CONFORMALLY EINSTEIN PRODUCTS AND NEARLY KÄHLER MANIFOLDS

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Abstract. In the first part of this note we study compact Riemannian manifolds (M, g) whose Riemannian product with $\mathbb{R}$ is conformally Einstein. We then consider 6-dimensional almost Hermitian manifolds of type $W_1 + W_4$ in the Gray-Hervella classification admitting a parallel vector field and show that (under some mild assumption) they are obtained as Riemannian cylinders over compact Sasaki-Einstein 5-dimensional manifolds.

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1. Introduction

The study of conformally Einstein metrics goes back to Brinkmann who determined in [3] the necessary and sufficient conditions for a Riemannian manifold to be mapped conformally on an Einstein manifold, and considered in [4] the special case of conformal mappings between Einstein manifolds.

More recently, Listing [16, 17] and Gover and Nurowski [10] have found tensorial obstructions for (semi-)Riemannian metrics to be conformally Einstein under some non-degeneracy hypothesis for the Weyl tensor.

Motivated by a problem coming from almost Hermitian geometry, we study conformally Einstein metrics from a different point of view. More precisely, we look for conformally Einstein metrics of product type $g + dt^2$ on cylinders $M \times \mathbb{R}$. In Theorem 2.1 we classify all such metrics in the positive scalar curvature case, assuming that M is compact. We show that (M, g) has to be Einstein with positive scalar curvature and, moreover, that the conformal Einstein factor on $M \times \mathbb{R}$ can be explicitly determined and only depends on the $\mathbb{R}$-coordinate.

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Although the problem is local, we had to assume compactness in order to solve completely the system of equations it leads to. We do not know whether the compactness hypothesis can be removed.

In the second part of this paper we turn our attention to nearly Kähler geometry, a subject which appears to be very important in contemporary theoretical physics (cf., for example, [9], [22]). After recalling some basic facts about nearly Kähler manifolds in Section 3 and reviewing the Gray-Hervella classification in Section 4, we address the question of the existence of (6-dimensional) nearly Kähler conformal structures on cylinders over compact manifolds. The link between this and the conformal Einstein problem is provided by the fact that nearly Kähler manifolds are automatically Einstein in dimension 6.

Our main result roughly says that a Riemannian cylinder $N^5 \times \mathbb{R}$ is conformally nearly Kähler if and only if the basis is Sasaki-Einstein (see Theorem 5.1 for a precise statement).

2. Conformally Einstein products

This section is devoted to the study of conformally Einstein metrics of cylindrical type. Notice that the special case where the conformal factor only depends on the coordinate of the generator, corresponds to warped products with one-dimensional basis and was studied in [2, 9.109] and [15, Lemma 13].

Theorem 2.1. *Let (M^n, g) be a compact Riemannian manifold not isometric to a round sphere. Suppose that the Riemannian cylinder over M is conformally Einstein with positive scalar curvature, that is, there exists a smooth function f on $M \times \mathbb{R}$, such that the Ricci tensor of the Riemannian manifold $(M \times \mathbb{R}, e^{2f}(g + dt^2))$ is a positive multiple of the metric. Then (M, g) is an Einstein manifold with positive scalar curvature and the conformal factor satisfies $e^{2f(x,t)} = \alpha^2 \cosh^{-2}(\beta t + \gamma)$ for some real constants α, β, γ.*

Proof. Let us denote by N and $\tilde{N}$ the Riemannian manifolds $(M \times \mathbb{R}, (g + dt^2))$ and $(M \times \mathbb{R}, e^{2f}(g + dt^2))$ respectively. We view the function f on N as a smooth 1-parameter family of functions on M by $f_t(x) := f(x, t)$. In this way, the exterior derivative df satisfies $df = df_t + f_t' dt$, where df_t denotes the derivative of f_t on M and $f_t' = \frac{\partial f}{\partial t}$. Similarly the Laplace operators of M and N are related by $\Delta^N f = -f_t'' + \Delta^M f_t$. In the sequel we shall denote by $\partial_t := \frac{\partial}{\partial t}$ the "vertical" vector field on N and by X, Y, vector fields on M, identified with their canonical extension to N commuting with ∂_t. The formula for the conformal change of the Ricci tensor (see *e.g.* [2, 1.59]),

$$\mathrm{Ric}^{\tilde{N}} = \mathrm{Ric}^N - (n-1)(\nabla^N df - df \otimes df) + (\Delta^N f - (n-1)|df|^2)(g + dt^2), \qquad (1)$$

yields in particular

$$\mathrm{Ric}^{\tilde{N}}(X, \partial_t) = -(n-1)(X(f_t') - X(f_t) f_t'), \qquad \forall\, X \in TM. \qquad (2)$$

