THE DIRAC SPECTRUM ON MANIFOLDS WITH GRADIENT CONFORMAL VECTOR FIELDS

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Abstract. We show that the Dirac operator on a spin manifold does not admit L^2 eigenspinors provided the metric has a certain asymptotic behaviour and is a warped product near infinity. These conditions on the metric are fulfilled in particular if the manifold is complete and carries a non-complete vector field which outside a compact set is gradient conformal and non-vanishing.

1. Introduction

The Dirac operator on a closed spin manifold is essentially self-adjoint as an unbounded operator in L^2, and has purely discrete spectrum. Its eigenvalues grow at a certain speed determined by the volume of the manifold and its dimension. Hence, although determining the eigenvalues can be a daunting task, the nature of the spectrum is rather well understood.

On non-compact manifolds, the spectrum of the Dirac operator can behave in a variety of ways. For instance, the Dirac operator on $\mathbb{R}^n$ has purely absolutely continuous spectrum, so in particular there are no L^2 eigenspinors. In contrast, Bär [2] showed that on complete spin hyperbolic manifolds of finite volume, the spectrum is purely discrete if the induced Dirac operators on the ends are invertible. In this situation even the classical Weyl law for the distribution of the eigenvalues holds [11]. Otherwise, if the limiting Dirac operator is not invertible, then the essential spectrum is the whole real axis. Similar results appear in [7] for the Laplace operator on forms and for magnetic Schrödinger operators.

In this paper we show that for a class of – possibly incomplete – spin Riemannian manifolds (X, g) which includes certain hyperbolic manifolds, the Dirac operator D does not carry L^2 eigenspinors of real eigenvalue. In particular, we deduce that the L^2 index of the Dirac operator on (X, g) vanishes.

Our main result (Theorem 2.1) makes special assumptions on the metric of X. Geometrically, these assumptions imply the existence of a non-complete vector field on X which is gradient conformal on an open subset U of X. Conversely, we show in Section 4 that the existence of such vector fields implies the hypothesis of Theorem 2.1 provided that X is complete and $X \setminus U$

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is compact. As a corollary, we obtain that on a complete spin manifold X which carries a non-complete vector field which is gradient conformal outside a compact subset of X, the Dirac operator has purely continuous spectrum (Theorem 4.1).

2. The main result

Let $(\bar{X}^n, h)$ be a connected spin manifold with boundary with interior X. Assume that there exists a boundary component M so that h is a product in a neighbourhood of M. Denote by $x : \bar{X} \to [0, \infty)$ the distance to M, so $h = dx^2 + h_M$ near M. Note that M inherits a spin structure from $\bar{X}$.

Let $f : X \to (0, \infty)$ be a smooth conformal factor which depends only on x in a neighbourhood of M.

Theorem 2.1. *Assume that M is at infinite distance from X with respect to the conformal metric*

$$g := f^2 h = f(x)^2(dx^2 + h_M). \tag{1}$$

Moreover assume that the Dirac operator D_M on (M, h_M) is essentially self-adjoint. Then the Dirac operator of (X, g) does not have any distributional L^2 eigenspinors of real eigenvalue.

Lott [9] proved that there is no L^2 harmonic spinor under somewhat similar assumptions. Namely, h could be any metric smooth up to M, and f could vary in the M directions. However, Lott assumes that f^{-1} extends to a locally Lipschitz function on $\bar{X}$ which eventually must be locally bounded by a multiple of x, while our hypothesis only asks that

$$\int_0^\epsilon f(x)dx = \infty. \tag{2}$$

In particular, unlike in [9], the function f^{-1} may be unbounded near M.

If we assume that (X, g) is complete as in [9], we deduce that (M, h_M) is complete so D_M is essentially self-adjoint. With this assumption we also know that D_g is essentially self-adjoint, so its spectrum is real. But we do not need to make this assumption as the statement is "local near M". This seems to be also the case in [9], although it is not claimed explicitly. However, there are many instances of *incomplete* manifolds whose Dirac operator is essentially self-adjoint, cf. Example 2.2.

The rest of this section is devoted to the proof of Theorem 2.1.

