ON PLURICANONICAL LOCALLY CONFORMALLY KÄHLER MANIFOLDS

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Abstract. We prove that compact pluricanonical locally conformally Kähler manifolds have parallel Lee form.

1. Introduction

A locally conformally Kähler (lcK) manifold is a complex manifold (M, J) together with a Hermitian metric g which is conformal to a Kähler metric in the neighbourhood of every point. The logarithmic differentials of the conformal factors glue up to a globally defined closed 1-form θ, called the *Lee form*, such that the fundamental 2-form $\Omega := g(J\cdot, \cdot)$ satisfies

$$\mathrm{d}\Omega = \theta \wedge \Omega. \tag{1}$$

If θ vanishes identically, the manifold (M, g, J) is Kähler. We will implicitly assume in the whole paper that the lcK structure is proper, i.e. that θ is not identically zero.

When θ is parallel with respect to the Levi-Civita connection ∇ of g, the lcK manifold (M, J, g) is called *Vaisman*.

G. Kokarev introduced in the context of harmonic maps [7] the seemingly larger class of *pluricanonical lcK manifolds*, defined as those lcK manifolds (M, g, J) satisfying

$$(\nabla\theta)^{1,1} = 0. \tag{2}$$

In their recent preprint [11], L. Ornea and M. Verbitsky announce the proof of the following result:

Theorem 1. *Every compact pluricanonical lcK manifold (M, J, g) is Vaisman.*

The arguments given in [11] are based on an impressive amount of previous results by numerous authors. Among these we mention: the classification of complex surfaces by Kodaira, the classification of complex surfaces of Kähler rank 1 by Chiose and Toma [3] and Brunella [2], some results by M. Kato concerning subvarieties of Hopf manifolds [6], the classification of surfaces carrying Vaisman metrics by Belgun [1], as well as several previous results by Ornea, Kamishima [4] and Ornea, Verbitsky, [9], [10].

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As a matter of fact, while were not able to follow their arguments in detail, we discovered instead that Theorem 1 can be proved in a more direct way. Our idea is based on the observation that on a pluricanonical manifold, the flows of the metric duals ξ and $J\xi$ of θ and $J\theta$ commute, and this eventually shows that some partial Laplacian (in the analists sense) of the square norm of $\nabla\xi$ is larger than or equal to the square norm of $\nabla\xi \circ \nabla\xi$. By looking at a point where $|\nabla\xi|^2$ is maximal, this shows that $\nabla\xi \equiv 0$. The details of the proof are given in Section 3.

In Section 4 we show, with similar arguments, that for every complete lcK manifold M which is pluricanonical but not Vaisman, there exists a holomorphic and isometric injective immersion $\Phi : \Sigma \to M$ with vanishing second fundamental form, where Σ is a Riemannian surface isometric either to the Euclidean plane or to a flat cylinder.

In the final section we explain that due to Theorem 1, Kokarev's strong rigidity [7, Theorem 5.1] is essentially an empty result. This fact was pointed out by the anonymous referee, whom we also thank for other useful suggestions.

Bibliographical remark. After the first version of this note was made public, the manuscript [11] was withdrawn and replaced with two other preprints [12], [13], where the proof of Theorem 1 is no longer provided, but referred instead to our present work. We acknowledge, however, the decisive influence of [11], which constituted the original impetus for our study.

2. Preliminaries on lcK metrics

Whenever a Riemannian metric g is given on a Riemannian manifold, it induces the so-called *musical isomorphisms* $\sharp : \mathrm{T}^*M \to \mathrm{T}M$ and $\flat : \mathrm{T}M \to \mathrm{T}^*M$, parallel with respect to the Levi-Civita connection of g and inverse to each other, defined by $g(\alpha^\sharp, \cdot) := \alpha$ and $X^\flat := g(X, \cdot)$ for every $\alpha \in \mathrm{T}^*M$ and $X \in \mathrm{T}M$.

