Parallel and Killing Spinors on Spin^c Manifolds

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Abstract: We describe all simply connected Spin^c manifolds carrying parallel and real Killing spinors. In particular we show that every Sasakian manifold (not necessarily Einstein) carries a canonical Spin^c structure with Killing spinors.

1 Introduction

The classification of irreducible simply connected spin manifolds with parallel spinors was obtained by M. Wang in 1989 [15] in the following way: the existence of a parallel spinor means that the spin representation of the holonomy group has a fixed point. Moreover, it requires the vanishing of the Ricci tensor, so the only symmetric spaces with parallel spinors are the flat ones. Then looking into Berger's list of possible holonomy groups for Riemannian manifolds and using some representation theory one finally obtains that the only suitable manifolds are those with holonomy 0, SU(n), Sp(n), Spin$_7$ and G_2. One can give the geometrical description of such a holonomy reduction in each of these cases [15]. For an earlier approach to this problem, see also [10].

The geometrical description of simply connected spin manifolds carrying real Killing spinors is considerably more complicated, and was obtained in 1993 by C. Bär [1] after a series of partial results of Th. Friedrich, R. Grunewald, I. Kath and O. Hijazi (cf. [4], [5], [6], [7], [9]). The main idea of C. Bär was to consider the cone over a manifold with Killing spinors and to show that the spin representation of the holonomy of the cone has a fixed point for a suitable scalar renormalisation of the metric on the base (actually this construction was already used in 1987 by R. Bryant [3]). By the previous discussion, this means that the cone carries a parallel spinor. Then one just has to translate in terms of the base the geometric data obtained using Wang's classification.

The problem of describing the Spin^c manifolds with parallel and real Killing spinors has recently been considered by S. Maier [13], who asserts that all these manifolds have flat auxiliary bundle, so the classification problem reduces to the above one. Unfortunately, as Th. Friedrich pointed out, his proof has an essential gap, and in fact his statement is not valid, as one easily sees from the following example. Let M be a Kähler manifold and consider its canonical Spin^c

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structure. Then the associated spinor bundle can be identified with $\Lambda^{0,*}M$, which obviously has a parallel section, and whose auxiliary bundle is not flat if M has non-vanishing Ricci curvature. In this paper we will give the complete description of simply connected Spin^c manifolds carrying parallel and real Killing spinors.

It came to us as a surprise that the above example of Spin^c manifold with parallel spinors is essentially the only one, excepting those with flat auxiliary bundle (i.e. spin structures). The result is the following

Theorem 1.1 *A simply connected Spin^c manifold carrying a parallel spinor is isometric to the Riemannian product between a simply connected Kähler manifold and a simply connected spin manifold carrying a parallel spinor.*

We then turn our attention to Spin^c manifolds with real Killing spinors, and prove that the cone over such a manifold inherits a canonical Spin^c structure such that the Killing spinor on the base induces a parallel spinor on the cone. Then using the above theorem and the fact that the cone over a complete Riemannian manifold is irreducible or flat (cf. [8]), we obtain that the only simply connected Spin^c manifolds with real Killing spinors with non-flat auxiliary bundle are the (non-Einstein) Sasakian manifolds. The importance of such a result comes from the fact that it gives a spinorial interpretation of Sasakian structures, just as in the case of Einstein–Sasakian and 3–Sasakian structures.

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2 Spin and Spin^c structures

Consider an oriented Riemannian manifold (M^n, g) and let $P_{SO(n)}M$ denote the bundle of oriented orthonormal frames on M.

Definition 2.1 *The manifold M is called spin if the there exists a 2–fold covering $P_{Spin_n}M$ of $P_{SO(n)}M$ with projection $\theta : P_{Spin_n}M \to P_{SO(n)}M$ satisfying the following conditions :*

i) $P_{Spin_n}M$ is a principal bundle over M with structure group $Spin_n$;

ii) If we denote by ϕ the canonical projection of $Spin_n$ over $SO(n)$, then for every $u \in P_{Spin_n}M$ and $a \in Spin_n$ we have

$$\theta(ua) = \theta(u)\phi(a).$$

A Riemannian manifold M is spin iff the second Stiefel-Whitney class of M, $w_2(M)$, vanishes.