Since $\tilde N$ is Einstein, (2) shows that $X(f_t') = X(f_t)f_t'$, which can be rewritten as $X(\partial_t(e^{-f})) = 0$ for every $X \in TM$. Consequently, there exist smooth functions $a \in \mathcal{C}^\infty(\mathbb{R})$ and $b \in \mathcal{C}^\infty(M)$ such that $e^{-f(x,t)} = a(t) + b(x)$, in other words

$$f(x,t) = -\ln(a(t)+b(x)), \qquad \forall\, x \in M,\ \forall\, t \in \mathbb{R}.$$

We now readily compute

$$df = df_t + f_t' dt = -\frac{db + a'dt}{a+b}, \qquad f_t'' = -\frac{a''(a+b)-(a')^2}{(a+b)^2}, \tag{3}$$

$$\Delta^N f = -f_t'' + \Delta^M f_t = \frac{a''(a+b)-(a')^2}{(a+b)^2} - \frac{\Delta^M b}{a+b} - \frac{|db|^2}{(a+b)^2}, \tag{4}$$

and

$$\begin{aligned}
\nabla^N df &= = -\frac{\nabla^N(db+a'dt)}{a+b} + \frac{1}{(a+b)^2} d(a+b)\otimes(db+a'dt)\\
&= -\frac{\nabla^M(db)+a''dt^2}{a+b} + \frac{1}{(a+b)^2}(db+a'dt)\otimes(db+a'dt)\\
&\overset{(3)}{=} -\frac{H(b)+a''dt^2}{a+b} + df\otimes df,
\end{aligned}$$

where $H(b)$ denotes the Hessian of b on M. Let r denote the Einstein constant of $\tilde N$. Plugging the relations above back into (1) yields

$$\begin{aligned}
\frac{r(g+dt^2)}{(a+b)^2} = \mathrm{Ric}^{\tilde N} &= \mathrm{Ric}^M + \frac{n-1}{a+b}(H(b)+a''dt^2)\\
&+ \frac{(a''-\Delta^M b)(a+b) - n(a')^2 - n|db|^2}{(a+b)^2}(g+dt^2),
\end{aligned}$$

which is equivalent to the system

$$\begin{cases} r = (na'' - \Delta^M b)(a+b) - n(a')^2 - n|db|^2 \\ rg = ((a''-\Delta^M b)(a+b) - n(a')^2 - n|db|^2)g + (a+b)^2\mathrm{Ric}^M + (n-1)(a+b)H(b). \end{cases}$$

Subtracting the first equation from the second one, the system becomes

$$\begin{cases} r = (na'' - \Delta^M b)(a+b) - n(a')^2 - n|db|^2 \\ (n-1)a''g = (a+b)\mathrm{Ric}^M + (n-1)H(b). \end{cases} \tag{5}$$

We distinguish three cases:

Case 1: b is constant on M. By a suitable change of coordinates, the metric becomes a warped product and the conclusion could be directly derived from [2, 9.109]. We will nevertheless provide the direct argument. Replacing a by $a-b$, we may assume that $b = 0$, so by the first equation in (5), a satisfies the ODE

$$a''a - (a')^2 = \frac{r}{n}.$$

The general solution of this equation is

$$a(t) = \sqrt{\frac{r}{n\beta^2}}\cosh(\beta t + \gamma).$$

Thus $e^{2f(t)} = \frac{1}{a^2} = \alpha^2 \cosh^{-2}(\beta t + \gamma)$, with $\alpha := \frac{n\beta^2}{r}$. The second relation in (5) shows that M is Einstein, with positive Einstein constant β^2.

Case 2: a is constant on $\mathbb{R}$. Again, replacing b by $b - a$, we may assume that $a = 0$, so the first equation of (5) becomes

$$n|db|^2 + b\Delta^M b + r = 0.$$

Integrating over M yields

$$0 = \int_M (n|db|^2 + b\Delta^M b + r)dv = \int_M ((n+1)|db|^2 + r)dv > 0,$$

showing that this case is impossible.

