THE DIRAC OPERATOR ON GENERALIZED TAUB-NUT SPACES

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Abstract. We find sufficient conditions for the absence of harmonic L^2 spinors on spin manifolds constructed as cone bundles over a compact Kähler base. These conditions are fulfilled for certain perturbations of the Euclidean metric, and also for the generalized Taub-NUT metrics of Iwai-Katayama, thus proving a conjecture of Vişinescu and the second author.

1. Introduction

The Taub-NUT metrics on $\mathbb{R}^4$ and their generalizations by Iwai-Katayama [9] provide a fruitful framework for the study of classical and quantum anomalies in the presence of conserved quantities, see e.g. [7]. To describe these metrics, consider the sphere S^3 as the unit sphere inside the quaternions. There exist then three orthogonal unit vector fields I, J, K given by left translation with the unit quaternions i, j, k. The Berger metrics g_λ on S^3 are defined by setting the length of I, J to be 1, and that of K to be λ. The Iwai-Katayama metrics on $\mathbb{R}^4 \setminus \{0\} \simeq \mathbb{R}^+ \times S^3$ have the form

$$g_{IK} = \gamma^2(t)(dt^2 + 4t^2 g_{\lambda(t)}) \tag{1.1}$$

where

$$\gamma(t) = \sqrt{\tfrac{a+bt}{t}}, \qquad \lambda(t) = \tfrac{1}{\sqrt{1+ct+dt^2}}$$

for positive constants a, b, c, d. The apparent singularity at the origin is removable.

We are interested here in the axial quantum anomaly already studied in [6, 16]. It was found in [6] that the axial anomaly, i.e., the difference between the number of null states of positive and of negative chirality on a ball or annular domain, may become non-zero for suitable choices of the parameters of the metric and of the domain when we impose the Atiyah-Patodi-Singer spectral condition at the boundary. Remarkably, when the radius of the ball is sufficiently large the index was always 0. It was further proved in [16] that on the whole space, although the Dirac operator is not Fredholm, it only has a finite number of null states. The method of proving the finiteness of the index in [16] relied on a general index formula due to Vaillant [19], and on a comparison between harmonic spinors for a pair of conformal metrics. On the standard Taub-NUT space, which is hyperkähler and therefore scalar-flat, it is easy to see that there are no harmonic L^2 spinors using the Lichnerowicz identity and the infiniteness of the volume. It was somehow natural to

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conjecture in [16] that the L^2 index of the Dirac operator corresponding to the generalized Taub-NUT metric is also zero. The motivation of the present work is to prove the above conjecture:

Theorem 1.1. *There do not exist L^2 harmonic spinors on $\mathbb{R}^4$ for the generalized Taub-NUT metrics. In particular, the L^2 index of the Dirac operator vanishes.*

As we just mentioned, for the standard Taub-NUT metric this has been proved in [16]. Our approach here is less analytic, and more geometric, than the previous attempt described above. We exploit the rich symmetries of the metric to decompose the spinors in terms of frequencies along the fibers as in e.g. [18], and then further in terms of eigenvalues of an associated spin^c Dirac operator on S^2. We obtain a system of ordinary differential equations which we show does not admit any L^2 solutions. There are similarities with [11], [15] in the analysis of this system, but the essential difference is that large time behavior is not enough to rule out L^2 harmonic spinors and we must use also the behavior near the origin. The method is more general and we can prove our results for a wider class of manifolds, constructed from a circle fibration over a compact Hodge base. Although the one-point completion of such a manifold will not be in general a topological manifold, we consider it as a singular complete metric space, the appropriate condition on spinors being boundedness in the L^∞ norm near the singular point. Our main result (Theorem 5.1) applies both to the Iwai-Katayama metrics and to Euclidean metrics, and to certain perturbations thereof.

The paper is organized as follows: In Section 2 we introduce the class of metrics studied in the rest of the paper. In Section 3 we relate geometric objects – like the Levi-Civita connection and the Dirac operator – of a circle-fibered space to the corresponding objects on the base, and we introduce the announced splitting into frequencies along the fibers. Section 4 contains similar computations in the case of warped products, introducing an extra variable corresponding to the radial direction, and computing the corresponding spin^c Dirac operator. The main analytic result is stated and proved in Section 5 by reducing the problem to a linear system of ordinary differential equations on the positive real half-line, and a careful analysis both near infinity and 0 to exclude L^2 solutions. Finally in Section 6 we extend the result in a rather formal way to include the Iwai-Katayama generalized Taub-NUT metrics.

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2. Circle fibered warped products

Let (B, g_B, Ω) be a compact Kähler manifold of real dimension $2m$. Let h denote the warped product metric $dt^2 + \alpha^2(t) g_B$ on $N := \mathbb{R}^+ \times B$ and let p denote the projection $N \to B$.

Assume that (B, g_B, Ω) is a Hodge manifold, i.e., $[\Omega] \in 2\pi H^2(B, \mathbb{Z})$. The classical isomorphism of Čech cohomology groups $H^1(B, S^1) \simeq H^2(B, \mathbb{Z})$ shows the existence of a Hermitian line bundle $L_0 \to B$ with first Chern class $c_1(L_0) = -[\Omega]/2\pi$. Let M_0 denote the circle bundle of L_0. The projection $q : M_0 \to B$ can be viewed as a principal S^1-bundle. By Chern-Weil theory (cf. [14], Ch. 16 for instance) there exists an imaginary-valued connection 1-form $i\xi$ on M_0 such that $d\xi = q^*\Omega$.

We define $L := \mathbb{R}^+ \times L_0$ and $\pi := \mathrm{id} \times p$ the projection of L onto N. Then $\pi : L \to N$ is a Hermitian line bundle over N whose circle bundle, denoted by M, is just $M := \mathbb{R}^+ \times M_0$.

We endow M with the Riemannian metric $g := dt^2 + \alpha^2(t)(p \circ \pi)^* g_B + \beta^2(t)\xi \otimes \xi$ for some positive functions α and β defined on $\mathbb{R}^+$.

The Riemannian manifold (M, g) obtained in this way will be referred to as the *circle-fibered warped product* (CFWP) over the Hodge manifold (B, g_B, Ω), with warping functions α and β. Notice that a CFWP can be viewed either as a *generalized cylinder* of a family of metrics on the S^1-bundle M_0 over B (cf. Proposition 2.3 below) or as a Riemannian submersion with 1-dimensional fibres over a warped product $\mathbb{R}^+ \times_\alpha B$. It is the latter point of view which will be useful in order to relate spinors on M and B.

Example 2.1. The flat space $\mathbb{C}^{m+1} \setminus \{0\}$ is the CFWP over the complex projective space $(\mathbb{CP}^m, g_{FS}, \Omega_{FS})$ endowed with the Fubini-Study metric, with warping functions $\alpha(t) = \frac{t}{\sqrt{2}}$, $\beta(t) = t$. The normalization g_{FS} of the Fubini-Study metric used here is the one with scalar curvature equal to $2m(m+1)$ or, equivalently, the one for which the projection $\mathbb{S}^{2m+1} \to (\mathbb{CP}^m, \frac{1}{2}g_{FS})$ is a Riemannian submersion (cf. [14], Ch. 13).

