weight is the sum of the highest weights of $V$ and $\mathrm{Sym}^p_0 V$.

Next we want to describe projections and embeddings for the first two summands. The projection $\pi_1 : V \otimes \mathrm{Sym}^p_0 V \to \mathrm{Sym}^{p+1}_0 V$ onto the first summand is defined as

$$\pi_1(v \otimes K) \;:=\; (v \cdot K)_0 \;\overset{(3)}{=}\; v \cdot K \;-\; \tfrac{1}{n+2(p-1)}\, \mathrm{L}\,(v \lrcorner K) \;. \tag{5}$$

The adjoint map $\pi_1^* : \mathrm{Sym}^{p+1}_0 V \to V \otimes \mathrm{Sym}^p_0 V$ is easily computed to be $\pi_1^*(K) = \sum e_i \otimes (e_i \lrcorner K)$. Note that for any vector $v \in V$ the symmetric tensor $v \lrcorner K$ is again trace-free, because $v \lrcorner$ commutes with $\Lambda$. Since $\pi_1\, \pi_1^* = (p+1)\,\mathrm{id}$ on $\mathrm{Sym}^{p+1}_0 V$, we conclude that

$$p_1 \;:=\; \tfrac{1}{p+1}\, \pi_1^*\, \pi_1 : V \otimes \mathrm{Sym}^p_0 V \;\to\; \mathrm{Sym}^{p+1}_0 V \;\subset\; V \otimes \mathrm{Sym}^p_0 V \tag{6}$$

is the projection onto the irreducible summand of $V \otimes \mathrm{Sym}^p_0 V$ isomorphic to $\mathrm{Sym}^{p+1}_0 V$.

Similarly the projection $\pi_2 : V \otimes \mathrm{Sym}^p_0 V \to \mathrm{Sym}^{p-1}_0 V$ onto the second summand in the decomposition (4) is given by the contraction map $\pi_2(v \otimes K) := v \lrcorner K$. In this case the adjoint map $\pi_2^* : \mathrm{Sym}^{p-1}_0 V \to V \otimes \mathrm{Sym}^p_0 V$ is computed to be

$$\pi_2^*(K) \;=\; \sum e_i \otimes (e_i \cdot K)_0 \;=\; \sum e_i \otimes (e_i \cdot P - \tfrac{1}{n+2(p-2)}\, \mathrm{L}\,(e_i \lrcorner P)) \;.$$

It follows that $\pi_2\, \pi_2^* = (n + p - 1)\,\mathrm{id} \;-\; \frac{2(p-2)}{n+2(p-2)}\,\mathrm{id} = \frac{(n+2p-2)(n+p-3)}{n+2p-4}\,\mathrm{id}$ . Thus the projection onto the irreducible summand in $V \otimes \mathrm{Sym}^p_0 V$ isomorphic to $\mathrm{Sym}^{p-1}_0 V$ is given by

$$p_2 \;:=\; \tfrac{n+2p-4}{(n+2p-2)(n+p-3)}\, \pi_2^*\, \pi_2 : V \otimes \mathrm{Sym}^p_0 V \;\to\; \mathrm{Sym}^{p-1}_0 V \;\subset\; V \otimes \mathrm{Sym}^p_0 V \;. \tag{7}$$

The projection $p_3$ onto the third irreducible summand in $V \otimes \mathrm{Sym}^p_0 V$ is obviously given by $p_3 = \mathrm{id} - p_1 - p_2$ .

Let $(M^n, g)$ be a Riemannian manifold with Levi-Civita connection $\nabla$. All the algebraic considerations above extend to vector bundles over $M$, e.g. the $\mathrm{O}(n)$-representation $\mathrm{Sym}^p V$

defines the real vector bundle $\mathrm{Sym}^p\,\mathrm{T}M$. The $\mathrm{O}(n)$-equivariant maps L and $\Lambda$ define bundle maps between the corresponding bundles. The same is true for the symmetric product and the contraction $\iota$, as well as for the maps $\pi_1, \pi_2$ and their adjoints, and the projection maps $p_1, p_2, p_3$. We will use the same notation for the bundle maps on $M$.

Next we will define first order differential operators on sections of $\mathrm{Sym}^p\,\mathrm{T}M$. We have

$$\mathrm{d} : \Gamma(\mathrm{Sym}^p\,\mathrm{T}M) \to \Gamma(\mathrm{Sym}^{p+1}\,\mathrm{T}M), \quad K \mapsto \sum e_i \cdot \nabla_{e_i} K \ ,$$

where $\{e_i\}$ denotes from now on a local orthonormal frame. The formal adjoint of d is the divergence operator $\delta$ defined by

$$\delta : \Gamma(\mathrm{Sym}^{p+1}\,\mathrm{T}M) \to \Gamma(\mathrm{Sym}^p\,\mathrm{T}M), \quad K \mapsto -\sum e_i \lrcorner\, \nabla_{e_i} K \ ,$$

An immediate consequence of the definition is

**Lemma 2.1.** *The operator* d *acts as a derivation on the algebra of symmetric tensors, i.e. for any* $A \in \Gamma(\mathrm{Sym}^p\,\mathrm{T}M)$ *and* $B \in \Gamma(\mathrm{Sym}^q\,\mathrm{T}M)$ *the following equation holds*

$$\mathrm{d}(A \cdot B) \ = \ (\mathrm{d}A) \cdot B \ + \ A \cdot (\mathrm{d}B) \ .$$

An easy calculation proves that the operators d and $\delta$ satisfy the commutator relations:

$$[\,\Lambda,\, \delta\,] \ = \ 0 \ = \ [\,\mathrm{L},\, \mathrm{d}\,], \quad [\,\Lambda,\, \mathrm{d}\,] \ = \ -2\delta, \quad [\,\mathrm{L},\, \delta\,] \ = \ 2\,\mathrm{d} \ . \tag{8}$$

**Lemma 2.2.** *Let* $K = K_0 + \mathrm{L}K_1 + \ldots$ *be the standard decomposition of a section of* $\mathrm{Sym}^p\,\mathrm{T}M$, *where* $K_i \in \mathrm{Sym}^{p-2i}_0\,\mathrm{T}M$. *Then there exist real constants* $a_i$ *such that*

$$\mathrm{d}K_i - a_i \mathrm{L}\delta K_i \in \mathrm{Sym}^{p-2i+1}_0\,\mathrm{T}M \ .$$

*The constants are given explicitly by* $a_i := -\frac{1}{n+2(p-2i-1)}$. *In particular, if* $K$ *is a section of* $\mathrm{Sym}^p_0\,\mathrm{T}M$, *it holds that*

$$(\mathrm{d}K)_0 \ = \ \mathrm{d}K \ + \ \tfrac{1}{n+2p-2}\,\mathrm{L}\,\delta K \ . \tag{9}$$

*Proof.* We write $K = \sum_{i \geq 0} \mathrm{L}^i\, K_i$, where $K_i$ is a section of $\mathrm{Sym}^{p-2i}_0\,\mathrm{T}M$. Then $\mathrm{d}K_i - a_i\mathrm{L}\delta K_i$ is a section of $\mathrm{Sym}^{p-2i+1}_0\,\mathrm{T}M$ if and only if

$$0 \ = \ \Lambda(\mathrm{d}K_i - a_i \mathrm{L}\delta K_i) \ = \ -2\,\delta\,K_i \ - \ a_i\,(2n + 4(p-2i-1))\,\delta\,K_i \ .$$

Thus the constants $a_i$ are as stated above. In particular we have for $i = 0$ that the expression $\mathrm{d}K_0 + \frac{1}{n+2p-2}\mathrm{L}\delta K_0$ is trace-free. This proves the last statement. □

The operators d and $\delta$ can be considered as components of the covariant derivative $\nabla$ acting on sections of $\mathrm{Sym}^p\,\mathrm{T}M$. To make this more precise we first note that

$$\pi_1(\nabla K) \ = \ (\mathrm{d}K)_0 \qquad \text{and} \qquad \pi_2(\nabla K) \ = \ -\,\delta K \ , \tag{10}$$

which follows from $\nabla K = \sum e_i \otimes \nabla_{e_i} K$ and the definitions above.

Let $K$ be a section of $\mathrm{Sym}^p_0 \mathrm{T}M$. Then $\nabla K$ is a section of $\mathrm{T}M \otimes \mathrm{Sym}^p_0 \mathrm{T}M$ and we may decompose $\nabla K$ corresponding to (4), i.e. $\nabla K = P_1(K) + P_2(K) + P_3(K)$, where we use the notation $P_i(K) := p_i(\nabla K), i = 1, 2, 3$. Substituting the definition of the operators $P_i$ and applying the resulting equation to a tangent vector $X$ we obtain

$$\begin{aligned} \nabla_X K &= \tfrac{1}{p+1}\, \pi_1^*(\mathrm{d}K)_0(X) \;-\; \tfrac{n+2p-4}{(n+2p-2)(n+p-3)}\, \pi_2^*(\delta K)(X) \;+\; P_3(K)(X) \\ &= \tfrac{1}{p+1}\, X \lrcorner (\mathrm{d}K)_0 \;-\; \tfrac{n+2p-4}{(n+2p-2)(n+p-3)}\, (X \cdot \delta K)_0 \;+\; P_3(K)(X) \end{aligned}$$

Using (3) and (9) we rewrite the formula for $\nabla_X K$ in terms of $\mathrm{d}K$ and $\delta K$ and obtain

$$\begin{aligned} \nabla_X K &= \tfrac{1}{p+1}\, X \lrcorner \mathrm{d}K \;+\; \left(\tfrac{1}{(n+2p-2)(p+1)} + \tfrac{1}{(n+2p-2)(n+p-3)}\right) \mathrm{L}\,(X \lrcorner \delta K) \\ &\qquad + \left(\tfrac{2}{(n+2p-2)(p+1)} - \tfrac{n+2p-4}{(n+2p-2)(n+p-3)}\right) X \cdot \delta K \;+\; P_3(K)(X) \\ &= \tfrac{1}{p+1}\, X \lrcorner \mathrm{d}K \;+\; \tfrac{1}{(p+1)(n+p-3)}\, \mathrm{L}\,(X \lrcorner \delta K) \;-\; \tfrac{p-1}{(p+1)(n+p-3)}\, X \cdot \delta K \;+\; P_3(K)(X)\;. \end{aligned}$$

Here we applied the commutator formula $X \lrcorner \mathrm{L}K = \mathrm{L}(X \lrcorner K) + 2X \cdot K$. For later use we still note the formulas

$$P_1(K)(X) \;=\; \tfrac{1}{p+1}\, X \lrcorner (\mathrm{d}K)_0 \qquad \text{and} \qquad P_2(K)(X) \;=\; -\tfrac{n+2p-4}{(n+2p-2)(n+p-3)}\, (X \cdot \delta K)_0\;.$$

At the end of this section we want to clarify the relations between $\mathrm{d}, \delta$ and $P_1, P_2$. For convenience we introduce the notation $\mathrm{d}_0 K := (\mathrm{d}K)_0$. The relation between $\mathrm{d}_0$ and $\mathrm{d}$ is given in (9). An easy calculation shows that $\delta^* \;=\; \mathrm{d}_0$ .

**Lemma 2.3.** *On sections of* $\mathrm{Sym}^p_0 \mathrm{T}M$ *the following equations hold:*

$$\mathrm{d}_0^*\, \mathrm{d}_0 \;=\; (p+1)\, P_1^* P_1 \qquad \textit{and} \qquad \delta^*\, \delta \;=\; \tfrac{(n+2p-2)(n+p-3)}{n+2p-4}\, P_2^* P_2\;.$$

*Proof.* Let $EM$ be a vector bundle associated to the frame bundle via a $\mathrm{SO}(n)$ representation E. The Levi-Civita connection induces a covariant derivative $\nabla$ acting on sections of $EM$. If T denotes the tangent representation, defining the tangent bundle, we have a decomposition into irreducible summands: $E \otimes \mathrm{T} = E_1 \oplus \ldots \oplus E_N$. Here the spaces $E_i$ are subspaces of $E \otimes \mathrm{T}$ but often they appear also in other realizations, like the spaces $\mathrm{Sym}^{p+1}_0 \mathrm{T}$ and $\mathrm{Sym}^{p-1}_0 \mathrm{T}$ in the decomposition of $\mathrm{Sym}^p_0 \mathrm{T} \otimes \mathrm{T}$ considered above.

Assume that $\tilde{E}_i$ are $\mathrm{SO}(n)$-representations isomorphic to $E_i$ and that $\pi_i : E \otimes \mathrm{T} \to \tilde{E}_i$ are representation morphisms with $\pi_i \circ \pi_i^* = c_i\, \mathrm{id}$ for some non-zero constants $c_i$. Then we can define projections $p_i : E \otimes \mathrm{T} \to E_i \subset E \otimes \mathrm{T}$ as above by $p_i := \frac{1}{c_i}\, \pi_i^* \pi_i$. From the condition on $\pi_i$ we obtain $p_i^2 = p_i^* \circ p_i = p_i$. Now we define two sets of operators on sections of $EM$: $\mathrm{d}_i := \pi_i \circ \nabla : \Gamma(EM) \to \Gamma(\tilde{E}_i M)$ and $P_i := p_i \circ \nabla : \Gamma(EM) \to \Gamma(EM \otimes \mathrm{T}M)$.

