INVARIANT FOUR-FORMS AND SYMMETRIC PAIRS

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Abstract. We give criteria for real, complex and quaternionic representations to define s-representations, focusing on exceptional Lie algebras defined by spin representations. As applications, we obtain the classification of complex representations whose second exterior power is irreducible or has an irreducible summand of co-dimension one, and we give a conceptual computation-free argument for the construction of the exceptional Lie algebras of compact type.

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1. Introduction

The initial impetus for this text was given by an observation by John Baez [2, p. 200] in his celebrated paper *The Octonions*, concerning the construction of the exceptional Lie algebra $\mathfrak{e}_8$ as the direct sum of $\mathfrak{spin}(16)$ and the real half-spin representation Σ_{16}^+ of Spin(16). After showing that the verification of the Jacobi identity reduces to the case where all three vectors lie in Σ_{16}^+, he writes

"...unfortunately, at this point a brute-force calculation seems to be required. For two approaches that minimize the pain, see the books by Adams [1] *and by Green, Schwartz and Witten* [6]*. It would be nice to find a more conceptual approach."*

This problem can be rephrased in terms of so-called s-representations, introduced by Kostant [9] and studied by Heintze and Ziller [8] and Eschenburg and Heintze [7] among others. Roughly speaking, an s-representation is a real representation of some Lie algebra of compact type $\mathfrak{h}$ which can be realized as the isotropy representation of a symmetric space. It turns out that the obstruction for a given representation $\mathfrak{m}$ to be an s-representation is encoded in some invariant element in the fourth exterior power of $\mathfrak{m}$, defined by the image of the Casimir element from the universal enveloping algebra of $\mathfrak{h}$.

In particular this obstruction automatically vanishes when $(\Lambda^4\mathfrak{m})^{\mathfrak{h}} = 0$. Now, this condition could seem *a priori* quite restrictive. For instance, it can never hold if the

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representation carries a complex (or all the more quaternionic) structure. This is due to the existence of universal elements in $(\Lambda^4\mathfrak{m})^{\mathfrak{h}}$ which are inherent to the structure of $\mathfrak{m}$. Nevertheless, if these are the only invariant elements, one can adapt our construction by adding a $\mathfrak{u}(1)$ or $\mathfrak{sp}(1)$ summand to $\mathfrak{h}$, depending on whether $\mathfrak{m}$ is complex or quaternionic, in order to "kill" the obstruction given by the Casimir element of $\mathfrak{h}$, and turn $\mathfrak{m}$ into an s-representation of $\mathfrak{h}\oplus\mathfrak{u}(1)$ or $\mathfrak{h}\oplus\mathfrak{sp}(1)$ respectively. These results are summarized in Propositions 2.8 and 2.9 below.

The idea of adding a summand to $\mathfrak{h}$ in order to obtain s-representations already appears in [8], in a setting which presents many similarities with ours. However, the criteria for s-representations obtained by Heintze and Ziller are somewhat complementary to ours. In the complex setting, for example, Theorem 2 in [8] can be stated as follows: If $\mathfrak{m}$ is a faithful complex representation of $\mathfrak{h}$ such that the orthogonal complement of $\mathfrak{h}\subset[\![\Lambda^{1,1}\mathfrak{m}]\!]$ is irreducible, then $\mathfrak{m}$ is an s-representation with respect to $\mathfrak{h}\oplus\mathfrak{u}(1)$. In contrast, in Theorem 3.3 we prove a similar statement, but under the different assumption that $\Lambda^2\mathfrak{m}$ is irreducible.

As applications of our results we then obtain in Theorem 3.3 a geometrical proof of the classification by Dynkin of complex representations with irreducible second exterior power as well as a classification result in Theorem 3.4 for representations with quaternionic structure whose second exterior power decomposes in only two irreducible summands (cf. also [12, Prop. 6.8]). This classification is based on a correspondence between such representations and s-representations already pointed out by Wang and Ziller in [16, p. 257] where a framework relating symmetric spaces and strongly isotropy irreducible homogeneous spaces is introduced. These results can also be compared to the classification by Calabi and Vesentini of complex representations whose symmetric power has exactly two irreducible components based on the classification of symmetric spaces, reproved by Wang and Ziller [17] using representation theoretical methods.

In the last section we apply the above ideas in order to give a purely conceptual proof for the existence of $\mathfrak{e}_8$ in Proposition 4.1. The same method, using spin representations, also works for the other exceptional simple Lie algebras except $\mathfrak{g}_2$. Conversely, we show that most spin representations which are isotropy representations of equal rank homogeneous spaces are actually s-representations and define the exceptional Lie algebras. Note that an alternative geometrical approach to the construction of $\mathfrak{f}_4$ and $\mathfrak{e}_8$ using the so-called Killing superalgebra was recently proposed by J. Figueroa-O'Farrill [5].

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2. A characterization of s-representations

Let $(\mathfrak{h}, B_{\mathfrak{h}})$ be a real Lie algebra of compact type endowed with an $\mathrm{ad}_{\mathfrak{h}}$-invariant Euclidean product and let $\rho : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$ be a faithful irreducible representation of $\mathfrak{h}$ over $\mathbb{R}$, endowed with an invariant Euclidean product $B_{\mathfrak{m}}$ (defined up to some positive constant). In order to simplify the notation we will denote $\rho(a)(v)$ by av for all $a \in \mathfrak{h}$ and $v \in \mathfrak{m}$. Our first goal is to find necessary and sufficient conditions for the existence of a Lie algebra structure on $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{m}$ compatible with the above data.

Lemma 2.1. *There exists a unique* ($\mathbb{R}$*-linear*) *bracket* $[.,.] : \Lambda^2\mathfrak{g}^* \to \mathfrak{g}$ *on* $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{m}$ *such that*

1. $[.,.]$ *restricted to* $\mathfrak{h}$ *is the usual Lie algebra bracket.*
2. $[a, v] = -[v, a] = av$ *for all* $a \in \mathfrak{h}$ *and* $v \in \mathfrak{m}$.
3. $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.
4. $B_{\mathfrak{h}}(a, [v, w]) = B_{\mathfrak{m}}(av, w)$ *for all* $a \in \mathfrak{h}$ *and* $v, w \in \mathfrak{m}$.

Proof. The uniqueness is clear. For the existence we just need to check that the restriction of $[.,.]$ to $\mathfrak{m} \otimes \mathfrak{m}$ given by (4) is skew-symmetric. This follows from the $\mathrm{ad}_{\mathfrak{h}}$-invariance of $B_{\mathfrak{m}}$. □

Definition 2.2. (*cf.* [9]) An irreducible representation $\mathfrak{m}$ of a normed Lie algebra $(\mathfrak{h}, B_{\mathfrak{h}})$ such that the bracket given by Lemma 2.1 defines a Lie algebra structure on $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{m}$ is called an *s-representation.* The Lie algebra $\mathfrak{g}$ is called the *augmented* Lie algebra of the s-representation $\mathfrak{m}$.

Note that the above construction was studied in greater generality by Kostant [10], who introduced the notion of *orthogonal representation of Lie type*, satisfying conditions (1), (2) and (4) in Lemma 2.1. One can compare his characterization of representations of Lie type ([10], Thm. 1.50 and 1.59) with Proposition 2.5 below.

Remark 2.3. If $\mathfrak{g}$ is the augmented Lie algebra of an s-representation of $(\mathfrak{h}, B_{\mathfrak{h}})$ on $\mathfrak{m}$, then the involution $\sigma := \mathrm{id}_{\mathfrak{h}} - \mathrm{id}_{\mathfrak{m}}$ is an automorphism of $\mathfrak{g}$, and $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is a symmetric pair of compact type. Conversely, every irreducible symmetric pair of compact type can be obtained in this way.

In the sequel $\{e_i\}$ will denote a $B_{\mathfrak{m}}$-orthonormal basis of $\mathfrak{m}$ and $\{a_k\}$ a $B_{\mathfrak{h}}$-orthonormal basis of $\mathfrak{h}$.