Example 2.2. The Taub-NUT metric on $\mathbb{C}^2$ is conformal to the one-point completion of the CFWP over the standard 2-sphere of radius $1/\sqrt{2}$ with warping functions $\alpha(t) = \sqrt{2}t$, $\beta(t) = \frac{2t}{1+bt}$. More generally, the *generalized* Taub-NUT metrics of Iwai-Katayama on $\mathbb{C}^2$ are conformal to the one-point completion of the CFWP over $(\mathbb{CP}^1, g_{FS})$, i.e., the standard 2-sphere of radius $1/\sqrt{2}$, with warping functions $\alpha(t) = \sqrt{2}t$, $\beta(t) = \frac{2t}{\sqrt{1+ct+dt^2}}$ for some positive constants c and d (cf. [16], p. 6576):

$$g_{IK} = \tfrac{a+bt}{t}\left(dt^2 + \alpha^2(t)\pi^* g_{FS} + \beta^2(t)\xi \otimes \xi\right). \tag{2.1}$$

By Remark 2.4 below, these are actually examples of CFWP's.

Proposition 2.3. *Let (M, g) be the CFWP over a Hodge manifold (B^{2m}, g_B, Ω) with warping functions α and β and assume that $\lim_{t\to 0} \alpha(t) = \lim_{t\to 0} \beta(t) = 0$. Let d denote the distance on M induced by g. Then the metric completion $(\hat{M}, d)$ of (M, d) has exactly one extra point. If g extends to a smooth metric on $\hat{M}$, then (B, g_B, Ω) is the complex projective space endowed with the Fubini-Study metric, and*

$$\lim_{t\to 0} \tfrac{\alpha(t)}{t} - \tfrac{1}{\sqrt{2}} = \lim_{t\to 0} \tfrac{\beta(t)}{t} - 1 = 0.$$

Proof. The Riemannian manifold (M, g) will be viewed as a generalized cylinder (cf. [4]) of the family of metrics $g_t := \alpha^2(t)q^* g_B + \beta^2(t)\xi \otimes \xi$ on the S^1-bundle M_0 over B (which is a compact manifold). The first statement follows immediately from the fact that for every $x \in M_0$, the rays $\mathbb{R}^+ \times \{x\}$ are geodesics parametrized by arc-length on M. Assume now that g extends smoothly to $\hat{M}$. The Gauss Lemma applied to a neighborhood of the origin $t = 0$ in $\hat{M}$ shows that the distance spheres (M_0, g_t) (renormalized by a factor $1/t^2$) tend to the standard sphere S^{2m+1} in the Gromov-Hausdorff topology. In other words, there exist non-zero constants α_0, β_0 such that $\lim_{t\to 0} \alpha(t)/t = \alpha_0$, $\lim\limits_{t\to 0} \beta(t)/t = \beta_0$ and $\alpha_0^2 q^* g_B + \beta_0^2 \xi \otimes \xi$ is the standard metric on S^{2m+1}. On the other hand, this metric is by definition a Riemannian submersion over $(B, \alpha_0^2 g_B)$ with totally geodesic fibres of length $2\pi\beta_0$. Since the length of every closed geodesic on S^{2m+1} is 2π we get $\beta_0 = 1$. The manifold $(B, \alpha_0^2 g_B)$ is then the quotient of the sphere by an isometric S^1 action, so $B = \mathbb{C}\mathrm{P}^m$ and $\alpha_0^2 g_B = \frac{1}{2} g_{FS}$ (cf. [14], Ch.13). The constant α_0 is determined by the normalization condition $c_1(L_0) = -[\Omega_B]/2\pi$. Indeed, since $L_0 = \mathbb{R}^{2m+2} \setminus \{0\}$ is clearly the tautological bundle $\Omega(-1)$ of $\mathbb{C}\mathrm{P}^m$, its first Chern class is equal to $-[\Omega_{FS}]/2\pi$, whence $\alpha_0 = 1/\sqrt{2}$. □

The converse holds under some extra smoothness assumption on α and β at $t = 0$ but we will not need this in the sequel.

Remark 2.4. A metric conformal to a CFWP is itself a CFWP provided that the conformal factor is a radial function (i.e., it only depends on t). Indeed, if

$$g = \gamma(t)^2(dt^2 + \alpha^2(t)\pi^* g_B + \beta^2(t)\xi \otimes \xi),$$

in the new coordinate $s := s(t)$ defined by $s = \int_0^t \gamma(u)du$, g reads

$$g = ds^2 + \alpha^2(t(s))\pi^* g_B + \beta^2(t(s))\xi \otimes \xi.$$

The generalized Taub-NUT metrics from Example 2.2 are thus particular cases of CFWP. We will analyze these metrics in more detail in Section 6.

Our main goal in this paper will be to study the L^2-index of the Dirac operator on a CFWP (M, g) when M is a spin manifold. As we will see below, this is automatically the case when B has a spin^c structure whose auxiliary bundle is some tensor power of L_0, i.e., if the second Stiefel-Whitney class of B satisfies $w_2(B) = 0$ or $w_2(B) \equiv c_1(L_0)$ mod 2. In the next two sections we will relate spinors on M to spin^c spinors on N and then further to spin^c spinors on B. The results are quite general and can be viewed as a natural extension of the theory of projectable spinors introduced in [12] to the case of submersions with non-totally geodesic fibres.

3. Spinors on circle fibrations

Let $\pi : (M, g) \to (N, h)$ be a Riemannian submersion with 1-dimensional fibres of length $2\pi\beta$ for some function $\beta : N \to \mathbb{R}^+$. The fibers of π are totally geodesic if and only if β is constant, but we will mostly be interested in examples with non-constant β in the sequel.

We can view M as a principal S^1-fibration over N. Indeed, the flow φ_t of the vertical Killing vector field V on M of length $\pi^*\beta$ closes up at time $t = 2\pi$, i.e., $\varphi_{2\pi} = id_M$, thus it defines a free S^1-action on M whose orbit space is N. We denote by $P_{\mathrm{U}(1)}N$ this principal S^1-bundle with total space M. The Riemannian metric g can be written as $g = \pi^* h + \beta^2(t)\xi \otimes \xi$, where ξ is the 1-form on M defined by $\xi(V) = 1$ and $\ker \xi = V^\perp$.

The 2-form $d\xi$ is basic, i.e., there exists some 2-form F on N such that $d\xi = \pi^*F$. This follows immediately from the Cartan formula and the fact that V is Killing, or alternately since $i\xi$ is a connection 1-form in the principal bundle $P_{\mathrm{U}(1)}N$ (cf. Section 2).

The following result holds without restriction on the dimension of N but we will state it only for the case we will need in the sequel.