The bundle $P_{Spin_n}M$ is called a spin structure. The representation theory shows that the Clifford algebra $Cl(n)$ has (up to equivalence) exactly one irreducible complex representation Σ_n for n even and two irreducible complex representations $\Sigma_n^{\pm}$ for n odd. In the last case, these two representations are equivalent when restricted to Spin_n, and this restriction is denoted by Σ_n. For n even, there is a splitting of ΣM with respect to the action of the volume element in $\Sigma_n := \Sigma_n^+ \oplus \Sigma_n^-$ and one usually calls elements of Σ_n^+ (Σ_n^-) positive (respectively negative) half-spinors. For arbitrary n, Σ_n is called the complex spin representation, and it defines a complex vector bundle associated to the spin structure, called the complex spinor bundle ΣM.

Definition 2.2 *A $Spin^c$ structure on M is given by a $U(1)$ principal bundle $P_{U(1)}M$ and a $Spin_n^c$ principal bundle $P_{Spin_n^c}M$ together with a projection $\theta : P_{Spin_n^c}M \to P_{SO(n)}M \times P_{U(1)}M$ satisfying*

$$\theta(\tilde{u}\tilde{a}) = \theta(\tilde{u})\xi(\tilde{a}),$$

for every $\tilde{u} \in P_{Spin_n^c}M$ and $\tilde{a} \in Spin_n^c$, where ξ is the canonical 2–fold covering of $Spin_n^c$ over $SO(n) \times U(1)$.

Recall that $\mathrm{Spin}_n^c = \mathrm{Spin}_n \times_{\mathbf{Z}_2} U(1)$, and that ξ is given by $\xi([u,a]) = (\phi(u), a^2)$. The complex representations of Spin_n^c are obviously the same as those of Spin_n, so to every Spin^c manifold is associated a spinor bundle just like the for spin manifolds. If M^{2m} is Kähler, there is a parallel decomposition $\Sigma M = \Sigma_0 M \oplus ... \oplus \Sigma_m M$, corresponding to the action of the Kähler form by Clifford multiplication. The bundles $\Sigma_k M$ lie in $\Sigma^+ M$ ($\Sigma^- M$) for k even (odd) (cf. [11]).

If M is spin, the Levi–Civita connection on $P_{SO(n)}M$ induces a connection on the spin structure $P_{Spin_n}M$, and thus a covariant derivative on ΣM denoted by ∇. If M has a Spin^c structure, then every connection form A on $P_{U(1)}M$ defines in a similar way (together with the Levi-Civita connection of M) a covariant derivative on ΣM denoted by ∇^A.

In general, by Spin^c manifold we will understand a set (M, g, S, L, A), where (M, g) is an oriented Riemannian manifold, S is a Spin^c structure, L is the complex line bundle associated to the auxiliary bundle of S and A is a connection form on L.

Lemma 2.1 *A $Spin^c$ structure on a simply connected manifold M with trivial auxiliary bundle is canonically identified with a spin structure. Moreover, if the connection defined by A is flat, then by this identification ∇^A corresponds to ∇ on the spinor bundles.*

Proof. One first remarks that since the $U(1)$ bundle associated to L is trivial, we can exhibit a global section of it, that we will call σ. Denote by $P_{Spin_n}M$

the inverse image by θ of $P_{SO(n)}M \times \sigma$. It is straightforward to check that this defines a spin structure on M, and that the connection on $P_{Spin_n^c}M$ restricts to the Levi-Civita connection on $P_{Spin_n}M$ if σ can be choosen parallel, i.e. if A defines a flat connection.

Q.E.D.

Consequently, all results concerning Spin^c structures obtained below are also valid for usual spin structures.

3 Parallel Spinors

In this section we classify all simply connected Spin^c manifolds (M, g, S, L, A) admitting parallel spinors. The curvature form of A can be viewed as an imaginary–valued 2–form on M, and will be denoted by $i\omega := dA$.