Lemma 2.4. *If an irreducible s-representation* $(\mathfrak{m}, B_{\mathfrak{m}})$ *of* $\mathfrak{h}$ *has a* $\mathfrak{h}$*-invariant orthogonal complex structure, then* $\mathfrak{h}$ *is not semi-simple.*

Proof. If J denotes the complex structure of $\mathfrak{m}$ commuting with the $\mathfrak{h}$-action, then for all $\alpha \in \mathfrak{h}$ and $v, w \in \mathfrak{m}$ we have

$$B_{\mathfrak{h}}(a, [Jv, w]) = B_{\mathfrak{m}}(aJv, w) = B_{\mathfrak{m}}(Jav, w) = -B_{\mathfrak{m}}(av, Jw) = -B_{\mathfrak{h}}(a, [v, Jw]),$$

whence

$$[Jv, w] = -[v, Jw]. \tag{1}$$

We now consider the element

$$a_J := \sum_i [e_i, Je_i] \in \mathfrak{h} \tag{2}$$

which clearly belongs to the center of $\mathfrak{h}$. In order to show that it does not vanish, we compute using the Jacobi identity

$$a_J v = [a_J, v] = \sum_i [[e_i, Je_i], v] = -\sum_i ([[Je_i, v], e_i] + [[v, e_i], Je_i]) = -2\sum_i [[Je_i, v], e_i].$$

We take $v = e_j$, make the scalar product with Je_j and sum over j. Using (1) we get

$$\sum_j B_{\mathfrak{m}}(a_J e_j, Je_j) = -2\sum_{i,j} B_{\mathfrak{h}}([Je_i, e_j], [e_i, Je_j]) = 2\sum_{i,j} B_{\mathfrak{h}}([Je_i, e_j], [Je_i, e_j]). \tag{3}$$

If $a_J = 0$ this equation would imply that the bracket vanishes on $\mathfrak{m}$, whence

$$0 = B_{\mathfrak{h}}(a, [v, w]) = B_{\mathfrak{m}}(av, w), \qquad \forall a \in \mathfrak{h},\ \forall v, w \in \mathfrak{m},$$

which is clearly impossible. ☐

The element a_J defined in the proof above plays an important rôle in the theory of Hermitian symmetric spaces. For now, let us remark that because of the irreducibility of $\mathfrak{m}$ and of the fact that a_J belongs to the center of $\mathfrak{h}$, there exists a non-zero constant μ, depending on the choice of $B_{\mathfrak{m}}$, such that

$$a_J v = \mu J v, \qquad \forall v \in \mathfrak{m}, \tag{4}$$

in other words a_J acts like μJ on $\mathfrak{m}$.

Using the $\mathfrak{h}$-invariant scalar product $B_{\mathfrak{m}}$, the representation ρ induces a Lie algebra morphism $\tilde\rho : \mathfrak{h} \to \Lambda^2\mathfrak{m} \subset \Lambda^{even}\mathfrak{m}$, $a \mapsto \tilde\rho(a) := \tilde a$ where

$$\tilde a(u, v) = B_{\mathfrak{m}}(au, v) = B_{\mathfrak{h}}(a, [u, v]). \tag{5}$$

For later use, we recall that the induced Lie algebra action of $\mathfrak{h}$ on exterior 2-forms is $a_*(\tau)(u, v) := -\tau(au, v) - \tau(u, av)$, for all $\tau \in \Lambda^2\mathfrak{m}$, whence, in particular, the following formula holds:

$$a_*\tilde b = \widetilde{[a, b]}, \qquad \forall a, b \in \mathfrak{h}. \tag{6}$$

The Lie algebra morphism $\tilde\rho$ extends to an algebra morphism $\tilde\rho : \mathrm{U}(\mathfrak{h}) \to \Lambda^{even}\mathfrak{m}$, where $\mathrm{U}(\mathfrak{h})$ denotes the enveloping algebra of $\mathfrak{h}$. This morphism maps the Casimir element $\mathrm{Cas}_{\mathfrak{h}} = \sum_k (a_k)^2$ of $\mathfrak{h}$ to an invariant element $\tilde\rho(\mathrm{Cas}_{\mathfrak{h}}) \in (\Lambda^4\mathfrak{m})^{\mathfrak{h}}$. It was remarked by Kostant [9] and Heintze and Ziller [8] that this element is exactly the obstruction for $\mathfrak{m}$ to be an s-representation. We provide the proof of this fact below for the reader's convenience.

Proposition 2.5 ([8], [9]). *An irreducible representation* $(\mathfrak{m}, B_\mathfrak{m})$ *of* $\mathfrak{h}$ *is an* s*-representation if and only if* $\tilde\rho(\mathrm{Cas}_\mathfrak{h}) = 0$.

Proof. We need to check that the Jacobi identity for the bracket defined in Lemma 2.1 on $\mathfrak{h} \oplus \mathfrak{m}$ is equivalent to the vanishing of $\tilde\rho(\mathrm{Cas}_\mathfrak{h})$. Note that the Jacobi identity is automatically satisfied by $[.,.]$ whenever one of the three entries belongs to $\mathfrak{h}$.

We now take four arbitrary vectors $u, v, w, z \in \mathfrak{m}$ and compute the obstruction

$$\mathcal{J}(u,v,w) := [[u,v],w] + [[v,w],u] + [[w,u],v]$$

using (5) as follows:

$$\begin{aligned}
B_\mathfrak{m}(\mathcal{J}(u,v,w),z) &= B_\mathfrak{m}([[u,v],w] + [[v,w],u] + [[w,u],v], z)\\
&= B_\mathfrak{h}([u,v],[w,z]) + B_\mathfrak{h}([v,w],[u,z]) + B_\mathfrak{h}([w,u],[v,z])\\
&= \sum_k \big(B_\mathfrak{h}(a_k,[u,v])B_\mathfrak{h}(a_k,[w,z]) + B_\mathfrak{h}(a_k,[v,w])B_\mathfrak{h}(a_k,[u,z])\\
&\qquad + B_\mathfrak{h}(a_k,[w,u])B_\mathfrak{h}(a_k,[v,z])\big)\\
&= \sum_k (\tilde a_k(u,v)\tilde a_k(w,z) + \tilde a_k(v,w)\tilde a_k(u,z) + \tilde a_k(w,u)\tilde a_k(v,z))\\
&= \frac{1}{2}\sum_k (\tilde a_k \wedge \tilde a_k)(u,v,w,z) = \frac{1}{2}\tilde\rho(\mathrm{Cas}_\mathfrak{h})(u,v,w,z).
\end{aligned}$$

□

The above result yields a simple criterion for s-representations:

Corollary 2.6. *If* $(\Lambda^4\mathfrak{m})^\mathfrak{h} = 0$*, then* $\mathfrak{m}$ *is an* s*-representation.*

Conversely, one could ask whether every s-representation arises in this way. One readily sees that this is not the case, since the condition $(\Lambda^4\mathfrak{m})^\mathfrak{h} = 0$ can only hold if $\mathfrak{h}$ is simple and $\mathfrak{m}$ is a purely real representation (*cf.* Lemma 2.7 below). Nevertheless, under these restrictions, the converse to Corollary 2.6 also holds, *cf.* Proposition 2.10 below.

Lemma 2.7. *Let* $\mathfrak{m}$ *be an irreducible real representation of* $(\mathfrak{h}, B_\mathfrak{h})$ *with* $\dim_\mathbb{R}(\mathfrak{m}) \geq 4$. *Then* $(\Lambda^4\mathfrak{m})^\mathfrak{h}$ *is non-zero if either* $\mathfrak{m}$ *has a complex structure or* $\mathfrak{h}$ *is not simple.*

Proof. If J is a $\mathfrak{h}$-invariant complex structure on $\mathfrak{m}$, then $B_\mathfrak{m}(J.,J.)$ is a positive definite $\mathfrak{h}$-invariant scalar product on $\mathfrak{m}$ so by the irreducibility of $\mathfrak{m}$ there is some positive constant ν such that $B_\mathfrak{m}(Ju,Jv) = \nu B_\mathfrak{m}(u,v)$ for every $u, v \in \mathfrak{m}$. Applying this relation to Ju, Jv yields $\nu^2 = 1$, so $\nu = 1$, *i.e.* J is orthogonal. The corresponding 2-form $\omega \in \Lambda^2\mathfrak{m}$ defined by

$$\omega(u,v) := B_\mathfrak{m}(Ju,v) \tag{7}$$

is $\mathfrak{h}$-invariant. Moreover, since $\dim_\mathbb{R}(\mathfrak{m}) \geq 4$, the four-form $\omega\wedge\omega$ is a non-zero element in $(\Lambda^4\mathfrak{m})^\mathfrak{h}$.