Lemma 3.1. *Let $P_{\mathrm{U}(1)}N \to N$ be the principal S^1-bundle over the $2m+1$-dimensional manifold N defined by the Riemannian submersion $\pi : M \to N$. Let $L \to N$ be the complex line bundle associated to $P_{\mathrm{U}(1)}N$ with respect to the canonical representation of S^1 on $\mathbb{C}$. Then every spin^c structure $P_{\mathrm{Spin}^c(2m+1)}N$ on N with auxiliary bundle $L^{\otimes k}$, $k \in \mathbb{Z}$ induces a spin structure on M and all these spin structures are isomorphic.*

Proof. By enlargement of the structure groups, the two-fold covering

$$\theta : P_{\mathrm{Spin}^c(2m+1)}N \to P_{\mathrm{SO}(2m+1)}N \times P_{\mathrm{U}(1)}N$$

gives a two-fold covering

$$\theta : P_{\mathrm{Spin}^c(2m+2)}N \to P_{\mathrm{SO}(2m+2)}N \times P_{\mathrm{U}(1)}N,$$

which, by pull-back through π, gives rise to a Spin^c structure on M:

$$\begin{array}{ccc}
P_{\mathrm{Spin}^c(2m+2)}M & \xrightarrow{\pi} & P_{\mathrm{Spin}^c(2m+2)}N \\
{\scriptstyle \pi^*\theta}\big\downarrow & & {\scriptstyle \theta}\big\downarrow \\
P_{\mathrm{SO}(2m+2)}M \times P_{\mathrm{U}(1)}M & \xrightarrow{\pi} & P_{\mathrm{SO}(2m+2)}N \times P_{\mathrm{U}(1)}N \\
\big\downarrow & & \big\downarrow \\
M & \xrightarrow{\pi} & P.
\end{array}$$

This construction actually yields a *spin* structure on M. Indeed, the pull back $P_{\mathrm{U}(1)}M$ to M of $P_{\mathrm{U}(1)}N$ is trivial since it carries a tautological global section $\sigma(u) = (u, u)$, $\forall u \in M = P_{\mathrm{U}(1)}N$. Correspondingly, the pull-back to M of every associated bundle $L^{\otimes k}$ is trivial. $\square$

From now on we assume that N carries some spin^c structure with auxiliary bundle $L^{\otimes k}$, and we study M with the spin structure induced by the previous lemma. In particular, we will consider the flat connection on the trivial bundle $P_{\mathrm{U}(1)}M$, rather than the pull-back connection from $P_{\mathrm{U}(1)}N$, in order to define covariant derivatives of spinors on M. The following result, first proved in [13], relates an arbitrary connection on a principal bundle $\pi : M = P_{\mathrm{U}(1)}N \to N$ and the flat connection on $\pi^*M = P_{\mathrm{U}(1)}M \to M$.

$$\begin{array}{ccc}
\pi^*M = P_{\mathrm{U}(1)}M \simeq M \times S^1 & \xrightarrow{\pi} & M = P_{\mathrm{U}(1)}N \\
{\scriptstyle \pi^*\pi}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\
M & \xrightarrow{\pi} & N
\end{array}$$

Lemma 3.2. *The connection form A_0 of the flat connection on $P_{\mathrm{U}(1)}M$ can be related to an arbitrary connection A on $P_{\mathrm{U}(1)}N$ by*

$$A_0((\pi^*s)_*(U)) = -A(U),$$

$$A_0((\pi^*s)_*(X^*)) = A(s_*X),$$

where U is a vertical vector field on M, X^* is the horizontal lift (with respect to A) of a vector field X on N, and s is a local section of $M \to N$.

Proof. The identification $M \times \mathrm{U}(1) \simeq \pi^*M$ is given by $(u, a) \mapsto (u, ua)$, for all $(u, a) \in M \times \mathrm{U}(1)$. For some fixed $u \in M$, take a path u_t in the fiber over $x := \pi(u)$ such that $u_0 = u$ and $\dot{u}_0 = U$. We define $a_t \in \mathrm{U}(1)$ by $u_t = s(x)a_t$, so via the above identification we have

$$(\pi^*s)(u_t) = (u_t, s(x)) = (u_t, (a_t)^{-1}),$$

and thus

$$A_0((\pi^*s)_*(U)) = -a_0^{-1}\dot{a}_0 = -A(\dot{u}_0) = -A(U).$$

Similarly, for $x \in N$ and $X \in T_xN$, take a path x_t in N such that $x_0 = x$ and $\dot{x}_0 = X$. Let $u \in \pi^{-1}(x)$ and u_t the horizontal lift of x_t such that $u_0 = u$. We define $a_t \in \mathrm{U}(1)$ by $s(x_t) = u_t a_t$, which by derivation gives $s_*(X) = R_{a_0}\dot{u}_0 + u_0\dot{a}_0$. Then

$$(\pi^*s)(u_t) = (u_t, s(x_t)) = (u_t, a_t),$$

and thus, using the fact that $\dot{u}_0$ is horizontal,

$$A_0((\pi^*s)_*(X^*)) = a_0^{-1}\dot{a}_0 = A(s_*(X)).$$ □

Recall that the complex Clifford representation $\Sigma_{2m+2} = \Sigma^+_{2m+2} \oplus \Sigma^-_{2m+2}$ can be identified with $\Sigma_{2m+1} \oplus \Sigma_{2m+1}$ by defining in an orthonormal basis

$$e_j \cdot (\psi, \phi) = \begin{cases} (e_j \cdot \phi, e_j \cdot \psi) & \text{for } j \leq 2m+1 \\ (-\phi, \psi) & \text{for } j = 2m+2. \end{cases}$$

Accordingly, we obtain identifications, denoted by $\pi^\pm$, of the pull back $\pi^*\Sigma N$ with $\Sigma^\pm M$. By a slight abuse of notation we will denote $\pi^\pm$ and $\Sigma^\pm M$ by π^ε and $\Sigma^\varepsilon M$ for $\varepsilon = \pm 1$. With respect to these identifications, if X is a vector and Ψ is a spinor on N, then

$$X^* \cdot \pi^\varepsilon \Psi = \pi^{-\varepsilon}(X \cdot \Psi), \tag{3.1}$$

$$\tfrac{1}{\beta} V \cdot (\pi^\varepsilon \Psi) = \varepsilon \pi^{-\varepsilon}\Psi, \tag{3.2}$$

where $\frac{1}{\beta}V$ is the unit vertical vector field defined at the beginning of this section, and X^* denotes the horizontal lift to M of a vector field X on N.

We consider now a spin^c structure $P_{\mathrm{Spin}^c(2m+1)}N$ on (N, h) with auxiliary bundle $L^{\otimes k}$ and denote by ∇^N the covariant derivative induced on ΣN by the connection form $i\xi$ of $P_{\mathrm{U}(1)}N$. By Lemma 3.1, the pull-back to M of $P_{\mathrm{Spin}^c(2m+1)}N$ induces by enlargement a spin structure on (M, g), where we recall that $g = \pi^*h + \beta^2\xi \otimes \xi$.

Proposition 3.3. *Let ∇^M denote the covariant derivative on $\Sigma^\varepsilon M$ induced by the Levi-Civita connection on (M, g) and the flat connection on $\pi^*P_{\mathrm{U}(1)}N$. Let ∇^N denote the* *spin^c covariant derivative on ΣN induced by the Levi-Civita connection on (N,h) and the connection form $A = i\xi$ on $P_{\mathrm{U}(1)}N$. Then ∇^M and ∇^N are related by*

$$\nabla^M_{X^*}(\pi^\varepsilon\Psi) = \pi^\varepsilon(\nabla^N_X\Psi - \tfrac{\varepsilon\beta}{4}T(X)\cdot\Psi), \qquad \forall X \in TM, \tag{3.3}$$

$$\nabla^M_V(\pi^\varepsilon\Psi) = \pi^\varepsilon\left(\tfrac{\beta^2}{4}F\cdot\Psi + \tfrac{\varepsilon}{2}d\beta\cdot\Psi - \tfrac{ki}{2}\Psi\right), \tag{3.4}$$

where T is the endomorphism of TN defined by $d\xi(X^*,Y^*) = F(X,Y) = h(TX,Y)$.