Lemma 3.1 *Suppose there exists a parallel spinor ψ on M^n*

$$\nabla_X^A \psi = 0 \qquad \forall X. \tag{1}$$

Then the following equation holds

$$\mathrm{Ric}(X) \cdot \psi = iX \lrcorner\, \omega \cdot \psi, \qquad \forall X. \tag{2}$$

Proof. Let $\{e_1, ...e_n\}$ be a local orthonormal frame. From (1) we easily obtain

$$\mathcal{R}_{X,Y}^A \psi = 0. \tag{3}$$

A local computation shows that the curvature operator on the spinor bundle is given by the formula

$$\mathcal{R}^A = \mathcal{R} + \frac{i}{2}\omega, \tag{4}$$

where

$$\mathcal{R}_{X,Y} = \frac{1}{2}\sum_{j<k} R(X, Y, e_j, e_k) e_j \cdot e_k \cdot \tag{5}$$

Using the first Bianchi identity for the curvature tensor one obtains ([2], p.16)

$$\sum_i e_i \cdot \mathcal{R}_{e_i,X} = \frac{1}{2}\mathrm{Ric}(X), \tag{6}$$

so by (3), (4) and (6),

$$\begin{aligned} 0 &= \sum_j e_j \cdot \mathcal{R}_{e_j,X}^A \psi \\ &= \sum_j e_j \cdot (\mathcal{R}_{e_j,X}\psi + \frac{i}{2}\omega(e_j, X)\psi) \\ &= \frac{1}{2}\mathrm{Ric}(X) \cdot \psi - \frac{i}{2} X \lrcorner\, \omega \cdot \psi. \end{aligned}$$

Q.E.D.

We consider Ric as an (1,1) tensor on M and denote for every $x \in M$ by $K(x)$ the image of Ric, i.e.

$$K(x) = \{\mathrm{Ric}(X) \mid X \in T_xM\}$$

and by $L(x)$ the orthogonal complement of $K(x)$ in T_xM, which by (2) can be written as

$$L(x) = \{X \in T_xM \mid \mathrm{Ric}(X) = 0\} = \{X \in T_xM \mid X \lrcorner \omega = 0\}.$$

Since ψ is parallel, $TM \cdot \psi$ and $iTM \cdot \psi$ are two parallel sub-bundles of ΣM. This shows that their intersection is also a parallel sub-bundle of ΣM. Let E be the inverse image of $TM \cdot \psi \cap iTM \cdot \psi$ by the isomorphism $\Phi : TM \to TM \cdot \psi$, given by $\Phi(X) = X \cdot \psi$. The fibre at some $x \in M$ of E can be expressed as

$$E_x = \{X \in T_xM \mid \exists Y \in T_xM,\ X \cdot \psi = iY \cdot \psi\}.$$

By the preceding discussion, E and $E^\perp$ are well-defined parallel distributions of M. Moreover, (2) shows that $K(x) \subset E_x$ for all x, so $E_x^\perp \subset L(x)$, i.e. the restriction of ω to $E^\perp$ vanishes. We now use the de Rham decomposition theorem and obtain that M is isometric to a Riemannian product $M = M_1 \times M_2$, where M_1 and M_2 are arbitrary integral manifolds of E and $E^\perp$ respectively. The Spin^c structure of M induces Spin^c structures on M_1 and M_2 with canonical line bundles whose curvature is given by the restriction of ω to E and $E^\perp$, and (using the correspondence between parallel spinors and fixed points of the spin holonomy representation) the parallel spinor ψ induces parallel spinors ψ_1 and ψ_2 on M_1 and M_2, which satisfy (2). It is clear that (since the restriction of ω to $E^\perp$ vanishes) the canonical line bundle of the Spin^c structure on M_2 is trivial and has vanishing curvature, so by Lemma 2.1, ψ_2 is actually a parallel spinor of a *spin* structure on M_2.

On the other hand, by the very definition of E one easily obtains that the equation

$$X \cdot \psi = iJ(X) \cdot \psi \tag{7}$$

defines an almost complex structure J on M_1.

Lemma 3.2 *The almost complex structure J defined by the above formula is parallel, so (M_1, J) is a Kähler manifold.*

Proof. Taking the covariant derivative (on M_1) in (7) in an arbitrary direction Y and using (1) gives

$$\nabla_Y X \cdot \psi = i\nabla_Y (J(X)) \cdot \psi. \tag{8}$$

On the other hand, replacing X by $\nabla_Y X$ in (7) and substracting from (8) yields $\nabla_Y(J(X))\cdot\psi = J(\nabla_Y X)\cdot\psi$, so $((\nabla_Y J)(X))\cdot\psi = 0$, and finally $(\nabla_Y J)(X) = 0$ since ψ never vanishes on M_1. As X and Y were arbitrary vector fields we deduce that $\nabla J = 0$.