We then have the general formula $\mathrm{d}_i^* \, \mathrm{d}_i \;=\; c_i \, P_i^* \, P_i$ . Indeed we have

$$\mathrm{d}_i^* \, \mathrm{d}_i \;=\; \nabla^* \, \pi_i^* \, \pi_i \nabla \;=\; c_i \, \nabla^* \, p_i \, \nabla \;=\; c_i \, \nabla^* \, p_i^* \, p_i \, \nabla \;=\; c_i \, P_i^* \, P_i \; .$$

The statement of the lemma now follows from (10) together with (6)–(7). □

## 3. Basics on Killing and conformal Killing tensors

**Definition 3.1.** A symmetric tensor $K \in \Gamma(\mathrm{Sym}^p \, \mathrm{T}M)$ is called *conformal Killing tensor* if there exists some symmetric tensor $k \in \Gamma(\mathrm{Sym}^{p-1} \, \mathrm{T}M)$ with $\mathrm{d}K = \mathrm{L}(k)$.

**Lemma 3.2.** *The defining equation for conformal Killing tensors is conformally invariant. More precisely, a section $K$ of $\mathrm{Sym}^p \, \mathrm{T}M$ is a conformal Killing tensor with respect to the metric $g$, if and only if it is a conformal Killing tensor with respect to every conformally related metric $g' = e^{2f} g$.*

*Proof.* Let $X, Y$ be any vector fields. Then the Levi-Civita connection $\nabla'$ for $g'$ is given by

$$\nabla'_X Y = \nabla_X Y + \mathrm{d}f(X) Y + \mathrm{d}f(Y) X - g(X, Y) \, \mathrm{grad}_g(f)$$

where $\mathrm{grad}_g(f)$ is the gradient of $f$ with respect to $g$ (cf. [2], Th. 1.159). It immediately follows that for any section $K$ of $\mathrm{Sym}^p \, \mathrm{T}M$ we have

$$\nabla'_X K \;=\; \nabla_X K \;+\; p \, \mathrm{d}f(X) \, K \;+\; X \cdot \mathrm{grad}_g(f) \lrcorner K \;-\; \mathrm{grad}_g(f) \cdot X \lrcorner K \; .$$

Hence we obtain for the differential $\mathrm{d}'K = \sum_i e'_i \cdot \nabla'_{e'_i} K = e^{-2f} \sum_i e_i \cdot \nabla'_{e_i} K$ the equation

$$\begin{aligned} \mathrm{d}'K \;&=\; e^{-2f} (\mathrm{d}K + p \, \mathrm{grad}_g(f) \cdot K + \mathrm{L} \, (\mathrm{grad}_g(f) \lrcorner K) - p \, \mathrm{grad}_g(f) \cdot K) \\ &=\; e^{-2f} \mathrm{d}K + \mathrm{L}' \, (\mathrm{grad}_g(f) \lrcorner K) \; . \end{aligned}$$

Hence if $K$ is conformal Killing tensor with respect to the metric $g$, i.e. $\mathrm{d}K = \mathrm{L}k$ for some section $k$ of $\mathrm{Sym}^{p-2} \, \mathrm{T}M$, then $K$ is a conformal Killing tensor with respect to the metric $g'$, too. Indeed $\mathrm{d}'K = \mathrm{L}'(k + \mathrm{grad}_g(f) \lrcorner K)$. □

Note that $K$ is conformal Killing if and only if its trace-free part is conformal Killing. Indeed, since d and L commute, if $K = \sum_{i \geq 0} \mathrm{L}^i \, K_i$, with $K_i \in \Gamma(\mathrm{Sym}_0^{p-2i} \, \mathrm{T}M)$ is the standard decomposition of $K$, then $\mathrm{d}K = \sum_{i \geq 0} \mathrm{L}^i \, \mathrm{d}K_i$, so $\mathrm{d}K$ is in the image of L if and only if $\mathrm{d}K_0$ is in the image of L. It is thus reasonable to consider only trace-free conformal Killing tensors.

**Lemma 3.3.** *Let $K \in \Gamma(\mathrm{Sym}^p \, \mathrm{T}M)$, then $K$ is a conformal Killing tensor if and only if*

$$\mathrm{d}K_0 \;=\; -\tfrac{1}{n+2p-2} \, \mathrm{L} \, \delta K_0$$

*or, equivalently, if and only if $(\mathrm{d}K_0)_0 = 0$. In particular, a trace-free symmetric tensor $K \in \Gamma(\mathrm{Sym}_0^p \, \mathrm{T}M)$ is a conformal Killing tensor if and only if $P_1(K) = 0$.*

*Proof.* We write $K = \sum_{i\geq 0} \mathrm{L}^i K_i$, where $K_i$ is a section of $\mathrm{Sym}_0^{p-2i}\,\mathrm{T}M$. Because of $[\mathrm{L},\mathrm{d}]=0$ we have

$$\mathrm{d}K = \sum_{i\geq 0} \mathrm{L}^i\,\mathrm{d}\,K_i = \mathrm{d}\,K_0 + \sum_{i\geq 1}\mathrm{L}^i \mathrm{d}K_i = (\mathrm{d}K_0 + \tfrac{1}{n+2p-2}\,\mathrm{L}\,\delta\,K_0) - \tfrac{1}{n+2p-2}\,\mathrm{L}\,\delta\,K_0 + \sum_{i\geq 1}\mathrm{L}^i \mathrm{d}K_i$$

We know from (9) that the bracket on the right hand-side is the trace-free part of $\mathrm{d}K_0$. Thus $\mathrm{d}K_0 = -\frac{1}{n+2p-2}\,\mathrm{L}\,\delta K_0$ holds if and only if $\mathrm{d}K = \mathrm{L}(k)$ for some section $k$ of $\mathrm{Sym}^{p-1}\,\mathrm{T}M$, i.e. if and only if $K$ is a conformal Killing tensor. □

**Remark 3.4.** Since $P_1(K)$ is the projection of the covariant derivative $\nabla K$ onto the Cartan summand $\mathrm{Sym}_0^{p+1}\,\mathrm{T}M \subset \mathrm{T}M\otimes \mathrm{Sym}_0^p\,\mathrm{T}M$, it follows that the defining differential equation for trace-free conformal Killing tensors is of finite type, also called strongly elliptic (cf. [18]). In particular the space of trace-free conformal Killing tensors is finite dimensional on any connected manifold. Moreover one can show that a conformal Killing $p$-tensor has to vanish identically on $M$ if its first $2p$ covariant derivatives vanish in some point of $M$, cf. [7].

**Definition 3.5.** A symmetric tensor $K\in\Gamma(\mathrm{Sym}^p\,\mathrm{T}M)$ is called *Killing tensor* if $\mathrm{d}K=0$ . A trace-free Killing tensor is called *Stäckel tensor.*

**Lemma 3.6.** *A symmetric tensor $K\in\Gamma(\mathrm{Sym}^p\,\mathrm{T}M)$ is a Killing tensor if and only if the complete symmetrization of $\nabla K$ vanishes. A Killing tensor is in particular a conformal Killing tensor.*

*Proof.* Recall that $\mathrm{d}:\Gamma(\mathrm{Sym}^p\,\mathrm{T}M)\to\Gamma(\mathrm{Sym}^{p+1}\,\mathrm{T}M)$ was defined as $\mathrm{d}K = \sum_i e_i\cdot\nabla_{e_i}K$, where $\{e_i\}$ is some local orthonormal frame. Thus

$$\begin{aligned}\mathrm{d}K(X_1,\ldots,X_{p+1}) &= \sum_i\sum_{\sigma\in S_p} g(e_i,X_{\sigma(1)})\,(\nabla_{e_i}K)(X_{\sigma(2)},\ldots,X_{\sigma(p+1)})\\ &= \sum_{\sigma\in S_p}(\nabla_{X_{\sigma(1)}}K)(X_{\sigma(2)},\ldots,X_{\sigma(p+1)})\ .\end{aligned}$$

□

**Proposition 3.7.** *Let $K=\sum_{i\geq 0}\mathrm{L}^iK_i\in\Gamma(\mathrm{Sym}^p\,\mathrm{T}M)$, with $K_i\in\Gamma(\mathrm{Sym}_0^{p-2i}\,\mathrm{T}M)$ be a symmetric tensor, $p=2l+\epsilon$ with $\epsilon=0$ or $1$. Then $K$ is a Killing tensor if and only if the following system of equations holds:*

$$\begin{aligned}\mathrm{d}K_0 &= a_0\mathrm{L}\delta K_0\ ,\\ \mathrm{d}K_1 &= a_1\mathrm{L}\delta K_1 - a_0\delta K_0\ ,\\ \vdots\ &= \qquad\vdots\\ \mathrm{d}K_l &= a_l\mathrm{L}\delta K_l - a_{l-1}\delta K_{l-1}\ ,\\ 0 &= \delta K_l\ ,\end{aligned}$$

*where $a_i$ are the constants of Lemma 2.2. In particular, the trace-free part $K_0$ is a conformal Killing tensor.*

*Proof.* We write $K = \sum_{i\geq 0} \mathrm{L}^i K_i$ with $K_i \in \Gamma(\mathrm{Sym}_0^{p-2i}\,\mathrm{T}M)$, then $\mathrm{d}K = 0$ if and only if

$$0 = \sum_{i\geq 0} \mathrm{L}^i\, \mathrm{d}K_i = \sum_{i\geq 0} \mathrm{L}^i\,(\mathrm{d}K_i - a_i \mathrm{L}\delta K_i) + a_i \mathrm{L}^{i+1}\delta K_i = \sum_{i\geq 0} \mathrm{L}^i(\mathrm{d}K_i - a_i\mathrm{L}\delta K_i + a_{i-1}\delta K_{i-1})\ ,$$

where we set $a_{-1} = 0$ and $K_{-1} = 0$ by convention. From Lemma 2.2 and $[\Lambda, \delta] = 0$ it follows that $\mathrm{d}K_i - a_i\mathrm{L}\delta K_i + a_{i-1}\delta K_{i-1}$ is trace-free. We conclude that $\mathrm{d}K = 0$ if and only if $\mathrm{d}K_i - a_i\mathrm{L}\delta K_i + a_{i-1}\delta K_{i-1} = 0$ for all $i$. □

**Example 3.8.** For $p = 2$ and $K \in \Gamma(\mathrm{Sym}^2\,\mathrm{T}M)$ we have $K = K_0 + 2fg$, for some function $f = K_1 \in C^\infty(M)$. Then $K$ is a Killing tensor if and only if

$$\mathrm{d}K_0 \;=\; -\tfrac{1}{n+2}\,\mathrm{L}\,\delta\,K_0 \qquad \text{and} \qquad \mathrm{d}f \;=\; \tfrac{1}{n+2}\,\delta\,K_0\ . \tag{11}$$

The second equation can equivalently be written as $\mathrm{d}\,\mathrm{tr}K = 2\delta K$.

**Example 3.9.** For $p = 3$ and $K \in \Gamma(\mathrm{Sym}^3\,\mathrm{T}M)$ we have $K = K_0 + \mathrm{L}\xi$, for some vector field $\xi = K_1$. Then $K$ is a Killing tensor if and only if

$$\mathrm{d}K_0 = -\tfrac{1}{n+4}\,\mathrm{L}\delta K_0\ , \quad \mathrm{d}\xi = \tfrac{1}{n+4}\,\delta K_0 \quad \text{and} \quad \delta\xi = 0\ .$$

**Corollary 3.10.** *If $K \in \Gamma(\mathrm{Sym}^p\,\mathrm{T}M)$ is a trace-free Killing tensor, then $\delta K = 0$. In other words, Stäckel tensors satisfy the equations $\mathrm{d}K = 0 = \delta K$, or equivalently the equations $P_1(K) = 0 = P_2(K)$.*

Conversely we may ask what is possible to say about the components of divergence-free Killing tensors. The result here is

**Proposition 3.11.** *Let $K \in \Gamma(\mathrm{Sym}^p\,\mathrm{T}M)$ be a divergence-free Killing tensor. Then all components $K_i \in \Gamma(\mathrm{Sym}_0^{p-2i}\,\mathrm{T}M)$ of its standard decomposition $K = \sum_{i\geq 0} \mathrm{L}^i\, K_i$ are Stäckel tensors.*

*Proof.* We first remark that iteration of the commutator formula (8) gives $[\delta, \mathrm{L}^i] = -2i\mathrm{L}^{i-1}\mathrm{d}$ for $i \geq 1$. Then $\delta K = 0$ implies $0 = \delta K_0 + \sum_{i\geq 1} \delta \mathrm{L}^i K_i = \delta K_0 - 2\mathrm{d}K_1 + \mathrm{L}k_1$ for some symmetric tensor $k_1$. Substituting $\mathrm{d}K_1$ by the second equation of Proposition 3.7 we obtain $0 = (1 + 2a_0)\delta K_0 + \mathrm{L}k_2$ for some symmetric tensor $k_2$. Since $\delta K_0$ is trace-free and since the coefficient $(1 + 2a_0)$ is different from zero for $n > 2$ it follows that $\delta K_0 = 0$. But then the first equation of Proposition 3.7 gives $\mathrm{d}K_0 = 0$. Thus $K_0$ is a Stäckel tensor.