Case 3: Neither a nor b are constant functions. We differentiate the second relation of (5) twice, first with respect to t, then with respect to some arbitrary vector $X \in TM$ and obtain

$$a''' X(b) = \frac{1}{n} a' X(\Delta^M b). \tag{6}$$

Taking some $x \in M$ and $X \in T_xM$ such that $X_x(b) \neq 0$, this relation shows that $a''' = \delta a'$ for some $\delta \in \mathbb{R}$. Similarly, taking some $t \in \mathbb{R}$ such that $a'(t) \neq 0$ gives some $\delta' \in \mathbb{R}$ such that $X(\Delta^M b) = \delta' X(b)$ for all $X \in TM$. Plugging these two relations back into (6) yields $\delta' = n\delta$. Summarizing, we have

$$\begin{cases} a'' = \delta a + \varepsilon \\ \Delta^M b = n\delta b + \varepsilon' \end{cases} \tag{7}$$

for some real constants $\varepsilon, \varepsilon'$. If $\delta = 0$, the second relation yields (by integration over M) $\varepsilon' = 0$, so b is constant, a contradiction. Thus $\delta \neq 0$. Replacing a by $a + \frac{\varepsilon}{\delta}$ (and correspondingly replacing b by $b - \frac{\varepsilon}{\delta}$), we may assume $\varepsilon = 0$. The second relation in (7) also shows that $n\delta$ is an eigenvalue of the Laplace operator (corresponding to the eigenfunction $b + \frac{\varepsilon'}{n\delta}$), whence $\delta > 0$. The second equation of the system (5) now becomes

$$\begin{cases} \mathrm{Ric}^M = (n-1)\delta g \\ H(b) = -b\delta g \end{cases} \tag{8}$$

Since b is non-zero, the Obata theorem (see [20, Theorem 3]) implies that M is isometric to a round sphere, a contradiction, which shows that this case is impossible as well. $\square$

We will give a concrete application of this theorem in Section 5, after reviewing some special classes of almost Hermitian manifolds in the next two sections.

3. Basics on nearly Kähler geometry

A. Gray was led to define nearly Kähler manifolds (also known as almost Tachibana spaces) by his research on weak holonomy of U_n-structures. An almost Hermitian manifold (N, h, J), with fundamental two-form Ω and Levi-Civita connection ∇ is called *nearly Kähler* if $\nabla\Omega$ is totally skew-symmetric.

From the viewpoint of the representations of U_n on the space of tensors with the same symmetries as $\nabla\Omega$, nearly Kähler manifolds appear in the class W_1 of the Gray-Hervella classification (see [13] and Section 4 below). It is also known that nearly Kähler manifolds with integrable almost complex structure are necessarily Kähler. The specific, non-trivial, case is then the so-called *strict* nearly Kähler, when $\nabla J \neq 0$ at every point of M.

The local structure of nearly Kähler manifolds was first discussed by Gray in [12] and was recently completely understood by P.A. Nagy in [19]: any nearly Kähler manifold is locally a product of 6-dimensional strict nearly Kähler manifolds, locally homogeneous manifolds and twistor spaces of positive quaternionic Kähler manifolds. According to this result, what remains to be studied are strict nearly Kähler structures in dimension 6. This is, in fact, the first interesting case, since 4-dimensional nearly Kähler manifolds are automatically Kähler. On the other hand, the dimension 6 is particularly important also because of the following result:

Proposition 3.1. [18] *A strict* 6*-dimensional nearly Kähler manifold is Einstein with positive scalar curvature.*

In fact, in dimension 6, a strict nearly Kähler structure is equivalent with the local existence of a non-trivial real Killing spinor, cf. [14].

Nearly Kähler structures are closely related to G_2 structures *via* the cone construction (see [1] for example):

Proposition 3.2. *A Riemannian manifold (N^6, h) carries a nearly Kähler structure if and only if its cone $(N^6 \times \mathbb{R}^+, t^2h + dt^2)$ has holonomy contained in G_2.*

For later use, let us also recall the following related result:

Proposition 3.3. [1] *A Riemannian manifold (M^{2m+1}, g) carries a Sasaki-Einstein structure if and only if its cone $(M \times \mathbb{R}^+, t^2g + dt^2)$ has holonomy contained in* $\mathrm{SU}_{\mathrm{m+1}}$.