Assume now that $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ is not simple. Then $\mathfrak{m} = \mathfrak{m}_1 \otimes \mathfrak{m}_2$ is the tensor product of irreducible representations $\mathfrak{m}_i$ of $\mathfrak{h}_i$. We endow each $\mathfrak{m}_i$ with an $\mathfrak{h}_i$-invariant scalar product and identify $\mathfrak{m}$ with the representation $L(\mathfrak{m}_1, \mathfrak{m}_2)$ of linear maps between $\mathfrak{m}_1$ and $\mathfrak{m}_2$. If $u \in L(\mathfrak{m}_1, \mathfrak{m}_2)$ we denote by $u^* \in L(\mathfrak{m}_2, \mathfrak{m}_1)$ its adjoint. We now consider the element $R \in \mathrm{Sym}^2(\Lambda^2\mathfrak{m})$ given by

$$R(u, v, w, z) := \mathrm{tr}((uv^* - vu^*)(wz^* - zw^*)),$$

and the four-form $\Omega := \beta(R)$, where β is the Bianchi map $\beta : \mathrm{Sym}^2(\Lambda^2(\mathfrak{m})) \to \Lambda^4(\mathfrak{m})$ defined by

$$\beta(T)(u, v, w, z) := T(u, v, w, z) + T(v, w, u, z) + T(w, u, v, z).$$

It is clear that Ω belongs to $(\Lambda^4\mathfrak{m})^{\mathfrak{h}}$ (it is actually $\mathfrak{so}(\mathfrak{m}_1) \oplus \mathfrak{so}(\mathfrak{m}_2)$-invariant). To see that it is non-zero, take orthonormal bases $\{x_i\}$, $\{y_j\}$ of $\mathfrak{m}_1$ and $\mathfrak{m}_2$ and check that $\Omega(z_{11}, z_{12}, z_{21}, z_{22}) = 2$ for $z_{ij} := x_i \otimes y_j$. □

In view of Lemma 2.7, it would be interesting to relax the condition $(\Lambda^4\mathfrak{m})^{\mathfrak{h}} = 0$ in Corollary 2.6 in order to obtain a criterion which could cover also the cases of complex or quaternionic representations. Let us first clarify the terminology. It is well-known that if $\rho : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$ is an irreducible $\mathbb{R}$-representation of $\mathfrak{h}$, the centralizer of $\rho(\mathfrak{h})$ in $\mathrm{End}(\mathfrak{m})$ is an algebra isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Correspondingly, we will say that $\mathfrak{m}$ has real, complex or quaternionic type respectively.

Remark that if a real representation $\mathfrak{m}$ of a semi-simple Lie algebra $(\mathfrak{h}, B_{\mathfrak{h}})$ of compact type has a complex structure I, then it can not be an s-representation by Lemma 2.4. Nevertheless, it turns out that the natural extension of ρ to $\mathfrak{h} \oplus \mathfrak{u}(1)$ defined on the generator $\mathfrak{i} \in \mathfrak{u}(1)$ by $\rho(\mathfrak{i}) = I \in \mathrm{End}(\mathfrak{m})$ can be an s-representation provided the space of invariant four-forms on $\mathfrak{m}$ is one-dimensional. More precisely, we have the following:

Proposition 2.8. *Let $\mathfrak{m}$ be a real representation of complex type of a semi-simple Lie algebra $(\mathfrak{h}, B_{\mathfrak{h}})$ of compact type and consider the representation of $\mathfrak{h} \oplus \mathfrak{u}(1)$ on $\mathfrak{m}$ induced by the complex structure. If $\dim_{\mathbb{R}}(\Lambda^4\mathfrak{m})^{\mathfrak{h}\oplus\mathfrak{u}(1)} = 1$, then there exists a unique positive real number r such that $\mathfrak{m}$ is an s-representation of $\mathfrak{h} \oplus \mathfrak{u}(1)$ with respect to the scalar product $B_{\mathfrak{h}} + rB_{\mathfrak{u}(1)}$. We denote here by $B_{\mathfrak{u}(1)}$ the scalar product on $\mathfrak{u}(1)$ satisfying $B_{\mathfrak{u}(1)}(\mathfrak{i}, \mathfrak{i}) = 1$.*

Proof. For every $a, b \in \mathfrak{h}$ we have $\mathrm{tr}(abI) = 0$ since a is skew-symmetric and bI is symmetric as endomorphisms of $\mathfrak{m}$. Consequently $\mathrm{tr}([a, b]I) = \mathrm{tr}(abI) - \mathrm{tr}(baI) = 0$. Since $\mathfrak{h}$ is semi-simple we have $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$, so $\mathrm{tr}(aI) = 0$ for all $a \in \mathfrak{h}$.

Let $\omega \in \Lambda^2\mathfrak{m}$ be the two-form corresponding to I by (7). An orthonormal basis of $(\mathfrak{h} \oplus \mathfrak{u}(1), B_{\mathfrak{h}} + rB_{\mathfrak{u}(1)})$ is $\{a_k, \frac{1}{\sqrt{r}}\mathfrak{i}\}$. The element in $\Lambda^2\mathfrak{m}$ induced by $\mathfrak{i}$ being $\tilde\rho(\mathfrak{i}) = \omega$, the image of the Casimir element corresponding to $B_{\mathfrak{h}} + rB_{\mathfrak{u}(1)}$ in $\Lambda^4\mathfrak{m}$ is $\tilde\rho(\mathrm{Cas}_{\mathfrak{h}}) + \frac{1}{r}\omega \wedge \omega$. Both summands are clearly $\mathfrak{h}$-invariant. To see that they are $\mathfrak{u}(1)$-invariant, note that by (6), both ω and the 2-forms $\tilde a \in \Lambda^2\mathfrak{m}$ for $a \in \mathfrak{h}$ are invariant under the induced action of $\mathfrak{u}(1)$ on $\Lambda^2\mathfrak{m}$. The hypothesis thus shows that there exists some real constant c with

$$\tilde\rho(\mathrm{Cas}_{\mathfrak{h}}) = c\,\omega \wedge \omega. \tag{8}$$

It remains to show that c is negative (since then one can apply Proposition 2.5 for $r = -1/c$).

Let $\lambda : \Lambda^k\mathfrak{m} \to \Lambda^{k-2}\mathfrak{m}$ denote the metric adjoint of the wedge product with ω. It satisfies

$$\lambda(\tau) = \tfrac{1}{2}\sum_i Ie_i \lrcorner e_i \lrcorner \tau$$

for every $\tau \in \Lambda^k\mathfrak{m}$, where $\lrcorner$ denotes the inner product. Let $2n \geq 4$ be the real dimension of $\mathfrak{m}$. Then $\lambda(\omega) = n$ and $\lambda(\omega \wedge \omega) = (2n-2)\omega$. In terms of λ, the relation $\mathrm{tr}(aI) = 0$ obtained above just reads $\lambda(\tilde{a}) = 0$ for all $a \in \mathfrak{h}$. We then get

$$\lambda(\tilde{\rho}(\mathrm{Cas}_\mathfrak{h})) = \lambda(\sum_k (\tilde{a}_k \wedge \tilde{a}_k)) = \sum_{i,k}(a_k e_i \wedge a_k I e_i),$$

whence

$$\lambda^2(\tilde{\rho}(\mathrm{Cas}_\mathfrak{h})) = -\frac{1}{2}\sum_{i,k} B_\mathfrak{m}(Ia_k e_i, Ia_k e_i).$$

From (8) we thus find $c(2n^2 - 2n) = -\frac{1}{2}\sum_{i,k} B_\mathfrak{m}(Ia_k e_i, Ia_k e_i)$, so c is negative. □

We consider now the quaternionic case. It turns out that a real representation $\mathfrak{m}$ of quaternionic type is never an s-representation. Indeed, if $\mathfrak{m}$ is an s-representation, it follows from the proof of Lemma 2.4 that the three elements a_I, a_J and a_K defined from the quaternionic structure by (2) belong to the center of $\mathfrak{h}$, so in particular a_I and a_J commute. On the other hand, (4) shows that a_I and a_J anti-commute when acting on $\mathfrak{m}$.

However, like in the complex case, there are situations when one may turn $\mathfrak{m}$ into an s-representation by adding an extra summand $\mathfrak{sp}(1)$ to $\mathfrak{h}$, and making it act on $\mathfrak{m}$ via the quaternionic structure.

