

# ON PLURICANONICAL LOCALLY CONFORMALLY KÄHLER MANIFOLDS

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ABSTRACT. We prove that compact pluricanonical locally conformally Kähler manifolds have parallel Lee form.

## 1. INTRODUCTION

A locally conformally Kähler (lcK) manifold is a complex manifold  $(M, J)$  together with a Hermitian metric  $g$  which is conformal to a Kähler metric in the neighbourhood of every point. The logarithmic differentials of the conformal factors glue up to a globally defined closed 1-form  $\theta$ , called the *Lee form*, such that the fundamental 2-form  $\Omega := g(J\cdot, \cdot)$  satisfies

$$(1) \quad d\Omega = \theta \wedge \Omega.$$

If  $\theta$  vanishes identically, the manifold  $(M, g, J)$  is Kähler. We will implicitly assume in the whole paper that the lcK structure is proper, i.e. that  $\theta$  is not identically zero.

When  $\theta$  is parallel with respect to the Levi-Civita connection  $\nabla$  of  $g$ , the lcK manifold  $(M, J, g)$  is called *Vaisman*.

G. Kokarev introduced in the context of harmonic maps [7] the seemingly larger class of *pluricanonical lcK manifolds*, defined as those lcK manifolds  $(M, g, J)$  satisfying

$$(2) \quad (\nabla\theta)^{1,1} = 0.$$

In their recent preprint [11], L. Ornea and M. Verbitsky announce the proof of the following result:

**Theorem 1.** *Every compact pluricanonical lcK manifold  $(M, J, g)$  is Vaisman.*

The arguments given in [11] are based on an impressive amount of previous results by numerous authors. Among these we mention: the classification of complex surfaces by Kodaira, the classification of complex surfaces of Kähler rank 1 by Chiose and Toma [3] and Brunella [2], some results by M. Kato concerning subvarieties of Hopf manifolds [6], the classification of surfaces carrying Vaisman metrics by Belgun [1], as well as several previous results by Ornea, Kamishima [4] and Ornea, Verbitsky, [9], [10].

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As a matter of fact, while we were not able to follow their arguments in detail, we discovered instead that Theorem 1 can be proved in a more direct way. Our idea is based on the observation that on a pluricanonical manifold, the flows of the metric duals  $\xi$  and  $J\xi$  of  $\theta$  and  $J\theta$  commute, and this eventually shows that some partial Laplacian (in the analysts sense) of the square norm of  $\nabla\xi$  is larger than or equal to the square norm of  $\nabla\xi \circ \nabla\xi$ . By looking at a point where  $|\nabla\xi|^2$  is maximal, this shows that  $\nabla\xi \equiv 0$ . The details of the proof are given in Section 3.

In Section 4 we show, with similar arguments, that for every complete lcK manifold  $M$  which is pluricanonical but not Vaisman, there exists a holomorphic and isometric injective immersion  $\Phi : \Sigma \rightarrow M$  with vanishing second fundamental form, where  $\Sigma$  is a Riemannian surface isometric either to the Euclidean plane or to a flat cylinder.

In the final section we explain that due to Theorem 1, Kokarev's strong rigidity [7, Theorem 5.1] is essentially an empty result. This fact was pointed out by the anonymous referee, whom we also thank for other useful suggestions.

*Bibliographical remark.* After the first version of this note was made public, the manuscript [11] was withdrawn and replaced with two other preprints [12], [13], where the proof of Theorem 1 is no longer provided, but referred instead to our present work. We acknowledge, however, the decisive influence of [11], which constituted the original impetus for our study.

## 2. PRELIMINARIES ON LCK METRICS

Whenever a Riemannian metric  $g$  is given on a Riemannian manifold, it induces the so-called *musical isomorphisms*  $\sharp : T^*M \rightarrow TM$  and  $\flat : TM \rightarrow T^*M$ , parallel with respect to the Levi-Civita connection of  $g$  and inverse to each other, defined by  $g(\alpha^\sharp, \cdot) := \alpha$  and  $X^\flat := g(X, \cdot)$  for every  $\alpha \in T^*M$  and  $X \in TM$ .