Returning to nearly Kähler geometry, the only known compact examples in dimension 6 are homogeneous: the sphere S^6, the 3-symmetric space $S^3 \times S^3$ and the twistor spaces, $\mathbb{C}\mathrm{P}^3$ and $F(1,2)$, of the 4-dimensional self-dual Einstein manifolds S^4 and $\mathbb{C}\mathrm{P}^2$. On the other hand, Butruille proved recently that every compact homogeneous strictly nearly Kähler manifold must be one of these (cf. [5]).

4. A review of the Gray-Hervella classification

Nearly Kähler manifolds can be understood better in terms of the classification of almost Hermitian structures [13].

For each almost Hermitian manifold (N^{2m}, h, J), with fundamental form $\Omega := h(J., .)$, the Nijenhuis tensor, viewed as a tensor of type $(3, 0)$ *via* the metric, splits in two components $N = N_1 + N_2$, where N_1 is totally skew-symmetric and N_2 satisfies the Bianchi identity. Similarly, the covariant derivative of J with respect to the Levi-Civita connection of h splits in four components under the action of the structure group $\mathrm{U_m}$ (see [13]):

$$\nabla J = (\nabla J)_1 + (\nabla J)_2 + (\nabla J)_3 + (\nabla J)_4.$$

The first component corresponds to N_1, and also to the $(3, 0) + (0, 3)$-part of $d\Omega$. The second component can be identified with N_2, while the two other components correspond respectively to the primitive part of $d\Omega^{(2,1)+(1,2)}$ and to the contraction $\Omega \lrcorner d\Omega$ which is a 1-form called the *Lee form*. The manifold N is called of type $W_1 + W_4$ if $(\nabla J)_2$ and $(\nabla J)_3$ vanish identically. Similar definitions apply for every subset of subscripts in $\{1, 2, 3, 4\}$. For example a manifold of type $W_3 + W_4$ is Hermitian, a manifold of type W_2 is symplectic, and a manifold of type W_1 is nearly Kähler.

From the definition it is more or less obvious that if the metric h is replaced by a conformally equivalent metric $\tilde{h} := e^{2f}h$, the first three components of ∇J are invariant and the fourth component satisfies $(\tilde{\nabla} J)_4 = (\nabla J)_4 + df$. Therefore, the Lee form of a manifold of type $W_1 + W_4$ is closed (resp. exact), if and only if the manifold is locally (resp. globally) conformal nearly Kähler. In dimension 6, Butruille proved in [6] that all manifolds in class $W_1 + W_4$ have *closed* Lee form, hence they have to be locally conformal nearly Kähler. But recently, Cleyton and Ivanov proved, [7, Lemma 8], that every locally conformal nearly Kähler structure is actually globally conformal. Combining these results, we may state:

Theorem 4.1. [6, 7] *Let M be a 6-dimensional almost Hermitian manifold of type W_1+W_4. Then its Lee form is closed, and M is either globally conformal nearly Kähler or locally conformal Kähler.*

This generalizes the well-known fact that for $m \geq 3$, every almost Hermitian manifold of type W_4 is locally conformal Kähler (lck).

5. Conformally nearly Kähler cylinders

The aim of this section is to classify all compact 5-dimensional Riemannian manifolds (M, g) with the property that the Riemannian cylinder $M \times \mathbb{R}$ carries an almost Hermitian structure of type $W_1 + W_4$.

Theorem 5.1. *If the Riemannian cylinder $N := (M \times \mathbb{R}, g + dt^2)$ over a compact 5-dimensional Riemannian manifold (M, g) carries an almost Hermitian structure of type*

$W_1 + W_4$ which is not of type W_4, then (M, g) is Sasaki-Einstein. Conversely, if (M, g) is Sasaki-Einstein, then its cylinder N carries a structure of type $W_1 + W_4$, besides its canonical Vaisman structure (locally conformal Kähler, i.e. *W_4, with parallel Lee form, cf.* [21]).

