INFINITESIMAL EINSTEIN DEFORMATIONS OF NEARLY KÄHLER METRICS

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Abstract. It is well known that every 6-dimensional strictly nearly Kähler manifold (M, g, J) is Einstein with positive scalar curvature $\mathrm{scal} > 0$. Moreover, one can show that the space E of co-closed primitive $(1, 1)$-forms on M is stable under the Laplace operator Δ. Let $E(\lambda)$ denote the λ-eigenspace of the restriction of Δ to E. If M is compact, and has normalized scalar curvature $\mathrm{scal} = 30$, we prove that the moduli space of infinitesimal Einstein deformations of the nearly Kähler metric g is naturally isomorphic to the direct sum $E(2)\oplus E(6)\oplus E(12)$. From [5], the last summand is itself isomorphic with the moduli space of infinitesimal nearly Kähler deformations.

2000 *Mathematics Subject Classification*: Primary 58E30, 53C10, 53C15.

Keywords: Einstein deformations, nearly Kähler manifolds, Gray structures.

1. Introduction

Nearly Kähler manifolds, introduced by Alfred Gray in the 70s in the framework of weak holonomy, are defined as almost Hermitian manifolds (M, g, J) which are not far from being Kähler in the sense that the covariant derivative of J with respect to the Levi-Civita connection of g is totally skew-symmetric.

The class of nearly Kähler manifolds is clearly stable under Riemannian products. Using the generalization by Richard Cleyton and Andrew Swann of the Berger-Simons holonomy theorem to the case of connections with torsion [3], Paul-Andi Nagy showed in [7] that every nearly Kähler manifold is locally a Riemannian product of Kähler manifolds, 3-symmetric spaces, twistor spaces over positive quaternion-Kähler manifolds and 6-dimensional nearly Kähler manifolds. This result shows, in particular, that genuine nearly Kähler geometry only occurs in dimension 6. It turns out that in this dimension, strict (*i.e.* non-Kähler) nearly Kähler manifolds have several other remarkable features: They carry a real Killing spinor – so they are in particular Einstein manifolds with positive scalar curvature – and they have a SU_3 structure whose intrinsic torsion is parallel with respect to the minimal connection (cf. [3]). A strict nearly Kähler structure on a compact 6-dimensional manifold with normalized scalar curvature $\mathrm{scal} = 30$ is called a Gray structure.

Date: May 9, 2015.

In [5] we have studied the moduli space $\mathcal{G}$ of infinitesimal deformations of Gray structures on compact 6-dimensional manifolds, and showed that this space is isomorphic to the space $E(12)$, where $E(\lambda)$ denotes the intersection of the λ-eigenspace of the Laplace operator and the space of co-closed primitive $(1,1)$-forms.

In the present paper we consider the related problem of describing the moduli space $\mathcal{E}$ of Einstein deformations of a Gray structure. Since every Gray structure is in particular Einstein, one has *a priori* $\mathcal{E} \supset \mathcal{G}$. Our main result (Theorem 5.1) gives a canonical isomorphism between $\mathcal{E}$ and the direct sum $E(2) \oplus E(6) \oplus E(12)$.

The main idea is the following. The space of infinitesimal Einstein deformation on every compact manifold consists of trace-free symmetric bilinear tensors in a certain eigenspace of a second order elliptic operator called the Lichnerowicz Laplacian Δ_L. On a 6-dimensional nearly Kähler manifold, one can decompose every infinitesimal Einstein deformation H (viewed as symmetric endomorphism) into its parts h and S commuting resp. anti-commuting with J. Under the SU_3 representation, the space of symmetric endomorphisms commuting with J is isomorphic to the space of $(1,1)$-forms and that of symmetric endomorphisms anti-commuting with J is isomorphic to the space of primitive $(2,1)+(1,2)$-forms and one may interpret the eigenvalue equation for Δ_L in terms of the forms φ and σ corresponding to h and S. The problem is that Δ_L does not commute with the isomorphisms above, because J is not parallel with respect to the Levi-Civita connection. It is thus natural to introduce a modified Lichnerowicz operator $\bar\Delta_L$, corresponding to the canonical Hermitian connection, better adapted to the nearly Kähler setting. It turns out that the eigenvalue equation for Δ_L translates, *via* $\bar\Delta_L$, into a differential system for φ and σ involving the usual form Laplacian, which eventually yields the claimed result.

2. Preliminaries

2.1. **Notation.** In this section we introduce our objects of study and derive several lemmas which will be needed later. Here and henceforth, (M^{2m}, g, J) will denote an almost Hermitian manifold with tangent bundle TM, cotangent bundle T^*M and tensor bundle $\mathcal{T}M$. We denote as usual by $\Lambda^{(p,q)+(q,p)}M$ the projection of the complex bundle $\Lambda^{(p,q)}M$ onto the real bundle $\Lambda^{p+q}M$. The bundle of g-symmetric endomorphisms $\mathrm{Sym}M$ splits in a direct sum $\mathrm{Sym}M = \mathrm{Sym}^+M \oplus \mathrm{Sym}^-M$, of symmetric endomorphisms commuting resp. anti-commuting with J. The trace of every element in Sym^-M is automatically 0, and Sym^+M decomposes further $\mathrm{Sym}^+M = \mathrm{Sym}^+_0M \oplus \langle \mathrm{id} \rangle$ into its trace-free part and multiples of the identity.

2.2. **Nearly Kähler manifolds.** An almost Hermitian manifold (M^{2m}, g, J) is called *nearly Kähler* if

$$(\nabla_X J)(X) = 0, \qquad \forall\ X \in TM, \tag{1}$$

where ∇ denotes the Levi-Civita connection of g. The canonical Hermitian connection $\bar\nabla$, defined by

$$\bar\nabla_X Y := \nabla_X Y - \frac{1}{2}J(\nabla_X J)Y, \qquad \forall\ X \in TM,\ Y \in \chi(M), \tag{2}$$

is a U_m connection on M (*i.e.* $\bar\nabla g = 0$ and $\bar\nabla J = 0$) with torsion $\bar T_X Y = -J(\nabla_X J)Y$. A fundamental observation, which – although not explicitly stated – goes back to Gray, is the fact that $\bar\nabla\bar T = 0$ on every nearly Kähler manifold (see [1]).