We write $K = K_0 + \mathrm{L}\tilde{K}$ with $\tilde{K} = \sum_{i\geq 1} \mathrm{L}^{i-1} K_i$. Since d commutes with L and multiplication with L is injective the equations $\mathrm{d}K = 0 = \mathrm{d}K_0$ imply $\mathrm{d}\tilde{K} = 0$. Similarly we obtain $0 = \delta(\mathrm{L}\,\tilde{K}) = \mathrm{L}\delta\tilde{K} - 2\mathrm{d}\tilde{K} = \mathrm{L}\delta\tilde{K}$. Thus $\delta\tilde{K} = 0$ and we can iterate the argument above. □

**Remark 3.12.** The above proof shows that the map L preserves the space of divergence-free Killing tensors.

In [26] the class of *special conformal Killing tensors* was introduced. These tensors were defined as symmetric 2-tensors $K \in \Gamma(\mathrm{Sym}^2 \mathrm{T}M)$ satisfying the equation

$$(\nabla_X K)(Y,Z) \;=\; k(Y)\, g(X,Z) \;+\; k(Z)\, g(X,Y) \tag{12}$$

for some 1-form $k$. It follows that $k = \frac{1}{2}\, \mathrm{d}\, \mathrm{tr}(K)$ and that $\mathrm{d}K = \mathrm{L}(k)$. Thus solutions of Equation (12) are automatically conformal Killing tensors. The tensor $k$ also satisfies $\delta K = -(n+1)\, k$ and it is easily proved that $\hat{K} := K - \mathrm{tr}(K)\, g$ is a Killing tensor, which is called *special Killing tensor* in [26]. Moreover the map $K \mapsto \hat{K}$ is shown to be injective and equivariant with respect to the action of the isometry group.

We will now generalize these definitions and statements to Killing tensors of arbitrary degree. We start with

**Definition 3.13.** A symmetric tensor $K \in \Gamma(\mathrm{Sym}^p \mathrm{T}M)$ is called *special conformal Killing tensor* if the equation $\nabla_X K = X \cdot k$ holds for all vector fields $X$ and some symmetric tensor $k \in \Gamma(\mathrm{Sym}^{p-1} \mathrm{T}M)$.

For $p = 2$, this is equivalent to Equation (12). Immediately from the definition it follows that $k = -\frac{1}{n+1} \delta K$ and that $\mathrm{d}K = \mathrm{L}k$. Hence $K$ is in particular a conformal Killing tensor. Using the standard decomposition $K = \sum_{j \geq 0} \mathrm{L}^j\, K_j$ and $k = \sum_{j \geq 0} \mathrm{L}^j\, k_j$ we can reformulate the defining equation for special conformal Killing tensors into a system of equations for the components $K_j$ and $k_j$. Let $K$ and $k$ be symmetric tensors as above then $\nabla_X K = \sum_{j \geq 0} \mathrm{L}^j\, \nabla_X K_j$ and by (3) we have:

$$X \cdot k \;=\; \sum_{j \geq 0} \mathrm{L}^j\, (X \cdot k_j) \;=\; \sum_{j \geq 0} \Big( \mathrm{L}^j\, (X \cdot k_j)_0 \;+\; \tfrac{1}{n+2(p-2-2j)} \mathrm{L}^{j+1} (\, X \lrcorner\, k_j) \Big) \;.$$

Comparing coefficients of powers of L, we conclude that $K$ is a special conformal Killing tensor if and only if the following system of equations is satisfied

$$\nabla_X K_j \;=\; (X \cdot k_j)_0 \;+\; \tfrac{1}{n+2(p-2j)}\, X \lrcorner\, k_{j-1}, \qquad j \geq 0 \;. \tag{13}$$

With the convention $k_{-1} = 0$ this contains the equation $\nabla_X K_0 = (X \cdot k_0)_0$ for $j = 0$.

**Definition 3.14.** A symmetric tensor $\hat{K} \in \Gamma(\mathrm{Sym}^p \mathrm{T}M)$ is called *special Killing tensor* if it is a Killing tensor satisfying the additional equation $\nabla_X \hat{K} = X \cdot \hat{k} + X \lrcorner\, \hat{l}$ for all vector fields $X$ and some symmetric tensors $\hat{k} \in \Gamma(\mathrm{Sym}^{p-1} \mathrm{T}M)$ and $\hat{l} \in \Gamma(\mathrm{Sym}^{p+1} \mathrm{T}M)$.

From the definition it follows directly that the tensors $\hat{k}$ and $\hat{l}$ are related by the equations

$$\hat{l} \;=\; -\tfrac{1}{p+1} \mathrm{L}\hat{k} \qquad \text{and} \qquad \delta \hat{K} = -(n+p-1)\hat{k} - \Lambda \hat{l} \;.$$

Hence $\hat{K}$ is a special Killing tensor if and only if for all vector fields $X$ we have

$$\nabla_X \hat{K} \;=\; X \cdot \hat{k} - \tfrac{1}{p+1} X \lrcorner \mathrm{L} \hat{k} \;=\; \tfrac{p-1}{p+1} X \cdot \hat{k} - \tfrac{1}{p+1} \mathrm{L} X \lrcorner \hat{k}.$$

As for special conformal Killing tensors we can reformulate the defining equation for special Killing tensors as a system of equations for the components $\hat{K}_j$ and $\hat{k}_j$. We find

$$\begin{aligned}
\sum_{j\geq 0} \mathrm{L}^j \nabla_X \hat{K}_j &= \sum_{j\geq 0} \tfrac{p-1}{p+1}\, X \cdot \mathrm{L}^j\, \hat{k}_j \;-\; \tfrac{1}{p+1}\, \mathrm{L}\, X \lrcorner \mathrm{L}^j\, \hat{k}_j \\
&= \sum_{j\geq 0} \tfrac{p-1-2j}{p+1}\, \mathrm{L}^j\, X \cdot \hat{k}_j \;-\; \tfrac{1}{p+1}\, \mathrm{L}^{j+1} X \lrcorner \hat{k}_j \\
&= \sum_{j\geq 0} \tfrac{p-1-2j}{p+1}\, \mathrm{L}^j\, (X \cdot \hat{k}_j)_0 \;+\; \left( \tfrac{p-1-2j}{(p+1)(n+2(p-2j-2))} \;-\; \tfrac{1}{p+1} \right) \mathrm{L}^{j+1} X \lrcorner \hat{k}_j \\
&= \sum_{j\geq 0} \tfrac{p-1-2j}{p+1}\, \mathrm{L}^j\, (X \cdot \hat{k}_j)_0 \;-\; \tfrac{n+p-2j-3}{(p+1)(n+2(p-2j-2))}\, \mathrm{L}^{j+1} X \lrcorner \hat{k}_j
\end{aligned}$$

Hence $\hat{K}$ is a special Killing tensor if and only if the components $\hat{K}_j$ and $\hat{k}_j$ of the standard decomposition satisfy the equations

$$\nabla_X \hat{K}_j \;=\; \tfrac{p-1-2j}{p+1}\, (X \cdot \hat{k}_j)_0 \;-\; \tfrac{n+p-2j-1}{(p+1)(n+2(p-2j))}\, X \lrcorner \hat{k}_{j-1} \qquad j \geq 0 \;. \tag{14}$$

With the convention $\hat{k}_{-1} = 0$ we have for $j = 0$ the equation $\nabla_X \hat{K}_0 = \frac{p-1}{p+1}(X \cdot \hat{k}_0)_0$.

For a given special conformal Killing tensor $K$ we now want to modify its components $K_j$ by scalar factors in order to obtain a special Killing tensor $\hat{K}$. This will generalize the correspondence between special conformal and special Killing 2-tensors obtained in [26].

We are looking for constants $a_j$, such that $\hat{K} = \sum_{j\geq 0} \hat{K}_j$ with $\hat{K}_j = a_j K_j$ is a special Killing tensor, where $\hat{k} = \sum_{j\geq 0} \hat{k}_j$ with $\hat{k}_j = b_j k_j$ for a other set of constants $b_j$. Considering first the special Killing equation for $j = 0$ we have $\nabla_X a_0 K_0 = \frac{p-1}{p+1}(X \cdot b_0 k_0)_0$. Hence we can define $a_0 = 1$ and $b_0 = \frac{p+1}{p-1}$. Then writing (14) with the modified tensors $\hat{K}_j$ and $\hat{k}_j$ and comparing it with (13) multiplied by $a_j$, we get the condition $a_j = \frac{p-1-2j}{p+1} b_j$ for the first summand on the right hand side and

$$\tfrac{a_j}{n+2(p-2j)} \;=\; -\,\tfrac{(n+p-2j-1)}{(p+1)(n+2(p-2j))}\, b_{j-1} \;=\; -\,\tfrac{(n+p-2j-1)}{(p+1)(n+2(p-2j))}\, \tfrac{p+1}{p+1-2j}\, a_{j-1} \;,$$

for the second summand. Hence $a_j = -\,\frac{n+p-2j-1}{p+1-2j}\, a_{j-1}$ and in particular $a_1 = -\,\frac{n+p-3}{p-1}$. By this recursion formula the coefficients $a_j$ and $b_j$ are completely determined and indeed $\hat{K}$ defined as above is a special Killing tensor.

As an example we consider the case $p = 2$. Let $K = K_0 + \mathrm{L}K_1$ be a special conformal Killing 2-tensor. Then $\hat{K} = K_0 + a_1\mathrm{L}K_1 = K_0 - (n-1)\mathrm{L}K_1 = K - n\mathrm{L}K_1 = K - \mathrm{tr}(K)g$. Indeed $\mathrm{tr}(K) = \Lambda K = 2nK_1$ and $\mathrm{L} = 2g$. Hence we obtain for special conformal Killing 2-tensors the same correspondence as in [26].

Special conformal Killing 2-tensors have the important property of being integrable. Indeed their associated Nijenhuis tensor vanishes (cf. [26], Prop. 6.5). Recall that if $A$ is any endomorphism field on $M$, its *Nijenhuis tensor* is defined by

$$\begin{aligned} N_A(X,Y) &= -A^2[X,Y] + A[AX,Y] + A[X,AY] - [AX,AY] \\ &= A(\nabla_X A)Y - A(\nabla_Y A)X - (\nabla_{AX}A)Y + (\nabla_{AY}A)X . \end{aligned}$$

**Lemma 3.15.** *Any special conformal Killing 2-tensor has vanishing Nijenhuis tensor and thus is integrable.*

*Proof.* Let $A$ be a special conformal Killing 2-tensor. Then $\nabla_X A = X \cdot \xi$ for some vector field $\xi$. Then $(\nabla_X A)Y = g(X,Y)\,\xi + g(\xi,Y)X$ and it follows

$$\begin{aligned} N_A(X,Y) &= g(X,Y)\,A\xi + g(\xi,Y)AX - g(Y,X)\,A\xi - g(\xi,X)AY \\ &\quad - g(AX,Y)\,\xi - g(\xi,Y)\,AX + g(AY,X)\,\xi + g(\xi,X)\,AY = 0 . \end{aligned}$$

□

It is easy to check that special Killing 2-tensors associated to special conformal Killing tensors as above, have non-vanishing Nijenhuis tensor, therefore the statement in [26], Prop. 6.5 is not valid for special Killing tensors.

From Proposition 3.7 it follows that Stäckel tensors are characterized among trace-free tensors by the vanishing of the two operators $P_1$ and $P_2$. It is natural to consider trace-free symmetric tensors with other vanishing conditions. Here we mention two other cases:

**Lemma 3.16.** *For a trace-free symmetric tensor $K \in \Gamma(\mathrm{Sym}^p_0\,\mathrm{T}M)$ the relations $P_1(K) = 0$ and $P_3(K) = 0$ hold if and only if there exists a section $k$ of $\mathrm{Sym}^{p-1}\,\mathrm{T}M$ with $\nabla_X K = (X \cdot k)_0$. In this case $k$ is uniquely determined and $\nabla_X K = -\frac{n+2p-4}{(n+2p-2)(n+p-3)}\,(X \cdot \delta K)_0$.*

It follows from (13) for $j = 0$ that the trace-free part $K_0$ of a special conformal Killing tensor $K$ satisfies $P_1(K_0) = 0$ and $P_3(K_0) = 0$.

For $p = 2$ consider a section $K_0$ of $\mathrm{Sym}^2_0\,\mathrm{T}M$ with $P_1(K_0) = 0$ and $P_3(K_0) = 0$. Then there exists a 1-form $k$ with $\nabla_X K_0 = (X \cdot k)_0 = X \cdot k - \frac{1}{n+2p-4}\,\mathrm{L}(X \lrcorner\, k) = X \cdot k - \frac{2}{n}\,k(X)\,g$. Substituting two tangent vectors $Y$ and $Z$ we obtain the equation

$$(\nabla_X K_0)(Y,Z) = g(X,Y)\,k(Z) + g(X,Z)\,k(Y) - \tfrac{2}{n}\,k(X)\,g(Y,Z) . \tag{15}$$

For the sake of completeness, we also consider the case of trace-free symmetric tensors $K \in \Gamma(\mathrm{Sym}^p_0\,\mathrm{T}M)$ satisfying the vanishing conditions $P_2(K) = 0 = P_3(K)$. Equivalently the covariant derivative $\nabla K$ is completely symmetric, i.e. $\nabla K \in \Gamma(\mathrm{Sym}_0^{p+1} \mathrm{T}M)$. This can also be written as $(\nabla_X K)(Y, \ldots) = (\nabla_Y K)(X, \ldots)$ for all tangent vectors $X$, $Y$ or as $\mathrm{d}^\nabla K = 0$, where $K$ is considered as a 1-form with values in $\mathrm{Sym}_0^{p-1} \mathrm{T}M$, i.e. a section of $\mathrm{T}^*M \otimes \mathrm{Sym}_0^{p-1} \mathrm{T}M$. For $p = 2$, a tensor $K$ with $\mathrm{d}^\nabla K = 0$ is called a *Codazzi tensor.*

## 4. Examples of manifolds with Killing tensors

Any parallel tensor is tautologically a Killing tensor. By the conformal invariance of the conformal Killing equation, a parallel tensor defines conformal Killing tensors for any conformally related metric. These are in general no Killing tensors. There are several explicit constructions of symmetric Killing tensors which we will describe in the following subsections.