It is known since Hitchin [8] that the Dirac operator has a certain conformal invariance property. More precisely, if $g = f^2 h$, the Dirac operator D_g is conjugated to $f^{-1} D_h$ by the Hilbert space isometry

$$L^2(X, \Sigma, \mathrm{vol}_g) \to L^2(X, \Sigma, f\mathrm{vol}_h), \qquad \psi \mapsto f^{\frac{n-1}{2}}\psi, \tag{3}$$

(see [12, Proposition 1]). Let $f^{-\frac{n-1}{2}}\phi \in L^2(X, \Sigma, \mathrm{vol}_g)$ be an eigenspinor of D_g (in the sense of distributions) of eigenvalue $l \in \mathbb{R}$. Then

$$(f^{-1} D_h - l)\phi = 0. \tag{4}$$

By elliptic regularity, ϕ is in fact a smooth spinor on X so (4) is equivalent to

$$D_h \phi = l f \phi. \tag{5}$$

Let c^0 denote the Clifford multiplication by the unit normal vector ∂_x with respect to h. Then D_h decomposes near M as follows:

$$D_h = c^0(\partial_x + A)$$

where for each $x > 0$, A is a differential operator on the sections of Σ over $M \times \{x\}$; moreover A is independent of x. Note that $\Sigma\big|_M$ is either the spinor bundle $\Sigma(M)$ of M with respect to the induced spin structure (if n is odd), or two copies of $\Sigma(M)$ if n is even. We can describe A in terms of the Dirac operator D_M on M with respect to the metric h_M as follows:

$$A = \begin{cases} D_M & \text{for } n \text{ odd,} \\ \begin{bmatrix} D_M & 0 \\ 0 & -D_M \end{bmatrix} & \text{for } n \text{ even.} \end{cases} \tag{6}$$

In both cases, A is symmetric and elliptic. Since D_M is essentially self-adjoint, so is A and we use the same symbol for its unique self-adjoint extension.

The case where A has pure-point spectrum. For the sake of clarity we make temporarily the assumption that $L^2(M, \Sigma, \mathrm{vol}_{h_M})$ admits an orthonormal basis made of eigenspinors of A of real eigenvalue (equivalently, A has pure-point spectrum). Since A is essentially 1 or 2 copies of D_M, this happens if M is compact, but also more generally.

Example 2.2. Suppose that either (M, h_M) is conformal to a closed cusp metric such that the induced Dirac operators on the ends are invertible (see [11]), or that (M, h_M) is compact with isolated conical singularities and the Dirac eigenvalues on the cone section do not belong to $(-\frac{1}{2}, \frac{1}{2})$ (see *e.g.* [6]). Then the operator D_M is essentially self-adjoint with purely discrete spectrum.

Note that since c^0 and A anti-commute, we have

$$A\phi_\lambda = \lambda\phi_\lambda \Longrightarrow Ac^0\phi_\lambda = -\lambda c^0\phi_\lambda$$

and so the spectrum of A is symmetric around 0. For each x, decompose ϕ onto the positive eigenspaces of A as follows:

$$\phi = \sum_{\substack{\lambda \in \mathrm{Spec}\, A \\ \lambda > 0}} \left(a_\lambda(x)\phi_\lambda + b_\lambda(x)c^0\phi_\lambda\right) + \phi_0(x)$$

where $\phi_0(x) \in \ker A$ for all $x > 0$, $\lambda > 0$ is an eigenvalue of A, and ϕ_λ is an eigenspinor of eigenvalue λ and norm 1 in $L^2(M, \Sigma, \mathrm{vol}_{h_M})$. We compute

$$\begin{aligned} \|\phi\|^2_{L^2(X,\Sigma,f\mathrm{vol}_h)} = & \int_{\{x>\epsilon\}} |\phi|^2 f \mathrm{vol}_h \\ & + \int_0^\epsilon \int_M |\phi_0(x)|^2 \mathrm{vol}_{h_M} f(x)dx \\ & + \sum_{\substack{\lambda \in \mathrm{Spec}\, A \\ \lambda > 0}} \int_0^\epsilon \left(|a_\lambda(x)|^2 + |b_\lambda(x)|^2\right) f(x)dx. \end{aligned}$$