Q.E.D.

We finally remark that the restriction of the Spin^c structure of M to M_1 is just the canonical Spin^c structure of M_1, since (7) and (2) show that the restriction of ω to M_1 is the Ricci form of M_1.

Remark 3.1 *Of course, replacing J by $-J$ just means switching from the canonical to the anti-canonical Spin^c structure of M_1, but we solve this ambiguity by "fixing" the sign of J with the help of (7).*

Conversely, the Riemannian product of two Spin^c manifolds carrying parallel spinors is again a Spin^c manifold carrying parallel spinors, and as we already remarked in the first section, the canonical Spin^c structure of every Kähler manifold carries parallel spinors. Consequently we have proved the following:

Theorem 3.1 *A simply connected Spin^c manifold M carries a parallel spinor if and only if it is isometric to the Riemannian product $M_1 \times M_2$ between a simply connected Kähler manifold and a simply connected spin manifold carrying a parallel spinor, and the Spin^c structure of M is the product between the canonical Spin^c structure of M_1 and the spin structure of M_2.*

There are two natural questions that one may ask at this stage:

Question 1: What is the dimension of the space of parallel spinors on M?

Question 2: How many Spin^c structures on M carry parallel spinors?

We can of course suppose that M is irreducible, since otherwise we decompose M, endow each component with the induced Spin^c structure, and make the reasonning below for each component separately. Using Theorem 3.1, we can thus always suppose that M is either an irreducible spin manifold carying parallel spinors, or an irreducible Kähler manifold not Ricci flat (since these ones are already contained in the first class), endowed with the canonical Spin^c structure. We call such manifolds of type S (spin) and K (Kähler) respectively. Then the answers to the above questions are:

1. For manifolds of type S the answer is given by M. Wang's classification [15]. For manifolds of type K we will show that the dimension of the space of parallel spinors is 1. Suppose we have two parallel spinors ψ_1 and ψ_2 on M. Correspondingly we have two Kähler structures J_1 and J_2. Moreover, the Ricci forms of these Kähler structures are both equal to ω, so $J_1 = J_2$ when restricted to the image of the Ricci tensor. On the other hand, the vectors X for which $J_1(X) = J_2(X)$ form a parallel distribution on M, which, by irreducibility, is either the whole of TM or empty. In the first case we have $J_1 = J_2$, so ψ_1 and ψ_2 are both parallel sections of $\Sigma_0 M$ whose complex dimension is 1. The second case is impossible, since it would imply that the Ricci tensor vanishes.

2. This question has a meaning only for manifolds of type K. A slight modification in the above argument shows that on a manifold of type K there is no other Kähler structure but J and $-J$. On the other hand, we have seen that a parallel spinor with respect to some Spin^c structure induces on such a manifold M a Kähler structure whose canonical Spin^c structure is just the given one. It is now clear that we can have at most two different Spin^c structures with parallel spinors: the canonical Spin^c structures induced by J and $-J$. These are just the canonical and anti-canonical Spin^c structures on (M, J), and they both carry parallel spinors.

We can synthetise this as

Proposition 3.1 *The only Spin^c structures on an irreducible Kähler manifold not Ricci-flat which carry Killing spinors are the canonical and anti-canonical ones. The dimension of the space of parallel spinors is in each of these two cases equal to 1.*

4 Killing spinors

In this section we classify all simply connected Spin^c manifolds (M^n, g, S, L, A) carrying real Killing spinors, i.e. spinors ψ satisfying the equation

$$\nabla^A_X \psi = \lambda X \cdot \psi, \quad \forall X \in TM, \tag{9}$$

for some fixed real number $\lambda \neq 0$. By rescaling the metric if necessary, we can suppose without loss of generality that $\lambda = \pm\frac{1}{2}$. Moreover, for n even we can suppose that $\lambda = \frac{1}{2}$, by taking the conjugate of ψ if necessary.