4.1. **Killing tensors on the sphere.** Let $\mathrm{Curv}(n{+}1)$ denote the space of algebraic curvature tensors on $\mathbb{R}^{n+1}$. Then any symmetric Killing 2-tensor $K$ on $\mathbb{S}^n$ is given in a point $p \in \mathbb{S}^n$ by $K(X, Y) = R(X, p, p, Y)$ for some algebraic curvature tensor $R \in \mathrm{Curv}(n + 1)$. The subspace of Weyl curvature tensors in $\mathrm{Curv}(n + 1)$ corresponds to the trace-free (and hence divergence-free) Killing tensors on $\mathbb{S}^n$, cf. [19].

The dimension of the space of Killing tensors on $\mathbb{S}^n$ gives an upper bound for this dimension on an arbitrary Riemannian manifold [29], Theorem 4.7.

4.2. **Symmetric products of Killing tensors.** Let $(M^n, g)$ be a Riemannian manifold with two Killing vector fields $\xi_1, \xi_2$. We define a symmetric 2-tensor $h$ as the symmetric product $h := \xi_1 \cdot \xi_2$. Then $h$ is a Killing tensor. Indeed $\mathrm{d}(\xi_1 \cdot \xi_2) = (\mathrm{d}\xi_1) \cdot \xi_2 + \xi_1 \cdot (\mathrm{d}\xi_2) = 0$ holds because of Lemma 2.1. More generally the symmetric product of Killing tensors defines again a Killing tensor. Conversely, it is known that on manifolds of constant sectional curvature, any Killing tensor can be written as a linear combination of symmetric products of Killing vector fields, cf. [29], Theorem 4.7.

**Lemma 4.1.** *For any two Killing vector fields $\xi_1, \xi_2$ it holds that*

$$\delta(\xi_1 \cdot \xi_2) \;=\; \mathrm{d}\, g(\xi_1, \xi_2) \;.$$

*Proof.* We compute $\delta(\xi_1 \cdot \xi_2) = -\sum e_i \lrcorner \nabla_{e_i}(\xi_1 \cdot \xi_2) = -\sum e_i \lrcorner ((\nabla_{e_i}\xi_1) \cdot \xi_2 + \xi_1 \cdot (\nabla_{e_i}\xi_2))$. Since the Killing vector fields $\xi_1, \xi_2$ are divergence free we obtain $\delta(\xi_1 \cdot \xi_2) = -\nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1$. Using again that $\xi_1, \xi_2$ are Killing vector field we have

$$X(g(\xi_1, \xi_2)) \;=\; g(\nabla_X \xi_1, \xi_2) \;+\; g(\xi_1, \nabla_X \xi_2) \;=\; -\,g(\nabla_{\xi_2}\xi_1 \;+\; \nabla_{\xi_1}\xi_2, X) \;=\; g(\delta(\xi_1 \cdot \xi_2), X) \;.$$

□

**Corollary 4.2.** *Let $\xi_1, \xi_2$ be two Killing vector fields with constant scalar product $g(\xi_1, \xi_2)$. Then $\xi_1 \cdot \xi_2$ is a divergence free Killing tensor and*

$$h \;:=\; \xi_1 \cdot \xi_2 \;-\; g(\xi_1, \xi_2)\, \tfrac{2}{n}\, g$$

*is a trace-free, divergence-free Killing* 2*-tensor.*

*Proof.* Indeed the trace of the symmetric endomorphism $h = \xi_1 \cdot \xi_2 \in \Gamma(\mathrm{Sym}^2 \mathrm{T}M)$ is given as $\mathrm{tr}(h) = 2\, g(\xi_1, \xi_2)$. Hence the tensor $h$ defined as above is a trace-free and divergence-free Killing tensor. ☐

**Example 4.3.** Let $(M^n, g, \xi)$ be a Sasakian manifold. Then $\xi \cdot \xi - \frac{2}{n} g$ is a trace-free Killing 2-tensor. On a 3-Sasakian manifold one has three pairwise orthogonal Killing vector fields of unit length defining a six-dimensional space of trace-free Killing 2-tensors.

**Example 4.4.** On spheres $\mathbb{S}^n$ with $n \geq 3$ one has pairs of orthogonal Killing vector fields, defining trace-free Killing 2-tensors. Indeed, every pair of anti-commuting skew-symmetric matrices of dimension $n+1$ defines a pair of orthogonal Killing vector fields on $\mathbb{S}^n$.

4.3. **Killing tensors from Killing forms.** There is a well-known relation between Killing forms and Killing tensors (e.g. cf. [3]). Let $u \in \Omega^p(M)$ be a Killing form, i.e. a $p$-form satisfying the equation $X \lrcorner \nabla_X u = 0$ for any tangent vector $X$. We define a symmetric bilinear form $K^u$ by $K^u(X, Y) = g(X \lrcorner u, Y \lrcorner u)$. Then $K^u$ is a symmetric Killing 2-tensor (for $p = 2$ this fact was also remarked in [10], Rem. 2.1). Indeed it suffices to show $(\nabla_X K^u)(X, X) = 0$ for any tangent vector $X$, which is immediate:

$$(\nabla_X K^u)(X, X) \;=\; \nabla_X (K^u(X, X)) \;-\; 2\, K^u(\nabla_X X, X) \;=\; 2\, g(X \lrcorner \nabla_X u, X \lrcorner u) \;=\; 0\ .$$

Since Killing 1-forms are dual to Killing vector fields, this construction generalizes the the one described in Section 4.2. If $u$ is a Killing 2-form considered as skew-symmetric endomorphism, then the associated symmetric Killing tensor $K^u$ is just $-u^2$. In this case $K^u$ commutes with the Ricci tensor, since the same is true for the skew-symmetric endomorphisms corresponding to Killing 2-forms, cf. [3]. Examples of manifolds with Killing forms are: the standard sphere $\mathbb{S}^n$, Sasakian, 3-Sasakian, nearly Kähler or weak $\mathrm{G}_2$ manifolds [27].

A related construction appears in the work of V. Apostolov, D.M.J. Calderbank and P. Gauduchon [1], in particular Appendix B.4. They prove a 1–1 relation between symmetric $J$- invariant Killing 2-tensors on Kähler surfaces and Hamiltonian 2-forms. In contrast to Killing forms there are many examples known of Hamiltonian 2-forms, thus providing a rich source of symmetric Killing tensors.

4.4. **The Ricci curvature as Killing tensor.** Special examples of Killing tensors arise as Ricci curvature of a Riemannian metric. Notice that if Ric is a Killing tensor then the scalar curvature scal is constant. Riemannian manifolds whose Ricci tensor is Killing were studied in [2] Ch. 16.G, as a class of generalized Einstein manifolds. In the same context this condition was originally discussed by A. Gray in [12]. It can be shown shown that all D'Atri spaces, i.e. Riemannian manifolds whose local geodesic symmetries preserve, up to a sign, the volume element, have Killing Ricci tensor (cf. [2], [6]). This provides a wide class of examples, many of them with non-parallel Ricci tensor. In particular naturally reductive spaces are D'Atri, and thus have Killing Ricci tensor. Here we want to present a direct argument.

**Proposition 4.5.** *The Ricci curvature of a naturally reductive space is a Killing tensor.*

*Proof.* Naturally reductive spaces are characterized by the existence of a metric connection $\bar{\nabla}$ with skew-symmetric, $\bar{\nabla}$-parallel torsion $T$ and parallel curvature $\bar{R}$. This gives in particular the following equations

$$g(T_XY,Z)+g(Y,T_XZ)=0 \qquad \text{and} \qquad T_XY+T_YX=0\ .$$

The condition $\bar{\nabla}\bar{R}=0$ can be rewritten as the following equation for the Riemannian curvature $R$

$$(\nabla_XR)_{Y,Z} \ = \ [T_X,R_{Y,Z}] \ - \ R_{T_XY,Z} \ - \ R_{Y,T_XZ}\ .$$

The Ricci curvature is defined as $\mathrm{Ric}(X,Y)=\sum g(R_{X,e_i}e_i,Y)$, where $\{e_i\}$ is an orthonormal frame. Its covariant derivative is given by

$$(\nabla_Z\mathrm{Ric})(X,Y) \ = \ \sum g((\nabla_ZR)_{X,e_i}e_i,Y)\ .$$

The Ricci curvature $\mathrm{Ric}\in\Gamma(\mathrm{Sym}^2\,\mathrm{T}M)$ is a Killing tensor if and only if $(\nabla_X\mathrm{Ric})(X,X)=0$ for all tangent vectors $X$, which is equivalent to

$$\sum g((\nabla_XR)_{X,e_i}e_i,X) \ = \ 0\ .$$

If $R$ is the Riemannian curvature tensor of naturally reductive metric this curvature expression can be rewritten as

$$\begin{aligned}
\sum g((\nabla_XR)_{X,e_i}e_i,X) \ &= \ \sum g([T_X,R_{X,e_i}]\,e_i \ - \ R_{T_XX,e_i}e_i \ - \ R_{X,T_Xe_i}e_i,\ X)\\
&= \ \sum g(T_X\,R_{X,e_i}e_i,\ X) \ - \ g(R_{X,e_i}\,T_Xe_i,\ X) \ - \ g(R_{X,T_Xe_i}e_i,\ X)\\
&= \ \sum -g(R_{X,e_i}e_i,T_XX) \ - \ 2\,g(R_{X,e_i}\,T_Xe_i,\ X)\\
&= \ 0
\end{aligned}$$

Here we used $T_XX=0$ and the equation $g(R_{X,e_i}\,T_Xe_i,\ X)=0$, which holds because of

$$\begin{aligned}
g(R_{X,e_i}\,T_Xe_i,\ X) \ &= \ -\,g(R_{e_i,T_Xe_i}\,X,\ X) \ - \ g(R_{T_Xe_i,X}e_i,X) \ = \ g(R_{X,T_Xe_i}e_i,\ X)\\
&= \ -\,g(R_{X,e_i}\,T_Xe_i,\ X)\ .
\end{aligned}$$

□

We define a modified Ricci tensor $\widetilde{\mathrm{Ric}}\in\mathrm{Sym}^2\,\mathrm{T}M$ as $\widetilde{\mathrm{Ric}}:=\mathrm{Ric}-\frac{2\,\mathrm{scal}}{n+2}\,\mathrm{id}$ . If Ric is a Killing tensor then the same is true for $\widetilde{\mathrm{Ric}}$, but not conversely.

**Lemma 4.6.** *The modified Ricci tensor* $\widetilde{\mathrm{Ric}}$ *is Killing tensor if and only if it is a conformal Killing tensor. Moreover* $\widetilde{\mathrm{Ric}}$ *is Killing if and only if* $(\nabla_X\mathrm{Ric})(X,X)=\frac{2}{n+2}\,X(\mathrm{scal})\,g(X,X)$ *for all vector fields* $X$*.*

*Proof.* A symmetric 2-tensor is a Killing tensor if and only if the two equations of (11) are satisfied. The first equation characterizes conformal Killing 2-tensors. Hence we only have to show that the second equations holds for $\widetilde{\mathrm{Ric}}$, i.e. we have to show that $\mathrm{d}\,\mathrm{tr}(\widetilde{\mathrm{Ric}}) = 2\,\delta\,\widetilde{\mathrm{Ric}}$. Since $\mathrm{tr}(\widetilde{\mathrm{Ric}}) = \frac{2-n}{n+2}\,\mathrm{scal}$, the well-known relation $\delta\,\mathrm{Ric} = -\frac{1}{2}\,\mathrm{d}\,\mathrm{scal}$ implies

$$\delta\,\widetilde{\mathrm{Ric}} = -\tfrac{1}{2}\,\mathrm{d}\,\mathrm{scal} + \tfrac{2}{n+2}\,\mathrm{d}\,\mathrm{scal} = \tfrac{-n+2}{2(n+2)}\,\mathrm{d}\,\mathrm{scal}, \qquad \mathrm{d}\,\mathrm{tr}(\widetilde{\mathrm{Ric}}) = \tfrac{2-n}{n+2}\,\mathrm{d}\,\mathrm{scal}\ .$$

Hence $\mathrm{d}\,\mathrm{tr}(\widetilde{\mathrm{Ric}}) = 2\,\delta\,\widetilde{\mathrm{Ric}}$ and the modified Ricci tensor $\widetilde{\mathrm{Ric}}$ is a Killing tensor if it is a conformal Killing tensor. The other direction and the equation for Ric are obvious. □

**Remark 4.7.** A. Gray introduced in [12] the notation $\mathcal{A}$, for the class of Riemannian manifolds with Killing Ricci tensor, and $\mathcal{C}$ for the class of Riemannian manifolds with constant scalar curvature. The class of Riemannian manifolds whose modified Ricci tensor is Killing was studied by W. Jelonek in [15] under the name $\mathcal{A} \oplus \mathcal{C}^{\perp}$. Finally we note that there are manifolds with Killing Ricci tensors which are neither homogeneous nor D'Atri. Examples were constructed by H. Pedersen and P. Tod in [24] and by W. Jelonek in [14].