Proof. If V denotes as before the vertical vector field such that $\xi(V) = 1$, the Koszul formula and the fact that $[V,X^*] = 0$ for all vector fields X on N yield

$$g(\nabla^M_{X^*}Y^*,Z^*) = h(\nabla^N_X Y, Z) \tag{3.5}$$

$$\begin{aligned} g(\nabla^M_V X^*, Y^*) &= g(\nabla^M_{X^*}V, Y^*) = -\tfrac{1}{2}g(V,[X^*,Y^*]) = -\tfrac{\beta^2}{2}\xi([X^*,Y^*]) \\ &= \tfrac{\beta^2}{2}d\xi(X^*,Y^*) = \tfrac{\beta^2}{2}h(TX,Y) = \tfrac{\beta^2}{2}F(X,Y), \end{aligned} \tag{3.6}$$

and

$$g(\nabla^M_V X^*, V) = g(\nabla^M_{X^*}V,V) = \beta X(\beta), \tag{3.7}$$

for all vector fields X, Y and Z on N.

Consider a spinor field on N locally expressed as $\Psi = [\sigma,\psi]$, where $\psi : U \subset N \to \Sigma_{2m+1}$ is a vector-valued function, and σ is a local section of $P_{\mathrm{Spin}^c(2m+1)}N$ whose projection onto $P_{\mathrm{SO}(2m+1)}N$ is a local orthonormal frame $(X_1,...,X_{2m+1})$ and whose projection onto $P_{\mathrm{U}(1)}N$ is a local section s. Then $\pi^\varepsilon\Psi$ can be expressed as $\pi^\varepsilon\Psi = [\pi^*\sigma, \pi^*\xi]$. Moreover, the projection of $\pi^*\sigma$ onto $P_{\mathrm{SO}(2m+2)}M$ is the local orthonormal frame $(\frac{1}{\beta}V, X_1^*,...,X_{2m+1}^*)$ and its projection onto $P_{\mathrm{U}(1)}M$ is just π^*s.

Using the general formula for the covariant derivative on spinors, Lemma 3.2, and the fact that the bundle $L^{\otimes k}$ is associated to $P_{\mathrm{U}(1)}N$ via the representation $\rho^k(z) = z^k$ of S^1 on $\mathbb{C}$, we obtain

$$\begin{aligned} \nabla^M_{X^*}\pi^\varepsilon\Psi &= [\pi^*\sigma, X^*(\pi^*\psi)] + \tfrac{1}{2}\sum_{j<k} g(\nabla^M_{X^*}X_j^*, X_k^*)X_j^*\cdot X_k^*\cdot\pi^\varepsilon\Psi \\ &\quad + \tfrac{1}{2}\sum_j g(\nabla^M_{X^*}X_j^*, \tfrac{1}{\beta}V)X_j^*\cdot\tfrac{1}{\beta}V\cdot\pi^\varepsilon\Psi + \tfrac{1}{2}\rho^k_*(A_0((\pi^*s)_*X^*))\pi^\varepsilon\Psi \\ &= [\pi^*\sigma, \pi^*(X(\psi))] + \tfrac{1}{2}\sum_{j<k} h(\nabla^N_X X_j, X_k)\pi^\varepsilon(X_j\cdot X_k\cdot\Psi) + \tfrac{1}{2}\rho^k_*(A(s_*X))\pi^\varepsilon\Psi \\ &\quad - \tfrac{\varepsilon\beta}{4}\sum_j h(T(X),X_j)\pi^\varepsilon(X_j\cdot\Psi) = \pi^\varepsilon\left(\nabla^N_X\Psi - \tfrac{\varepsilon\beta}{4}T(X)\cdot\Psi\right). \end{aligned}$$

and similarly, since $A = i\xi$,

$$\begin{aligned}
\nabla^M_V(\pi^\varepsilon\Psi) &= [\pi^*\sigma, V(\pi^*\psi)] + \tfrac{1}{2}\sum_{j<k} g(\nabla^M_V X_j^*, X_k^*) X_j^*\cdot X_k^*\cdot\pi^\varepsilon\Psi \\
&\quad + \tfrac{1}{2}\sum_j g(\nabla^M_V X_j^*, \tfrac{1}{\beta}V) X_j^*\cdot\tfrac{1}{\beta}V\cdot\pi^\varepsilon\Psi + \tfrac{1}{2}\rho^k_*(A_0((\pi^*s)_*V))\pi^\varepsilon\Psi \\
&= \tfrac{\beta^2}{4}\sum_{j<k} F(X_j, X_k)\pi^\varepsilon(X_j\cdot X_k\cdot\Psi) + \varepsilon\tfrac{1}{2}\sum_j X_j(\beta)\pi^\varepsilon(X_j\cdot\Psi) - \tfrac{1}{2}\rho^k_*(A(V))\pi^\varepsilon\Psi \\
&= \tfrac{\beta^2}{4}\pi^\varepsilon(F\cdot\Psi) + \tfrac{\varepsilon}{2}\pi^\varepsilon(d\beta\cdot\Psi) - \tfrac{ki}{2}\pi^\varepsilon\Psi = \pi^\varepsilon\left(\tfrac{\beta^2}{4}F\cdot\Psi + \tfrac{\varepsilon}{2}d\beta\cdot\Psi - \tfrac{ki}{2}\Psi\right). \qquad \square
\end{aligned}$$

Corollary 3.4. *The Dirac operators on M and N are related by*

$$D^M(\pi^\varepsilon\Psi) = \pi^{-\varepsilon}\left(D^N\Psi - \tfrac{\varepsilon\beta}{4}F\cdot\Psi + \tfrac{d\beta}{2\beta}\cdot\Psi - \tfrac{\varepsilon ki}{2\beta}\Psi\right) \tag{3.8}$$

Proof. Simple computation using (3.1)–(3.4):

$$\begin{aligned}
D^M(\pi^\varepsilon\Psi) &= \sum_j X_j^*\cdot\nabla^M_{X_j^*}(\pi^\varepsilon\Psi) + \tfrac{1}{\beta}V\cdot\nabla^M_{\frac{1}{\beta}V}(\pi^\varepsilon\Psi) \\
&= \sum_j X_j^*\cdot\pi^\varepsilon\left(\nabla^N_{X_j}\Psi - \tfrac{\varepsilon\beta}{4}T(X_j)\cdot\Psi\right) + \tfrac{1}{\beta}V\cdot\tfrac{1}{\beta}\pi^\varepsilon\left(\tfrac{\beta^2}{4}F\cdot\Psi + \tfrac{\varepsilon}{2}d\beta\cdot\Psi - \tfrac{ki}{2}\Psi\right) \\
&= \pi^{-\varepsilon}\left(D^N\Psi - \tfrac{\varepsilon\beta}{4}X_j\cdot T(X_j)\cdot\Psi + \tfrac{\varepsilon\beta}{4}F\cdot\Psi + \tfrac{d\beta}{2\beta}\cdot\Psi - \tfrac{\varepsilon ki}{2\beta}\Psi\right) \\
&= \pi^{-\varepsilon}\left(D^N\Psi - \tfrac{\varepsilon\beta}{4}F\cdot\Psi + \tfrac{d\beta}{2\beta}\cdot\Psi - \tfrac{\varepsilon ki}{2\beta}\Psi\right). \qquad \square
\end{aligned}$$