We denote as usual the Kähler form of M by $\omega := g(J.,.)$. The tensor $\psi^+ := \nabla\omega$ is totally skew-symmetric by (1). Moreover, since $J^2 = -\mathrm{id}$, it is easy to check that $\psi^+(X, JY, JZ) = -\psi^+(X, Y, Z)$. In other words, ψ^+ is a form of type $(3, 0)+(0, 3)$. Let us now assume that the dimension of M is $2m = 6$ and that the nearly Kähler structure is strict, *i.e.* (M, g, J) is not Kähler. The form ψ^+ can be seen as the real part of a $\bar\nabla$-parallel complex volume form on M, so M carries an SU_3 structure whose minimal connection (cf. [3]) is exactly $\bar\nabla$.

Let $A \in \Lambda^1 M\otimes \mathrm{End}M$ denote the tensor $A_X := J(\nabla_X J) = -\psi^+_{JX}$. We will sometimes identify the endomorphism A_X with the corresponding form in $\Lambda^{(2.0)+(0,2)}M$, *e.g.* in formula (8) below. By definition, we have $\nabla_X = \bar\nabla_X + \frac{1}{2}A_X$ on TM. In fact this relation can be extended on the whole tensor bundle, provided we use the right extension for A_X.

2.3. **The induced action.** On a manifold M, every endomorphism A of TM extends as derivation to the tensor bundle $\mathcal{T}M$. In fact if we identify $\mathrm{End}(T_xM)$ with $\mathfrak{gl}(n,\mathbb{R})$ this is precisely the Lie algebra action on the defining representation of $\mathcal{T}M$. We denote by $A_\star$ this induced action. For example, we have

$$A_\star\tau = -\tau\circ A, \qquad A_\star f = A\circ f - f\circ A, \qquad \text{for } \tau\in T^*M,\ f\in \mathrm{End}M.$$

If (M, g, J) is almost Hermitian and $f \in \mathrm{Sym}^+M$, let $\tilde f$ denote the associated $(1, 1)$-form $\tilde f := g(Jf., .)$, so in particular $\tilde{\mathrm{id}} = \omega$, where ω denotes the Kähler form of M. We compute, for later use:

$$(f_\star\omega)(X, Y) = -\omega(fX, Y) - \omega(X, fY) = -g(JfX, Y) - g(JX, fY) = -2\tilde f(X, Y). \tag{3}$$

A similar calculation shows that

$$S_\star\omega = 0, \qquad \forall\ S\in \mathrm{Sym}^-M. \tag{4}$$

Notice that the map $\mathrm{End}M \to \mathrm{End}(\mathcal{T}M)$, $A \mapsto A_\star$ is a Lie algebra morphism, *i.e.*

$$[A, B]_\star = [A_\star, B_\star], \qquad \forall\ A, B\in \mathrm{End}M,$$

which can be expressed as

$$A_\star(B_\star T) = (A_\star B)_\star T + B_\star(A_\star T), \qquad \forall\ A, B\in \mathrm{End}M,\ T\in\mathcal{T}M. \tag{5}$$

A convenient way of writing the induced action of $A \in \mathrm{End}M$ on a p-form u is

$$A_\star u = -A^*(e_i)\wedge e_i\lrcorner u,$$

where A^* is the adjoint of A and $\{e_i\}$ is a local orthonormal basis of TM. Here, as well as in the remaining part of this paper, we adopt the Einstein convention of summation on the repeated subscripts.

Notice that by (2), the extensions of ∇ and $\bar\nabla$ to the tensor bundle $\mathcal{T}M$ are related by

$$\bar\nabla_X = \nabla_X - \frac{1}{2}(A_X)_\star. \tag{6}$$

2.4. **Algebraic results on nearly Kähler manifolds.** Assume that (M^6, g, J) is a strict nearly Kähler manifold and that the metric on M is normalized such that scal $=$ 30. From [4, Theorem 5.2] it follows that for every unit vector X, the endomorphism $\nabla_X J$ (which vanishes on the 2-plane spanned by X and JX) defines a complex structure on the orthogonal complement of that 2-plane. Then the same holds for A_X (because $A_X = -\nabla_{JX} J$).

The exterior bundle $\Lambda^2 M$ decomposes into irreducible SU_3 components as follows:

$$\Lambda^2 M \simeq \Lambda^{(2,0)+(0,2)} M \oplus \Lambda_0^{(1,1)} M \oplus \mathbb{R}\omega.$$

The map $X \mapsto X \lrcorner \psi^+$ identifies the first summand with TM, and $h \mapsto g(Jh.,.)$ defines an isomorphism between $\mathrm{Sym}_0^+ M$ and the second summand.

Similarly, one can decompose $\Lambda^3 M$ into irreducible SU_3 components

$$\Lambda^3 M \simeq \Lambda^{(3,0)+(0,3)} M \oplus \Lambda_0^{(2,1)+(1,2)} M \oplus \Lambda^1 M \wedge \omega.$$

The first summand is a rank 2 trivial bundle spanned by ψ^+ and its Hodge dual $*\psi^+$, and the isomorphism $S \mapsto S_\star \psi^+$ identifies $\mathrm{Sym}^- M$ with the second summand.

If $\{e_i\}$ denotes a local orthonormal basis of TM, it is straightforward to check the following formulas:

$$A_{e_i} A_{e_i}(X) = -4X, \qquad \forall\ X \in TM. \tag{7}$$

$$A_{e_i} \wedge A_{e_i} = 2\omega^2. \tag{8}$$

Lemma 2.1. (i) *For every $X \in TM$, with corresponding 1-form $X^\flat$ one has*

$$A_{X\star}\psi^+ = -2X^\flat \wedge \omega. \tag{9}$$

(ii) *If S is a section of* $\mathrm{Sym}^- M$*, then the following formula holds for every $X \in TM$:*

$$A_{X\star}(S_\star\psi^+) = 2(SX)^\flat \wedge \omega. \tag{10}$$

Proof. (i) An easy computation shows:

$$\begin{aligned} A_{X\star}\psi^+ &= A_X e_i \wedge e_i \lrcorner \psi^+ = A_X Je_i \wedge Je_i \lrcorner \psi^+ = A_{Je_i} X \wedge A_{e_i} \\ &= A_{e_i}(JX) \wedge A_{e_i} = \frac{1}{2} JX \lrcorner (A_{e_i} \wedge A_{e_i}) \overset{(8)}{=} -2X \wedge \omega. \end{aligned}$$