Consider the cone $(\bar{M}, \bar{g})$ over M given by $\bar{M} = M \times_{r^2} \mathbf{R}^+$, with the metric $\bar{g} = r^2 g + dr^2$. Let us denote by ∂_r the vertical unit vector field. Then the covariant derivative $\bar{\nabla}$ of the Levi-Civita connection of $\bar{g}$ satisfies the formulas of warped products ([14], p.206)

$$\bar{\nabla}_{\partial_r} \partial_r = 0 \ , \tag{10}$$

$$\bar{\nabla}_{\partial_r} X = \bar{\nabla}_X \partial_r = \frac{1}{r} X \ , \tag{11}$$

$$\bar{\nabla}_X Y = \nabla_X Y - r\, g(X,Y) \partial_r \ , \tag{12}$$

where here and in the formulas below, X, Y are vector fields on M, identified with their canonical extensions to $\bar{M}$. Using this, one easily computes the curvature tensor $\bar{R}$ of $\bar{M}$ (cf. [14], p.210)

$$\bar{R}(X,\partial_r)\partial_r = \bar{R}(X,Y)\partial_r = \bar{R}(X,\partial_r)Y = 0, \tag{13}$$

$$\bar{R}(X,Y)Z = R(X,Y)Z + g(X,Z)Y - g(Y,Z)X. \tag{14}$$

In particular, if $\bar{M}$ is flat, then M is a space form.

For later use, let us recall an important result concerning the holonomy of such cone metrics, originally due, as far as we know, to S. Gallot ([8], Prop. 3.1).

Lemma 4.1 *If M is complete, then $\bar{M}$ is irreducible or flat.*

The use of the cone over M is the key of the classification, as it is the case for C. Bär's description of manifolds carrying Killing spinors with respect to usual spin structures. The principal ingredient is the following

Proposition 4.1 *Denote by π the projection $\bar{M} \to M$. Then, every Spin^c structure (S, L, A) on M induces a canonical Spin^c structure $(\bar{S}, \pi^*L, \pi^*A)$ on $\bar{M}$. Moreover, if the dimension of M is even (odd), every spinor ψ on M induces a spinor $\pi^*\psi$ (respectively positive and negative half spinors $(\pi^*\psi)^\pm$) on $\bar{M}$ satisfying*

$$\bar{\nabla}^A_X(\pi^*\psi) = \pi^*(\nabla^A_{\pi_* X}\psi - \frac{1}{2}(\pi_* X)\cdot\psi) \qquad \forall X \in T\bar{M}, \tag{15}$$

$$\bar{\nabla}^A_X(\pi^*\psi)^\pm = \pi^*(\nabla^A_{\pi_* X}\psi \mp \frac{1}{2}(\pi_* X)\cdot\psi) \qquad \forall X \in T\bar{M}, \tag{16}$$

Proof. By enlarging the structure groups, the two-fold covering

$$\theta : P_{Spin^c_n}M \to P_{SO(n)}M \times P_{U(1)}M,$$

gives a two-fold covering

$$\theta : P_{Spin^c_{n+1}}M \to P_{SO(n+1)}M \times P_{U(1)}M,$$

which, by pull-back through π, gives rise to a Spin^c structure on $\bar{M}$

$$\begin{array}{ccc}
P_{Spin^c_{n+1}}\bar{M} & \xrightarrow{\pi} & P_{Spin^c_{n+1}}M \\
\pi^*\theta \downarrow & & \theta \downarrow \\
P_{SO(n+1)}\bar{M} \times P_{U(1)}\bar{M} & \xrightarrow{\pi} & P_{SO(n+1)}M \times P_{U(1)}M \\
\downarrow & & \downarrow \\
\bar{M} & \xrightarrow{\pi} & M
\end{array}$$

The verification of the fact that the pull back of $P_{SO(n+1)}M$ is indeed isometric to the oriented orthonormal frame bundle of $\bar{M}$ and that $\pi^*\theta$ is Spin^c–equivariant is left to the reader. The Levi-Civita connection of $\bar{M}$ and the pull-back connection on $P_{U(1)}\bar{M}$ induce a connection on $P_{Spin^c_{n+1}}\bar{M}$. We will now relate the (complex) spinor bundles associated to the Spin^c structures on M and $\bar{M}$.