In particular, $\phi \in L^2(X, \Sigma, f\mathrm{vol}_h)$ implies that

$$\int_0^\epsilon \int_M |\phi_0(x)|^2 \mathrm{vol}_{h_M} f(x) dx < \infty \tag{7}$$

and

$$a_\lambda, b_\lambda \in L^2((0,\epsilon), fdx). \tag{8}$$

The eigenspinor equation (5) becomes

$$\begin{aligned}
0 &= (D_h - lf)\phi \\
&= c^0(\partial_x + A + c^0 lf)\phi \\
&= c_0 \sum_{\substack{\lambda\in \mathrm{Spec}\, A \\ \lambda>0}} \big((a'_\lambda + \lambda a_\lambda - lfb_\lambda)\phi_\lambda + (b'_\lambda - \lambda b + lfa_\lambda)c^0\phi_\lambda\big) \\
&\quad + c^0\phi'_0 - lf\phi_0.
\end{aligned}$$

So we get

$$\phi'_0 = -lfc^0\phi_0 \tag{9}$$

for the part of ϕ which for all fixed x lives in $\ker A$, while for $\lambda > 0$,

$$\begin{cases} a'_\lambda = -\lambda a_\lambda + lfb_\lambda \\ b'_\lambda = -lfa_\lambda + \lambda b_\lambda. \end{cases} \tag{10}$$

First, since l is real and c^0 is skew-adjoint, Eq. (9) implies that the function $|\phi_0(x)|^2$ is constant in x. Together with (2) and (7), we see that $\int_M |\phi_0(x_0)|^2 dh_M = 0$ so $\phi_0 \equiv 0$.

We show now that for all $\lambda > 0$, the system (10) does not have nonzero solutions satisfying (8), *i.e.*, in $L^2((0,\epsilon), fdx)$.

Remark 2.3. The Wronskian of (10) is constant in x, so by (2), the two fundamental solutions cannot belong simultaneously to $L^2((0,\epsilon), fdx)$. However, this fact alone does not stop *one* solution from being in L^2!

Fix $0 < \lambda \in \mathrm{Spec}\, A$ and set

$$a(x) := e^{\lambda x} a_\lambda(x), \qquad b(x) := e^{-\lambda x} b_\lambda(x).$$

Then (10) becomes

$$\begin{cases} a'(x) = le^{2\lambda x} f(x) b(x) \\ b'(x) = -le^{-2\lambda x} f(x) a(x). \end{cases} \tag{11}$$

Note that the system (11) has real coefficients (here we use the hypothesis that l is real) so by splitting into real and imaginary parts, we can assume that a, b are also real.

Since $e^{\pm 2\lambda x}$ is bounded for $0 \leq x \leq \epsilon < \infty$, condition (8) implies that $a, b \in L^2((0,\epsilon), fdx)$. If $l = 0$ then a, b are constant functions, which by (2) do not belong to $L^2((0,\epsilon), fdx)$ unless they are 0. So in that case a_λ and b_λ vanish identically.

Let L^1 denote the space of integrable functions on $(0,\epsilon)$ with respect to the Lebesgue measure dx. Then (8) implies that $fab \in L^1$. So from (11),

$$(a^2)' = 2le^{2\lambda x} f(x) a(x) b(x) \in L^1.$$

Hence
$$\lim_{x\to 0} a^2(x) = a^2(x_0) - \int_0^{x_0} (a^2)'(t)dt$$
exists, in other words a^2 extends continuously in $x = 0$. The same argument shows that b^2 (and so also a, b) are continuous in 0.

The case $l = 0$ was treated above so we can assume that $l \neq 0$. We claim that $b(0) = 0$. Otherwise, by continuity, $b(x) \neq 0$ for $0 \leq x \leq x_0$ so from (11), there exists $C > 0$ with $|a'(x)| > Cf(x)$ (in particular, by continuity a' has constant sign) and therefore
$$|a(0) - a(x_0)| = \lim_{x\to 0} \int_x^{x_0} |a'(x)|dx > C \int_0^{x_0} f(t)dt = \infty$$
which is a contradiction. So $b(0) = 0$ and similarly $a(0) = 0$.

We pull now our final trick. Recall that $\lambda > 0$. Consider the function
$$F(x) := e^{-4\lambda x} a^2(x) + b^2(x) \geq 0.$$
We just showed that F is continuous in 0 and $F(0) = 0$. From (11) we compute
$$F'(x) = -4\lambda e^{-4\lambda x} a^2(x) \leq 0.$$
By collecting what we know about F, we note:

(1) $F(x) \geq 0$;
(2) $F(0) = 0$;
(3) $F'(x) \leq 0$ for $\epsilon > x > 0$

Together these facts imply that $F \equiv 0$ on $(0, \epsilon)$. This is equivalent to saying $a(x) = b(x) = 0$ for all $0 \leq x \leq \epsilon$.