## 5. Conformal Killing tensors on Riemannian products

Let $(M_1, g_1)$ and $(M_2, g_2)$ be two compact Riemannian manifolds. The aim of this section is to prove the following result, which reduces the study of conformal Killing 2-tensors on Riemannian products $M := M_1 \times M_2$ to that of Killing tensors on the factors.

**Theorem 5.1.** *Let $h \in \Gamma(\mathrm{Sym}^2_0(\mathrm{T}M))$ be a trace-free conformal Killing tensor. Then there exist Killing tensors $K_i \in \Gamma(\mathrm{Sym}^2(\mathrm{T}M_i))$, $i = 1, 2$, and Killing vector fields $\xi_1, \ldots, \xi_k$ on $M_1$ and $\zeta_1, \ldots, \zeta_k$ on $M_2$ such that*

$$h = (K_1 + K_2)_0 + \sum_{i=1}^{k} \xi_i \cdot \zeta_i\ .$$

*Conversely, every such tensor on $M$ is a trace-free conformal Killing tensor.*

*Proof.* We denote by $n$, $n_1$ and $n_2$ the dimensions of $M$, $M_1$ and $M_2$. Consider the natural decomposition $h = h_1 + h_2 + \phi$, where $h_1$, $h_2$ and $\phi$ are sections of $\pi_1^*\mathrm{Sym}^2(\mathrm{T}M_1)$, $\pi_2^*\mathrm{Sym}^2(\mathrm{T}M_2)$, and $\pi_1^*\,\mathrm{T}M_1 \otimes \pi_2^*\,\mathrm{T}M_2$ respectively. We consider the lifts to $M$ of the operators $\mathrm{d}_i$, $\delta_i$, $\mathrm{L}_i$, $\Lambda_i$ on the factors. Clearly two such operators commute if they have different subscripts, and satisfy the relations (1) and (8) if they have the same subscript. We define $f_i := \Lambda_i(h_i)$. Since $h$ is trace-free, we have $f_1 + f_2 = 0$. The conformal Killing equation

$$\mathrm{d}h = -\frac{1}{n+2}\mathrm{L}\delta h$$

reads

$$\begin{aligned} \mathrm{d}_1 h_1 + \mathrm{d}_1 h_2 + \mathrm{d}_1\phi + \mathrm{d}_2 h_1 + \mathrm{d}_2 h_2 + \mathrm{d}_2\phi &= -\frac{1}{n+2}\mathrm{L}_1(\delta_1 h_1 + \delta_1\phi + \delta_2 h_2 + \delta_2\phi) \\ &\quad -\frac{1}{n+2}\mathrm{L}_2(\delta_1 h_1 + \delta_1\phi + \delta_2 h_2 + \delta_2\phi) \ . \end{aligned}$$

Projecting this equation onto the different summands of $\mathrm{Sym}^3(\mathrm{T}M)$ yields the following system:

$$\begin{cases} \mathrm{d}_1 h_1 & = -\frac{1}{n+2}\mathrm{L}_1(\delta_1 h_1 + \delta_2\phi) \\ \mathrm{d}_2 h_2 & = -\frac{1}{n+2}\mathrm{L}_2(\delta_2 h_2 + \delta_1\phi) \\ \mathrm{d}_1 h_2 + \mathrm{d}_2\phi & = -\frac{1}{n+2}\mathrm{L}_2(\delta_1 h_1 + \delta_2\phi) \\ \mathrm{d}_2 h_1 + \mathrm{d}_1\phi & = -\frac{1}{n+2}\mathrm{L}_1(\delta_2 h_2 + \delta_1\phi) \end{cases} \tag{16}$$

Applying $\Lambda_1$ to the first equation of (16) and using (8) gives

$$-2\delta_1 h_1 + \mathrm{d}_1 f_1 = -\frac{2(n_1+2)}{n+2}(\delta_1 h_1 + \delta_2\phi) \ ,$$

whence

$$\delta_1 h_1 = \frac{n_1+2}{n_2}\delta_2\phi + \frac{n+2}{2n_2}\mathrm{d}_1 f_1 \ . \tag{17}$$

Similarly, applying $\Lambda_2$ to the second equation of (16) gives

$$\delta_2 h_2 = \frac{n_2+2}{n_1}\delta_1\phi + \frac{n+2}{2n_1}\mathrm{d}_2 f_2 \ . \tag{18}$$

Replacing $\delta_i h_i$ in the right hand side of (16) using (17) and (18), yields

$$\begin{cases} \mathrm{d}_1 h_1 & = -\frac{1}{2n_2}\mathrm{L}_1(2\delta_2\phi + \mathrm{d}_1 f_1) \\ \mathrm{d}_2 h_2 & = -\frac{1}{2n_1}\mathrm{L}_2(2\delta_1\phi + \mathrm{d}_2 f_2) \\ \mathrm{d}_1 h_2 + \mathrm{d}_2\phi & = -\frac{1}{2n_2}\mathrm{L}_2(2\delta_2\phi + \mathrm{d}_1 f_1) \\ \mathrm{d}_2 h_1 + \mathrm{d}_1\phi & = -\frac{1}{2n_1}\mathrm{L}_1(2\delta_1\phi + \mathrm{d}_2 f_2) \end{cases} \tag{19}$$

We now apply $\delta_2$ to the third equation of (19) and use (8) and (18) together with the fact that $f_2 = -f_1$ to compute:

$$\begin{aligned} \delta_2\mathrm{d}_2\phi &= -\delta_2\mathrm{d}_1 h_2 + \frac{1}{n_2}\mathrm{d}_2(2\delta_2\phi + \mathrm{d}_1 f_1) \\ &= -\mathrm{d}_1\left(\frac{n_2+2}{n_1}\delta_1\phi + \frac{n+2}{2n_1}\mathrm{d}_2 f_2\right) + \frac{2}{n_2}\mathrm{d}_2\delta_2\phi + \frac{1}{n_2}\mathrm{d}_2\mathrm{d}_1 f_1 \\ &= -\frac{n_2+2}{n_1}\mathrm{d}_1\delta_1\phi + \frac{2}{n_2}\mathrm{d}_2\delta_2\phi + \frac{n(n_2+2)}{2n_1 n_2}\mathrm{d}_2\mathrm{d}_1 f_1 \ . \end{aligned}$$

Similarly, applying $\delta_1$ to the fourth equation of (19) and using (8) and (17) yields:

$$\delta_1 \mathrm{d}_1 \phi = -\frac{n_1+2}{n_2}\mathrm{d}_2\delta_2\phi + \frac{2}{n_1}\mathrm{d}_1\delta_1\phi + \frac{n(n_1+2)}{2n_1n_2}\mathrm{d}_2\mathrm{d}_1 f_2 \ . \tag{20}$$

In order to eliminate the terms involving $f_1$ and $f_2$ in these last two formulas, we multiply the first one with $n_1+2$ and add it to the second one multiplied with $n_2+2$, which yields

$$(n_1+2)\delta_2\mathrm{d}_2\phi + (n_2+2)\delta_1\mathrm{d}_1\phi + (n_2+2)\mathrm{d}_1\delta_1\phi + (n_1+2)\mathrm{d}_2\delta_2\phi = 0 \ ,$$

which after a scalar product with $\phi$ and integration over $M$ gives $\mathrm{d}_1\phi = \mathrm{d}_2\phi = 0$ and $\delta_1\phi = \delta_2\phi = 0$. Plugging this back into (20) also shows that $\mathrm{d}_1\mathrm{d}_2 f_2 = 0$. In other words, there exist functions $\varphi_i \in C^\infty(M_i)$ such that $f_2 = \varphi_1+\varphi_2$, and correspondingly $f_1 = -\varphi_1-\varphi_2$ (we identify here $\varphi_i$ with their pullbacks to $M$ in order to simplify the notations).

The system (19) thus becomes:

$$\begin{cases} \mathrm{d}_1 h_1 &= -\frac{1}{2n_2}\mathrm{L}_1\mathrm{d}_1 f_1 = \frac{1}{2n_2}\mathrm{L}_1\mathrm{d}_1\varphi_1 \\ \mathrm{d}_2 h_2 &= -\frac{1}{2n_1}\mathrm{L}_2\mathrm{d}_2 f_2 = -\frac{1}{2n_1}\mathrm{L}_2\mathrm{d}_2\varphi_2 \\ \mathrm{d}_1 h_2 &= -\frac{1}{2n_2}\mathrm{L}_2\mathrm{d}_1 f_1 = \frac{1}{2n_2}\mathrm{L}_2\mathrm{d}_1\varphi_1 \\ \mathrm{d}_2 h_1 &= -\frac{1}{2n_1}\mathrm{L}_1\mathrm{d}_2 f_2 = -\frac{1}{2n_1}\mathrm{L}_1\mathrm{d}_2\varphi_2 \end{cases} \tag{21}$$

This system shows that the tensors

$$K_1 := h_1 + \frac{1}{2n_1}\mathrm{L}_1\varphi_2 - \frac{1}{2n_2}\mathrm{L}_1\varphi_1 \qquad \text{and} \qquad K_2 := h_2 + \frac{1}{2n_1}\mathrm{L}_2\varphi_2 - \frac{1}{2n_2}\mathrm{L}_2\varphi_1$$

verify $\mathrm{d}_1 K_2 = \mathrm{d}_2 K_1 = 0$ and $\mathrm{d}_1 K_1 = \mathrm{d}_2 K_2 = 0$. The first two equations show that $K_1$ and $K_2$ are pull-backs of symmetric tensors on $M_1$ and $M_2$, and the last two equations show that these tensors are Killing tensors of the respective factors $M_1$ and $M_2$. Moreover, we have

$$K_1 + K_2 = h_1 + h_2 + \mathrm{L}\left(\frac{1}{2n_1}\varphi_2 - \frac{1}{2n_2}\varphi_1\right)$$

and thus $(K_1+K_2)_0 = h_1+h_2$.

Finally, we claim that $\mathrm{d}_1\phi = 0$ and $\mathrm{d}_2\phi = 0$ imply that $\phi$ has the form stated in the theorem. Let $T_1 := \pi_1^*(\mathrm{T}M_1)$ and $T_2 := \pi_2^*(\mathrm{T}M_2)$ denote the pull-backs on $M$ of the tangent bundles of the factors. Then $\phi$ is a section of $T_1 \otimes T_2$, and $\mathrm{d}_1\phi = 0$ and $\mathrm{d}_2\phi = 0$ imply that $\nabla\phi = \nabla^{M_1}\phi + \nabla^{M_2}\phi$ is a section of $\Lambda^2 T_1 \otimes T_2 + T_1 \otimes \Lambda^2 T_2$. In some sense, one can view $\phi$ as a Killing vector field on $M_1$ twisted with $T_2$, and also as a Killing vector field on $M_2$ twisted with $T_1$. Let us denote by $\phi_1 := \nabla^{M_1}\phi \in \Gamma(\Lambda^2 T_1 \otimes T_2)$, $\phi_2 := \nabla^{M_2}\phi \in \Gamma(T_1 \otimes \Lambda^2 T_2)$ and $\phi_3 := \nabla^{M_1}\nabla^{M_2}\phi = \nabla^{M_2}\nabla^{M_1}\phi \in \Gamma(\Lambda^2 T_1 \otimes \Lambda^2 T_2)$. The usual Kostant formula for Killing vector fields immediately generalizes to

$$\nabla_{X_1}\phi_1 = R_{X_1}\phi, \qquad \nabla_{X_2}\phi_2 = R_{X_2}\phi, \qquad \nabla_{X_1}\phi_3 = R_{X_1}\phi_2, \qquad \nabla_{X_2}\phi_3 = R_{X_2}\phi_1,$$

where for any vector bundle $F$, $i \in \{1,2\}$, and $X \in T_i$, $R_X : T_i \otimes F \to \Lambda^2 T_i \otimes F$ is defined by $R_X(Y \otimes \sigma) := R_{X,Y} \otimes \sigma$.

We thus get a parallel section $\Phi := (\phi, \phi_1, \phi_2, \phi_3)$ of the vector bundle

$$E := (T_1 \otimes T_2) \oplus (\Lambda^2 T_1 \otimes T_2) \oplus (T_1 \otimes \Lambda^2 T_2) \oplus (\Lambda^2 T_1 \otimes \Lambda^2 T_2)$$

with respect to the connection defined on vectors $X_i \in T_i$ by

$$\tilde{\nabla}_{X_1}(\phi, \phi_1, \phi_2, \phi_3) := (\nabla_{X_1}\phi - \phi_1(X_1), \nabla_{X_1}\phi_1 - R_{X_1}\phi, \nabla_{X_1}\phi_2 - \phi_3(X_1), \nabla_{X_1}\phi_3 - R_{X_1}\phi_2)$$

and

$$\tilde{\nabla}_{X_2}(\phi, \phi_1, \phi_2, \phi_3) := (\nabla_{X_2}\phi - \phi_2(X_2), \nabla_{X_2}\phi_1 - \phi_3(X_2), \nabla_{X_2}\phi_2 - R_{X_2}\phi, \nabla_{X_2}\phi_3 - R_{X_2}\phi_1).$$

We now define for $i = 1, 2$ the connections $\tilde{\nabla}^i$ on $E_i := \mathrm{T}M_i \oplus \Lambda^2 \mathrm{T}M_i$ by

$$\tilde{\nabla}^i_{X_i}(\alpha_i, \beta_i) := (\nabla^{M_i}_{X_i}\alpha_i - \beta_i(X_i), \nabla^{M_i}_{X_i}\beta_i - R^{M_i}_{X_i}(\alpha_i)),$$

and notice that $E = \pi_1^*(E_1) \otimes \pi_2^*(E_2)$ and that $\tilde{\nabla}$ coincides with the tensor product connection induced by $\tilde{\nabla}^1$ and $\tilde{\nabla}^2$ on $E$. It follows that the space of $\tilde{\nabla}$-parallel sections of $E$ is the tensor product of the spaces of $\tilde{\nabla}^1$-parallel sections of $E_1$ and of $\tilde{\nabla}^2$-parallel sections of $E_2$. Taking the first component of these sections yields the desired result. $\square$

**Remark 5.2.** Note that the compactness assumption in Theorem 5.1 is essential. There are many non compact products, with trace-free conformal Killing 2-tensors which are not defined by Killing tensors of the factors. The simplest example is the flat space $\mathbb{R}^n$.