Note that Equation (3.4) is just the Bourguignon-Gauduchon formula [5] for the Lie derivative of a spinor field with respect to a Killing vector field:

$$\nabla_V\Phi = \mathcal{L}_V\Phi + \tfrac{1}{4}dV^\flat\cdot\Phi.$$

Incidentally, this formula shows that if $\Phi = \pi^\varepsilon\Psi$ is the pull-back of a spin^c spinor on N corresponding to a spin^c structure with auxiliary bundle $L^{\otimes k}$, then $\mathcal{L}_V\Phi = -\frac{ik}{2}\Phi$. Yet another way to understand this fact is the following. A section s of the complex line bundle $L \to N$ can be identified with a complex function f_s on $M \subset L$ with "frequency" -1 by $s(\pi(x)) = xf_s(x)$ (clearly $f_s(\varphi_t(x)) = e^{-it}f_s(x)$ so $\mathcal{L}_V(f_s) = -if_s$). Now, a spin^c bundle on N with auxiliary bundle $L^{\otimes k}$ is the tensor product between two locally defined bundles: the spin bundle of N and a square root of $L^{\otimes k}$. It is now clear that the pull-back to M of its sections are spinors on M with "frequency" $-k/2$.

Lemma 3.5. *Let $\mathcal{H}$ be the Hilbert space of L^2 spinors on M and let $\mathcal{H}_n$ be the Hilbert space of L^2 sections of the spin^c bundle on N with auxiliary bundle $L^{\otimes n}$. If the spin structure of M is induced as before by a spin^c structure on N with auxiliary bundle $L^{\otimes k}$, $\mathcal{H}$ decomposes in a Hilbertian orthogonal direct sum*

$$\mathcal{H} = \bigoplus_{n\in\mathbb{Z},\varepsilon=\pm 1}\pi^\varepsilon(\mathcal{H}_{k+2n}).$$

If $\Phi \in \mathcal{H}$ is smooth, the same holds for its components $\Phi_n \in \mathcal{H}_{k+2n}$ and the length of Φ_n at any $x \in M$ is bounded by the maximum of the lengths of Φ along the S^1-orbit of x.

Proof. The space of L^2 functions on M decomposes in a Hilbertian orthogonal direct sum

$$L^2(M) = \bigoplus_{n\in\mathbb{Z}} L^2_n(M),$$

where $L^2_n(M) := \{f \in L^2(M) \mid \mathcal{L}_V(f) = in \cdot f\}$ is the space of L^2 functions on M of frequency n, identified as before with the space of L^2 sections of $(L^*)^{\otimes n}$. The desired decomposition now follows immediately from the fact that the tensor product between $L^{\otimes n}$ and the spin^c bundle on N with auxiliary bundle $L^{\otimes k}$ is the spin^c bundle on N with auxiliary bundle $L^{\otimes(k+2n)}$.

The last assertion follows from the fact that

$$\Phi_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}((\varphi_t)_*\Phi)(x)dt. \qquad \square$$

A similar decomposition of the space of spinors on total space of S^1 fibrations was used by Ammann [1] and Ammann and Bär [3] in order to study the properties of the spectrum of the Dirac operator when the fibres collapse, and also by Nistor [18] in his study of the S^1-equivariant index.

Note that Eqs. (3.3) and (3.4) already appeared (in a slightly different form because of different spinor identifications) as Lemma 3.2 and Eq. (2) respectively in Ammann [1]. However, since the proofs of these formulas appear only in Ammann's thesis [2], we have chosen for the reader's convenience to include here the full details of the proofs.

4. Spinors on warped products

Let now (B, g_B) be a Riemannian manifold and assume that (N, h) is the warped product $(\mathbb{R}^+ \times B, dt^2 + \alpha(t)^2 g_B)$ for some positive function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$. We denote by $p : N \to B$ the standard projection. Let ∇^N and ∇^B denote the Levi-Civita covariant derivatives on N and B and let ∂_t denote the (unit) radial vector field on N. Every vector field X on B defines a "horizontal" vector field also denoted by X on N such that $[X, \partial_t] = 0$.

The warped product formulae for the covariant derivatives ([17], p.206) are

$$\nabla^N_{\partial_t}\partial_t = 0, \tag{4.1}$$

$$\nabla^N_{\partial_t}X = \nabla^N_X\partial_t = \tfrac{\alpha'}{\alpha}X, \tag{4.2}$$

$$\nabla^N_X Y = \nabla^B_X Y - \alpha\alpha' g_B(X, Y)\partial_t. \tag{4.3}$$

Consider a spin^c structure on B with auxiliary line bundle $L_0^{\otimes k}$ and the induced pull-back spin^c structure on N with auxiliary line bundle $L^{\otimes k}$, where $L = p^*L_0$ is the pull-back of L_0. We continue to denote by ∇^B and ∇^N the spin^c covariant derivatives induced by some connection on L_0.

Assume now that B has even dimension. The spinor bundle ΣN can be canonically identified with $\pi^*(\Sigma B)$ such that the Clifford product satisfies

$$\tfrac{1}{\alpha}X\cdot(p^*\Psi) = p^*(X\cdot\Psi), \qquad \forall X \in TB, \tag{4.4}$$

and

$$\partial_t\cdot(p^*\Psi) = ip^*(\bar\Psi), \tag{4.5}$$

where $\bar\Psi := \Psi_+ - \Psi_-$ is the "conjugate" of $\Psi = \Psi_+ + \Psi_-$ with respect to the chiral decomposition $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$. From (4.4) we easily obtain

$$\alpha^q(p^*\omega)\cdot(p^*\Psi) = p^*(\omega\cdot\Psi) \tag{4.6}$$

for every q-form ω on B. Using the warped product formulae one can easily relate the spin^c covariant derivatives ∇^N and ∇^B like before:

$$\nabla^N_X(p^*\Psi) = p^*(\nabla^B_X\Psi - \tfrac{1}{2}i\alpha' X\cdot\bar\Psi), \qquad \forall X \in TB, \tag{4.7}$$

and

$$\nabla^N_{\partial_t}(p^*\Psi) = 0. \tag{4.8}$$

In particular, the Dirac operators on N and B are related by

$$D^N(p^*\Psi) = \tfrac{1}{\alpha}p^*(D^B\Psi + im\alpha'\bar\Psi). \tag{4.9}$$

Indeed, if $(X_1, \ldots, X_{2m})$ is a local orthonormal basis on B, then $(\frac{1}{\alpha}X_1, \ldots, \frac{1}{\alpha}X_{2m}, \partial_t)$ is a local orthonormal basis on N, whence

$$\begin{aligned} D^N(p^*\Psi) &= \sum_j \tfrac{1}{\alpha}X_j\cdot\nabla^N_{\frac{1}{\alpha}X_j}(p^*\Psi) + \partial_t\cdot\nabla^N_{\partial_t}(p^*\Psi) \\ &= \sum_j \tfrac{1}{\alpha}X_j\cdot\left(\tfrac{1}{\alpha}p^*(\nabla^B_{X_j}\Psi - \tfrac{1}{2}i\alpha' X_j\cdot\bar\Psi)\right) \\ &= \tfrac{1}{\alpha}p^*(D^B\Psi + im\alpha'\bar\Psi). \end{aligned}$$