So we showed that ϕ vanishes near M. The eigensections of D have the unique continuation property, which implies that ϕ vanishes on X.

Remark 2.4. Recall that the spectrum of A is symmetric around 0, and a_λ, b_λ are the coefficients in ϕ of the eigensections ϕ_λ, $c^0\phi_\lambda$ of eigenvalue λ, respectively $-\lambda$. It may seem that starting the decomposition using positive λ was a fortunate choice, otherwise the last argument would not hold. But in fact, the argument works for $-\lambda$ by choosing a different function $\tilde{F}(x) := a^2(x) + e^{4\lambda x} b^2(x)$.

The general case. Let us remove the assumption that the spectrum of A is purely discrete. We will model the proof on the argument given above, which is now loaded with technical subtleties.

By assumption, a neighbourhood of the infinity in X is isometric to $(0, \epsilon) \times M$ with the metric (1).

After the unitary transformation (3), the eigenspinor equation reads as before
$$(\partial_x + A)\phi = -c^0 f^{-1} l\phi \tag{12}$$
where l is real, and ϕ is smooth (by elliptic regularity) and square-integrable. To make this last condition precise, let I_ϵ denote the interval $(0, \epsilon)$ with the measure $f(x)dx$. Denote by $\mathcal{H}$ the Hilbert space $L^2(M, \Sigma, \mathrm{vol}_{h_M})$, then
$$\phi \in L^2(X_\epsilon, \Sigma, f\mathrm{vol}_h) = L^2(I_\epsilon, \mathcal{H}). \tag{13}$$
In particular for almost all $x \in I_\epsilon$, we have $\phi_x \in \mathcal{H}$.

Let χ be the characteristic function of the interval $[-N, N]$ for some $N \in \mathbb{R}$. Let $\chi(A)$ be the corresponding spectral projection. Since A anti-commutes with c^0 and χ is even, it follows that $\chi(A)$ commutes with c^0.

Let $\mathcal{H}^1 \subset \mathcal{H}$ be the domain of A and $\mathcal{H}^{-1} \supset \mathcal{H}$ its dual inside distributions, *i.e.*, the space of those distributions which extend continuously to $\mathcal{H}^1$. Since χ has compact support, we deduce that $\chi(A)$ acts continuously from $\mathcal{H}$ to $\mathcal{H}^1$, and also from $\mathcal{H}^{-1}$ to $\mathcal{H}$.

From (13) we deduce $A\phi \in L^2(I_\epsilon, \mathcal{H}^{-1})$, $\chi(A)\phi \in L^2(I_\epsilon, \mathcal{H}^1)$ and

$$\chi(A)(A\phi) = A(\chi(A)\phi) \in L^2(I_\epsilon, \mathcal{H}).$$

Similarly, $\partial_x\phi \in H^{-1}_{loc}(I_\epsilon, \mathcal{H})$ and

$$\chi(A)(\partial_x\phi) = \partial_x(\chi(A)\phi) \in H^{-1}_{loc}(I_\epsilon, \mathcal{H}^1).$$

It follows that $\tilde{\phi} := \chi(A)\phi$ satisfies (in distributions) the eigenspinor equation (12). Denote by $\mathcal{H}_N$ the range of the projection $\chi(A)$, then $\tilde{\phi} \in L^2(I_\epsilon, \mathcal{H}_N)$. Most importantly for us, A acts as a self-adjoint *bounded* operator on $\mathcal{H}_N$.

Lemma 2.5. *Let H be a Hilbert space, $A : H \to H$ a bounded self-adjoint operator, and c^0 a skew-adjoint involution of H which anti-commutes with A. Then for every $l \in \mathbb{R}$, the equation* (12) *does not have* (*distributional*) *solutions in $L^2(I_\epsilon, H)$ other than 0.*