## 6. Weitzenböck formulas

Let $(M^n, g)$ be an oriented Riemannian manifold with Riemannian curvature tensor $R$. The curvature operator $\mathcal{R} : \Lambda^2 \mathrm{T}M \to \Lambda^2 \mathrm{T}M$ is defined by $g(\mathcal{R}(X \wedge Y), Z \wedge V) = R(X, Y, Z, V)$. With this convention we have $\mathcal{R} = -\,\mathrm{id}$ on the standard sphere.

Let $P = P_{\mathrm{SO}(n)}$ be the frame bundle and $EM$ a vector bundle associated to $P$ via a $\mathrm{SO}(n)$-representation $\rho : \mathrm{SO}(n) \to \mathrm{Aut}(E)$. Then the curvature endomorphism $q(R) \in \mathrm{End}\, EM$ is defined as

$$q(R) \;:=\; \frac{1}{2} \sum_{i,j} (e_i \wedge e_j)_* \circ \mathcal{R}(e_i \wedge e_j)_* \;.$$

Here $\{e_i\}, i = 1, \dots n$, is a local orthonormal frame and for $X \wedge Y \in \Lambda^2 \mathrm{T}M$ we define $(X \wedge Y)_* = \rho_*(X \wedge Y)$, where $\rho_* : \mathfrak{so}(n) \to \mathrm{End}E$ is the differential of $\rho$. In particular, the standard action of $\Lambda^2 \mathrm{T}M$ on $\mathrm{T}M$ is written as $(X \wedge Y)_* Z \;=\; g(X,\, Z)\, Y \;-\; g(Y,\, Z)\, X = (Y \cdot X \lrcorner - X \cdot Y \lrcorner) Z$. This is compatible with

$$g((X \wedge Y)_* Z, V) \;=\; g(X \wedge Y, Z \wedge V) \;=\; g(X, Z)\, g(Y, V) \;-\; g(X, V)\, g(Y, Z) \;.$$

For any section $\varphi \in \Gamma(EM)$ we have $\mathcal{R}(X \wedge Y)_* \varphi \;=\; R_{X,Y}\, \varphi$. It is easy to check that $q(R)$ acts as the Ricci tensor on tangent vectors. The definition of $q(R)$ is independent of the orthonormal frame of $\Lambda^2 \mathrm{T}M$, i.e. $q(R)$ can be written as $q(R) = \sum \omega_{i\,*} \circ \mathcal{R}(\omega_i)_*$ for any orthonormal frame of $\Lambda^2\,\mathrm{T}M$. Moreover it is easy to verify that $q(R)$ is a symmetric endomorphism of the vector bundle $EM$.

The action of $q(R)$ on a symmetric $p$-tensors $K$ can be written as $q(R)K = \sum e_j \cdot e_i \lrcorner R_{e_i\,e_j} K$. On symmetric 2-tensors $h$ the curvature endomorphism $q(R)$ is related to the classical curvature endomorphism $\mathring{R}$ (cf. [2, p. 52]), which is defined by

$$(\mathring{R}h)(X,Y) \;=\; \sum h(R_{X,e_i}Y, e_i)\;.$$

If $h$ is considered as a symmetric endomorphism the action of $\mathring{R}$ on $h$ can be written as $\mathring{R}(h)(X) = -\sum R_{X,e_i}\,h(e_i)$ .

The action of Ric is extended to symmetric 2-tensors $h$ as a derivation, i.e. it is defined as $\mathrm{Ric}(h)(X,Y) = -h(\mathrm{Ric}X, Y) - h(X, \mathrm{Ric}Y)$ . Then the following formula holds on $\mathrm{Sym}^2\,\mathrm{T}M$:

$$q(R) \;=\; 2\,\mathring{R} \;-\; \mathrm{Ric}\;. \tag{22}$$

If $h$ is the metric $g$ then $(\mathring{R}g)(X,Y) = -\mathrm{Ric}(X,Y)$ and $\mathrm{Ric}(g)(X,Y) = -2\,\mathrm{Ric}(X,Y)$ .

As seen above, the covariant derivative $\nabla$ on $\mathrm{Sym}^p_0\,\mathrm{T}M$ decomposes into three components defining three first order differential operators: $P_i(K) := p_i(\nabla K),\ i = 1,2,3$, where $p_i$ are the orthogonal projections onto the three summands in the decomposition (4). The operators $P_i^*P_i$ , $i = 1,2,3$ are then second order operators on sections of $\mathrm{Sym}^p_0\,\mathrm{T}M$. These three operators are linked by a Weitzenböck formula:

**Proposition 6.1.** *Let $K$ be any section of* $\mathrm{Sym}^p_0\,\mathrm{T}M$*, then:*

$$q(R)\,K \;=\; -\,p\,P_1^*P_1\;K \;+\; (n+p-2)\,P_2^*P_2\;K \;+\; P_3^*P_3\;K\;.$$

*Proof.* The stated Weitzenböck formula can be obtained as a special case of a general procedure described in [28]. However it is easy to check it directly using the following remarks.

Let $E$ be any $\mathrm{SO}(n)$-representation defining a vector bundle $EM$ and let $\mathrm{T}$ be the standard representation defining the tangent bundle $\mathrm{T}M$. Then any $p \in \mathrm{End}(\mathrm{T}\otimes E)$ can be interpreted as an element in $\mathrm{Hom}(\mathrm{T}\otimes\mathrm{T}\otimes E, E)$ defined as $p(a\otimes b\otimes e) = (a\lrcorner\otimes\,\mathrm{id})\,p(b\otimes e)$, for $a,b\in\mathrm{T}$ and $e \in E$. Important examples of such endomorphisms are the orthogonal projections $p_i$, $i = 1,\ldots,N$, onto the summands in a decomposition $\mathrm{T}\otimes E = V_1\oplus\ldots\oplus V_N$. Another example is the so-called *conformal weight operator* $B \in \mathrm{Hom}(\mathrm{T}\otimes\mathrm{T}\otimes E, E)$ defined as $B(a\otimes b\otimes e) = (a\wedge b)_*e$. As an element in $\mathrm{End}(\mathrm{T}\otimes E)$, the conformal weight operator can be written as $B(b\otimes e) = \sum e_i\otimes(e_i\wedge b)_*e$ .

Let $K$ be a section of $EM$, then $\nabla^2 K = \sum e_i\otimes e_j\otimes\nabla^2_{e_i,e_j}K$ is a section of the bundle $\mathrm{Hom}(\mathrm{T}M\otimes\mathrm{T}M\otimes EM, EM)$. Using the remark above we can apply elements of the bundle $\mathrm{End}(\mathrm{T}M\otimes EM)$ to $\nabla^2 K$. It is then easy to check that

$$B(\nabla^2K) \;=\; q(R)\,K, \qquad \mathrm{id}(\nabla^2K) \;=\; -\,\nabla^*\nabla K, \qquad p_i(\nabla^2K) \;=\; -\,P_i^*P_i\,K$$

where $P_i$, $i = 1,\ldots,N$ are the first order differential operators $P_i(K) := p_i(\nabla K)$. Hence in order to prove the Weitzenböck formula above it is enough to verify the following equation for endomorphisms of $\mathrm{T} \otimes E$ in the case $E = \mathrm{Sym}^p\,\mathrm{T}$:

$$
\begin{aligned}
B \;&=\; p\,p_1 \;-\; (n+p-2)\,p_2 \;-\; p_3 \;=\; (p+1)\,p_1 \;-\; (n+p-3)\,p_2 \;-\; \mathrm{id} \\
&=\; \pi_1^*\,\pi_1 \;-\; \tfrac{n+2p-4}{n+2p-2}\,\pi_2^*\,\pi_2 \;-\; \mathrm{id}\;.
\end{aligned}
$$

This is an easy calculation using the explicit formulas for $\pi_i^*$ and $\pi_i$, $i = 1, 2$ given above. □

6.1. **Eigenvalue estimates for the Lichnerowicz Laplacian.** The Lichnerowicz Laplacian $\Delta_L$ is a Laplace-type operator acting on sections of $\mathrm{Sym}^p\,\mathrm{T}M$. It can be defined by $\Delta_L := \nabla^*\,\nabla + q(R)$. On symmetric 2-tensors it is usually written as $\Delta_L = \nabla^*\nabla + 2\,\mathring{R} - \mathrm{Ric}$, which is the same formula, by (22).

**Proposition 6.2.** *Let $(M^n, g)$ be a compact Riemannian manifold. Then $\Delta_L \geq 2\,q(R)$ holds on the space of divergence-free symmetric tensors. Equality $\Delta_L h = 2q(R)h$ holds for a divergence free tensor $h$ if and only if $h$ is a Killing tensor.*

*Proof.* Directly from the definition we calculate

$$
\mathrm{d}\delta h = -\sum e_i \cdot \nabla_{e_i}(e_j \lrcorner \nabla_{e_j} h) = -\sum e_i \cdot (e_j \lrcorner \nabla_{e_i}\nabla_{e_j} h)\;.
$$

Similarly we have

$$
\begin{aligned}
\delta\mathrm{d}h \;&=\; -\sum e_i \lrcorner \nabla_{e_i}(e_j \cdot \nabla_{e_j} h) \;=\; -\sum e_i \lrcorner (e_j \cdot \nabla_{e_i}\nabla_{e_j} h) \\
&=\; -\sum \nabla_{e_i}\nabla_{e_i} h \;-\; \sum e_j \cdot (e_i \lrcorner \nabla_{e_i}\nabla_{e_j} h) \\
&=\; \nabla^*\nabla h \;-\; \sum e_i \cdot (e_j \lrcorner \nabla_{e_j}\nabla_{e_i} h)
\end{aligned}
$$

Taking the difference we immediately obtain

$$
\delta\mathrm{d}h \;-\; \mathrm{d}\delta h \;=\; \nabla^*\nabla h \;-\; \sum e_i \cdot e_j \lrcorner R_{e_j,e_i} h \;=\; \nabla^*\nabla h \;-\; q(R)h \;=\; \Delta_L h - 2q(R)h\;.
$$

Thus, if $h$ is divergence free we have $(\Delta_L - 2q(R))h = \delta\mathrm{d}h$ and the inequality follows after taking the $L^2$ product with $h$. The equality case is clearly characterized by $\mathrm{d}h = 0$. □

**Remark 6.3.** For symmetric 2-tensors this estimate for $\Delta_L$ was proven in [11].

**Remark 6.4.** As a consequence of Proposition 6.2 we see that divergence free Killing tensors on compact Riemannian manifolds are characterized by the equation $\nabla^*\nabla h = q(R)h$. This generalizes the well known characterization of Killing vector fields as divergence free vector fields $\xi$ with $\nabla^*\nabla\xi = \mathrm{Ric}(\xi)$.

**Remark 6.5.** Recall that $q(R)$ is a symmetric endomorphism. The eigenvalues of $q(R)$ are constant on homogeneous spaces. On symmetric spaces $M = G/K$ the Lichnerowicz Laplacian $\Delta_L$ can be identified with the Casimir operator $\mathrm{Cas}_G$ of the group $G$ and $q(R)$ with the Casimir operator $\mathrm{Cas}_K$ of the group $K$.

6.2. **Non-existence results.** In [5] Dairbekov and Sharafutdinov show the non-existence of trace-free conformal Killing tensors on manifolds with negative sectional curvature. In this section we will give a short new proof of this result.

**Proposition 6.6.** *On a compact Riemannian manifold $(M, g)$ of non-positive sectional curvature any trace-free conformal Killing tensor has to be parallel. If in addition there exists a point in $M$ where the sectional curvature of every two-plane is strictly negative, then $M$ does not carry any (non-identically zero) trace-free conformal Killing tensor.*

*Proof.* On sections of $\mathrm{Sym}^p_0\,\mathrm{T}M$ we consider the Weitzenböck formula of Proposition 6.1:

$$q(R) \;=\; -p\,P_1^*P_1 \;+\; (n+p-2)\,P_2^*P_2 \;+\; P_3^*P_3\ .$$

Trace-free conformal Killing tensors are characterized by the equation $P_1K = 0$. In particular we obtain for the $\mathrm{L}^2$-scalar product:

$$(q(R)K, K)_{\mathrm{L}^2} = (n+p-2)\|P_2K\|^2 + \|P_3K\|^2 \geq 0\ , \tag{23}$$

where $K$ is a trace-free conformal Killing tensor. We will show that $(q(R)K, K)_{\mathrm{L}^2} \leq 0$ holds on a manifold with non-positive sectional curvature. This together with (23) immediately implies that $P_2K = 0$ and $P_3K = 0$ and thus that $K$ has to be parallel.