(ii) The symmetric endomorphism $A_{X\star}S = A_X \circ S - S \circ A_X$ commutes with J and is trace-free. Consequently, by Schur's Lemma (cf. [5] for a more detailed argument)

$$(A_{X\star}S)_\star\psi^+ = 0. \tag{11}$$

Notice that if $X^\flat$ is the 1-form corresponding to X (which we usually identify with X), then $f_\star X^\flat = -(fX)^\flat$ for every symmetric endomorphism f. We then compute:

$$\begin{aligned} A_{X\star}(S_\star\psi^+) &\overset{(5)}{=} (A_{X\star}S)_\star\psi^+ + S_\star(A_{X\star}\psi^+) \overset{(9),(11)}{=} -2S_\star(X^\flat\wedge\omega) \\ &= 2(SX)^\flat\wedge\omega - 2X\wedge(S_\star\omega) \overset{(4)}{=} 2(SX)^\flat\wedge\omega. \end{aligned}$$

□

3. The curvature operator

Let (M^n, g) be a Riemannian manifold. The curvature operator $\mathcal{R} : \Lambda^2M \to \Lambda^2M$ is defined by the equation $g(\mathcal{R}(X\wedge Y), U\wedge V) = g(R_{X,Y}V, U)$, for any vector fields X, Y, U, V on M, identified with the corresponding 1-forms *via* the metric. In a local orthonormal frame $\{e_i\}$ it can be written as

$$\mathcal{R}(e_i\wedge e_j) = \frac{1}{2}R_{ijkl}e_l\wedge e_k = -\frac{1}{2}e_k\wedge R_{e_i,e_j}e_k. \tag{12}$$

Using the identification of 2-vectors and (skew-symmetric) endomorphisms given by $(X\wedge Y)(Z) := g(X,Z)Y - g(Y,Z)X$, formula (12) yields $\mathcal{R}(X\wedge Y)(Z) = -R_{X,Y}Z$. Notice that a manifold with curvature operator $\mathcal{R} = c\mathrm{id}$ has Ricci curvature $c(n-1)$ and in particular the curvature operator of the sphere is a positive multiple of the identity.

Let EM be the vector bundle associated to the bundle of orthonormal frames *via* some representation $\pi : SO(n) \to \mathrm{Aut}(E)$. Every orthogonal automorphism f of TM defines in a canonical way an automorphism of EM, denoted, by a slight abuse of notation, $\pi(f)$. The differential of π maps skew-symmetric endomorphisms of TM (or equivalently elements of Λ^2M) to endomorphisms of EM. The Levi-Civita connection of M induces a connection on EM whose curvature R^E satisfies $R^E(X,Y) = \pi_*(R(X,Y)) = -\pi_*(\mathcal{R}(X\wedge Y))$. Notice that $\pi_*(A)$ is exactly $A_\star$ in the notation of Section 2.

We now define the curvature endomorphism $q(R) \in \mathrm{End}(EM)$ as

$$q(R) := -\frac{1}{2}(e_i\wedge e_j)_\star\mathcal{R}(e_i\wedge e_j)_\star. \tag{13}$$

For example the curvature endomorphism $q(R)$ on the form bundle $EM = \Lambda^pM$ satisfies

$$q(R) = (e_j\wedge e_i\lrcorner)\circ(R_{e_i,e_j}e_k\wedge e_k\lrcorner). \tag{14}$$

In particular we have $q(R) = \mathrm{Ric}$ on 1-forms, and $q(R) = -\mathrm{Ric}_\star - 2\mathcal{R}$ on 2-forms.

It is easy to check that the action of $q(R)$ is compatible with the identification of Λ^2M with the space of skew-symmetric endomorphisms (and, more generally, with all SO_n equivariant isomorphisms):

Lemma 3.1. *Let $\varphi \in \Lambda^2M$ be a 2-form with associated skew-symmetric endomorphism A,* i.e. *$\varphi(Y,Z) = g(AY,Z)$ for any vector fields Y, Z. Then*

$$(q(R)\varphi)(Y,Z) = g((q(R)A)Y,Z).$$

We now return to the case of a 6-dimensional strict nearly Kähler manifold (M^6, g, J) with scalar curvature $\mathrm{scal} = 30$.

Let R be the curvature of the Levi-Civita connection and let $\bar{R}$ be the curvature of the canonical Hermitian connection $\bar{\nabla}$. The following relation between R and $\bar{R}$ is implicitly contained in [4].

Lemma 3.2. *For any tangent vectors W, X, Y, Z one has*

$$\begin{aligned} R_{WXYZ} &= \bar{R}_{WXYZ} - \frac{1}{4}g(Y,W)g(X,Z) + \frac{1}{4}g(X,Y)g(Z,W) \\ &\quad + \frac{3}{4}g(Y,JW)g(JX,Z) - \frac{3}{4}g(Y,JX)g(JW,Z) - \frac{1}{2}g(X,JW)g(JY,Z). \end{aligned}$$

Proof. The stated formula follows using equation (3.1) and the polarization of equation (5.1) from [4]. Note that there is a different sign convention for the curvature tensors in [4]. □

The Ricci curvature of $\bar{R}$ satisfies $\overline{\mathrm{Ric}} = 4g$. This follows from the formula above and the fact that (M^6, g, J) is Einstein with $\mathrm{Ric} = 5g$.

Replacing R by $\bar{R}$ in formula (14) yields a curvature endomorphism $q(\bar{R})$. It is easy to check that the curvature operator with respect to $\bar{\nabla}$, denoted by $\bar{\mathcal{R}}$, is a section of $\mathrm{Sym}(\Lambda^{(1,1)}_0 M)$ so we can express $q(\bar{R}) = -\sum \alpha_{i\star}\bar{\mathcal{R}}(\alpha_i)_\star$ for any orthonormal basis α_i of $\Lambda^{(1,1)}_0 M$. Since $\Lambda^{(1,1)}_0\mathbb{R}^6 \simeq \mathfrak{su}_3$, we see that $q(\bar{R})$ preserves all tensor bundles associated to SU_3 representations. Moreover, a straightforward computation using the fact that $\alpha_\star J = 0$ and $\alpha_\star\psi^+ = 0$ for every $\alpha \in \mathfrak{su}_3$ yields:

$$g((q(\bar{R})h)J.,.) = q(\bar{R})\varphi, \qquad \text{and} \qquad q(\bar{R})(S_\star\psi^+) = (q(\bar{R})S)_\star\psi^+, \tag{15}$$

for every sections $h \in \mathrm{Sym}^+ M$ and $S \in \mathrm{Sym}^- M$, where φ denotes the $(1,1)$-form defined by $\varphi = g(hJ.,.)$.