Proof. Let ϕ be a solution of (12), square-integrable with respect to the measure $f(x)dx$ on I_ϵ. By elliptic regularity, ϕ is smooth in x. Since $\exp(xA)c^0 = c^0\exp(-xA)$, we get $\partial_x(\exp(xA)\phi) = -lfc^0\exp(-xA)\phi$, hence the family of H-norms $x \mapsto \|\partial_x(\exp(xA)\phi_x)\|$ is square-integrable with respect to the measure $f^{-1}dx$. Since $x \mapsto \|\phi_x\|$ is L^2 with respect to fdx, and $\exp(xA)$ is uniformly bounded, by the Cauchy-Schwartz inequality we see that the function

$$x \mapsto \frac{d}{dx}\|\exp(xA)\phi_x\|^2$$

is integrable with respect to the Lebesgue measure dx. Thus

$$\|\exp(xA)\phi_x\|^2 = \|\exp(x_0A)\phi_{x_0}\|^2 + \int_{x_0}^{x}\frac{d}{dx}\|\exp(xA)\phi_x\|^2dx$$

has a finite limit as $x \searrow 0$. We claim that this limit is 0. Otherwise, since $\lim_{x\searrow 0}\exp(xA) = 1$, we would have $\lim_{x\searrow 0}\|\phi_x\|^2 > 0$ which, together with (2), contradicts the fact that ϕ is square-integrable with respect to fdx.

Thus ϕ_x tends in norm to 0 in H as $x \searrow 0$. Let now $|A|$ be the absolute value of A, and define

$$F(x) := \|\exp(-x|A|)\phi_x)\|^2.$$

We notice that c^0 commutes with $|A|$ since it commutes with A^2. A direct computation shows, using that c^0 is skew-adjoint,

$$\frac{dF}{dx} = -2\langle(A + |A|)\exp(-x|A|)\phi_x, \exp(-x|A|)\phi_x\rangle \leq 0.$$

Hence F is decreasing, on the other hand it vanishes at $x = 0$ and it is non-negative, so in conclusion it vanishes identically. Since $\exp(-x|A|)$ is invertible, we conclude that $\phi \equiv 0$. ☐

We apply this lemma to the eigenspinor $\tilde{\phi}$ constructed above with $H = \mathcal{H}_N$. Therefore $\chi(A)\phi = 0$ for all $N \in \mathbb{R}$. But as $N \to \infty$ we have $\chi(A)\phi \to \phi$. By the uniqueness of the limit, ϕ must be identically zero on $(0, \epsilon) \times M$ which is an open subset of X. By the unique continuation property, it follows that ϕ vanishes on X as claimed. This ends the proof of Theorem 2.1.

Remark 2.6. Theorem 2.1 holds for Dirac operators twisted with a Hermitian bundle E whose connection ∇^E is flat in the direction of the conformal gradient vector field $\xi = \partial_x$, *i.e.*, such that the contraction of the curvature of ∇^E with the field ξ vanishes. We only need to replace in equation (6) the operator D_M by the twisted operator D_M^E. The flatness condition ensures that this operator is independent of x. The rest of the proof remains unchanged.

3. A formal extension of Theorem 2.1

For applications, it might be useful to view the metric g given by (15) in different coordinates. We state below the most general reformulation of Theorem 2.1.

Theorem 3.1. *Let $\bar{X}$ be a smooth manifold with boundary, let M be a boundary component, and let $[0, \epsilon) \times M \hookrightarrow \bar{X}$ be a collar neighbourhood of M. Take a smooth L^1 function $\rho : (0, \epsilon) \to (0, \infty)$ and let $\tilde{h}$ be a possibly incomplete Riemannian metric on X which is a warped product near M:*

$$\tilde{h} = dx^2 + \rho^{-2}(x) h_M.$$

Let $f : X \to (0, \infty)$ be a smooth conformal factor depending only on x near M and satisfying $\int_0^\epsilon f(x)dx = \infty$. Assume that the Dirac operator D_M on (M, h_M) with the induced spin structure is essentially self-adjoint. Then the Dirac operator $D_{\tilde{g}}$ of the metric $\tilde{g} := f^2 \tilde{h}$ does not carry square-integrable eigenspinors of real eigenvalue.