For any $x \in M$ and any fixed tangent vector $X \in \mathrm{T}_xM$ we consider the symmetric bilinear form $B_X(Y, Z) := g(R_{X,Y}X, Z)$, defined on tangent vectors $Y, Z \in \mathrm{T}_xM$. Since the sectional curvature is non-positive, this bilinear form is positive semi-definite. Hence there is an orthonormal basis $e_1, \ldots, e_n$ of $\mathrm{T}_xM$ (depending on $X$) with $B_X(e_i, e_j) = 0$ for $i \neq j$ and $B_X(e_i, e_i) = a_i(X) \geq 0$ for all $i$.

A symmetric tensor $K \in \Gamma(\mathrm{Sym}^p_0TM)$ can also be considered as a polynomial map on $\mathrm{T}_M$ by the formula $K(X) := g(K, X^p)$. In particular we have for the Riemannian curvature $(R_{Y,Z}K)(X) = (\sum R_{Y,Z}e_k \cdot e_k \lrcorner K)(X) = p\sum g(R_{Y,Z}e_k, X)\,(e_k \lrcorner K)(X)$.

Let $T$ be the tangent space $T = \mathrm{T}_xM$ for some $x \in M$. Then we can define a scalar product on $\mathrm{Sym}^p_0T$ by $\tilde{g}(K_1, K_2) := \int_{S_T} K_1(X)K_2(X)\mathrm{d}\mu$, where $S_T$ is the unit sphere in $T$ and $\mathrm{d}\mu$ denotes the standard Lebesgue measure on $S_T$. From Schur's Lemma it follows the existence of a non-zero constant $c$ such that $\tilde{g}(K_1, K_2) = \frac{1}{c}\,g(K_1, K_2)$ holds for all $K_1, K_2 \in \mathrm{Sym}^p_0T$. Since both scalar products are positive definite the constant $c$ has to be positive. We now compute the scalar product $g(q(R)K, K)$ at some point $x \in M$. From the remarks above we obtain:

$$\begin{aligned}
g(q(R)K,K) &= \sum g(e_j \cdot e_i \lrcorner R_{e_i,e_j}K, K) \;=\; \sum g(R_{e_i,e_j}K, e_i \cdot e_j \lrcorner K) \\
&= c\int_{S_T} \sum (R_{e_i,e_j}K)(X) \cdot (e_i \cdot e_j \lrcorner K)(X)\, \mathrm{d}\mu \\
&= c\,p^2 \int_{S_T} \sum g(R_{e_i,e_j}e_k, X) \cdot (e_k \lrcorner K)(X) \cdot g(e_i, X) \cdot (e_j \lrcorner K)(X)\, \mathrm{d}\mu \\
&= -\,c\,p^2 \int_{S_T} \sum g(R_{X,e_j}X, e_k) \cdot (e_k \lrcorner K)(X) \cdot (e_j \lrcorner K)(X)\, \mathrm{d}\mu \\
&= -\,c\,p^2 \int_{S_T} \sum B_X(e_j, e_k) \cdot (e_k \lrcorner K)(X) \cdot (e_j \lrcorner K)(X)\, \mathrm{d}\mu \\
&= -\,c\,p^2 \int_{S_T} \sum a_j(X)\,((e_j \lrcorner K)(X))^2\, \mathrm{d}\mu \\
&\leq 0\ .
\end{aligned}$$

This proves that for every trace-free symmetric tensor $K$, on a manifold with non-positive sectional curvature, the inequality $g(q(R)K,K) \leq 0$ holds at every point. By the above arguments, if $K$ is conformal Killing, then $K$ has to be parallel.

If in addition there is a point $x \in M$ where all sectional curvatures are negative, then the symmetric form $B_X$ is positive definite for all $X \in S_T$, so its eigenvalues are positive: $a_j(X) > 0$. The computation above shows that $(Y \lrcorner K)(X) = 0$ for every $X \in S_T$ and for all tangent vectors $Y$ orthogonal to $X$. This is equivalent to

$$0 = g(K, (|X|^2 Y - \langle X, Y\rangle X) \cdot X^{p-1}) = |X|^2 g(K, Y \cdot X^{p-1}) - \langle X, Y\rangle g(K, X^p) \tag{24}$$

for all tangent vectors $X, Y \in \mathrm{T}_xM$. On the other hand, from (2) we immediately get for every $X, Y \in \mathrm{T}_xM$

$$g(Y \cdot K, X^{p+1}) = (p+1)\langle X, Y\rangle g(K, X^p)$$

and

$$g(\mathrm{L} \cdot (Y \lrcorner K), X^{p+1}) = p(p+1)|X|^2 g(K, Y \cdot X^{p-1})\ .$$

From (24) we thus obtain

$$\mathrm{L} \cdot (Y \lrcorner K) = p\, Y \cdot K, \qquad \forall\, Y \in \mathrm{T}_xM\ .$$

Applying $\Lambda$ and using (1) and (2) together with the fact that $\Lambda K = 0$, yields

$$(2n + 4(p-1))Y \lrcorner K = 2p\, Y \lrcorner K$$

at $x$, whence $K_x = 0$. As $K$ is parallel from the first part of the proof, this shows that $K \equiv 0$. ☐

**Corollary 6.7.** *Let $\Sigma_g$ be a compact Riemannian surface of genus $g \geq 2$. Then $M$ admits no trace-free conformal Killing tensors. More generally, there are no trace-free conformal Killing tensors on compact quotients of symmetric spaces of non-compact type.*

**Remark 6.8.** Note that this result was also obtained by D.J.F. Fox in [9], Corollary 3.1.

## 7. Killing tensors with two eigenvalues

Let $K \in \Gamma(\mathrm{Sym}^2_0 \mathrm{T}M)$ be a non-trivial trace-free Killing (i.e. Stäckel) tensor on a connected Riemannian manifold $(M^n, g)$. We assume throughout this section that $K$ has at most two eigenvalues at every point of $M$.

**Lemma 7.1.** *The multiplicities of the eigenvalues of $K$ are constant on $M$, so the eigenspaces of $K$ define two distributions $\mathrm{T}M = E_1 \oplus E_2$. If $n_1, n_2$ denote the dimensions of $E_1, E_2$ and $\pi_i$ denote the orthogonal projection onto $E_i$ for $i = 1, 2$, then $K$ is a constant multiple of $n_2\pi_1 - n_1\pi_2$.*

*Proof.* Since $K$ is trace-free, the eigenvalues of $K$ are distinct at every point $p \in M$ where $K_p \neq 0$. Every such point $p$ has a neighborhood $U$ on which the multiplicities of the eigenvalues of $K$ are constant. The eigenspaces of $K$ define two orthogonal distributions $E_1$ and $E_2$ along $U$ such that $\mathrm{T}M|_U = E_1 \oplus E_2$. Then the Killing tensor $K$ can be written as $K = f\pi_1 + h\pi_2$. Since $K$ is trace-free, we have $0 = n_1 f + n_2 h$. The covariant derivative of $K$ can be written as

$$
\begin{aligned}
g((\nabla_X K)Y, Z) &= g(\nabla_X(KY) - K(\nabla_X Y), Z) \\
&= g(X(f)\pi_1(Y) + f(\nabla_X \pi_1)Y + X(h)\pi_2(Y) + h(\nabla_X \pi_2)Y, Z) \ .
\end{aligned}
$$

Note that for any vector $X$ and vector fields $X_i, Y_i \in E_i$ for $i = 1, 2$ we have

$$
g((\nabla_X \pi_1)X_i, Y_i) \ = \ 0 \tag{25}
$$

and similarly for $\pi_2$. For $X \in E_1$, the Killing tensor equation gives $(\nabla_X K)(X, X) = 0$ and it follows from the formula above that $g(X(f)\pi_1(X), X) = 0$. Thus $X(f) = 0$ for all $X \in E_1$ and similarly $X(h) = 0$ for all $X \in E_2$. It follows that $f$ and $h$ are constant on $U$, since $f$ and $h$ are related via $n_1 f + n_2 h = 0$. The eigenvalues of $K$ are thus constant on $U$. Since this is true on some neighbourhood of every point $p$ where $K_p \neq 0$, we deduce that the eigenvalues of $K$, and their multiplicities, are constant on $M$. This proves the lemma. □

We will now characterize orthogonal splittings of the tangent bundle which lead to trace-free Killing tensors.

**Proposition 7.2.** *Let $E_1$ and $E_2$ be orthogonal complementary distributions on $M$ of dimensions $n_1$ and $n_2$ respectively. Then the trace-free symmetric tensor $K = n_2\pi_1 - n_1\pi_2$ is Killing if and only if the following conditions hold:*

$$
\nabla_{X_1} X_1 \in \Gamma(E_1) \qquad \forall\, X_1 \in \Gamma(E_1) \qquad \textit{and} \qquad \nabla_{X_2} X_2 \in \Gamma(E_2) \qquad \forall\, X_2 \in \Gamma(E_2) \ . \tag{26}
$$

*Proof.* Assume first that $K = n_2\pi_1 - n_1\pi_2$ is a Killing tensor. Since $\pi_1 + \pi_2 = \mathrm{id}$ is parallel, we see that $\pi_1$ and $\pi_2$ are Killing tensors too. Let $X_1 \in \Gamma(E_1)$ and $X_2 \in \Gamma(E_2)$. As $\pi_1$ is a

Killing tensor, we get from (25):

$$\begin{aligned} 0 &= g((\nabla_{X_1}\pi_1)X_1, X_2) + g((\nabla_{X_1}\pi_1)X_2, X_1) + g((\nabla_{X_2}\pi_1)X_1, X_1) \\ &= 2g((\nabla_{X_1}\pi_1)X_1, X_2) \\ &= 2g(\nabla_{X_1}X_1, X_2)\ , \end{aligned}$$

and similarly $0 = g(\nabla_{X_2}X_2, X_1)$, thus proving (26).

Conversely, if (26) holds, then for every vector field $X$ on $M$ we can write $X = X_1 + X_2$, where $X_i := \pi_i(X)$ for $i = 1, 2$ and compute using (25) again:

$$\begin{aligned} g((\nabla_X\pi_1)X, X) &= 2g((\nabla_X\pi_1)X_1, X_2) = 2g(\nabla_X X_1, X_2) \\ &= 2g(\nabla_{X_1}X_1, X_2) + 2g(\nabla_{X_2}X_1, X_2) \\ &= 2g(\nabla_{X_1}X_1, X_2) - 2g(\nabla_{X_2}X_2, X_1) = 0\ . \end{aligned}$$

□

Pairs of distributions with this property were studied in [21] by A. Naveira under the name of almost product structures of type $D_1$.

Note that Killing tensors with two eigenvalues were intensively studied by W. Jelonek and also by B. Coll et al. in [4]. The results above about Stäckel tensors with at most two eigenvalues also follow from the work of W. Jelonek, cf. Theorem 1.2 and Theorem 2.1. in [15] and Lemma 5 in [16], or Lemma 2 in [17]. In fact, combining [15] and [17], a more general classification result can be proved for general (not necessarily trace-free) Killing 2-tensors, which are characterized by Equation (1.5) in [15].

**Example 7.3.** If $M \to N$ is a Riemannian submersion with totally geodesic fibers and $\mathcal{V}$ and $\mathcal{H}$ denote its vertical and horizontal distributions, then $E_1 := \mathcal{V}$ and $E_2 := \mathcal{H}$ satisfy (26), by the O'Neill formulas. It turns out that this generalizes several examples of Killing tensors appearing in the physics literature, e.g. in [11].

**Remark 7.4.** Note that (26) does not imply the integrability of $E_1$ or $E_2$. However, assuming that (26) holds and that one of the distributions, say $E_1$, is integrable, then there exists locally a Riemannian submersion with totally geodesic fibers whose vertical and horizontal distributions are $E_1$ and $E_2$ respectively.

## 8. Conformal Killing tensors on hypersurfaces

In this last section we give a short proof, using the formalism developed above, of a vanishing result of Dairbekov and Sharafutdinov:

**Theorem 8.1** ([5]). *Let $(M^n, g)$ be a connected Riemannian manifold and let $H \subset M$ be a hypersurface. If a trace-free conformal Killing tensor $K$ vanishes along $H$, then $K$ vanishes identically on $M$.*

*Proof.* Let $K \in \Gamma(\mathrm{Sym}^p_0 \mathrm{T}M)$ be a trace-free conformal Killing tensor vanishing along a hypersurface $H \subset M$. Starting with $K_0 := K$ we recursively define tensors $K_l := \nabla K_{l-1}$, which are sections of $\mathrm{T}M^{\otimes l} \otimes \mathrm{Sym}^p_0 \mathrm{T}M$. We claim that all tensors $K_i$ vanish along $H$. Since the conformal Killing equation is of finite type this will imply that $K$ is identically zero on $M$.