The following lemma describes the difference $q(R) - q(\bar{R})$. It is an immediate consequence of the curvature formula in Lemma 3.2. We will denote with $\mathrm{Cas} = \frac{1}{2}(e_i \wedge e_j)_\star(e_i \wedge e_j)_\star$ the Casimir operator of $\mathfrak{so}(n)$ acting on the representation E and at the same time the corresponding endomorphism of EM.

Lemma 3.3. *The difference $q(R) - q(\bar{R}) \in \mathrm{End}(EM)$ is given as*

$$q(R) - q(\bar{R}) = -\frac{1}{4}\mathrm{Cas} + \frac{3}{8}(e_i \wedge e_j)_\star(Je_i \wedge Je_j)_\star - \frac{1}{8}(e_i \wedge Je_i)_\star(e_k \wedge Je_k)_\star$$

In the remaining part of this section we will apply Lemma 3.3 in order to compute $q(R) - q(\bar{R})$ on certain spaces of endomorphisms and forms.

On the bundle $\mathrm{End}M$ we define projections $\mathrm{pr}_\pm$ by $\mathrm{pr}_\pm(\mathrm{H}) = \frac{1}{2}(\mathrm{H} \mp \mathrm{JHJ})$. Then $H = \mathrm{pr}_+(\mathrm{H}) + \mathrm{pr}_-(\mathrm{H})$ is the decomposition of the endomorphism H in a part commuting resp. anti-commuting with J. Using this notation we find for the first sum in the above equation

$$(e_i \wedge e_j)_\star (Je_i \wedge Je_j)_\star = \begin{cases} -2\mathrm{id} & \text{on } TM \\ -8\mathrm{pr}_- & \text{on } \mathrm{Sym}M \\ -8\mathrm{pr}_+ & \text{on } \Lambda^2_0 M \end{cases}$$

Indeed for any tangent vector $v \in TM$ we have

$$(e_i \wedge e_j)_\star (Je_i \wedge Je_j)_\star v = (e_i \wedge e_j)_\star (g(Je_i, v)Je_j - g(Je_j, v)Je_i) = -2(Jv \wedge e_j)_\star Je_j = -2v.$$

We now recall that for every skew-symmetric endomorphism $A \in \mathfrak{so}(TM) \cong \Lambda^2 M$ and for every $H \in \mathrm{End}M$ we have $A_\star H = [A, H]$. Hence, we obtain

$$\begin{aligned} (e_i \wedge e_j)_\star (Je_i \wedge Je_j)_\star H &= [(e_i \wedge e_j), [(Je_i \wedge Je_j), H]] \\ &= (e_i \wedge e_j)(Je_i \wedge Je_j)H + H(e_i \wedge e_j)(Je_i \wedge Je_j) \\ &\quad -2(e_i \wedge e_j)H(Je_i \wedge Je_j) \\ &= -4H - 2(e_i \wedge e_j)H(Je_i \wedge Je_j). \end{aligned}$$

It remains to compute the endomorphism $B := (e_i \wedge e_j)H(Je_i \wedge Je_j)$. Applying it to a vector Y and taking the scalar product with a vector Z we find

$$\begin{aligned} g(BY, Z) &= -g(H(Je_i \wedge Je_j)Y, (e_i \wedge e_j)Z) \\ &= -g(H[g(Je_i, Y)Je_j - g(Je_j, Y)Je_i], [g(e_i, Z)e_j - g(e_j, Z)e_i]) \\ &= -2(g(JZ, Y)g(HJe_j, e_j) + g(HJZ, JY)). \end{aligned}$$

Hence the sum $B = (e_i \wedge e_j)H(Je_i \wedge Je_j)$ equals $2JHJ$ if the endomorphism H is symmetric and $-2JHJ$ if H is skew-symmetric and $\mathrm{tr}(HJ) = 0$.

Next we compute the second sum of Lemma 3.3 on tangent vectors and endomorphisms. Since $2J = (e_i \wedge Je_i)$ we immediately obtain

$$(e_i \wedge Je_i)_\star (e_j \wedge Je_j)_\star = \begin{cases} -4\mathrm{id} & \text{on } TM \\ -16\mathrm{pr}_- & \text{on } \mathrm{End}M \end{cases}$$

We next determine the two sums of Lemma 3.3 on the space of 3-forms. Recall the type decomposition

$$\Lambda^3 M = \Lambda^{(3,0)+(0,3)} M \oplus \Lambda^{(2,1)+(1,2)} M,$$

which coincides with the eigenspace decomposition of $(J_\star)^2$, with eigenvalue -9 on the first and eigenvalue -1 on the second summand. For any 3-form α we define a new 3-form $\hat{\alpha}$ by the formula $\hat{\alpha}(X, Y, Z) = \alpha(JX, JY, Z) + \alpha(JX, Y, JZ) + \alpha(X, JY, JZ)$. Then $(J_\star)^2 \alpha = -3\alpha + 2\hat{\alpha}$ and the two components of $\Lambda^3 M$ may also be characterized by

$$\alpha \in \Lambda^{(3,0)+(0,3)} M \qquad \text{if and only if} \qquad \hat{\alpha} = -3\alpha,$$

and similarly

$$\alpha \in \Lambda^{(2,1)+(1,2)} M \qquad \text{if and only if} \qquad \hat{\alpha} = \alpha.$$

Let $\mathrm{pr}_{3,0}$ and $\mathrm{pr}_{2,1}$ denote the projections onto the two summands of $\Lambda^3 M$. Then for $\alpha \in \Lambda^3 M$

$$\mathrm{pr}_{3,0}(\alpha) = \tfrac{1}{4}(\alpha - \hat{\alpha}) \qquad \text{and} \qquad \mathrm{pr}_{2,1}(\alpha) = \tfrac{1}{4}(3\alpha + \hat{\alpha}).$$

For any 3-form α we have $(e_i \wedge Je_i)_\star (e_k \wedge Je_k)_\star \alpha = 4(J_\star)^2 \alpha = -12\alpha + 8\hat{\alpha}$, which gives the second sum of Lemma 3.3 and a simple calculation yields $(e_i \wedge e_j)_\star (Je_i \wedge Je_j)_\star \alpha = -6\alpha - 4\hat{\alpha}$ for the first sum.