Assume from now on that  $(M, g, J, \theta)$  is an lcK manifold. It is well known that on Hermitian manifolds, the exterior derivative of the fundamental 2-form  $\Omega := g(J\cdot, \cdot)$  determines its covariant derivative. The formula for the covariant derivative of  $J$  determined by (1) is (see e.g. [8]):

$$(3) \quad \nabla_X J = \frac{1}{2} (X \wedge J\theta + JX \wedge \theta), \quad \forall X \in TM,$$

where if  $\alpha$  is a 1-form,  $J\alpha$  denotes the 1-form defined by  $(J\alpha)(Y) := -\alpha(JY)$  for all tangent vectors  $Y$ , and  $X \wedge \alpha$  is the endomorphism of the tangent bundle defined by

$$(X \wedge \alpha)(Y) := g(X, Y)\alpha^\sharp - \alpha(Y)X.$$

Note that in [8], a different normalization is used in the definition of the Lee form (1), which introduces a factor  $\frac{1}{2}$  in (3), compared to the corresponding formula in [8].

Let  $\xi := \theta^\sharp$  denote the metric dual of the Lee form  $\theta$ . Consider the bilinear form  $\sigma := \nabla\theta$  and the associated endomorphism  $S := \nabla\xi$ . They are related by the formula

$\sigma(\cdot, \cdot) = g(S\cdot, \cdot)$ . Since the Lee form  $\theta$  is closed, we get for every vector fields  $X, Y$  on  $M$ :

$$0 = d\theta(X, Y) = (\nabla_X\theta)(Y) - (\nabla_Y\theta)(X) = \sigma(X, Y) - \sigma(Y, X),$$

thus showing that  $\sigma$  is a symmetric bilinear form, and correspondingly  $S$  is a symmetric endomorphism with respect to the metric  $g$ .

The pluricanonicity condition (2) is equivalent to

$$0 = \nabla\theta(X, Y) + \nabla\theta(JX, JY) = g(SX, Y) + g(SJX, JY) = g(SX - JSJX, Y),$$

for every vector fields  $X, Y$  on  $M$ . We thus see that an lcK structure is pluricanonical if and only if the tensor  $S := \nabla(\theta^\sharp)$  satisfies  $S = JSJ$ , or equivalently

$$(4) \quad SJ = -JS.$$

### 3. PROOF OF THEOREM 1

Assume from now on that  $(M, g, J, \theta)$  is a pluricanonical lcK manifold. We need to show that, under the compactness assumption, the relation (4) implies the vanishing of  $S$ . From the definition of  $S$ , together with (3), we have

$$(5) \quad \nabla_X\xi = SX, \quad \nabla_X(J\xi) = JSX + \frac{1}{2}(\theta(X)J\xi + \theta(JX)\xi - |\theta|^2JX),$$

which by lowering the indices also reads

$$(6) \quad \nabla_X\theta = (SX)^\flat, \quad \nabla_X(J\theta) = (JSX)^\flat + \frac{1}{2}X \lrcorner (\theta \wedge J\theta - |\theta|^2\Omega).$$

By (4), the endomorphism  $JS$  is symmetric. From (6) we thus get

$$(7) \quad d(J\theta) = \theta \wedge J\theta - |\theta|^2\Omega,$$

$$(8) \quad \mathcal{L}_\xi g = 2g(S\cdot, \cdot), \quad \mathcal{L}_{J\xi} g = 2g(JS\cdot, \cdot).$$

Taking a further exterior derivative in (7) and using (1) yields

$$0 = d^2(J\theta) = -\theta \wedge d(J\theta) - d(|\theta|^2) \wedge \Omega - |\theta|^2 d\Omega = -d(|\theta|^2) \wedge \Omega,$$

whence  $|\theta|^2$  is constant on  $M$  (this constancy property of pluricanonical metrics was already noticed in [11]). We thus obtain for every tangent vector  $X$ :

$$0 = X(|\xi|^2) = 2g(\nabla_X\xi, \xi) = 2g(SX, \xi) = 2g(S\xi, X),$$

showing that  $S\xi = 0$  (and therefore also  $SJ\xi = 0$  from (4)). Using (5) we thus get  $\nabla_{J\xi}\xi = \nabla_\xi(J\xi) = \nabla_{J\xi}(J\xi) = \nabla_\xi\xi = 0$ , and in particular

$$(9) \quad [\xi, J\xi] = 0.$$

We note for later use that the distribution  $\{\xi, J\xi\}$  is integrable, and its integral leaves are totally geodesic.