More generally, one can identify a spinor Ψ on N with a 1-parameter family Ψ_t of spinors on B, and (4.9) becomes

$$D^N\Psi = p^*\left(\tfrac{1}{\alpha}D^B\Psi_t + \tfrac{im\alpha'}{\alpha}\bar\Psi_t + i\dot{\bar\Psi}_t\right). \tag{4.10}$$

5. Harmonic spinors on CFWP's

We now have all necessary ingredients in order to prove the main result of this paper:

Theorem 5.1. *Let (M, g), $g = dt^2 + \alpha^2(t)\pi^* g_B + \beta^2(t)\xi\otimes\xi$ be a circle-fibered warped product (CFWP) over the Hodge manifold (B^{2m}, g_B, Ω), endowed with the spin structure defined by some spin^c structure on B as before. Assume that the positive warping functions α and β satisfy the conditions*

(a) $\displaystyle\int_x^\infty e^{-\int_x^t \frac{1}{\sqrt{2}\alpha(s)}ds}dt = \infty$ *for all* $x > 0$*;*

(b) $\lim_{t\to 0} \alpha(t) = 0$*;*

(c) $2\alpha^2(t) \geq \beta^2(t) > \frac{m-1}{m}2\alpha^2(t)$ *for all* $t > 0$*.*

Then (M, g) *carries no non-trivial harmonic* L^2 *spinors which are bounded near the singularity* $t = 0$*.*

Proof. Assume that Φ is a non-zero harmonic L^2 spinor, bounded near $t = 0$. By elliptic regularity, Φ is smooth. We can of course assume that Φ is chiral, i.e., it is a section of $\Sigma^\varepsilon M$ for some $\varepsilon = \pm 1$. By Lemma 3.5, and using the fact that the Dirac operator commutes with the Lie derivative $\mathcal{L}_V$, we can also assume that $\Phi = \pi^\varepsilon\Psi$ is the pull-back of some L^2 section Ψ of the spin^c structure on N with auxiliary bundle $L^{\otimes k}$ (the integer k is even or odd, depending on whether the spin structure on M projects onto a spin structure on N or not).

Using (3.8) we infer

$$D^N\Psi = \tfrac{1}{4\beta}(\varepsilon\beta^2 F\cdot\Psi - 2d\beta\cdot\Psi + \varepsilon 2ik\Psi). \tag{5.1}$$

We now view Ψ as a family Ψ_t of spin^c spinors on B. Recalling that $F = p^*\Omega$ and taking (5.1), (4.6) and (4.10) into account, we get

$$\left(\tfrac{1}{\alpha}D^B\Psi_t + \tfrac{im\alpha'}{\alpha}\bar\Psi_t + i\dot{\bar\Psi}_t\right) = \tfrac{1}{4\beta}\left(\varepsilon\tfrac{\beta^2}{\alpha^2}\Omega\cdot\Psi_t - 2i\beta'\bar\Psi_t + \varepsilon 2ik\Psi_t\right) \tag{5.2}$$

The spin^c bundle ΣB decomposes in a direct sum (cf. [10])

$$\Sigma B = \bigoplus_{l=0}^{m} \Sigma^l B$$

of eigenspaces of the operator of Clifford multiplication by the Kähler form Ω, i.e.,

$$\Sigma^l B = \{\Psi \in \Sigma B \mid \Omega\cdot\Psi = i(2l - m)\Psi\}.$$

One has $\Sigma^l B \subset \Sigma^+ B$ if l is even and $\Sigma^l B \subset \Sigma^- B$ if l is odd. Moreover D^B maps sections of $\Sigma^l B$ to sections of $\Sigma^{l-1}B \oplus \Sigma^{l+1}B$ and each $\Sigma^l B$ is stable by $(D^B)^2$. This easily shows that every eigenspinor of D^B is a finite sum of eigenspinors of D^B in $C^\infty(\Sigma^l B \oplus \Sigma^{l+1}B)$ for $0 \leq l \leq m-1$.

Since B is compact and D^B is elliptic, the space of L^2 spinors on B is the Hilbertian direct sum of the eigenspaces of D^B. By the above, there exits $l \in \{0, \ldots, m-1\}$, $\lambda \in \mathbb{R}$ and a spinor $\Phi = \Phi_l + \Phi_{l+1} \in C^\infty(\Sigma^l B \oplus \Sigma^{l+1}B)$ with $D^B\Phi = \lambda\Phi$ such that the functions

$$u(t) := \int_B \langle\Psi_t, \Phi_l\rangle dv_B, \qquad v(t) := \int_B \langle\Psi_t, \Phi_{l+1}\rangle dv_B$$

do not vanish identically.

Taking the scalar product with Φ_l and Φ_{l+1} in (5.2) and integrating over B yields

$$\begin{aligned}
&\tfrac{\lambda}{\alpha}v + (-1)^l\tfrac{im\alpha'}{\alpha}u + (-1)^l iu' = \varepsilon\tfrac{i(2l-m)\beta^2}{4\beta\alpha^2}u - (-1)^l\tfrac{i\beta'}{2\beta}u + \varepsilon\tfrac{ik}{2\beta}u\\
&\tfrac{\lambda}{\alpha}u - (-1)^l\tfrac{im\alpha'}{\alpha}v - (-1)^l iv' = \varepsilon\tfrac{i(2(l+1)-m)\beta^2}{4\beta\alpha^2}v + (-1)^l\tfrac{i\beta'}{2\beta}v + \varepsilon\tfrac{ik}{2\beta}v
\end{aligned}$$

which can be written after setting $w := iv$:

$$u' = \left(\tfrac{\varepsilon(-1)^l(2l-m)\beta^2-2\alpha^2\beta'+\varepsilon(-1)^l 2\alpha^2k-4m\alpha\alpha'\beta}{4\beta\alpha^2}\right)u + (-1)^l\tfrac{\lambda}{\alpha}w$$
$$w' = (-1)^l\tfrac{\lambda}{\alpha}u + \left(\tfrac{\varepsilon(-1)^l(m-2(l+1))\beta^2-2\alpha^2\beta'-\varepsilon(-1)^l 2\alpha^2k-4m\alpha\alpha'\beta}{4\beta\alpha^2}\right)w.$$

Denoting $U := u\beta^{\frac{1}{2}}\alpha^m$ and $W := w\beta^{\frac{1}{2}}\alpha^m$ this simplifies to

$$\begin{aligned}(-1)^lU' &= \varepsilon\left(\tfrac{(2l-m)\beta^2+2\alpha^2k}{4\beta\alpha^2}\right)U + \tfrac{\lambda}{\alpha}W\\ (-1)^lW' &= \tfrac{\lambda}{\alpha}U + \varepsilon\left(\tfrac{(m-2(l+1))\beta^2-2\alpha^2k}{4\beta\alpha^2}\right)W.\end{aligned} \tag{5.3}$$

We have shown that if (M, g) carries a non-trivial harmonic spinor, then (5.3) has a non-trivial solution (U, W) for some $l \in \{0, \dots, m-1\}$, $k \in \mathbb{Z}$, $\lambda \in \mathrm{Spec}(D^B) \subset \mathbb{R}$, and $\varepsilon \in \{\pm 1\}$. Moreover, since the volume form on (M, g) is

$$dv_M = \alpha^{2m}\beta dt \wedge \xi \wedge dv_B$$

and B is compact, Fubini's theorem shows that the original spinor Φ is L^2 on M if and only if

$$\int_0^\infty |\alpha^m\beta^{\frac{1}{2}}\Psi_t(x)|^2 dt < \infty, \qquad\qquad \forall x \in B.$$

We thus get that U and W are L^2 functions on $\mathbb{R}^+$ and satisfy $U, W \in O(\alpha^m\beta^{\frac{1}{2}})$ at $t = 0$.