Finally we may substitute the Casimir eigenvalues and our explicit expressions into the formula of Lemma 3.3. Recall that in the normalization with $\{e_i \wedge e_j\}$ as orthonormal basis of $\Lambda^2 M \cong \mathfrak{so}(TM)$, the Casimir operator acts as $-p(n-p)\mathrm{id}$ on $\Lambda^p M$, and as $-2n\mathrm{id}$ on $\mathrm{Sym} M$. Hence we obtain for $n = 6$:

Proposition 3.4.

$$q(R) - q(\bar{R}) = \begin{cases} \mathrm{id} & \text{on } TM \\ 3\mathrm{pr}_+ + 2\,\mathrm{pr}_- & \textit{on } \mathrm{Sym} M \\ -\mathrm{pr}_+ + 4\,\mathrm{pr}_- & \textit{on } \Lambda^2_0 M \\ -\mathrm{pr}_{2,1} + 9\,\mathrm{pr}_{3,0} & \textit{on } \Lambda^3 M \end{cases}$$

Corollary 3.5.

$$q(R) - q(\bar{R}) = \begin{cases} 3\,\mathrm{id} & \textit{on } \mathrm{Sym}^+ M \\ 2\,\mathrm{id} & \textit{on } \mathrm{Sym}^- M \\ -\mathrm{id} & \textit{on } \Lambda^{(1,1)}_0 M \\ -\mathrm{id} & \textit{on } \Lambda^{(2,1)+(1,2)} M \end{cases}$$

4. Comparing rough Laplacians

Let (M, g, J) be a strict nearly Kähler manifold with Levi-Civita connection ∇ and canonical Hermitian connection $\bar{\nabla}$. In this section we compare the actions of the rough Laplacians $\nabla^*\nabla$ and $\bar{\nabla}^*\bar{\nabla}$ on several tensor bundles.

We will perform all calculations below at some fixed point $x \in M$ using a local orthonormal frame $\{e_i\}$ which is ∇-parallel at x. On any tensor bundle on M we can write $\nabla^*\nabla = -\nabla_{e_i}\nabla_{e_i}$ and because $\bar{\nabla}_{e_i} e_i = \nabla_{e_i} e_i - \frac{1}{2}J(\nabla_{e_i} J)e_i = 0$, we also have $\bar{\nabla}^*\bar{\nabla} = -\bar{\nabla}_{e_i}\bar{\nabla}_{e_i}$. We are interested in the operator $P := \nabla^*\nabla - \bar{\nabla}^*\bar{\nabla}$. Using (6) and the fact that the tensor $A := J\nabla J$ is $\bar{\nabla}$-parallel, we have

$$\begin{aligned} P &= -\nabla_{e_i}\nabla_{e_i} + \bar{\nabla}_{e_i}\bar{\nabla}_{e_i} = -(\bar{\nabla}_{e_i} + \frac{1}{2}A_{e_i\star})(\bar{\nabla}_{e_i} + \frac{1}{2}A_{e_i\star}) + \bar{\nabla}_{e_i}\bar{\nabla}_{e_i} \\ &= -\frac{1}{4}A_{e_i\star}A_{e_i\star} - A_{e_i\star}\bar{\nabla}_{e_i}. \end{aligned}$$

We now compute the action of the two operators occurring in the previous formula on several tensor bundles which are of interest in the deformation problem.

Lemma 4.1. *Let φ, σ, h and S be sections of $\Lambda_0^{(1,1)}M$, $\Lambda_0^{(2,1)+(1,2)}M$, Sym^+M and Sym^-M respectively. Then*

$$A_{e_i\star}(A_{e_i\star}\varphi) = -4\varphi. \tag{16}$$

$$A_{e_i\star}(A_{e_i\star}\sigma) = -4\sigma. \tag{17}$$

$$A_{e_i\star}(A_{e_i\star}h) = -12h. \tag{18}$$

$$A_{e_i\star}(A_{e_i\star}S) = -8S. \tag{19}$$

Proof. Let $X \in TM$ be a tangent vector. Since A_X and φ are 2-forms of type $(2,0)+(0,2)$ and $(1,1)$ respectively, we get

$$A_X(e_k, e_j)\varphi(e_k, e_j) = 2\langle A_X, \varphi\rangle = 0, \qquad \forall\ X \in TM. \tag{20}$$

We then compute

$$\begin{aligned}
A_{e_i\star}(A_{e_i\star}\varphi) &= A_{e_i}e_j \wedge e_j \lrcorner (A_{e_i}e_k \wedge e_k \lrcorner \varphi)\\
&= A_{e_i}e_j A_{e_i}(e_k, e_j) \wedge (e_k \lrcorner \varphi) - A_{e_i}e_j \wedge A_{e_i}e_k\varphi(e_k, e_j)\\
&\overset{(20)}{=} A_{e_i}^2(e_k) \wedge (e_k \lrcorner \varphi) - e_j \lrcorner (A_{e_i} \wedge A_{e_i}e_k\varphi(e_k, e_j))\\
&\overset{(7)}{=} -4e_k \wedge (e_k \lrcorner \varphi) - \frac{1}{2}e_j \lrcorner e_k \lrcorner (A_i \wedge A_i\varphi(e_k, e_j))\\
&\overset{(8)}{=} -8\varphi - e_j \lrcorner e_k \lrcorner (\omega^2\varphi(e_k, e_j)) = -8\varphi - 2e_j \lrcorner (Je_k \wedge \omega\varphi(e_k, e_j))\\
&= -8\varphi - 2\omega\varphi(e_k, Je_k) + 2Je_k \wedge Je_j\varphi(e_k, e_j))\\
&= -8\varphi + 2e_k \wedge e_j\varphi(e_k, e_j)) = -8\varphi + 4\varphi = -4\varphi.
\end{aligned}$$

In order to prove (17), we express σ as $\sigma = S_\star\psi^+$ for some section S of Sym^-M. Using (10) we obtain

$$\begin{aligned}
A_{e_i\star}(A_{e_i\star}\sigma) &= 2A_{e_i\star}(Se_i \wedge \omega) = 2Se_i \wedge (A_{e_i\star}\omega) = 2Se_i \wedge J\psi^+_{e_i}e_j \wedge Je_j\\
&= -2Se_i \wedge \psi^+_{e_i}Je_j \wedge Je_j = 4Se_i \wedge \psi^+_{e_i} = -4S_\star\psi^+ = -4\sigma.
\end{aligned}$$