Proof. The metric $\tilde{h}$ is conformal to $(\rho(x)dx)^2 + h_M$. Set

$$t(x) := \int_0^x \rho(s)ds.$$

Since ρ is in L^1, it follows that t is well-defined and $t(0) = 0$. Since ρ is positive, $x \mapsto t(x)$ is an increasing diffeomorphism from $(0, \epsilon)$ to $(0, t(\epsilon))$ which extends to a homeomorphism between $[0, \epsilon)$ and $[0, t(\epsilon))$. We write $x = x(t)$ for its inverse. Define $\tilde{f}(t) := \frac{f(x(t))}{\rho(x(t))}$. Clearly, $dt = \rho dx$ so

$$\tilde{g} = \tilde{f}^2(t) \left(dt^2 + h_M \right).$$

Note that

$$\int_0^{t(\epsilon)} \tilde{f}(t)dt = \int_0^\epsilon f(x)dx = \infty$$

so we can apply Theorem 2.1. □

4. Gradient conformal vector fields

Let (X^n, g) be a Riemannian manifold. A *gradient conformal vector field* (or GCVF) on X is a conformal vector field ξ which is at the same time the gradient of a function on X:

$$\begin{cases} \mathcal{L}_\xi g = \alpha g, & \text{for some } \alpha \in \mathcal{C}^\infty(X), \\ \xi = \nabla^g F, & \text{for some } F \in \mathcal{C}^\infty(X). \end{cases} \tag{14}$$

Gradient conformal vector fields were studied intensively in the 70s (see [4] and references therein). More recently, they turned out to be a very useful tool in understanding other geometric objects, like closed twistor 2-forms on compact Riemannian manifolds [10]. The aim of this section is to prove the following:

Theorem 4.1. *Let (X, g) be a complete spin manifold. Assume that X carries a non-complete vector field which outside some compact subset is nowhere-vanishing and GCVF. Then the Dirac operator of (X, g) does not carry square-integrable eigenspinors.*

This theorem is a direct consequence of Theorem 2.1 and Proposition 4.3 below.

We first recall some basic properties of GCVFs.

Lemma 4.2. *Let ξ be a GCVF satisfying the system* (14). *Then the following assertions hold:*

(i) *The covariant derivative of ξ depends on only the co-differential of ξ:*

$$\nabla_Y \xi = \phi Y, \qquad \forall\, Y \in TX, \tag{15}$$

where the function ϕ equals $-\frac{1}{n}\delta\xi$.

(ii) *Let X_0 be the set of points where ξ does not vanish. The distribution $\xi^\perp$ defined on X_0 is involutive and its maximal integral leaves are exactly the connected components of the level sets of F on X_0.*

(iii) *The length of ξ is constant along the integral leaves of $\xi^\perp$.*

(iv) *The integral curves of ξ are geodesics and F is strictly increasing along them.*

(v) *Each point p of X_0 has a neighbourhood isometric to*

$$((-\varepsilon, \varepsilon) \times V, f^2(x)(dx^2 + h)),$$

where (V, h) is a local integral leaf of $\xi^\perp$ through p and $f : (-\varepsilon, \varepsilon) \to \mathbb{R}^+$ is some positive function. In these coordinates ξ corresponds to $\partial/\partial x$ and $f(x)$ is the norm of ξ on the leaf $\{x\} \times V$.

Proof. (i) The first equation in (14) is equivalent to the vanishing of the trace-free symmetric part of $\nabla\xi$. The second equation of (14) implies that the skew-symmetric part of $\nabla\xi$ vanishes too. We are left with $\nabla_X \xi = \phi X$ for some function ϕ. Taking the scalar product with X and the sum over an orthonormal basis $X = e_i$ yields $\phi = -\frac{1}{n}\delta\xi$.

(ii) By definition, the distribution $\xi^\perp$ on X_0 is exactly the kernel of the 1-form dF, so it is involutive. Let M be a maximal integral leaf of $\xi^\perp$. Clearly F is constant on M, which is connected, so M is a subset of some level set $F^{-1}(y)$. Moreover M is open in $F^{-1}(y)$, as can be seen in local charts. Thus

M is a connected component of $F^{-1}(y)$. (iii) For every $Y \in TM$ one can write

$$Y(|\xi|^2) = 2g(\nabla_Y \xi, \xi) \overset{(15)}{=} \phi g(Y, \xi) = 0,$$

so $|\xi|^2$ is constant on M.

(iv) Let φ_t denote the local flow of ξ and let $\gamma_t := \varphi_t(p)$ for some $p \in M$. Taking $X = \xi$ in (15) yields

$$\nabla_{\dot\gamma}\dot\gamma = \nabla_\xi \xi = \phi\xi = \phi\dot\gamma,$$

which shows that γ_t is a (non-parametrized) geodesic. Furthermore,

$$\frac{d}{dt}F(\gamma_t) = \dot\gamma_t(F) = \xi(F) = |\xi^2| > 0,$$

so $F(\gamma_t)$ is increasing.