From conditions (b), (c) and the definition of U, V we get

$$\lim_{t\to 0} U(t) = \lim_{t\to 0} W(t) = 0. \tag{5.4}$$

The system (5.3) reads

$$\begin{cases} U' = \rho U + \sigma W \\ W' = \sigma U + \tau W \end{cases} \tag{5.5}$$

where

$$\rho := \ \varepsilon(-1)^l\left(\tfrac{(2l-m)\beta^2+2\alpha^2k}{4\beta\alpha^2}\right), \qquad \tau := \ \varepsilon(-1)^l\left(\tfrac{(m-2(l+1))\beta^2-2\alpha^2k}{4\beta\alpha^2}\right), \qquad \sigma := \ (-1)^l\tfrac{\lambda}{\alpha}.$$

Notice that the coefficients of the system (5.5) are real functions, thus we can assume that U, W are real by considering separately their real and imaginary parts.

Lemma 5.2. *If a linear combination of the functions U and W is monotonous, it must vanish identically.*

Proof. Using (5.4) we see that if $aU + bW$ does not vanish identically, then $|aU + bW|$ is bounded from below by a non-zero constant on $[x_0, \infty)$ for some $x_0 > 0$, so it cannot be L^2. □

The previous lemma together with (5.5) show that $\lambda \neq 0$: indeed, for $\lambda = 0$ the system (5.5) uncouples into two first-order linear ODE's, whose nontrivial solutions never vanish by uniqueness, hence they have constant sign and so Lemma 5.2 applies. By changing U to $-U$ if necessary, we can therefore assume that $\sigma(t) > 0$ for all $t > 0$.

Lemma 5.3. *If $\sigma(t) > 0$ for all $t > 0$ then we must have $(UW)(t) \leq 0$ for all $t \in \mathbb{R}^+$.*

Proof. Assume that $UW > 0$ on some open interval I. From (c) we easily infer

$$\tau + \rho = -\varepsilon(-1)^l \tfrac{\beta}{2\alpha^2} \geq -\tfrac{1}{\sqrt{2}\alpha}, \tag{5.6}$$

so (5.5) yields

$$(UW)' = (\tau + \rho)UW + \sigma(U^2 + W^2) \geq -\tfrac{1}{\sqrt{2}\alpha}UW. \tag{5.7}$$

Consider the maximal interval $J := (x_0, x_1)$ containing I on which $UW > 0$. For every $x_0 < x \leq t < x_1$, (5.7) implies

$$(UW)(t) \geq (UW)(x)e^{-\int_x^t \frac{1}{\sqrt{2}\alpha(s)}ds}. \tag{5.8}$$

If $x_1 < \infty$ then by continuity $(UW)(x_1) \geq (UW)(x)e^{-\int_x^{x_1} \frac{1}{\sqrt{2}\alpha(s)}ds} > 0$, contradicting the maximality of J. Therefore $x_1 = \infty$, so $UW(t) > 0$ for all $t > x_0$. By integration, (5.8) implies

$$\int_x^\infty (UW)(t)dt \geq (UW)(x)\int_x^\infty e^{-\int_x^t \frac{1}{\sqrt{2}\alpha(s)}ds}dt.$$

By hypothesis (a), the last integral is infinite, however $U, W \in L^2(\mathbb{R}^+, dt)$ implies that $\int_x^\infty (UW)(t)dt < \infty$, contradiction. □

Lemma 5.4. $(UW)(t) < 0$ *for all* $t > 0$.

Proof. Assume for instance that $U(x_0) = 0$. The Cauchy-Lipschitz theorem gives $W(x_0) \neq 0$ and the first equation in (5.5) shows that $U(x)$ has the same sign as $W(x)$ for every x in some small interval $(x_0, x_0 + \delta)$, contradicting Lemma 5.3. The same argument works when $W(x_0) = 0$ by considering the second equation in (5.5). □

We proved so far that U and W have opposite signs and σ is positive. Condition (c) implies that τ and ρ have constant signs on $\mathbb{R}^+$ since $0 \leq l \leq m - 1$. If $\tau \leq 0$, it means that τ and σ have opposite signs, and since also U and W have opposite signs by Lemma 5.4, it follows from the second equation in (5.5) that W' has constant sign. By Lemma 5.2 we get a contradiction. This shows that $\tau > 0$ and similarly we prove $\rho > 0$. By condition (c), this can only happen for $k = 0$, $m = 2l + 1$ and $(-1)^l = -\varepsilon$.

Assuming this to be the case, the system (5.5) reads

$$\begin{aligned} U' &= \tfrac{\beta}{4\alpha^2}U + \tfrac{|\lambda|}{\alpha}W \\ W' &= \tfrac{|\lambda|}{\alpha}U + \tfrac{\beta}{4\alpha^2}W. \end{aligned} \tag{5.9}$$

The difference $D := U - W$ is thus a non-vanishing function satisfying

$$D' = \left(\tfrac{\beta}{4\alpha^2} - \tfrac{|\lambda|}{\alpha}\right)D, \tag{5.10}$$

so for every $t_0 > 0$,

$$D(t) = D(t_0)e^{\int_{t_0}^t \frac{\beta(s)}{4\alpha(s)^2} - \frac{|\lambda|}{\alpha(s)} ds}.$$

To conclude the proof of the theorem we distinguish two cases. If $\lambda \leq 2^{-3/2}$ we get

$$|D(t)| > |D(t_0)|e^{-\int_{t_0}^t \frac{1}{2\sqrt{2}\alpha(s)} ds}$$

so D cannot be square-integrable because of hypothesis (a).

If $\lambda \geq 2^{-3/2}$, (5.6) together with (5.10) show that D is decreasing, contradicting Lemma 5.2. □

6. Axial anomaly for generalized Taub-NUT metrics on $\mathbb{R}^4$

6.1. **Radial perturbations of the Euclidean metric on $\mathbb{R}^{2m+2}$.** Recall from Example 2.1 that the Euclidean space is the metric completion of the CFWP with $\alpha = \frac{t}{\sqrt{2}}$, $\beta = t$ and with basis $B = \mathbb{C}\mathrm{P}^m$ endowed with the Fubini-Study metric. Note that by elliptic regularity, bounded spinors which are harmonic on a punctured ball $B_0(\epsilon) \setminus \{0\}$ are actually smooth and harmonic on $B_0(\epsilon)$, while conversely harmonic spinors on $\mathbb{R}^{2m+2}$ are clearly bounded near 0. Theorem 5.1 applies therefore to the Euclidean metric on $\mathbb{R}^{2m+2}$, for all $m \geq 1$. It is of course well-known that there are no harmonic L^2 spinors on the Euclidean space. Our results generalize this to metrics which are radial perturbations of the standard Euclidean metric with any α, β satisfying the conditions of Theorem 5.1.