Cartan's formula shows that on every lcK manifold

$$(10) \quad \mathcal{L}_{J\xi}\Omega = d(J\xi \lrcorner \Omega) + J\xi \lrcorner d\Omega = -d\theta + J\xi \lrcorner (\theta \wedge \Omega) = 0.$$

Moreover, on pluricanonical manifolds, equation (7) gives

$$(11) \quad \mathcal{L}_\xi \Omega = d(\xi \lrcorner \Omega) + \xi \lrcorner d\Omega = d(J\theta) + \xi \lrcorner (\theta \wedge \Omega) = 0.$$

From (8) and (11) we infer

$$(12) \quad \mathcal{L}_\xi J = 2JS, \quad \mathcal{L}_{J\xi} J = -2S.$$

We notice that (9) implies  $[\mathcal{L}_\xi, \mathcal{L}_{J\xi}] = \mathcal{L}_{[\xi, J\xi]} = 0$ , and thus from (12):

$$(13) \quad \mathcal{L}_\xi S = -\frac{1}{2}\mathcal{L}_\xi \mathcal{L}_{J\xi} J = -\frac{1}{2}\mathcal{L}_{J\xi} \mathcal{L}_\xi J = -\mathcal{L}_{J\xi}(JS) = 2S^2 - J\mathcal{L}_{J\xi} S,$$

which (after composing with  $J$  on the left) also reads

$$(14) \quad \mathcal{L}_{J\xi} S = J\mathcal{L}_\xi S - 2JS^2.$$

Taking a further Lie derivative in (13) and using (12) yields

$$\begin{aligned} \mathcal{L}_{J\xi} \mathcal{L}_\xi S &= 2S\mathcal{L}_{J\xi} S + 2(\mathcal{L}_{J\xi} S)S + 2S\mathcal{L}_{J\xi} S - J\mathcal{L}_{J\xi}^2 S \\ &= 4S\mathcal{L}_{J\xi} S + 2(\mathcal{L}_{J\xi} S)S - J\mathcal{L}_{J\xi}^2 S. \end{aligned}$$

Similarly, from (14) and (12) we obtain:

$$\begin{aligned} \mathcal{L}_\xi \mathcal{L}_{J\xi} S &= 2JS\mathcal{L}_\xi S + J\mathcal{L}_\xi^2 S - 4JS^3 - 2J(\mathcal{L}_\xi S)S - 2JS\mathcal{L}_\xi S \\ &= J\mathcal{L}_\xi^2 S - 4JS^3 - 2J(\mathcal{L}_\xi S)S \\ &= J\mathcal{L}_\xi^2 S - 8JS^3 - 2(\mathcal{L}_{J\xi} S)S. \end{aligned}$$

Comparing the last two equations and using  $\mathcal{L}_\xi \mathcal{L}_{J\xi} = \mathcal{L}_{J\xi} \mathcal{L}_\xi$  we obtain

$$(15) \quad J(\mathcal{L}_\xi^2 S + \mathcal{L}_{J\xi}^2 S) = 4S\mathcal{L}_{J\xi} S + 4(\mathcal{L}_{J\xi} S)S + 8JS^3.$$