(v) The tangent bundle of X_0 has two involutive orthogonal distributions $\mathbb{R}\xi$ and $\xi^\perp$. The Frobenius integrability theorem shows that there exists a local coordinate system $(x, y_1, \ldots, y_{n-1})$ around every $p \in X_0$ such that $\xi = \partial_x$ and ∂_{y_i} span $\xi^\perp$. Let V denote the set $\{x = 0\}$ in these coordinates. The metric tensor can be written

$$g = f^2 dx^2 + \sum_{i,j=1}^{n-1} g_{ij} dy_i \otimes dy_j.$$

From (iii) we see that f only depends on x. Using the first equation in the system (14) we get

$$2g_{ij}(\log f)'(x) = \frac{\partial g_{ij}}{\partial x}, \qquad \forall\, i, j \le n-1,$$

which shows that $g_{ij}(x, y) = f^2(x)h_{ij}(y)$ for some metric tensor h on V. $\square$

It turns out that under some completeness assumptions, the last statement of the lemma also holds globally:

Proposition 4.3. *Let (X^n, g) be a complete Riemannian manifold. If ξ is a non-complete vector field on X which, outside a compact subset of X, is gradient conformal and non-vanishing, then there exists an open subset of X which is isometric to $((0, c) \times M, f^2(x)(dx^2 + h))$ for some complete Riemannian manifold (M^{n-1}, h) and smooth positive function $f : (0, c) \to \mathbb{R}^+$ with $\int_0^c f(x)dx = \infty$.*

Proof. Let φ_t denote the local flow of ξ and let K be a compact subset of X such that ξ is gradient conformal and nowhere-vanishing on $X \setminus K$. By definition, $\xi = \nabla F$ for some function F defined on $X \setminus K$. Consider the open set

$$K_\varepsilon := \{p \in X \mid d(p, K) < \varepsilon\}.$$

Since $\overline{K_\varepsilon}$ is compact, there exists some $\delta > 0$ such that φ_t is defined on K_ε for every $|t| < \delta$. Since ξ is non-complete, there exists some $p \in X$ and $a \in \mathbb{R}$ such that $\varphi_t(p)$ tends to infinity as t tends to a. By changing ξ to $-\xi$ if necessary, we can assume that $a > 0$. From the definition of δ we see that $\varphi_t(p) \in X \setminus K_\varepsilon$ for all $t \in [a - \delta, a)$. Since

$$\lim_{t\to a} \int_0^t |\xi_{\varphi_s(p)}| ds \ge \lim_{t\to a} d(p, \varphi_t(p)) = \infty,$$

the norm of ξ has to be unbounded along its integral curve through p in the positive direction. Therefore one can find a point $q := \varphi_{t_0}(p)$ ($t_0 \in [0, a)$) on this integral curve such that $|\xi_q|$ is larger than the supremum of the norm of ξ over $\overline{K_\varepsilon}$. Let M be the maximal leaf through q of the involutive distribution $\xi^\perp$ (defined on $X \setminus K$). Since the norm of ξ is constant on M, it is clear that M does not intersect K_ε. We notice that M is complete with respect to the induced Riemannian metric h. This does not follow directly from the completeness of (X, g) since the distribution $\xi^\perp$ is only defined and involutive on $X \setminus K$. Nevertheless, since M is a connected component of some level set of F, it is closed in X, and every closed submanifold of a complete Riemannian manifold is also complete with respect to the induced Riemannian metric.

From the definition of q, it is clear that the integral curve $\varphi_t(q)$ is defined for $t < a - t_0$. From Lemma 4.2, two integral curves of ξ which do not meet K, which are issued from points of the same maximal leaf, are geodesics and have the same length. Consequently, for every other point $q' \in M$, the integral curve of ξ in the positive direction is defined at least for all $t < a - t_0$. Moreover, the map

$$\psi : M \times (0, a - t_0) \to X, \qquad \psi(r, t) := \varphi_t(r)$$

is one-to-one since the vector field ξ does not have zeros on M.