Proposition 2.9. *Let $\mathfrak{m}$ be a real representation of quaternionic type of a Lie algebra $(\mathfrak{h}, B_\mathfrak{h})$ of compact type and consider the representation of $\mathfrak{h} \oplus \mathfrak{sp}(1)$ on $\mathfrak{m}$ induced by the quaternionic structure. If $\dim_\mathbb{R}(\Lambda^4\mathfrak{m})^{\mathfrak{h}\oplus\mathfrak{sp}(1)} = 1$, then there exists a unique positive real number r such that the induced representation of $(\mathfrak{h} \oplus \mathfrak{sp}(1), B_\mathfrak{h} + rB_{\mathfrak{sp}(1)})$ on $\mathfrak{m}$ is an s-representation, where $B_{\mathfrak{sp}(1)}$ denotes the scalar product of $\mathfrak{sp}(1)$ such that $\mathfrak{i}, \mathfrak{j}, \mathfrak{k}$ is an orthonormal basis.*

Proof. Let ω_I, ω_J and ω_K denote the elements in $\Lambda^2\mathfrak{m}$ induced by the quaternionic structure $\{\mathfrak{i}, \mathfrak{j}, \mathfrak{k}\}$ via (7). Like before, the image of the Casimir element corresponding to $B_\mathfrak{h} + rB_{\mathfrak{sp}(1)}$ in $\Lambda^4\mathfrak{m}$ is

$$\tilde{\rho}(\mathrm{Cas}_\mathfrak{h}) + \tfrac{1}{r}(\omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K).$$

Both terms are clearly $\mathfrak{h}$-invariant by (6). To see that they are $\mathfrak{sp}(1)$-invariant, we use (6) again to see that the induced action of $\mathfrak{sp}(1)$ on $\Lambda^2\mathfrak{m}$ satisfies

$$\mathfrak{i}_*(\tilde{a}) = 0 \qquad \forall a \in \mathfrak{h} \qquad \text{and} \qquad \mathfrak{i}_*(\omega_I) = 0,\ \mathfrak{i}_*(\omega_J) = 2\omega_K,\ \mathfrak{i}_*(\omega_K) = -2\omega_J,$$

whence $\mathfrak{i}_*(\tilde\rho(\mathrm{Cas}_\mathfrak{h}))=0$ and $\mathfrak{i}_*(\omega_I\wedge\omega_I+\omega_J\wedge\omega_J+\omega_K\wedge\omega_K)=4\omega_J\wedge\omega_K-4\omega_K\wedge\omega_J=0$. The invariance with respect to $\mathfrak{j}_*$ and $\mathfrak{k}_*$ can be proved in the same way. The hypothesis thus shows that there exists some real constant c with

$$\tilde\rho(\mathrm{Cas}_\mathfrak{h})=c(\omega_I\wedge\omega_I+\omega_J\wedge\omega_J+\omega_K\wedge\omega_K), \tag{9}$$

and again it remains to show that c is negative.

Let $\lambda_\mathfrak{i}:\Lambda^k\mathfrak{m}\to\Lambda^{k-2}\mathfrak{m}$ denote the metric adjoint of the wedge product with ω_I. From the computations in the complex case we have

$$\lambda_\mathfrak{i}^2(\tilde\rho(\mathrm{Cas}_\mathfrak{h}))=-\frac{1}{2}\sum_{i,k}B_\mathfrak{m}(Ia_ke_i,Ia_ke_i)\qquad\text{and}\qquad\lambda_\mathfrak{i}^2(\omega_I\wedge\omega_I)=2n(n-1),$$

where $2n$ denotes the real dimension of $\mathfrak{m}$. An easy computation gives

$$\lambda_\mathfrak{i}^2(\omega_J\wedge\omega_J)=\lambda_\mathfrak{i}^2(\omega_K\wedge\omega_K)=2n,$$

so from (9) we get $c(2n^2+2n)=-\frac{1}{2}\sum_{i,k}B_\mathfrak{m}(Ia_ke_i,Ia_ke_i)$, showing that c is negative. □

We can summarize Corollary 2.6 and Propositions 2.8, 2.9 by saying that a certain condition on the invariant part of $\Lambda^4\mathfrak{m}$ is sufficient for the existence of an s-representation on $\mathfrak{m}$. Conversely one might ask whether this condition is also necessary for a given s-representation. It turns out that this is always the case if $\mathfrak{h}$ is simple. More precisely, we have:

Proposition 2.10. *Let $(\mathfrak{h},B_\mathfrak{h})$ be a simple Lie algebra of compact type and $\mathfrak{m}$ an irreducible representation of $\mathfrak{h}$ over $\mathbb{R}$.*

1. *If $\mathfrak{m}$ is an s-representation representation of $\mathfrak{h}$, then $(\Lambda^4\mathfrak{m})^\mathfrak{h}=0$.*
2. *If $\mathfrak{m}$ has complex type and is an s-representation of $(\mathfrak{h}\oplus\mathfrak{u}(1),B_\mathfrak{h}+rB_{\mathfrak{u}(1)})$ for some positive real number r, then $\dim_\mathbb{R}(\Lambda^4\mathfrak{m})^{\mathfrak{h}\oplus\mathfrak{u}(1)}=1$.*
3. *If $\mathfrak{m}$ has quaternionic type and is an s-representation of $(\mathfrak{h}\oplus\mathfrak{sp}(1),B_\mathfrak{h}+rB_{\mathfrak{sp}(1)})$ for some positive real number r, then $\dim_\mathbb{R}(\Lambda^4\mathfrak{m})^{\mathfrak{h}\oplus\mathfrak{sp}(1)}=1$.*

Proof. The statement of (1) is already contained in [9]. For the proof one has to use the well-known fact that the dimension of $(\Lambda^4\mathfrak{m})^\mathfrak{h}$ is just the fourth Betti number of the corresponding symmetric space G/H. Now, since in the case of compact symmetric spaces the Poincaré polynomials are known explicitly, it is easy to check that $\mathrm{b}_4(G/H)$ vanishes. The proof in the cases (2) and (3) is similar, and is based on the computation of the fourth Betti numbers of Hermitian symmetric spaces and Wolf spaces. □

3. Applications to complex representations

In this section we will give some applications of our characterization of s-representations in order to classify complex representations whose exterior powers have certain irreducibility properties. From now on $\mathfrak{m}$ will denote a *complex* irreducible representation of some Lie algebra $\mathfrak{h}$ of compact type. We will study three instances, corresponding to the cases where $\mathfrak{m}$ has a real structure, $\mathfrak{m}$ is purely complex, or $\mathfrak{m}$ has a quaternionic structure.

3.1. **Representations with a real structure.** Our main result in this case is the classification of all complex representations $\mathfrak{m}$ with real structure whose fourth exterior power has no trivial summand. Let $[\![\mathfrak{m}]\!]$ denote the real part of $\mathfrak{m}$, *i.e.* the fix point set of the real structure. Then $[\![\mathfrak{m}]\!]$ is a real representation of $\mathfrak{h}$ and $\mathfrak{m} = [\![\mathfrak{m}]\!] \otimes_{\mathbb{R}} \mathbb{C}$.