There is an isomorphism $Cl(n) \simeq Cl^0(n+1)$ obtained by extending the mapping $\mathbf{R}^n \to Cl^0(n+1)$, $v \mapsto v \cdot e_{n+1}$ to $Cl(n)$, which gives an inclusion $i : \mathrm{Spin}_{n+1} \to Cl(n)$ and thus an embedding of Spin^c_{n+1} into the complex Clifford algebra $\mathbf{C}l(n)$ by $a \times e^{it} \mapsto i(a)e^{it}$. This makes the complex spin representation Σ_n into a Spin^c_{n+1}–representation. Recall that Σ_n is an irreducible $\mathbf{C}l(n)$–representation of complex dimension $2^{[\frac{n}{2}]}$, and that Spin^c_k has (up to equivalence) a unique complex representation of dimension $2^{[\frac{k}{2}]}$, namely Σ_k for k odd and two inequivalent complex representations of dimension $2^{[\frac{k}{2}]}$ (the so-called half-spin representations) $\Sigma^{\pm}_k$ for k even. By dimensional reasons, we deduce that the above constructed representation of Spin^c_{n+1} on Σ_n is equivalent to Σ_{n+1} for $n+1$ odd and to one of the half-spin representations for $n+1$ even. In order to decide which of them is obtained by this procedure for $n = 2k+1$, we recall that, on $\Sigma^{\pm}_{2k+2}$, the complex volume element $\omega_{\mathbf{C}}(2k+2) := i^{k+1}e_1 \cdot ... \cdot e_{2k+2}$ acts by ± 1 and that on Σ_{2k+1}, the complex volume element $\omega_{\mathbf{C}}(2k+1) := i^{k+1}e_1 \cdot ... \cdot e_{2k+1}$ acts by the identity. But $\omega_{\mathbf{C}}(2k+2) := i^{k+1}e_1 \cdot ... \cdot e_{2k+2} = i^{k+1}(e_1 \cdot e_{2k+2}) \cdot ... \cdot (e_{2k+1} \cdot e_{2k+2})$, which by the above acts on Σ_{2k+1} exactly as $\omega_{\mathbf{C}}(2k+1)$, i.e. by the identity.

Consequently, we have identified the pull back $\pi^*\Sigma M$ with $\Sigma\bar{M}$ $(\Sigma^+\bar{M})$ for n even (respectively odd), and with respect to this identification, if X and Y are vectors on M and ψ a spinor on M, then

$$\frac{1}{r}X \cdot \partial_r \cdot \pi^*\psi = \pi^*(X \cdot \psi). \tag{17}$$

$$\frac{1}{r}X \cdot \frac{1}{r}Y \cdot \pi^*\psi = \pi^*(X \cdot Y \cdot \psi). \tag{18}$$

Similarly, if we define (for n odd) an isomorphism $Cl(n) \simeq Cl^0(n+1)$ by extending to $Cl(n)$ the mapping $\mathbf{R}^n \to Cl^0(n+1)$, $v \mapsto -v \cdot e_{n+1}$, we obtain an identification of the pull back $\pi^*\Sigma M$ with $\Sigma^-\bar{M}$, and the formula (17) has to be replaced with

$$\frac{1}{r}X \cdot \partial_r \cdot \pi^*\psi = -\pi^*(X \cdot \psi). \tag{19}$$

The above constructed connection on $P_{Spin^c_{n+1}}\bar{M}$ defines a covariant derivative $\bar{\nabla}^A$ on $\Sigma\bar{M}$ $(\Sigma^+\bar{M})$ for n even (respectively odd). Consider the pull-back $\pi^*\psi$ of a spinor $\psi = [\sigma, \xi]$, where $\xi : U \subset M \to \Sigma_n$ is a vector valued function, and σ is a local section of $P_{Spin^c_n}M$ whose projection onto $P_{SO(n)}M$ is a local orthonormal frame $(X_1, ..., X_n)$ and whose projection onto $P_{U(1)}M$ is a local section s. Then $\pi^*\psi$ can be expressed as $\pi^*\psi = [\pi^*\sigma, \pi^*\xi]$, and it is easy to see that the projection of $\pi^*\sigma$ onto $P_{SO(n+1)}\bar{M}$ is the local orthonormal frame $(\frac{1}{r}X_1, ..., \frac{1}{r}X_n, \partial_r)$ and its projection onto $P_{U(1)}\bar{M}$ is just π^*s. In order to compute the covariant derivative of the spinor $\pi^*\psi$ at a point $(x, r) \in \bar{M}$ in terms of the covariant derivative of ψ on M we use the following formula which easily follows from (11):