Proof. Assume first that N carries a structure of type $W_1 + W_4$. By Theorem 4.1, N is either globally conformal nearly Kähler or lck. Since we assumed that N is not lck, there exists a function f on N such that $(N, e^{2f}(g + dt^2))$ is a strict nearly Kähler manifold. By Proposition 3.1, every such manifold in dimension 6 is Einstein with positive scalar curvature. We apply Theorem 2.1 and obtain that either M is the round sphere (which, in particular is Sasaki-Einstein), or $e^{2f} = \alpha^2 \cosh^{-2}(\beta t + \gamma)$ for some real constants α, β, γ.

Let us now consider the diffeomorphism

$$\varphi : M \times \mathbb{R} \to M \times (0, \pi), \qquad (x, t) \mapsto (x, 2\tan^{-1}(e^{\beta t + \gamma})).$$

A straightforward computation shows that

$$e^{2f}(g + dt^2) = \frac{\alpha^2}{\beta^2}\varphi^*(\beta^2 \sin^2 s\, g + ds^2). \tag{9}$$

We have obtained that the so-called *sine-cone* (see [8]) of $(M, \beta^2 g)$ has a nearly Kähler structure. The first part of the theorem then follows from the next lemma, which can be found (in a slightly different version) in [8].

Lemma 5.2. *Up to constant re-scalings, the sine-cone $(M \times (0, \pi), \sin^2 s\, g + ds^2)$ of a 5-dimensional Riemannian manifold (M, g) has a nearly Kähler structure if and only if M is Sasaki-Einstein.*

Proof of the lemma. In [8] the authors prove the result by an explicit calculation, using so-called *hypo structures* on Sasaki-Einstein manifolds. We provide here a different argument. The key idea is the fact that the Riemannian product of two cone metrics is again a cone metric, as shown by the formula

$$(t^2 g + dt^2) + (s^2 h + ds^2) = r^2(\sin^2\theta\, g + \cos^2\theta\, h + d\theta^2) + dr^2, \qquad (s, t) = (r\cos\theta, r\sin\theta).$$

In particular, taking $h = 0$ (the metric of a point), shows that the cylinder over the Riemannian cone of a metric g is isometric to the Riemannian cone over the sine-cone of g. By Proposition 3.3, if M is Sasaki-Einstein, its Riemannian cone $\bar{M}$ has holonomy in SU_3, so the cylinder $\bar{M} \times \mathbb{R}$ has holonomy in $\mathrm{SU}_3 \times \{1\} \subset G_2$. The previous remark, together with Proposition 3.2, shows that the sine-cone of M is nearly Kähler. Conversely, if this holds, then the Riemannian cone of the sine-cone has holonomy in G_2. Thus the holonomy of the cylinder $\bar{M} \times \mathbb{R}$ is a subgroup of G_2. But since $G_2 \cap (\mathrm{O}_6 \times \{1\}) = \mathrm{SU}_3 \times \{1\} \subset \mathrm{O}_7$, this means that $\bar{M}$ has holonomy in SU_3, so M is Sasaki-Einstein. The lemma, and the first part of the theorem are thus proved.

Conversely, let (M^5, g) be a Sasaki-Einstein manifold. We first notice that by Proposition 3.3, the Riemannian cone $\bar{M}$ is Kähler, so the cylinder $M \times \mathbb{R}$, which is conformal to $\bar{M}$, is lck (and even Vaisman, see [21]).

On the other hand, Lemma 5.2 shows that the sine-cone of M is nearly Kähler, and therefore the cylinder over M, which by (9) is conformal to the sine-cone, has a structure of type $W_1 + W_4$. □

Remark. As the referee pointed out, Theorem 5.1 above has the following interesting consequence:

Corollary 5.3. *If φ is an isometry of a compact 5-dimensional Riemannian manifold (M, g), the mapping torus of φ does not carry any compatible almost Hermitian structure of type $W_1 + W_4$.*

Proof. The mapping torus of φ is the quotient of the Riemannian product $(M\times\mathbb{R}, g+dt^2)$ by the discrete group of isometries generated by $(x, t) \mapsto (\varphi(x), t + 1)$. If the quotient carries a structure of type $W_1 + W_4$, the same holds for the cylinder $M \times \mathbb{R}$ (by pull-back). From Theorem 5.1 we thus obtain that M is Sasaki-Einstein. Moreover, the almost Hermitian structure on the cylinder can be easily made explicit, cf. [8, Theorem 3.6]. As it was pointed out by S. Ivanov, this almost complex structure is not invariant by any translation in the $\mathbb{R}$-direction, a contradiction. □

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