Now, for every endomorphism H of TM, we have

$$A_{e_i\star}(A_{e_i\star}H) = A_{e_i}^2H + HA_{e_i}^2 - 2A_{e_i}HA_{e_i} \overset{(7)}{=} -8H - 2A_{e_i}HA_{e_i}. \tag{21}$$

If $h \in \mathrm{Sym}^+M$, let $\varphi(.,.) = g(Jh.,.)$ be its associated $(1,1)$-form. By (16) we have for every tangent vectors X, Y

$$\begin{aligned}
-4\varphi(X, Y) &= A_{e_i\star}(A_{e_i\star}\varphi)(X, Y)\\
&= \varphi(A_{e_i}^2X, Y) + \varphi(X, A_{e_i}^2Y) + 2\varphi(A_{e_i}X, A_{e_i}Y)\\
&\overset{(7)}{=} -8\varphi(X, Y) + 2g(hJA_{e_i}X, A_{e_i}Y)\\
&= -8\varphi(X, Y) + 2g(JA_{e_i}hA_{e_i}X, Y),
\end{aligned}$$

whence $A_{e_i}hA_{e_i} = 2h$. This, together with (21), yields (18). If $S \in \mathrm{Sym}^-M$, using the fact that $A_{JX} = A_X \circ J = -J \circ A_X$ for every X, we can write

$$A_{e_i}SA_{e_i} = A_{Je_i}SA_{Je_i} = -A_{e_i}JSJA_{e_i} = -A_{e_i}SA_{e_i},$$

which together with (21) yields (19). □

Lemma 4.2. *The following relations hold:*

$$e_i \lrcorner (A_{e_i\star}\varphi) = 0, \qquad \forall\ \varphi \in \Lambda_0^{(1,1)}M. \tag{22}$$

$$(A_{e_i\star}h)(e_i) = 0, \qquad \forall\ h \in \mathrm{Sym}_0^+ M. \tag{23}$$

$$e_i \wedge (A_{e_i\star}\varphi) = 0, \qquad \forall\ \varphi \in \Lambda_0^{(1,1)}M. \tag{24}$$

$$e_i \lrcorner (A_{i\star}(S_\star\psi^+)) = 0, \qquad \forall\ S \in \mathrm{Sym}^- M. \tag{25}$$

Proof. Simple application of the Schur Lemma, taking into account the decomposition of the exterior bundles into irreducible components with respect to the SU_3 action. □

Lemma 4.3. *Let φ and S be sections of $\Lambda_0^{(1,1)}M$ and $\mathrm{Sym}^- M$ respectively. If h and σ are defined as usual by $g(Jh.,.) := \varphi(.,.)$ and $\sigma := S_\star\psi^+$, then*

$$A_{e_i\star}\bar\nabla_{e_i}\varphi = -(J\delta\varphi) \lrcorner \psi^+. \tag{26}$$

$$A_{e_i\star}\bar\nabla_{e_i}\sigma = -2\delta S \wedge \omega. \tag{27}$$

$$(A_{e_i\star}\bar\nabla_{e_i}h)_\star\psi^+ = -2\delta h \wedge \omega - 4d\varphi. \tag{28}$$

$$A_{e_i\star}\bar\nabla_{e_i}S = (\delta S \lrcorner \psi^+ + \delta(S_\star\psi^+)) \circ J. \tag{29}$$

Here δ denotes the co-differential on exterior forms and the divergence operator whenever applied to symmetric endomorphisms.

Proof. Since A_{e_i} anti-commutes with J and $\bar\nabla_{e_i}\varphi$ is of type $(1,1)$, it follows that $A_{e_i\star}\bar\nabla_{e_i}\varphi$ is a form of type $(2,0)+(0,2)$, so there exists a vector field α such that $A_{e_i\star}\bar\nabla_{e_i}\varphi = \alpha \lrcorner \psi^+$. In order to find α, we use the relation $(\alpha \lrcorner \psi^+) \wedge \psi^+ = \alpha \wedge \omega^2$ (see [5]) and compute:

$$\begin{aligned}
\alpha \wedge \omega^2 &= (\alpha \lrcorner \psi^+) \wedge \psi^+ = (A_{e_i\star}\bar\nabla_{e_i}\varphi) \wedge \psi^+ \\
&= A_{e_i\star}((\bar\nabla_{e_i}\varphi) \wedge \psi^+) - \bar\nabla_{e_i}\varphi \wedge (A_{e_i\star}\psi^+) \overset{(9)}{=} 2\bar\nabla_{e_i}\varphi \wedge e_i \wedge \omega \\
&= -\bar\nabla_{e_i}\varphi \wedge (Je_i \lrcorner \omega^2) = -Je_i \lrcorner (\bar\nabla_{e_i}\varphi \wedge \omega^2) + (Je_i \lrcorner \bar\nabla_{e_i}\varphi) \wedge \omega^2 \\
&= J(e_i \lrcorner \bar\nabla_{e_i}\varphi) \wedge \omega^2 \overset{(22)}{=} J(e_i \lrcorner \nabla_{e_i}\varphi) \wedge \omega^2 = -J\delta\varphi \wedge \omega^2,
\end{aligned}$$

so $\alpha = -J\delta\varphi$, thus proving (26). Using the fact that ψ^+ is $\bar\nabla$-parallel, we get:

$$\begin{aligned}
A_{e_i\star}\bar\nabla_{e_i}\sigma &= A_{e_i\star}(\bar\nabla_{e_i}S_\star\psi^+) \overset{(5)}{=} (A_{e_i\star}\bar\nabla_{e_i}S)_\star\psi^+ + \bar\nabla_{e_i}S_\star(A_{e_i\star}\psi^+) \\
&\overset{(9),(11)}{=} -2\bar\nabla_{e_i}S_\star(e_i \wedge \omega) \overset{(4)}{=} 2(\bar\nabla_{e_i}S)e_i \wedge \omega = -2\delta S \wedge \omega.
\end{aligned}$$