6.2. **Generalized Taub-NUT metrics.** The main application of Theorem 5.1 that we have in mind is the vanishing of the index for the generalized Tab-NUT metric of Iwai-Katayama. The difficulty of the problem resides in the non-Fredholmness of the Dirac operator as an unbounded operator in L^2. Nevertheless in [16] it was proved that the L^2 kernel of D is finite-dimensional, and vanishes for the standard Taub-NUT metric.

We cannot apply Theorem 5.1 directly because of the conformal factor $\gamma(t)$ in (1.1). As in Remark 2.4 we set $ds = \gamma(t)dt$. Notice that $s = s(t)$, $t = t(s)$ are diffeomorphisms of $\mathbb{R}^+$ onto itself provided

$$\int_0^1 \gamma(t)dt < \infty, \qquad \int_0^\infty \gamma(t)dt = \infty. \tag{6.1}$$

This condition clearly holds for the conformal factor in (1.1), which is asymptotically constant near infinity and of order $t^{1/2}$ near $t = 0$. Thus we obtain a CFWP metric $\gamma^2(t)g$ where g is itself a CFWP metric.

Lemma 6.1. *Let g be a CFWP metric with coefficients $\alpha(t), \beta(t)$, and $\gamma(t)$ a conformal factor satisfying* (6.1)*. Then the CFWP metric $\gamma^2 g$ satisfies the hypotheses of Theorem 5.1 if and only if*

(a') $\displaystyle\int_x^\infty \gamma(t)e^{-\int_x^t \frac{1}{\sqrt{2}\alpha(u)} du} dt = \infty$ *for all $x > 0$;*

(b') $\displaystyle\lim_{t\to 0} \gamma(t)\alpha(t) = 0$*;*

(c') $2\alpha^2(t) \geq \beta^2(t) > \frac{m-1}{m} 2\alpha^2(t)$ *for all $t > 0$.*

Proof. The coefficients of the CFWP metric $\gamma^2 g$ are

$$\tilde{\alpha}(s) = \gamma(t(s))\alpha(t(s)), \qquad \tilde{\beta}(s) = \gamma(t(s))\beta(t(s))$$

so conditions $(b), (c)$ from Theorem 5.1 for $\tilde{\alpha}, \tilde{\beta}$ are clearly equivalent to conditions (b'), (c'). Now $\frac{1}{\alpha(t)}dt = \frac{1}{\tilde{\alpha}(s)}ds$ and by definition $\gamma(t)dt = ds$ so by two changes of variables, condition (a') is equivalent to condition (a) for $\tilde{\alpha}$. □

As a corollary to Theorem 5.1 we deduce that the Iwai-Katayama metrics on $\mathbb{R}^4$ do not admit non-trivial L^2 harmonic spinors.

Proof of Theorem 1.1. It is straightforward to check that the conditions of Lemma 6.1 hold for the coefficients of Example 2.2, namely

$$m = 1, \qquad \alpha(t) = \sqrt{2}t, \qquad \beta(t) = \frac{2t}{\sqrt{1+ct+dt^2}}, \qquad \gamma(t) = \sqrt{\frac{a+bt}{t}}.$$

It follows from Theorem 5.1 that there do not exist non-trivial L^2 harmonic spinors on $(\mathbb{R}^4 \setminus \{0\}, g_{IK})$ bounded near the origin. Of course, the metric g_{IK} is smooth at the origin, as can be seen by the change of variable $r^2 = t$. In particular we have proved that there do not exist L^2 harmonic spinors on $(\mathbb{R}^4, g_{IK})$. □

We could have also used the conformal covariance of the Dirac operator (cf. [8], see also [18]) to related harmonic spinors for the metrics g and $\lambda^2(t)g = g_{IK}$. We do not give details since this approach is essentially equivalent to the above proof.

References

1. B. Ammann, *The Dirac operator on collapsing S^1-bundles*, Sémin. Théor. Spec. Géom., Univ. Grenoble, **16** (1998), 33–42.
2. B. Ammann, *Spin-Strukturen und das Spektrum des Dirac-Operators*, PhD Thesis, Freiburg, 1998.
3. B. Ammann, C. Bär, *The Dirac Operator on Nilmanifolds and Collapsing Circle Bundles*, Ann. Global Anal. Geom. **16** (1998), 221–253.
4. C. Bär, P. Gauduchon and A. Moroianu, *Generalized Cylinders in Semi-Riemannian and Spin Geometry,* Math. Z. **249** (2005), 545–580.
5. J.-P. Bourguignon, P. Gauduchon, *Spineurs, opérateurs de Dirac et variations de métriques*, Commun. Math. Phys. **144** (1992), 581–599.
6. I. I. Cotăescu, S. Moroianu, M. Vişinescu, *Gravitational and axial anomalies for generalized Euclidean Taub-NUT metrics,* J. Phys. A – Math. Gen. **38** (2005), 7005–7019.
7. I. I. Cotăescu, M. Vişinescu, *Runge-Lenz operator for Dirac field in Taub-NUT background,* Phys. Lett. B **502** (2001), 229–234.
8. N. Hitchin, *Harmonic spinors,* Adv. in Math. **14** (1974), 1–55.
9. T. Iwai, N. Katayama, *On extended Taub-NUT metrics,* J. Geom. Phys. **12** (1993), 55–75.
10. K.-D. Kirchberg, *An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature,* Ann. Global Anal. Geom. **3** (1986), 291–325.
11. J. Lott, *The Dirac operator and conformal compactification,* Internat. Math. Res. Notices **2001**, no. 4, 171–178.
12. A. Moroianu, *La première valeur propre de l'opérateur de Dirac sur les variétés kähleriennes compactes,* Commun. Math. Phys. **169** (1995), 373–384.
13. A. Moroianu, *Spin^c Manifolds and Complex Contact Structures*, Commun. Math. Phys. **193** (1998), 661–673.

14. A. Moroianu, *Lectures on Kähler Geometry,* LMS Student Texts **69**, Cambridge, 2007.
15. A. Moroianu, S. Moroianu, *The Dirac spectrum on manifolds with gradient conformal vector fields,* J. Funct. Analysis **253** (2007), 207–219.
16. S. Moroianu, M. Vişinescu, L^2*-index of the Dirac operator of generalized Euclidean Taub-NUT metrics*, J. Phys. A - Math. Gen. **39** (2006), 6575–6581.
17. B. O'Neill, *Semi-Riemannian geometry,* Acad. Press, New York, 1983.
18. V. Nistor, *On the kernel of the equivariant Dirac operator,* Ann. Global Anal. Geom. **17** (1999), 595–613.
19. B. Vaillant, *Index- and spectral theory for manifolds with generalized fibered cusps,* Dissertation, Bonner Math. Schriften **344** (2001), Rheinische Friedrich-Wilhelms-Universität Bonn.

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