Assume from now on that (M, g, J, θ) is an lcK manifold. It is well known that on Hermitian manifolds, the exterior derivative of the fundamental 2-form $\Omega := g(J\cdot, \cdot)$ determines its covariant derivative. The formula for the covariant derivative of J determined by (1) is (see e.g. [8]):

$$\nabla_X J = \frac{1}{2}\left(X \wedge J\theta + JX \wedge \theta\right), \qquad \forall X \in \mathrm{T}M, \tag{3}$$

where if α is a 1-form, $J\alpha$ denotes the 1-form defined by $(J\alpha)(Y) := -\alpha(JY)$ for all tangent vectors Y, and $X \wedge \alpha$ is the endomorphism of the tangent bundle defined by

$$(X \wedge \alpha)(Y) := g(X, Y)\alpha^\sharp - \alpha(Y)X.$$

Note that in [8], a different normalization is used in the definition of the Lee form (1), which introduces a factor $\frac{1}{2}$ in (3), compared to the corresponding formula in [8].

Let $\xi := \theta^\sharp$ denote the metric dual of the Lee form θ. Consider the bilinear form $\sigma := \nabla\theta$ and the associated endomorphism $S := \nabla\xi$. They are related by the formula

$\sigma(\cdot,\cdot) = g(S\cdot,\cdot)$. Since the Lee form θ is closed, we get for every vector fields X, Y on M:

$$0 = \mathrm{d}\theta(X,Y) = (\nabla_X\theta)(Y) - (\nabla_Y\theta)(X) = \sigma(X,Y) - \sigma(Y,X),$$

thus showing that σ is a symmetric bilinear form, and correspondingly S is a symmetric endomorphism with respect to the metric g.

The pluricanonicality condition (2) is equivalent to

$$0 = \nabla\theta(X,Y) + \nabla\theta(JX,JY) = g(SX,Y) + g(SJX,JY) = g(SX - JSJX, Y),$$

for every vector fields X, Y on M. We thus see that an lcK structure is pluricanonical if and only if the tensor $S := \nabla(\theta^\sharp)$ satisfies $S = JSJ$, or equivalently

$$SJ = -JS. \tag{4}$$

3. Proof of Theorem 1

Assume from now on that (M, g, J, θ) is a pluricanonical lcK manifold. We need to show that, under the compactness assumption, the relation (4) implies the vanishing of S. From the definition of S, together with (3), we have

$$\nabla_X\xi = SX, \qquad \nabla_X(J\xi) = JSX + \frac{1}{2}(\theta(X)J\xi + \theta(JX)\xi - |\theta|^2 JX), \tag{5}$$

which by lowering the indices also reads

$$\nabla_X\theta = (SX)^\flat, \qquad \nabla_X(J\theta) = (JSX)^\flat + \frac{1}{2}X\lrcorner(\theta\wedge J\theta - |\theta|^2\Omega). \tag{6}$$

By (4), the endomorphism JS is symmetric. From (6) we thus get

$$\mathrm{d}(J\theta) = \theta\wedge J\theta - |\theta|^2\Omega, \tag{7}$$

$$\mathcal{L}_\xi g = 2g(S\cdot,\cdot), \qquad \mathcal{L}_{J\xi} g = 2g(JS\cdot,\cdot). \tag{8}$$

Taking a further exterior derivative in (7) and using (1) yields

$$0 = \mathrm{d}^2(J\theta) = -\theta\wedge\mathrm{d}(J\theta) - \mathrm{d}(|\theta|^2)\wedge\Omega - |\theta|^2\mathrm{d}\Omega = -\mathrm{d}(|\theta|^2)\wedge\Omega,$$

whence $|\theta|^2$ is constant on M (this constancy property of pluricanonical metrics was already noticed in [11]) . We thus obtain for every tangent vector X:

$$0 = X(|\xi|^2) = 2g(\nabla_X\xi,\xi) = 2g(SX,\xi) = 2g(S\xi,X),$$

showing that $S\xi = 0$ (and therefore also $SJ\xi = 0$ from (4)). Using (5) we thus get $\nabla_{J\xi}\xi = \nabla_\xi(J\xi) = \nabla_{J\xi}(J\xi) = \nabla_\xi\xi = 0$, and in particular

$$[\xi, J\xi] = 0. \tag{9}$$

We note for later use that the distribution $\{\xi, J\xi\}$ is integrable, and its integral leaves are totally geodesic.