Theorem 3.1. *Let $\mathfrak{m}$ be a complex irreducible faithful representation with real structure of a Lie algebra $\mathfrak{h}$ of compact type such that $(\Lambda^4\mathfrak{m})^{\mathfrak{h}} = 0$. Then $\mathfrak{h}$ is simple, $\mathfrak{g} := \mathfrak{h} \oplus [\![\mathfrak{m}]\!]$ has a Lie algebra structure and $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair which belongs to the following list:*

Helgason's type	$\mathfrak{h}$	$[\![\mathfrak{m}]\!]$	$\mathfrak{g}$
	$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{h} \oplus \mathfrak{h}$
BD I	$\mathfrak{so}(n), n \neq 4$	$\mathbb{R}^n$	$\mathfrak{so}(n+1)$
A I	$\mathfrak{so}(n), n \neq 4$	$\mathrm{Sym}^2_0 \mathbb{R}^n$	$\mathfrak{su}(n)$
A II	$\mathfrak{sp}(n)$	$\Lambda^2_0 \mathbb{H}^n$	$\mathfrak{su}(2n)$
F II	$\mathfrak{spin}(9)$	Σ_9	F_4
E I	$\mathfrak{sp}(4)$	$\Lambda^4_0 \mathbb{H}^4$	E_6
E IV	F_4	V_{26}	E_6
E V	$\mathfrak{su}(8)$	$[\![\Lambda^4\mathbb{C}^8]\!]$	E_7
E VIII	$\mathfrak{spin}(16)$	Σ^+_{16}	E_8

where V_{26} denotes the real 26-dimensional irreducible representation of F_4 *and $[\![\mathfrak{m}]\!] = \mathfrak{h}$ in the first row denotes the adjoint representation of $\mathfrak{h}$.*

Proof. Since $(\Lambda^4\mathfrak{m})^{\mathfrak{h}} = (\Lambda^4_{\mathbb{R}}[\![\mathfrak{m}]\!])^{\mathfrak{h}} \otimes \mathbb{C}$, the hypothesis implies that $(\Lambda^4_{\mathbb{R}}[\![\mathfrak{m}]\!])^{\mathfrak{h}} = 0$. From Lemma 2.7 it follows that $\mathfrak{h}$ has to be simple, and Corollary 2.6 implies that $[\![\mathfrak{m}]\!]$ is an s-representation and thus $[\![\mathfrak{m}]\!]$ is the isotropy representation of an irreducible symmetric space G/H of compact type with H simple.

The list of possible pairs $(\mathfrak{h}, [\![\mathfrak{m}]\!])$ then follows from the list of irreducible symmetric spaces of compact type [3, p. 312-314]. Here the adjoint representation on $[\![\mathfrak{m}]\!] = \mathfrak{h}$ corresponds to the isotropy representation on symmetric spaces of the type II, *i.e.* of the form $(H \times H)/H$.

Conversely, the fourth exterior power of the representations in the table above have no invariant elements by Proposition 2.10. In some cases a direct proof can also be given, see Proposition 4.1 below. □

Remark 3.2. We will see later on in Proposition 4.2 that the real half-spin representation Σ^+_8 is also an s-representation, and has no invariant elements in its fourth exterior power (*cf.* also Proposition 2.10). One may thus wonder why it does not appear in the above table. The explanation is that it actually appears in a disguised form, as the standard representation of $\mathfrak{so}(8)$ on $\mathbb{R}^8$. To make this more precise, note that n-dimensional representations are usually classified up to isomorphism, *i.e.* up to composition with some element in the inner automorphism group of $\mathfrak{so}(n)$. On the other hand, if one wants to classify all pairs $(\mathfrak{h}, \mathfrak{m})$ with $(\Lambda^4\mathfrak{m})^{\mathfrak{h}} = 0$, then there is another group acting on the space of solutions: the outer automorphism group of $\mathfrak{so}(n)$. Our classification above is up to the action of this group. In particular, the triality phenomenon in dimension 8 can be interpreted by saying that the outer automorphism group of $\mathfrak{so}(8)$ is isomorphic to the permutation group of the set $\{\mathbb{R}^8, \Sigma_8^+, \Sigma_8^-\}$ of 8-dimensional representations of $\mathfrak{so}(8) \cong \mathfrak{spin}(8)$. The same remark also applies below to complex representations, where taking the conjugate of a representation can be viewed as composing it with the non-trivial outer automorphism of $\mathfrak{su}(n)$.

3.2. **Representations with irreducible second exterior power.** As another application of the above ideas, we will now obtain in a simple geometrical way Dynkin's classification [4, Thm. 4.7] of complex representations $\mathfrak{m}$ with $\Lambda^2\mathfrak{m}$ irreducible.

Theorem 3.3. *Let $\mathfrak{m}$ be a complex irreducible faithful representation of a Lie algebra $\mathfrak{h}$ of compact type such that $\Lambda^2\mathfrak{m}$ is irreducible. Then either $\mathfrak{h} = \mathfrak{h}_0$ is simple, or $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{u}(1)$ with $\mathfrak{h}_0$ simple, $\mathfrak{g} := \mathfrak{h}_0 \oplus \mathfrak{u}(1) \oplus \mathfrak{m}^{\mathbb{R}}$ has a Lie algebra structure and $(\mathfrak{g}, \mathfrak{h}_0 \oplus \mathfrak{u}(1))$ is a symmetric pair which belongs to the following list:*

Helgason's type	$\mathfrak{h}_0$	$\mathfrak{m}$	$\mathfrak{g}$
A III	$\mathfrak{su}(n)$	$\mathbb{C}^n$	$\mathfrak{su}(n+1)$
D III	$\mathfrak{su}(n)$	$\Lambda^2\mathbb{C}^n$	$\mathfrak{so}(2n)$
C I	$\mathfrak{su}(n)$	$\mathrm{Sym}^2\mathbb{C}^n$	$\mathfrak{sp}(n)$
BD I	$\mathfrak{so}(n), n \neq 4$	$\mathbb{R}^n \otimes \mathbb{C}$	$\mathfrak{so}(n+1)$
E III	$\mathfrak{spin}(10)$	Σ_{10}	E_6
E VII	E_6	V_{27}	E_7

where V_{27} denotes the 27-dimensional irreducible representation of E_6.

Proof. If $\mathfrak{h}$ is not simple, it may be written as the sum of two ideals, $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$, and $\mathfrak{m}$ is the tensor product representation $\mathfrak{m} = E \otimes F$. It follows that

$$\Lambda^2\mathfrak{m} = \Lambda^2(E \otimes F) \cong (\Lambda^2 E \otimes \mathrm{Sym}^2 F) \oplus (\mathrm{Sym}^2 E \otimes \Lambda^2 F).$$

Hence $\Lambda^2\mathfrak{m}$ can only be irreducible if one factor, say F, is one-dimensional. Since F is a faithful representation, one must have $\mathfrak{h}_1 = \mathfrak{u}(1)$. This argument shows that every ideal of $\mathfrak{h}$ has either dimension or co-dimension at most one. In particular, $\mathfrak{h}_0$ is simple.

Consider now the real representation $\mathfrak{m}^{\mathbb{R}}$ of $\mathfrak{h}_0$ obtained by forgetting the complex multiplication in $\mathfrak{m}$. The Lie algebra $\mathfrak{u}(1)$ acts on the fourth exterior power $\Lambda^4_{\mathbb{R}}\mathfrak{m}^{\mathbb{R}}$ by extending the action of the complex structure J from $\mathfrak{m}^{\mathbb{R}}$. We claim that the space of invariant elements $(\Lambda^4_{\mathbb{R}}\mathfrak{m}^{\mathbb{R}})^{\mathfrak{h}_0 \oplus \mathfrak{u}(1)}$ is one-dimensional. If we denote as usual by $\Lambda^{p,q}\mathfrak{m} := \Lambda^p\mathfrak{m} \otimes \Lambda^q\bar{\mathfrak{m}}$, then

$$\Lambda^4_{\mathbb{R}}\mathfrak{m}^{\mathbb{R}} = [\![\Lambda^{4,0}\mathfrak{m}]\!] \oplus [\![\Lambda^{3,1}\mathfrak{m}]\!] \oplus [\![\Lambda^{2,2}\mathfrak{m}]\!].$$

Since J^2 acts as $-(p-q)^2\mathrm{id}$ on $[\![\Lambda^{p,q}\mathfrak{m}]\!]$, it follows that the $\mathfrak{u}(1)$-invariant part of $\Lambda^4_{\mathbb{R}}\mathfrak{m}^{\mathbb{R}}$ is the third summand $[\![\Lambda^{2,2}\mathfrak{m}]\!] = [\![\Lambda^2\mathfrak{m}\otimes\Lambda^2\bar{\mathfrak{m}}]\!] = [\![\mathrm{End}(\Lambda^2\mathfrak{m})]\!]$. Consequently,

$$(\Lambda^4_{\mathbb{R}}\mathfrak{m}^{\mathbb{R}})^{\mathfrak{h}_0\oplus\mathfrak{u}(1)} = [\![\mathrm{End}(\Lambda^2\mathfrak{m})]\!]^{\mathfrak{h}_0} = [\![(\mathrm{End}(\Lambda^2\mathfrak{m}))^{\mathfrak{h}_0}]\!] \tag{10}$$

is one-dimensional since by assumption $\Lambda^2\mathfrak{m}$ is irreducible as representation of $\mathfrak{h}$, so also of $\mathfrak{h}_0$. We can therefore apply Proposition 2.8 to realize $\mathfrak{m}^{\mathbb{R}}$ as an s-representation of $\mathfrak{h}_0\oplus\mathfrak{u}(1)$, so $\mathfrak{m}^{\mathbb{R}}$ is the isotropy representation of some Hermitian symmetric space. Checking again the list in [3, pp. 312-314] we obtain the possible pairs $(\mathfrak{h}_0,\mathfrak{m})$ as stated above.