We compose with  $-SJ$  to the left and take the trace in the above equation:

$$\text{tr}(S(\mathcal{L}_\xi^2 S + \mathcal{L}_{J\xi}^2 S)) = -4\text{tr}(SJ\mathcal{L}_{J\xi} S) - 4\text{tr}(SJ(\mathcal{L}_{J\xi} S)S) + 8\text{tr}(S^4) = 8\text{tr}(S^4),$$

from the trace identity and the hypothesis  $SJ = -JS$ . Using this we compute:

$$\begin{aligned} (\mathcal{L}_\xi^2 + \mathcal{L}_{J\xi}^2)(\text{tr}(S^2)) &= \text{tr}((\mathcal{L}_\xi^2 S)S + 2(\mathcal{L}_\xi S)^2 + S(\mathcal{L}_\xi^2 S) + (\mathcal{L}_{J\xi}^2 S)S + 2(\mathcal{L}_{J\xi} S)^2 + S(\mathcal{L}_{J\xi}^2 S)) \\ &= 2\text{tr}((\mathcal{L}_\xi S)^2) + 2\text{tr}((\mathcal{L}_{J\xi} S)^2) + 2\text{tr}(S(\mathcal{L}_\xi^2 S) + S(\mathcal{L}_{J\xi}^2 S)) \\ &= 2\text{tr}((\mathcal{L}_\xi S)^2) + 2\text{tr}((\mathcal{L}_{J\xi} S)^2) + 16\text{tr}(S^4). \end{aligned}$$

By taking the Lie derivative with respect to  $\xi$  of the relation  $g(S\cdot, \cdot) = g(\cdot, S\cdot)$  and using (8) we immediately get  $g(\mathcal{L}_\xi S\cdot, \cdot) = g(\cdot, \mathcal{L}_\xi S\cdot)$ , i.e., the endomorphism  $\mathcal{L}_\xi S$  is symmetric. Taking now the Lie derivative of the relation  $SJ + JS = 0$  with respect to  $\xi$  and using (13) we obtain that  $\mathcal{L}_\xi S$  anti-commutes with  $J$ . Finally, (14) shows that the symmetric part of  $\mathcal{L}_{J\xi} S$  is  $J\mathcal{L}_\xi S$  and its skew-symmetric part is  $-2JS^2$ . The previous relation thus reads

$$\begin{aligned} (\mathcal{L}_\xi^2 + \mathcal{L}_{J\xi}^2)(\text{tr}(S^2)) &= 2\text{tr}((\mathcal{L}_\xi S)^2) + 2\text{tr}((\mathcal{L}_{J\xi} S)^2) + 16\text{tr}(S^4) \\ &= 2\text{tr}((\mathcal{L}_\xi S)^2) + 2\text{tr}((\mathcal{L}_\xi S)^2 - 4S^4) + 16\text{tr}(S^4) \\ &= 4\text{tr}((\mathcal{L}_\xi S)^2) + 8\text{tr}(S^4). \end{aligned}$$

We use now the compactness assumption: there exists a point  $x_{max} \in M$  where  $\text{tr}(S^2)$ , the square norm of  $S$ , attains its supremum. At  $x_{max}$  the left hand side of the equation above is non-positive, while the right hand side is non-negative (since we have seen that  $\mathcal{L}_\xi S$  is symmetric). We deduce that  $\text{tr}(S^4)$  – and thus  $S$  itself – both vanish at  $x_{max}$ , so  $S$  vanishes identically. This is the conclusion of Theorem 1.

#### 4. NON-COMPACT PLURICANONICAL MANIFOLDS

Our method of proof extends partially to the case where the pluricanonical manifold  $(M, J, g)$  is complete but not compact.

**Theorem 2.** *Let  $(M, J, g)$  be a complete pluricanonical manifold which is not Vaisman. Then there exists a Riemannian surface  $\Sigma$  isometric either to the Euclidean plane  $\mathbb{R}^2$  or to a flat cylinder  $\mathbb{R}^2/l\mathbb{Z}$  for some radius  $l > 0$ , and a holomorphic and isometric immersion  $\Phi : \Sigma \rightarrow M$  with vanishing second fundamental form.*

*Proof.* Each leaf  $F$  of the foliation tangent to the totally geodesic distribution  $\{\xi, J\xi\}$  is totally geodesic and, although not necessarily a submanifold in  $M$ , is a complete flat surface. More precisely, there exists a flat Riemannian surface  $\Sigma$  and a holomorphic and isometric injective immersion  $\Phi : \Sigma \rightarrow M$  with vanishing second fundamental form such that  $F = \Phi(\Sigma)$ . The universal cover of  $\Sigma$  is isometric to the Euclidean plane, hence  $\Sigma$  is isomorphic (as Kähler manifold) to either  $\mathbb{R}^2$ , a flat cylinder, or a flat torus.