Cartan's formula shows that on every lcK manifold

$$\mathcal{L}_{J\xi}\Omega = \mathrm{d}(J\xi\lrcorner\Omega) + J\xi\lrcorner\mathrm{d}\Omega = -\mathrm{d}\theta + J\xi\lrcorner(\theta\wedge\Omega) = 0. \tag{10}$$

Moreover, on pluricanonical manifolds, equation (7) gives

$$\mathcal{L}_\xi\Omega = \mathrm{d}(\xi\lrcorner\Omega) + \xi\lrcorner\mathrm{d}\Omega = \mathrm{d}(J\theta) + \xi\lrcorner(\theta\wedge\Omega) = 0. \tag{11}$$

From (8) and (11) we infer

$$\mathcal{L}_\xi J = 2JS, \qquad \mathcal{L}_{J\xi} J = -2S. \tag{12}$$

We notice that (9) implies $[\mathcal{L}_\xi, \mathcal{L}_{J\xi}] = \mathcal{L}_{[\xi,J\xi]} = 0$, and thus from (12):

$$\mathcal{L}_\xi S = -\frac{1}{2}\mathcal{L}_\xi\mathcal{L}_{J\xi}J = -\frac{1}{2}\mathcal{L}_{J\xi}\mathcal{L}_\xi J = -\mathcal{L}_{J\xi}(JS) = 2S^2 - J\mathcal{L}_{J\xi}S, \tag{13}$$

which (after composing with J on the left) also reads

$$\mathcal{L}_{J\xi}S = J\mathcal{L}_\xi S - 2JS^2. \tag{14}$$

Taking a further Lie derivative in (13) and using (12) yields

$$\begin{aligned}\mathcal{L}_{J\xi}\mathcal{L}_\xi S &= 2S\mathcal{L}_{J\xi}S + 2(\mathcal{L}_{J\xi}S)S + 2S\mathcal{L}_{J\xi}S - J\mathcal{L}^2_{J\xi}S\\ &= 4S\mathcal{L}_{J\xi}S + 2(\mathcal{L}_{J\xi}S)S - J\mathcal{L}^2_{J\xi}S.\end{aligned}$$

Similarly, from (14) and (12) we obtain:

$$\begin{aligned}\mathcal{L}_\xi\mathcal{L}_{J\xi}S &= 2JS\mathcal{L}_\xi S + J\mathcal{L}^2_\xi S - 4JS^3 - 2J(\mathcal{L}_\xi S)S - 2JS\mathcal{L}_\xi S\\ &= J\mathcal{L}^2_\xi S - 4JS^3 - 2J(\mathcal{L}_\xi S)S\\ &= J\mathcal{L}^2_\xi S - 8JS^3 - 2(\mathcal{L}_{J\xi}S)S.\end{aligned}$$

Comparing the last two equations and using $\mathcal{L}_\xi\mathcal{L}_{J\xi} = \mathcal{L}_{J\xi}\mathcal{L}_\xi$ we obtain

$$J(\mathcal{L}^2_\xi S + \mathcal{L}^2_{J\xi}S) = 4S\mathcal{L}_{J\xi}S + 4(\mathcal{L}_{J\xi}S)S + 8JS^3. \tag{15}$$

We compose with $-SJ$ to the left and take the trace in the above equation:

$$\mathrm{tr}(S(\mathcal{L}^2_\xi S + \mathcal{L}^2_{J\xi}S)) = -4\mathrm{tr}(SJS(\mathcal{L}_{J\xi}S)) - 4\mathrm{tr}(SJ(\mathcal{L}_{J\xi}S)S) + 8\mathrm{tr}(S^4) = 8\mathrm{tr}(S^4),$$

from the trace identity and the hypothesis $SJ = -JS$. Using this we compute:

$$\begin{aligned}(\mathcal{L}^2_\xi + \mathcal{L}^2_{J\xi})(\mathrm{tr}(S^2)) &= \mathrm{tr}\left((\mathcal{L}^2_\xi S)S + 2(\mathcal{L}_\xi S)^2 + S(\mathcal{L}^2_\xi S) + (\mathcal{L}^2_{J\xi}S)S + 2(\mathcal{L}_{J\xi}S)^2 + S(\mathcal{L}^2_{J\xi}S)\right)\\ &= 2\mathrm{tr}((\mathcal{L}_\xi S)^2) + 2\mathrm{tr}((\mathcal{L}_{J\xi}S)^2) + 2\mathrm{tr}\left(S(\mathcal{L}^2_\xi S) + S(\mathcal{L}^2_{J\xi}S)\right)\\ &= 2\mathrm{tr}((\mathcal{L}_\xi S)^2) + 2\mathrm{tr}((\mathcal{L}_{J\xi}S)^2) + 16\mathrm{tr}(S^4).\end{aligned}$$