Conversely, if $(\mathfrak{h}_0,\mathfrak{m})$ belongs to the above list, then $(\Lambda^4_{\mathbb{R}}\mathfrak{m}^{\mathbb{R}})^{\mathfrak{h}_0\oplus\mathfrak{u}(1)}$ is one-dimensional by Proposition 2.10, thus (10) shows that $\Lambda^2\mathfrak{m}$ is irreducible. □

3.3. **Representations with quaternionic structure.** As another application we will now consider complex representations $\mathfrak{m}$ of $\mathfrak{h}$ with quaternionic structure. Such representations can be characterized by the existence of an invariant element in $\Lambda^2\mathfrak{m}$, which is therefore never irreducible. Considering the $\mathfrak{h}$-invariant decomposition $\Lambda^2\mathfrak{m} = \Lambda^2_0\mathfrak{m}\oplus\mathbb{C}$, one can nevertheless ask whether $\Lambda^2_0\mathfrak{m}$ can be irreducible. The classification of such representations is given by the following:

Theorem 3.4. *Let $\mathfrak{m}$ be a complex irreducible faithful representation of a Lie algebra $\mathfrak{h}$ of compact type with a quaternionic structure, and let $\Lambda^2\mathfrak{m} = \Lambda^2_0\mathfrak{m}\oplus\mathbb{C}$ be the standard decomposition of the second exterior power of $\mathfrak{m}$. If the $\mathfrak{h}$-representation $\Lambda^2_0\mathfrak{m}$ is irreducible then $\mathfrak{h}$ is simple, $\mathfrak{g} := \mathfrak{h}\oplus\mathfrak{sp}(1)\oplus\mathfrak{m}^{\mathbb{R}}$ has a Lie algebra structure and $(\mathfrak{g},\mathfrak{h}\oplus\mathfrak{sp}(1))$ is a symmetric pair which belongs to the following list:*

Helgason's type	$\mathfrak{h}$	$\mathfrak{m}$	$\mathfrak{g}$
C II	$\mathfrak{sp}(n)$	$\mathbb{H}^n$	$\mathfrak{sp}(n+1)$
F I	$\mathfrak{sp}(3)$	$\Lambda^3_0\mathbb{H}^3$	F_4
G I	$\mathfrak{sp}(1)$	$\mathrm{Sym}^3\mathbb{H}$	G_2
E II	$\mathfrak{su}(6)$	$\Lambda^3\mathbb{C}^6$	E_6
E VI	$\mathfrak{spin}(12)$	Σ^+_{12}	E_7
E IX	E_7	V_{56}	E_8

where V_{56} is the 56-dimensional irreducible representation of E_7.

Proof. Let $\mathfrak{i}$ denote the complex structure of $\mathfrak{m}$ and let $\mathfrak{j}$ be the quaternionic structure, *i.e.* a real endomorphism of $\mathfrak{m}^{\mathbb{R}}$ anti-commuting with $\mathfrak{i}$ and satisfying $\mathfrak{j}^2 = -\mathrm{id}$. Like before, if $\mathfrak{h}$ is not simple, one can write $\mathfrak{h} = \mathfrak{h}_0\oplus\mathfrak{h}_1$, $\mathfrak{m} = E\otimes F$ and

$$\Lambda^2\mathfrak{m} \cong (\Lambda^2E\otimes\mathrm{Sym}^2F)\oplus(\mathrm{Sym}^2E\otimes\Lambda^2F).$$

If E and F have both dimension larger than one, then both summands in the above expression have the same property, which is impossible because of the hypothesis. Assume that one factor, say F, is one-dimensional. Since F is a faithful representation, one must have $\mathfrak{h}_1 = \mathfrak{u}(1)$. Let $a \neq 0$ be the endomorphism of $\mathfrak{m}$ determined by the generator of $\mathfrak{u}(1)$. By the Schur Lemma there exists $z\in\mathbb{C}^*$ such that $a = z\,\mathrm{id}$. Since a commutes with the quaternionic structure we must have $z \in \mathbb{R}$, which is impossible since a has to be a skew-symmetric endomorphism of $\mathfrak{m}^{\mathbb{R}}$ with respect to some $\mathfrak{h}$-invariant scalar product. Thus $\mathfrak{h}$ is simple.

The Lie algebra $\mathfrak{sp}(1)$ acts on the fourth exterior power $\Lambda^4_{\mathbb{R}}\mathfrak{m}^{\mathbb{R}}$ by extending the action of the (real) endomorphisms $\mathfrak{i}$ and $\mathfrak{j}$ of $\mathfrak{m}^{\mathbb{R}}$. We claim that the space of invariant elements $(\Lambda^4_{\mathbb{R}}\mathfrak{m}^{\mathbb{R}})^{\mathfrak{h}\oplus\mathfrak{sp}(1)}$ is one-dimensional. Using (10) we see that

$$(\Lambda^4_{\mathbb{R}}\mathfrak{m}^{\mathbb{R}})^{\mathfrak{h}\oplus\mathfrak{u}(1)} = [\![(\mathrm{End}(\Lambda^2\mathfrak{m}))^{\mathfrak{h}}]\!] = [\![(\mathrm{End}(\Lambda^2_0\mathfrak{m}\oplus\mathbb{C}))^{\mathfrak{h}}]\!] = [\![(\mathrm{End}(\Lambda^2_0\mathfrak{m}))^{\mathfrak{h}}]\!]\oplus\mathbb{R}$$

is two-dimensional since by assumption $\Lambda^2_0\mathfrak{m}$ is irreducible. The first summand is generated by $\Omega_1 := \omega_I\wedge\omega_I$, whereas the second one is generated by $\Omega_2 := \omega_J\wedge\omega_J+\omega_K\wedge\omega_K$. Using (6) we readily obtain $\mathfrak{j}_*\Omega_1 = -4\omega_K\wedge\omega_I$ and $\mathfrak{j}_*\Omega_2 = 4\omega_I\wedge\omega_K$, thus showing that $(\Lambda^4_{\mathbb{R}}\mathfrak{m}^{\mathbb{R}})^{\mathfrak{h}\oplus\mathfrak{sp}(1)}$ is one-dimensional and spanned by $\Omega_1+\Omega_2$. We can therefore apply Proposition 2.8 to realize $\mathfrak{m}^{\mathbb{R}}$ as an s-representation of $\mathfrak{h}\oplus\mathfrak{sp}(1)$. Consequently, $\mathfrak{m}$ is the isotropy representation of a Wolf space, and thus belongs to the above table by [3, pp. 312-314].

Conversely, it is standard fact that $\Lambda^2_0\mathbb{H}^n$ is an irreducible $\mathfrak{sp}(n)$-representation, and one can check (*e.g.* using the LiE software [13]) that for all other representations $\mathfrak{m}$ in this table, $\Lambda^2_0\mathfrak{m}$ is indeed irreducible. □

A similar result was obtained in [12, Prop. 6.8] by a different approach using algebraic geometry and the theory of minuscule representations.

4. Spin representations and exceptional Lie algebras

In this section we obtain a completely self-contained construction of exceptional simple Lie algebras based on the results in Section 2. We will only give the details for the construction of E_8 arising from the half-spin representation of Spin(16). The other exceptional simple Lie algebras (except G_2) can be constructed by similar methods using spin representations. Conversely, we give a short algebraic argument showing that there are no other spin representations which are s-representations but those giving rise to exceptional Lie algebras.

4.1. A computation-free argument for the existence of E_8.

As already mentioned in the introduction, the only non-trivial part in the construction of E_8 is to check that the natural bracket on $\mathfrak{spin}(16)\oplus\Sigma^+_{16}$ constructed in Lemma 2.1 satisfies the Jacobi identity. This follows directly from Corollary 2.6, together with the following:

Proposition 4.1. *The fourth exterior power of the real half-spin representation* Σ^+_{16} *has no trivial summand.*

Proof. One can use the plethysm function of the LiE software [13] to check that the fourth exterior power of Σ^+_{16} has nine irreducible summands, each of them being non-trivial. However, our purpose is exactly to replace such brute force computations by conceptual arguments!