This proves (27). We next use (23) to write

$$\delta h = -(\nabla_{e_i}h)e_i = -(\bar\nabla_{e_i}h)e_i \tag{30}$$

whence

$$\begin{aligned}(A_{e_i\star}\bar\nabla_{e_i}h)_\star\psi^+ &\overset{(5)}{=} A_{e_i\star}((\bar\nabla_{e_i}h)_\star\psi^+) - (\bar\nabla_{e_i}h)_\star(A_{e_i\star}\psi^+)\\ &\overset{(9)}{=} 2(\bar\nabla_{e_i}h)_\star(e_i\wedge\omega) = 2(\bar\nabla_{e_i}h)e_i\wedge\omega + 2e_i\wedge((\bar\nabla_{e_i}h)_\star\omega)\\ &\overset{(3)}{=} -2\delta h\wedge\omega - 4e_i\wedge\bar\nabla_{e_i}\varphi = -2\delta h\wedge\omega - 4d\varphi.\end{aligned}$$

In order to check (29) we first compute

$$\begin{aligned}\delta(S_\star\psi^+) &= -e_i\lrcorner\nabla_{e_i}(S_\star\psi^+) \overset{(25)}{=} -e_i\lrcorner\bar\nabla_{e_i}(S_\star\psi^+) = e_i\lrcorner\bar\nabla_{e_i}(S(e_j)\wedge\psi^+_{e_j})\\ &= e_i\lrcorner((\bar\nabla_{e_i}S)e_j\wedge\psi^+_{e_j}) = -g(\delta S, e_j)\psi^+_{e_j} - (\bar\nabla_{e_i}S)e_j\wedge(e_i\lrcorner\psi^+_{e_j})\\ &= -\delta S\lrcorner\psi^+ + (e_i\lrcorner\psi^+_{e_j})\wedge(\bar\nabla_{e_i}S)e_j.\end{aligned}$$

Let B denote the endomorphism of TM corresponding to the 2-form $\delta(S_\star\psi^+)+\delta S\lrcorner\psi^+$. By the calculation above we get

$$\begin{aligned}B(X) &= ((e_i\lrcorner\psi^+_{e_j})\wedge(\bar\nabla_{e_i}S)e_j)(X) = \psi^+(e_j,e_i,X)(\bar\nabla_{e_i}S)e_j - (\bar\nabla_{e_i}S)(e_j,X)(e_i\lrcorner\psi^+_{e_j})\\ &= (\bar\nabla_{e_i}S)(\psi^+_{e_i}X) + \psi^+_{e_i}((\bar\nabla_{e_i}S)X) = (\bar\nabla_{e_i}S)(A_{e_i}JX) + A_{e_i}(J(\bar\nabla_{e_i}S)X)\\ &= (\bar\nabla_{e_i}S)(A_{e_i}JX) - A_{e_i}((\bar\nabla_{e_i}S)JX) = -(A_{e_i\star}(\bar\nabla_{e_i}S))(JX).\end{aligned}$$

Replacing X by JX yields (29). ☐

From Lemma 4.1 and Lemma 4.3 we infer directly

Corollary 4.4. *Let φ and S be sections of $\Lambda^{(1,1)}_0M$ and Sym^-M respectively. If h and σ are defined by $g(Jh.,.) := \varphi(.,.)$ and $\sigma := S_\star\psi^+$, then*

$$(\nabla^*\nabla - \bar\nabla^*\bar\nabla)\varphi = \varphi + (J\delta\varphi)\lrcorner\psi^+. \tag{31}$$

$$(\nabla^*\nabla - \bar\nabla^*\bar\nabla)\sigma = \sigma + 2\delta S\wedge\omega. \tag{32}$$

$$(\nabla^*\nabla - \bar\nabla^*\bar\nabla)h = 3h + s, \qquad \text{where } s\in\mathrm{Sym}^-M, \text{ and } s_\star\psi^+ = 2\delta h\wedge\omega + 4d\varphi. \tag{33}$$

$$(\nabla^*\nabla - \bar\nabla^*\bar\nabla)S = 2S - (\delta S\lrcorner\psi^+ + \delta\sigma)\circ J. \tag{34}$$

Finally, we obtain the invariance of the space of primitive co-closed $(1,1)$-forms under the Laplace operator:

Proposition 4.5. *If φ is a co-closed section of $\Lambda^{(1,1)}_0M$, then the same holds for $\Delta\varphi$.*

Proof. The 2-form $\Delta\varphi$ is clearly co-closed since φ is co-closed. Using (31), Corollary 3.5 and the classical Weitzenböck formula on 2-forms yield

$$\Delta\varphi = (\nabla^*\nabla + q(R))\varphi = (\bar\nabla^*\bar\nabla + q(\bar R))\varphi.$$

The last term is a section of $\Lambda^{(1,1)}_0M$ since both $\bar\nabla$ and $q(\bar R)$ preserve this space. ☐

5. The moduli space of Einstein deformations

We now have all the ingredients for the main result of this paper:

Theorem 5.1. *Let (M^6, g, J) be a Gray manifold. Then the moduli space of infinitesimal Einstein deformations of g is isomorphic to the direct sum of the spaces of primitive co-closed $(1,1)$-eigenforms of the Laplace operator for the eigenvalues 2, 6 and 12.*

Proof. Let g be an Einstein metric with $\mathrm{Ric} = Eg$. From [2], Theorem 12.30, the space of infinitesimal Einstein deformations of g is isomorphic to the set of symmetric trace-free endomorphisms H of TM such that $\delta H = 0$ and such that $\Delta_L H = 2EH$, where $\Delta_L = \nabla^*\nabla + q(R)$ is the so-called Lichnerowicz Laplacian Δ_L. Remark that $q(R) = 2\overset{\circ}{R} + 2E\mathrm{id}$ in the notation of [2].