By taking the Lie derivative with respect to ξ of the relation $g(S\cdot,\cdot) = g(\cdot, S\cdot)$ and using (8) we immediately get $g(\mathcal{L}_\xi S\cdot,\cdot) = g(\cdot,\mathcal{L}_\xi S\cdot)$, i.e., the endomorphism $\mathcal{L}_\xi S$ is symmetric. Taking now the Lie derivative of the relation $SJ + JS = 0$ with respect to ξ and using (13) we obtain that $\mathcal{L}_\xi S$ anti-commutes with J. Finally, (14) shows that the symmetric part of $\mathcal{L}_{J\xi}S$ is $J\mathcal{L}_\xi S$ and its skew-symmetric part is $-2JS^2$. The previous relation thus reads

$$\begin{aligned}(\mathcal{L}^2_\xi + \mathcal{L}^2_{J\xi})(\mathrm{tr}(S^2)) &= 2\mathrm{tr}((\mathcal{L}_\xi S)^2) + 2\mathrm{tr}((\mathcal{L}_{J\xi}S)^2) + 16\mathrm{tr}(S^4)\\ &= 2\mathrm{tr}((\mathcal{L}_\xi S)^2) + 2\mathrm{tr}((\mathcal{L}_\xi S)^2 - 4S^4) + 16\mathrm{tr}(S^4)\\ &= 4\mathrm{tr}((\mathcal{L}_\xi S)^2) + 8\mathrm{tr}(S^4).\end{aligned}$$

We use now the compactness assumption: there exists a point $x_{max} \in M$ where $\mathrm{tr}(S^2)$, the square norm of S, attains its supremum. At x_{max} the left hand side of the equation above is non-positive, while the right hand side is non-negative (since we have seen that $\mathcal{L}_\xi S$ is symmetric). We deduce that $\mathrm{tr}(S^4)$ – and thus S itself – both vanish at x_{max}, so S vanishes identically. This is the conclusion of Theorem 1.

4. Non-compact pluricanonical manifolds

Our method of proof extends partially to the case where the pluricanonical manifold (M, J, g) is complete but not compact.

Theorem 2. *Let (M, J, g) be a complete pluricanonical manifold which is not Vaisman. Then there exists a Riemannian surface Σ isometric either to the Euclidean plane $\mathbb{R}^2$ or to a flat cylinder $\mathbb{R}^2/l\mathbb{Z}$ for some radius $l > 0$, and a holomorphic and isometric immersion $\Phi : \Sigma \to M$ with vanishing second fundamental form.*

Proof. Each leaf F of the foliation tangent to the totally geodesic distribution $\{\xi, J\xi\}$ is totally geodesic and, although not necessarily a submanifold in M, is a complete flat surface. More precisely, there exists a flat Riemannian surface Σ and a holomorphic and isometric injective immersion $\Phi : \Sigma \to M$ with vanishing second fundamental form such that $F = \Phi(\Sigma)$. The universal cover of Σ is isometric to the Euclidean plane, hence Σ is isomorphic (as Kähler manifold) to either $\mathbb{R}^2$, a flat cylinder, or a flat torus.

If Σ is compact, the endomorphism S vanishes over $F = \Phi(\Sigma)$ by the same argument as in the last paragraph of the proof of Theorem 1. So if (M, J, g) is not Vaisman, we must have at least one non-compact leaf, hence the conclusion of the theorem. □

We do not know whether there exist complete non-compact pluricanonical manifolds which are not Vaisman.

5. Final remark

Kokarev's original motivation for introducing pluricanonical lcK metrics was an attempt to generalize Siu's strong rigidity [14] to a wider class of manifolds. In [7, Theorem 5.1] he makes the following statement:

If M is a compact pluricanonical lcK manifold homotopic to a compact locally Hermitian symmetric space of non-compact type M' whose universal cover has no hyperbolic plane as a factor, then M is biholomorphic to M'.

This statement is in fact empty since by Theorem 1, every compact pluricanonical lcK manifold is Vaisman, thus its first Betti number is odd [5], [15], whereas the first Betti number of a compact locally Hermitian symmetric space is even.

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