Let $\langle .,. \rangle_{\mathrm{S}}$ and $\langle .,. \rangle_{\Sigma}$ be Spin(16)-invariant scalar products on $\mathfrak{spin}(16)$ and Σ_{16}^{+} respectively. We start by recalling that the second exterior power of the real half-spin representation in dimension $8k$ decomposes in irreducible summands as

$$\Lambda^2(\Sigma_{8k}^+) \simeq \bigoplus_{i=1}^{k} \Lambda^{4i-2}(\mathbb{R}^{8k}).$$

This isomorphism can also be proved in an elementary way. Indeed, the right hand term acts skew-symmetrically and faithfully by Clifford multiplication on Σ_{8k}^+ and thus can be identified with a sub-representation of $\Lambda^2(\Sigma_{8k}^+)$. On the other hand, its dimension is equal to

$$\dim \bigoplus_{i=1}^{k} \Lambda^{4i-2}(\mathbb{R}^{8k}) = \frac{1}{8}\left(2^{8k} - (1+i)^{8k} - (1-i)^{8k}\right) = 2^{4k-2}(2^{4k-1}-1) = \dim \Lambda^2(\Sigma_{8k}^+).$$

For $k = 2$ we thus get

$$\Lambda^2(\Sigma_{16}^+) \simeq \Lambda^2(\mathbb{R}^{16}) \oplus \Lambda^6(\mathbb{R}^{16}). \tag{11}$$

Recall the standard decomposition

$$\mathrm{Sym}^2(\Lambda^2(\Sigma_{16}^+)) \simeq \mathcal{R} \oplus \Lambda^4(\Sigma_{16}^+),$$

where $\mathcal{R}$ is the kernel of the Bianchi map $\beta : \mathrm{Sym}^2(\Lambda^2(\Sigma_{16}^+)) \to \Lambda^4(\Sigma_{16}^+)$. The trace element $R_1 \in \mathrm{Sym}^2(\Lambda^2(\Sigma_{16}^+))$ defined by

$$R_1(v, w, v', w') := \langle v \wedge w, v' \wedge w' \rangle_{\Sigma}$$

is invariant under the action of Spin(16) and belongs to $\mathcal{R}$ since $\beta(R_1) = 0$.

Assume for a contradiction that $\Lambda^4(\Sigma_{16}^+)$ contains some invariant element Ω and consider the invariant element $R_2 \in \mathrm{Sym}^2(\Lambda^2(\Sigma_{16}^+))$ defined by

$$R_2(v, w, v', w') := \langle [v, w], [v', w'] \rangle_{\mathrm{S}},$$

where $[.,.]$ is the bracket defined by Lemma 2.1. Since the two irreducible summands in (11) are not isomorphic, the space of invariant elements in $\mathrm{Sym}^2(\Lambda^2(\Sigma_{16}^+))$ has dimension two. Hence there exist real constants k, l such that $R_2 = kR_1 + l\Omega$. In particular we would have

$$|[v, w]|_{\mathrm{S}}^2 = k|v \wedge w|_{\Sigma}^2, \qquad \forall v, w \in \Sigma_{16}^+. \tag{12}$$

Since $\dim(\mathfrak{spin}(16)) = 120$ is strictly smaller than $\dim(\Sigma_{16}^+) - 1 = 127$, one can find non-zero vectors $v_0, w_0 \in \Sigma_{16}^+$ such that $v_0 \wedge w_0 \neq 0$ and $\langle v_0, aw_0 \rangle_{\Sigma} = 0$ for all $a \in \mathfrak{spin}(16)$. By the definition of the bracket in Lemma 2.1 (4), this implies $[v_0, w_0] = 0$, so using (12) for $v = v_0$ and $w = w_0$ yields $k = 0$. By (12) again, this would imply $[v, w] = 0$ for all $v, w \in \Sigma_{16}^+$, so we would have

$$0 = \langle a, [v, w] \rangle_{\mathrm{S}} = \langle av, w \rangle_{\Sigma}, \qquad \forall a \in \mathfrak{spin}(16),\ \forall v, w \in \Sigma_{16}^+,$$

which is clearly a contradiction. □

4.2. **The construction of F_4, E_6 and E_7.** Consider the following spin representations: Σ_9, which is real, Σ_{10} which is purely complex, and Σ_{12}^+ which is quaternionic. In order to show that they give rise to s-representations of $\mathfrak{spin}(9)$, $\mathfrak{spin}(10)\oplus\mathfrak{u}(1)$ and $\mathfrak{spin}(12)\oplus\mathfrak{sp}(1)$ respectively, we need to check that one can apply the criteria in Corollary 2.6, Proposition 2.8 and Proposition 2.9. Taking into account the results in Section 3, it suffices to show that $(\Lambda^4\Sigma_9)^{\mathfrak{spin}(9)} = 0$, and that $\Lambda^2\Sigma_{10}$ and $\Lambda^2_0\Sigma^+_{12}$ are irreducible. The first assertion can be proved like in Proposition 4.1, whereas the two other follow from the classical decompositions of the second exterior power of spin representations

$$\Lambda^2\Sigma_{10} \cong \Lambda^3(\mathbb{C}^{10}), \qquad \Lambda^2\Sigma^+_{12} \cong \Lambda^0(\mathbb{C}^{12}) \oplus \Lambda^4(\mathbb{C}^{12}).$$

4.3. **On spin representations of Lie type.** In this final part we will show that very few spin representations are of Lie type. To make things precise, recall that the real Clifford algebras Cl_n are of the form $\mathbb{K}(r)$ or $\mathbb{K}(r)\oplus\mathbb{K}(r)$ where:

$n:$	$8k+1$	$8k+2$	$8k+3$	$8k+4$	$8k+5$	$8k+6$	$8k+7$	$8k+8$
$r:$	2^{4k}	2^{4k}	2^{4k}	2^{4k+1}	2^{4k+2}	2^{4k+3}	2^{4k+3}	2^{4k+4}
$\mathrm{Cl}_n:$	$\mathbb{C}(r)$	$\mathbb{H}(r)$	$\mathbb{H}(r)\oplus\mathbb{H}(r)$	$\mathbb{H}(r)$	$\mathbb{C}(r)$	$\mathbb{R}(r)$	$\mathbb{R}(r)\oplus\mathbb{R}(r)$	$\mathbb{R}(r)$

The Clifford representation of the real Clifford algebra is by definition the unique irreducible representation of Cl_n for $n \neq 3 \mod 4$, and the direct sum of the two inequivalent representations for $n = 3 \mod 4$. The real spinor representation Σ_n is the restriction of the Clifford representation to $\mathfrak{spin}(n) \subset \mathrm{Cl}_{n-1}$ (note the shift from n to $n-1$). For $n \neq 0 \mod 4$ the spin representation is irreducible, and for $n = 0 \mod 4$ it decomposes as the direct sum of two irreducible representations $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$. We introduce the notation

$$\Sigma_n^{(+)} := \begin{cases} \Sigma_n^+ & \text{if } n = 0 \mod 4,\\ \Sigma_n & \text{if } n \neq 0 \mod 4.\end{cases}$$

The table above shows that the spin representation $\Sigma_n^{(+)}$ is of real type for $n = 0, 1, 7 \mod 8$, of complex type for $n = 2$ or $6 \mod 8$ and of quaternionic type for $n = 3, 4, 5 \mod 8$. We define the Lie algebras

$$\widetilde{\mathfrak{spin}}(n) := \begin{cases} \mathfrak{spin}(n) & \text{if } n = 0,1,7 \mod 8,\\ \mathfrak{spin}(n)\oplus\mathfrak{u}(1) & \text{if } n = 2 \text{ or } 6 \mod 8,\\ \mathfrak{spin}(n)\oplus\mathfrak{sp}(1) & \text{if } n = 3,4,5 \mod 8.\end{cases}$$

We can view $\Sigma_n^{(+)}$ as a $\widetilde{\mathfrak{spin}}(n)$-representation, where the $\mathfrak{u}(1)$ or $\mathfrak{sp}(1)$ actions are induced by the complex or quaternionic structure of the spin representation in the last two cases.