If  $\Sigma$  is compact, the endomorphism  $S$  vanishes over  $F = \Phi(\Sigma)$  by the same argument as in the last paragraph of the proof of Theorem 1. So if  $(M, J, g)$  is not Vaisman, we must have at least one non-compact leaf, hence the conclusion of the theorem.  $\square$

We do not know whether there exist complete non-compact pluricanonical manifolds which are not Vaisman.

#### 5. FINAL REMARK

Kokarev's original motivation for introducing pluricanonical lcK metrics was an attempt to generalize Siu's strong rigidity [14] to a wider class of manifolds. In [7, Theorem 5.1] he makes the following statement:

*If  $M$  is a compact pluricanonical lcK manifold homotopic to a compact locally Hermitian symmetric space of non-compact type  $M'$  whose universal cover has no hyperbolic plane as a factor, then  $M$  is biholomorphic to  $M'$ .*

This statement is in fact empty since by Theorem 1, every compact pluricanonical lcK manifold is Vaisman, thus its first Betti number is odd [5], [15], whereas the first Betti number of a compact locally Hermitian symmetric space is even.

## REFERENCES

- [1] F. A. Belgun, *On the metric structure of non-Kähler complex surfaces*, Math. Ann. **317** (2000), no. 1, 1–40.
- [2] M. Brunella, *A characterization of Inoue surfaces*, Comment. Math. Helv. **88** (2013), no. 4, 859–874.
- [3] I. Chiose, M. Toma, *On compact complex surfaces of Kähler rank one*, Amer. J. Math. **135** (2013), no. 3, 851–860.
- [4] Y. Kamishima, L. Ornea, *Geometric flow on compact locally conformally Kähler manifolds*, Tohoku Math. J. **57** (2005), no. 2, 201–221.
- [5] T. Kashiwada, S. Sato, *On harmonic forms in compact locally conformal Kähler manifolds with the parallel Lee form*, Ann. Fac. Sci. Univ. Nat. Zaïre (Kinshasa) Sect. Math.-Phys. **6** (1980), no. 1-2, 17–29.
- [6] M. Kato, *Some Remarks on Subvarieties of Hopf Manifolds*, Tokyo J. Math. **2** (1979), no. 1, 47–61.
- [7] G. Kokarev, *On pseudo-harmonic maps in conformal geometry*, Proc. London Math. Soc. **99** (2009), 168–194.
- [8] A. Moroianu, *Compact lcK manifolds with parallel vector fields*. Complex Manifolds **2** (2015), 26–33.
- [9] L. Ornea, M. Verbitsky, *Locally conformal Kähler manifolds with potential*, Math. Ann. **348** (2010), 25–33.
- [10] L. Ornea, M. Verbitsky, *Locally conformally Kähler metrics obtained from pseudoconvex shells*, Proc. Amer. Math. Soc. **144** (2016), 325–335.
- [11] L. Ornea, M. Verbitsky, *Compact pluricanonical manifolds are Vaisman*, arXiv:1512.00968.
- [12] L. Ornea, M. Verbitsky, *Hopf surfaces in locally conformally Kähler manifolds with potential*, arXiv:1601.07421.
- [13] L. Ornea, M. Verbitsky, *LCK rank of locally conformally Kähler manifolds with potential*, arXiv:1601.07413.
- [14] Y. T. Siu, *The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds*, Ann. Math. (2) **112** (1980), 73–111.
- [15] I. Vaisman, *Generalized Hopf manifolds*, Geom. Dedicata **13** (1982), no. 3, 231–255.

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