Consider the natural extensions $\mathrm{d} : \Gamma(\mathrm{T}M^{\otimes l} \otimes \mathrm{Sym}^p \mathrm{T}M) \to \Gamma(\mathrm{T}M^{\otimes l} \otimes \mathrm{Sym}^{p+1} \mathrm{T}M)$, $\delta : \Gamma(\mathrm{T}M^{\otimes l} \otimes \mathrm{Sym}^p \mathrm{T}M) \to \Gamma(\mathrm{T}M^{\otimes l} \otimes \mathrm{Sym}^{p-1} \mathrm{T}M)$ and $\nabla : \Gamma(\mathrm{T}M^{\otimes l} \otimes \mathrm{Sym}^p \mathrm{T}M) \to \Gamma(\mathrm{T}M^{\otimes (l+1)} \otimes \mathrm{Sym}^p \mathrm{T}M)$ of $\mathrm{d}$, $\delta$ and $\nabla$, defined on decomposable tensors by

$$\mathrm{d}(T \otimes K) := \sum_i e_i \cdot \nabla_{e_i}(T \otimes K) = T \otimes \mathrm{d}K + \sum_i \nabla_{e_i} T \otimes e_i \cdot K \ ,$$

$$\delta(T \otimes K) := -\sum_i e_i \lrcorner \nabla_{e_i}(T \otimes K) = T \otimes \delta K - \sum_i \nabla_{e_i} T \otimes e_i \lrcorner K \ ,$$

$$\nabla(T \otimes K) := \nabla T \otimes K + \sum_i (e_i \otimes T) \otimes \nabla_{e_i} K \ ,$$

where $\{e_i\}$ denotes as usual a local orthonormal basis of $\mathrm{T}M$. A straightforward computation shows that

$$[\mathrm{d}, \nabla] = \mathcal{R}^+, \qquad [\delta, \nabla] = \mathcal{R}^-, \tag{27}$$

where $\mathcal{R}^+ : \mathrm{T}M^{\otimes l} \otimes \mathrm{Sym}^p \mathrm{T}M \to \mathrm{T}M^{\otimes (l+1)} \otimes \mathrm{Sym}^{p+1} \mathrm{T}M$ is defined by

$$\mathcal{R}^+(T \otimes K) := \sum_{i,j} (e_i \otimes T) \otimes (e_j \cdot R_{e_j, e_i} K) + (e_i \otimes R_{e_j, e_i} T) \otimes (e_j \cdot K) \ ,$$

and $\mathcal{R}^- : \mathrm{T}M^{\otimes l} \otimes \mathrm{Sym}^p \mathrm{T}M \to \mathrm{T}M^{\otimes (l+1)} \otimes \mathrm{Sym}^{p-1} \mathrm{T}M$ is defined by

$$\mathcal{R}^-(T \otimes K) := -\sum_{i,j} (e_i \otimes T) \otimes (e_j \lrcorner R_{e_j, e_i} K) + (e_i \otimes R_{e_j, e_i} T) \otimes (e_j \lrcorner K) \ .$$

Since $K$ is trace-free conformal Killing, Lemma 3.3 shows that

$$\mathrm{d}K_0 \ = \ -\tfrac{1}{n+2p-2} \, \mathrm{L} \, \delta K_0 \ . \tag{28}$$

(Note that since $K$ is trace-free, the notation $K_0$ from Lemma 3.3 coincides with our notation $K = K_0$ above). We will prove by induction that there exist vector bundle morphisms $F_{i,l} : \mathrm{T}M^{\otimes i} \otimes \mathrm{Sym}^p \mathrm{T}M \to \mathrm{T}M^{\otimes l} \otimes \mathrm{Sym}^{p+1} \mathrm{T}M$ such that

$$\mathrm{d}K_l \ = \ -\tfrac{1}{n+2p-2} \, \mathrm{L} \, \delta K_l + \sum_{i=0}^{l-1} F_{i,l}(K_i) \ , \tag{29}$$

where here $\mathrm{L} : \mathrm{T}M^{\otimes i} \otimes \mathrm{Sym}^p \mathrm{T}M \to \mathrm{T}M^{\otimes i} \otimes \mathrm{Sym}^{p+2} \mathrm{T}M$ denotes the natural extension of $\mathrm{L}$, which of course commutes with $\nabla$. For $l = 0$, this is just (28). Assuming that the relation holds for some $l \geq 0$ we get from (27)

$$\begin{aligned}
\mathrm{d}K_{l+1} &= \mathrm{d}\nabla K_l = \nabla \mathrm{d}K_l + \mathcal{R}^+ K_l \\
&= \nabla\left(-\tfrac{1}{n+2p-2}\,\mathrm{L}\,\delta K_l + \sum_{i=0}^{l-1} F_{i,l}(K_i)\right) + \mathcal{R}^+ K_l \\
&= -\tfrac{1}{n+2p-2}\,\mathrm{L}\,\nabla\delta K_l + \sum_{i=0}^{l-1}\left((\nabla F_{i,l})(K_i) + (\mathrm{id}\otimes F_{i,l})(K_{i+1})\right) + \mathcal{R}^+ K_l \\
&= -\tfrac{1}{n+2p-2}\left(\mathrm{L}\,\delta K_{l+1} + \mathrm{L}\,\mathcal{R}^- K_l\right) + \sum_{i=0}^{l-1}\left((\nabla F_{i,l})(K_i) + (\mathrm{id}\otimes F_{i,l})(K_{i+1})\right) + \mathcal{R}^+ K_l\ ,
\end{aligned}$$

which is just (29) for $l$ replaced by $l+1$ and

$$F_{i,l+1} := \begin{cases} \nabla F_{i,l} + (\mathrm{id}\otimes F_{i-1,l}), & i \leq l-1 \\ -\tfrac{1}{n+2p-2}\,\mathrm{L}\,\mathcal{R}^- + (\mathrm{id}\otimes F_{l-1,l}) + \mathcal{R}^+, & i = l\ . \end{cases}$$

This proves (29) for all $l$.

Assume now that $K_0, \ldots, K_l$ vanish along $H$ for some $l \geq 0$. We claim that $K_{l+1}$ is also vanishing along $H$. Take any point $x \in H$ and choose a local orthonormal frame $\{e_i\}$ such that $e_1 =: N$ is normal to $H$ and $e_2, \ldots, e_n$ are tangent to $H$ at $x$. From (29) we have $\mathrm{d}K_l = -\frac{1}{n+2p-2}\,\mathrm{L}\,\delta K_l$ at $x$. Moreover, $\nabla_{e_i} K_l$ vanishes at $x$ for every $i \geq 2$. The previous relation thus reads

$$N \cdot \nabla_N K_l = \tfrac{1}{n+2p-2}\,\mathrm{L}\, N \lrcorner \nabla_N K_l\ . \tag{30}$$

Writing

$$\nabla_N K_l = \sum_{I \in \{1,\ldots,n\}^l} e_I \otimes S_I\ ,$$

with $e_I := e_{i_1} \otimes \ldots \otimes e_{i_l}$, and $S_I \in \mathrm{Sym}^p\,\mathrm{T}M$, (30) becomes

$$N \cdot S_I = \tfrac{1}{n+2p-2}\,\mathrm{L}\, N \lrcorner S_I \tag{31}$$

for every $I$. It is easy to check that this implies $S_I = 0$ for every $I$. Indeed, if $i_0 \in \{0, \ldots, p\}$ denotes the largest index $i$ such that the coefficient $C_i$ of $N^i$ in $S_I$ is non-zero, comparing the coefficients of $N^{i_0+1}$ in (31) yields $C_{i_0} = \frac{1}{n+2p-2} i_0 C_{i_0}$, which is clearly impossible for $n > 2$. This shows that $\nabla_N K_l = 0$ at $x$, and since we already noticed that $\nabla_{e_i} K_l$ vanishes at $x$ for every $i \geq 2$, we have $K_{l+1} = 0$ at $x$. As this holds for every $x \in H$, our claim is proved.

Consequently, if $K$ vanishes along $H$, then all covariant derivatives of $K$ vanish along $H$. Since the conformal Killing equation has finite type (cf. Remark 3.4), this implies that $K$ vanishes identically on $M$ (being a component of a parallel section of some vector bundle on $M$ which vanishes along $H$). This proves the theorem. □

## References


[1] V. Apostolov, D.M.J. Calderbank, P. Gauduchon, *Ambitoric geometry I: Einstein metrics and extremal ambikähler structures*, J. Reine Angew. Math., DOI: 10.1515/crelle-2014-0060, August 2014 arXiv:1302.6975

[2] A. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **10** Springer-Verlag, Berlin, 1987.

[3] B. Carter, R.G. McLenaghan, *Generalized total angular momentum operator for the Dirac equation in curved space-time*, Phys. Rev. D (3) **19** (1979), no. 4, 1093–1097.

[4] B. Coll, J.J. Ferrando, J.A. Sez, Juan Antonio, *On the geometry of Killing and conformal tensors*, J. Math. Phys. **47** (2006), no. 6, 062503, 14 pp.

[5] N. S. Dairbekov, V. A. Sharafutdinov, *Conformal Killing symmetric tensor fields on Riemannian manifolds*, Mat. Tr. **13** (2010), no. 1, 85–145. arXiv: 1103.3637.

[6] J. E. D'Atri, H. K. Nickerson, *Divergence-preserving geodesic symmetries*, J. Differential Geom. **3** (1969), 467–476.

[7] M. Eastwood, *Higher symmetries of the Laplacian*, Ann. of Math. **161** (2005), no. 3, 1645–1665.

[8] D.J.F. Fox, *Geometric structures modeled on affine hypersurfaces and generalizations of the Einstein Weyl and affine hypersphere equations*, arXiv:0909.1897.

[9] D.J.F. Fox, *Einstein-like geometric structures on surfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **12** (2013), no. 3, 499–585.

[10] P. Gauduchon, A. Moroianu, *Killing 2-forms in dimension 4*, arXiv:1506.04292.

[11] G.W. Gibbons, S.A. Hartnoll, C.N. Pope, *Bohm and Einstein-Sasaki metrics, black holes, and cosmological event horizons*, Phys. Rev. D (3) **67** (2003), no. 8, 084024, 24 pp.

[12] A. Gray, *Einstein-like manifolds which are not Einstein*, Geom. Dedicata **7** (1978), no. 3, 259–280.

[13] C. Guillarmou, G.P. Paternain, M. Salo, G. Uhlmann, *The X-ray transform for connections in negative curvature*, arXiv: 1502.04720.

[14] W. Jelonek, *$K$-contact $\mathcal{A}$-manifolds*, Colloq. Math. **75** (1998), no. 1, 9–103.

[15] W. Jelonek, *Killing tensors and Einstein-Weyl geometry*, Colloq. Math. **81** (1999), no. 1, 5–19.

[16] W. Jelonek, *Higher-dimensional Gray Hermitian manifolds*, J. Lond. Math. Soc. (2) **80** (2009), no. 3, 729–749

[17] W. Jelonek, *Compact conformally Kähler Einstein-Weyl manifolds*, Ann. Global Anal. Geom. **43** (2013), no. 1, 19–29.

[18] J. Kalina, A. Pierzchalski, P. Walczak, *Only one of generalized gradients can be elliptic.*, Ann. Polon. Math. **67** (1997), no. 2, 111–120.

[19] R. G. McLenaghan, R. Milson, R. G. Smirnov, *Killing tensors as irreducible representations of the general linear group*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 9, 621–624.

[20] A. Moroianu, U. Semmelmann, *Twistor forms on Kähler manifolds,* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **2** (2003), no. 4, 823–845.

[21] A. M. Naveira, *A classification of Riemannian almost-product manifolds*, Rend. Mat. (7) **3** (1983), no. 3, 577–592.

[22] D.N. Page, C.N. Pope, *Stability analysis of compactifications of $D = 11$ supergravity with $SU(3) \times SU(2) \times U(1)$ symmetry*, Phys. Lett. B **145** (1984), no. 5-6, 337–341.

[23] G.P. Paternain, M. Salo, G. Uhlmann, *Invariant distributions, Beurling transforms and tensor tomography in higher dimensions*, arXiv: 1404.7009.

[24] H. Pedersen, P. Tod, *The Ledger curvature conditions and D'Atri geometry*, Differential Geom. Appl. **11** (1999), no. 2, 155–162.

[25] R. Penrose, M. Walker, *On quadratic first integrals of the geodesic equations for type $\{22\}$ spacetimes*, Commun. Math. Phys. **18** (1970) 265–274.

[26] K. Schöbel, *The variety of integrable Killing tensors on the 3-sphere*, SIGMA Symmetry Integrability Geom. Methods Appl. **10** (2014), Paper 080, 48 pp., arXiv: 1205.6227.

[27] U. Semmelmann, *Conformal Killing forms on Riemannian manifolds*, Math. Z. **245** (2003), no. 3, 503–527.

[28] U. Semmelmann, G. Weingart, *The Weitzenböck machine*, Compositio Math. **146** (2010), no. 2, 507–540.

[29] G. Thompson, *Killing tensors in spaces of constant curvature*, J. Math. Phys. **27** (1986), no. 11, 2693–2699.

[30] N.M.J. Woodhouse, *Killing tensors and the separation of the Hamilton-Jacobi equation*, Commun. Math. Phys. **44** (1975), no. 1, 9–38.



Konstantin Heil, Institut für Geometrie und Topologie, Fachbereich Mathematik, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

*E-mail address*: `konstantin.heil@mathematik.uni-stuttgart.de`

Andrei Moroianu, Laboratoire de Mathématiques de Versailles, UVSQ, CNRS, Université Paris-Saclay, 78035 Versailles, France

*E-mail address*: `andrei.moroianu@math.cnrs.fr`

Uwe Semmelmann, Institut für Geometrie und Topologie, Fachbereich Mathematik, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

*E-mail address*: `uwe.semmelmann@mathematik.uni-stuttgart.de`