We study the following question: *for which $n \geq 5$ is $\Sigma_n^{(+)}$ a representation of Lie type of $\widetilde{\mathfrak{spin}}(n)$?* We will see that there are almost no other examples but the above examples which lead to the construction of exceptional Lie algebras. This is a consequence of the very special structure of the weights of the spin representations (see also [14], [15] for a much more general approach to this question).

Proposition 4.2. *For $n \geq 5$, the representation $\Sigma_n^{(+)}$ of $\widetilde{\mathfrak{spin}}(n)$ is of Lie type if and only if $n \in \{5, 6, 8, 9, 10, 12, 16\}$.*

Proof. The representation $\Sigma_n^{(+)}$ of $\widetilde{\mathfrak{spin}}(n)$ is of Lie type if and only if there exists a Lie algebra structure on $\mathfrak{g} := \widetilde{\mathfrak{spin}}(n) \oplus \Sigma_n^{(+)}$ satisfying conditions (1), (2) and (4) in Lemma 2.1 with respect to some $\mathrm{ad}_{\widetilde{\mathfrak{spin}}(n)}$ invariant scalar products on $\widetilde{\mathfrak{spin}}(n)$ and $\Sigma_n^{(+)}$. We will always consider some fixed Cartan subalgebra of $\widetilde{\mathfrak{spin}}(n)$, which is automatically a Cartan subalgebra of $\mathfrak{g}$ since the (half-)spin representations have no zero weight.

Consider first the case $n = 8k$. Since $\widetilde{\mathfrak{spin}}(8k) = \mathfrak{spin}(8k)$, the scalar products above are unique up to some constant. We choose the scalar product $\langle ., . \rangle$ on $\mathfrak{spin}(8k)$ such that in some orthonormal basis $\{e_1, \ldots, e_{4k}\}$ of the Cartan subalgebra of $\mathfrak{spin}(8k)$, the roots of $\mathfrak{spin}(8k)$ are

$$\mathcal{R} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4k\},$$

and the weights of the (complexified) half-spin representation $\Sigma_{8k}^+ \otimes \mathbb{C}$ are

$$\mathcal{W} = \{\tfrac{1}{2} \sum_{i=1}^{4k} \varepsilon_i e_i \mid \varepsilon_i = \pm 1,\ \varepsilon_1 \ldots \varepsilon_{4k} = 1\}.$$

The union $\mathcal{R} \cup \mathcal{W}$ is then the root system of $\mathfrak{g}$, which is a Lie algebra of compact type. In particular, the quotient

$$q(\alpha, \beta) := \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \tag{13}$$

is an integer satisfying $|q(\alpha, \beta)| \leq 3$ for all $\alpha, \beta \in \mathcal{R} \cup \mathcal{W}$ (cf. [1], p. 119). Taking $\alpha = e_1 + e_2$ and $\beta = \frac{1}{2} \sum_{i=1}^{4k} e_i$ we get $q(\alpha, \beta) = 2/k$, whence $k = 1$ or $k = 2$, so $n = 8$ or $n = 16$. Conversely, the real half-spin representations Σ_8^+ and Σ_{16}^+ are of Lie type (actually they are s-representations with augmented Lie algebras $\mathfrak{spin}(9) = \mathfrak{spin}(8) \oplus \Sigma_8^+$ and $\mathfrak{e}_8 = \mathfrak{spin}(16) \oplus \Sigma_{16}^+$).

If $n = 8k - 1$, a similar argument using the root $\alpha = e_1$ of $\mathfrak{spin}(8k - 1)$ and the weight $\beta = \frac{1}{2} \sum_{i=1}^{4k-1} e_i$ of the spin representation shows that $q(\alpha, \beta) = 2/(4k - 1)$ cannot be an integer.

If $n = 8k+1$, one has $q(\alpha, \beta) = 1/k$ for $\alpha = e_1$ and $\beta = \frac{1}{2} \sum_{i=1}^{4k} e_i$, so $k = 1$. Conversely, Σ_9 is an s-representation, as shown by the exceptional Lie algebra $\mathfrak{f}_4 = \mathfrak{spin}(9) \oplus \Sigma_9$.

Consider now the case when the spin representation is complex, *i.e.* $n = 4k + 2$, with $k \geq 1$. Assume that on $\mathfrak{g} := (\mathfrak{spin}(4k + 2) \oplus \mathfrak{u}(1)) \oplus \Sigma_{4k+2}$ there exists a Lie algebra structure satisfying conditions (1), (2) and (4) in Lemma 2.1 with respect to some $\mathrm{ad}_{\mathfrak{spin}(4k+2)\oplus\mathfrak{u}(1)}$ invariant scalar products on $\mathfrak{spin}(4k + 2) \oplus \mathfrak{u}(1)$ and Σ_{4k+2}. The latter scalar product is defined up to a scalar, whereas for the first one there is a two-parameter family of possible choices. By rescaling, we may assume that the restriction of the scalar product on the $\mathfrak{spin}(4k + 2)$ summand is such that in some orthonormal basis $\{e_1, \dots, e_{2k+1}\}$ of the Cartan subalgebra, the root system of $\mathfrak{spin}(4k+2)$ is

$$\mathcal{R} = \{\pm e_i \pm e_j \mid 1 \le i < j \le 2k+1\}.$$

There exists a unique vector $e_{2k+2} \in \mathfrak{u}(1)$ such that the set of weights of the representation $\Sigma_{4k+2} \otimes \mathbb{C} \cong \Sigma_{4k+2} \oplus \overline{\Sigma}_{4k+2}$ of $\mathfrak{spin}(4k+2) \oplus \mathfrak{u}(1)$ is

$$\mathcal{W} = \{\tfrac{1}{2} \sum_{i=1}^{2k+2} \varepsilon_i e_i \mid \varepsilon_i = \pm 1,\ \varepsilon_1 \dots \varepsilon_{2k+2} = 1\}.$$

We denote by $x := |e_{2k+2}|^2$ its square norm. The root system of $\mathfrak{g}$ is clearly $\mathcal{R}(\mathfrak{g}) = \mathcal{R} \cup \mathcal{W}$.

Recall that for every non-orthogonal roots α and β of $\mathfrak{g}$, their sum or difference is again a root [1]. On the other hand, neither the sum, nor the difference of the two roots $\alpha := \frac{1}{2}(\sum_{i=1}^{2k+2} e_i)$ and $\beta := \frac{1}{2}(\sum_{i=1}^{2k} e_i - e_{2k+1} - e_{2k+2})$ of $\mathfrak{g}$ belongs to $\mathcal{R}(\mathfrak{g}) = \mathcal{R} \cup \mathcal{W}$. Thus $\langle \alpha, \beta \rangle = 0$, which implies $x = 2k - 1$. Consider now the root $\gamma := e_1 + e_2$. The integer defined in (13) is

$$q(\gamma, \alpha) = \frac{2\langle \gamma, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \frac{2}{\frac{1}{4}(2k+1+x)} = \frac{2}{k},$$

showing that necessarily $k = 1$ or $k = 2$. Conversely, both cases do occur, since Σ_6 and Σ_{10} are s-representations of $\mathfrak{spin}(6) \oplus \mathfrak{u}(1) \cong \mathfrak{u}(4)$ and $\mathfrak{spin}(10) \oplus \mathfrak{u}(1)$ with augmented Lie algebras $\mathfrak{u}(5)$ and $\mathfrak{e}_6$ respectively.

Similar arguments (see also [14]) show that in the quaternionic case (when $n = 3, 4, 5$ mod 8) there are only two representations $\Sigma_n^{(+)}$ of $\mathfrak{spin}(n) \oplus \mathfrak{sp}(1)$ which are of Lie type, for $n = 5$ and $n = 12$. They are both s-representations and their augmented Lie algebras are $\mathfrak{spin}(5) \oplus \mathfrak{sp}(1) \oplus \Sigma_5 \cong \mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus \mathbb{H}^2 \cong \mathfrak{sp}(3)$ and $\mathfrak{spin}(12) \oplus \mathfrak{sp}(1) \oplus \Sigma_{12}^+ \cong \mathfrak{e}_7$. □

Note that J. Figueroa-O'Farrill has recently asked in [5, p. 673] about the existence of Killing superalgebra structures on spheres other than $\mathbb{S}^7$, $\mathbb{S}^8$ and $\mathbb{S}^{15}$. Proposition 4.2 can be interpreted as a negative answer to this question.

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