Finally, Lemma 4.2 (v) shows that ψ is an isometric embedding of $(M \times (0, a - t_0), f^2(x)(dx^2 + h))$ into (M, g), where $f(x)$ denotes the length of ξ on the maximal leaf $\varphi_x(M)$ of $\xi^\perp$. □

Remark 4.4. The incompleteness condition on ξ in Theorem 4.1 is necessary. Indeed, complete hyperbolic manifolds of finite volume are isometric outside a compact set to a disjoint union of *cusps*, *i.e.* cylinders $(0, \infty) \times T$ over some flat connected Riemannian manifold (T, h), with metric

$$dt^2 + e^{-2t} h = e^{-2t}((de^t)^2 + h).$$

The vector field $e^{-t}\partial/\partial t$ is GCVF and complete. These manifolds are known to have *purely discrete spectrum* if the spin structure on each cusp is non-trivial [2], which is the case for instance in dimension 2 or 3 when there is only one cusp. The eigenvalues then obey the Weyl asymptotic law [11]. On the contrary, when some cusps have non-trivial spin structures, the spectrum of D is the real line. In this case (like for the scalar Laplace operator) the existence of L^2 eigenspinors is generally unknown.

5. Applications

Real hyperbolic space. The Poincaré disk model of the hyperbolic space is conformally equivalent to the standard flat metric. In polar coordinates, this metric is a warped product so Theorem 3.1 shows that the Dirac operator on the hyperbolic space does not have point spectrum (the spectrum is real since $\mathbb{H}$ is complete). This was first studied with different methods by Bunke [5].

Hyperbolic manifolds. More generally, let (M^n, h_M) be a spin hyperbolic manifold whose Dirac operator is essentially self-adjoint. Taking $A = 0$ in Theorem 7.2 of [3] shows that the Riemannian manifold

$$(X_a, g) := ((a, \infty) \times M, dt^2 + \cosh(t)^2 h_M)$$

is a spin hyperbolic manifold of dimension $n+1$ for every $a \in \mathbb{R} \cup \{-\infty\}$. Setting $x := e^{-t}$ near $t = \infty$, the metric g becomes

$$g = x^{-2}\left(dx^2 + \frac{(1+x^2)^2}{4} h_M\right).$$

Theorem 3.1 thus shows that the Dirac operator on X_a (or on any spin Riemannian manifold containing X_a as an open set) does not have L^2 eigenspinors of real eigenvalue. This result was previously known when M is compact. Interesting non-compact cases are obtained when M is complete, or when M is compact with conical singularities with small angles [6].

Rotationally symmetric Riemannian manifolds. It is proved in [1] that on $\mathbb{R}^n$ with a metric which written in polar coordinates has the form $ds^2 = dr^2 + \psi(r)^2 d\theta^2$, there are no L^2 harmonic spinors. This metric is complete. By the change of variables

$$x(r) := \int_r^\infty \frac{ds}{\psi(s)},$$

Anghel's metric becomes a particular case of (1) with $\psi(r(x))$ in the rôle of $f(x)$ from Theorem 2.1, provided that $\int_1^\infty dr/\psi(r) < \infty$. The absence of harmonic spinors is guaranteed in this case by [9] if the resulting conformal factor is Lipschitz. By Theorem 2.1 we know, even without the Lipschitz hypothesis, not only that there cannot exist L^2 harmonic spinors, but also that there are no L^2 eigenspinors at all.

The L^2-index of the Dirac operator. Let D^+ denote the chiral component of D, viewed as an unbounded operator in L^2, acting on compactly supported smooth spinors on a spin Riemannian manifold as in Theorem 2.1. Denote by $\overline{D}^+$ its closure. The L^2-index is defined as

$$\operatorname{index}(\overline{D}^+) := \dim\ker(\overline{D}^+) - \dim\ker(\overline{D}^+)^*$$

where $(\overline{D}^+)^*$ is the adjoint of $\overline{D}^+$ (the definition makes sense whenever both kernels are finite-dimensional, even when $\overline{D}^+$ is not Fredholm). Here $\ker(\overline{D}^+)^*$ is precisely the distributional null-space of D^- inside L^2, while $\ker(\overline{D}^+)$ is a subspace of the distributional null-space of D^+ inside L^2. Both these spaces vanish by Theorem 2.1, so in particular it follows that $\operatorname{index}(\overline{D}^+) = 0$.

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