In our present situation the Einstein constant equals $E = 5$, so the space of infinitesimal Einstein deformations of g is isomorphic to the set of $H \in \mathrm{Sym}M$ with $\delta H = 0 = \mathrm{tr}H$ such that

$$(\nabla^*\nabla + q(R))H = 10H. \tag{35}$$

Let $h := \mathrm{pr}_+\mathrm{H}$ and $S := \mathrm{pr}\_\mathrm{H}$ denote the projections of H onto $\mathrm{Sym}^{\pm}M$. We define the primitive $(1,1)$-form $\varphi(.,.) := g(Jh.,.)$ and the 3-form $\sigma := S_\star\psi^+$. The key idea is to express (35) in terms of an exterior differential system for φ and σ. Using Corollary 3.5 and Corollary 4.4, (35) becomes

$$(\bar\nabla^*\bar\nabla + q(\bar R))(h+S) = 10(h+S) - (3h+s) - (2S - (\delta S \lrcorner \psi^+ + \delta\sigma)\circ J) - 3h - 2S,$$

where s is the section of Sym^-M defined in the second part of (33). Since the operator $(\bar\nabla^*\bar\nabla + q(\bar R))$ preserves the decomposition $\mathrm{Sym}M = \mathrm{Sym}^+M \oplus \mathrm{Sym}^-M$, the previous equation is equivalent to the system

$$\begin{cases} (\bar\nabla^*\bar\nabla + q(\bar R))h = 4h + \delta\sigma\circ J, \\ (\bar\nabla^*\bar\nabla + q(\bar R))S = 6S - s, \\ \delta S = 0. \end{cases} \tag{36}$$

Taking the composition with J and using (15), the first equation of (36) becomes

$$(\bar\nabla^*\bar\nabla + q(\bar R))\varphi = 4\varphi - \delta\sigma. \tag{37}$$

Similarly, taking the action on ψ^+ and using (15) and the definition of s, the second equation of (36) becomes

$$(\bar\nabla^*\bar\nabla + q(\bar R))\sigma = 6\sigma - 2\delta h\wedge\omega - 4d\varphi. \tag{38}$$

Notice that $\delta h = \delta H - \delta S = 0$, which can also be seen by examining the algebraic types in equation (38). From (30) we get

$$0 = \delta h = -(\bar\nabla_{e_i}h)e_i = (\bar\nabla_{e_i}J\varphi)e_i = J(\bar\nabla_{e_i}\varphi)e_i = -J\delta\varphi,$$

so finally the system (36) is equivalent to

$$\begin{cases} (\bar\nabla^*\bar\nabla + q(\bar R))\varphi = 4\varphi - \delta\sigma, \\ (\bar\nabla^*\bar\nabla + q(\bar R))\sigma = 6\sigma - 4d\varphi, \\ \delta\varphi = 0. \end{cases} \tag{39}$$

Using Corollary 3.5 together with the equations (31) and (32) (keeping in mind that $\delta S = 0$ and $\delta\varphi = 0$) we get the two identities $(\bar\nabla^*\bar\nabla + q(\bar R))\varphi = (\nabla^*\nabla + q(R))\varphi$ and $(\bar\nabla^*\bar\nabla + q(\bar R))\sigma = (\nabla^*\nabla + q(R))\sigma$. Hence the classical Weitzenböck formula for the Laplace operator on forms implies that (39) is equivalent to

$$\begin{cases} \Delta\varphi = 4\varphi - \delta\sigma, \\ \Delta\sigma = 6\sigma - 4d\varphi, \\ \delta\varphi = 0. \end{cases} \tag{40}$$

Lemma 5.2. *Let $E(\lambda)$ be the λ-eigenspace of Δ restricted to the space of co-closed primitive $(1,1)$-forms. Then the space of solutions of the system (40) is isomorphic to the direct sum $E(2)\oplus E(6)\oplus E(12)$. The isomorphism can be written explicitly as*

$$(\varphi,\sigma) \overset{\Psi}{\mapsto} (8\varphi + \delta\sigma, *d\sigma, 2\varphi - \delta\sigma) \in E(2)\oplus E(6)\oplus E(12)$$

with inverse

$$(\alpha,\beta,\gamma) \in E(2)\oplus E(6)\oplus E(12) \overset{\Phi}{\mapsto} \left(\frac{\alpha+\gamma}{10}, \frac{3d\alpha - 5*d\beta - 2d\gamma}{30}\right).$$

Proof. The first thing to check is the fact that Φ and Ψ take values in the right spaces.

Let $(\alpha,\beta,\gamma)\in E(2)\oplus E(6)\oplus E(12)$ and $(\varphi,\sigma) := \Phi(\alpha,\beta,\gamma)$. From Lemma 4.4 in [5], the exterior derivative of every co-closed primitive $(1,1)$-form is a primitive $(2,1)+(1,2)$-form. Thus $\varphi\in\Omega_0^{(1,1)}M$ and $\sigma\in\Omega_0^{(2,1)+(1,2)}M$. A simple calculation shows that (φ,σ) satisfy the system (40).

Conversely, let (φ,σ) be a solution of (40) and $(\alpha,\beta,\gamma) := \Psi(\varphi,\sigma)$. Clearly α, β and γ are co-closed. From Proposition 4.5 and the first equation of (40) we see that $\delta\sigma$ is a section of $\Lambda_0^{(1,1)}M$, so the same holds for α and γ. The fact that $*d\sigma$ is a section of the same bundle follows from Lemma 4.3 in [5]. A direct check shows that $\Delta\alpha = 2\alpha$, $\Delta\beta = 6\beta$ and $\Delta\gamma = 12\gamma$.

Finally, it is straightforward to check that Φ and Ψ are inverse to each other. This proves the lemma and the theorem. ☐

In order to apply this result, one should try to compute the spectrum of the Laplacian on 2-forms on some explicit compact nearly Kähler 6-dimensional manifolds. Besides the sphere S^6 – which has no infinitesimal Einstein deformations (cf. [2]) – the only known examples are 3-symmetric spaces $\mathbb{CP}^3 = \mathrm{SO}_5/\mathrm{U}_2$, $F(1,2) = \mathrm{SU}_3/U_1\times\mathrm{U}_1$ and $\mathrm{SU}_2\times\mathrm{SU}_2 = \mathrm{SU}_2\times\mathrm{SU}_2\times\mathrm{SU}_2/\Delta$.

Computations of the Laplace spectrum using the Peter-Weyl theorem show that $E(2)$ and $E(6)$ vanish on each of these spaces. Moreover, $E(12)$ vanishes on $\mathbb{C}\mathrm{P}^3$ and on $\mathrm{SU}_2 \times \mathrm{SU}_2$, and is 8-dimensional on $F(1,2)$ (cf. [6]). As a consequence of these facts, we deduce:

(1) every infinitesimal Einstein deformation of the 6-dimensional 3-symmetric spaces is an infinitesimal Gray deformation (cf. [5]);
(2) the nearly Kähler structure on $\mathbb{C}\mathrm{P}^3$ and on $\mathrm{SU}_2 \times \mathrm{SU}_2$ is rigid;
(3) there is an 8-dimensional space of infinitesimal deformations of the nearly Kähler structure on $F(1,2)$.

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