$$\bar{\nabla}_{\partial_r}\frac{1}{r}X = 0, \tag{20}$$

for every vector field X on M identified with the induced vector field on $\bar{M}$. We then obtain

$$\begin{aligned}
\bar{\nabla}^A_{\partial_r}\pi^*\psi &= [\pi^*\sigma, \partial_r(\pi^*\xi)] + \frac{1}{2}\sum_{j<k}\bar{g}(\bar{\nabla}_{\partial_r}(\frac{1}{r}X_j), \frac{1}{r}X_k)\frac{1}{r}X_j \cdot \frac{1}{r}X_k \cdot \pi^*\psi \\
&\quad + \frac{1}{2}\sum_j \bar{g}(\bar{\nabla}_{\partial_r}\frac{1}{r}X_j, \partial_r)\frac{1}{r}X_j \cdot \partial_r \cdot \pi^*\psi + \frac{1}{2}\pi^*A((\pi^*s)_*(\partial_r))\pi^*\psi \\
&= [\pi^*\sigma, \partial_r(\pi^*\xi)] + \frac{1}{2}A((\pi \circ \pi^*s)_*(\partial_r))\pi^*\psi \\
&= 0 + \frac{1}{2}A((s \circ \pi)_*(\partial_r))\pi^*\psi \\
&= 0,
\end{aligned}$$

and (using (11), (12), (17) and (18))

$$\begin{aligned}
\bar{\nabla}^A_X\pi^*\psi &= [\pi^*\sigma, X(\pi^*\xi)] + \frac{1}{2}\sum_{j<k}\bar{g}(\bar{\nabla}_X(\frac{1}{r}X_j), \frac{1}{r}X_k)\frac{1}{r}X_j \cdot \frac{1}{r}X_k \cdot \pi^*\psi \\
&\quad + \frac{1}{2}\sum_j \bar{g}(\bar{\nabla}_X\frac{1}{r}X_j, \partial_r)\frac{1}{r}X_j \cdot \partial_r \cdot \pi^*\psi + \frac{1}{2}\pi^*A((\pi^*s)_*(X))\pi^*\psi \\
&= [\pi^*\sigma, \pi^*(X(\xi))] + \frac{1}{2}\sum_{j<k}\frac{1}{r^2}(\bar{g}(\nabla_X X_j, X_k))\pi^*(X_j \cdot X_k \cdot \psi) \\
&\quad - \frac{1}{2}\sum_j g(X, X_j)\pi^*(X_j \cdot \psi) + \frac{1}{2}A((\pi \circ \pi^*s)_*(X))\pi^*\psi \\
&= \pi^*\Big([\sigma, (X(\xi))] + \frac{1}{2}\sum_{j<k} g(\nabla_X X_j, X_k)X_j \cdot X_k \cdot \psi \\
&\quad - \frac{1}{2}X \cdot \psi + \frac{1}{2}A(s_*X)\psi\Big) \\
&= \pi^*(\nabla^A_X\psi - \frac{1}{2}X \cdot \psi).
\end{aligned}$$

The computations are similar for $(\pi^*\psi)^\pm$.

Q.E.D.

The following result comes as a direct consequence of the above proposition.

Corollary 4.1 *The cone over a Spin^c manifold M^n carrying real Killing spinors has a Spin^c structure carrying parallel spinors. Moreover, for n odd, a Killing spinor with Killing constant $\frac{1}{2}$ generates a positive parallel half-spinor and a Killing spinor with Killing constant $-\frac{1}{2}$ generates a negative parallel half-spinor. The converse is also true.*

Definition 4.1 *A vector field ξ on a Riemannian manifold (M, g) is called a Sasakian structure if the following conditions are satisfied:*

1. ξ is a Killing vector field of unit length;

2. the tensors $\varphi := -\nabla\xi$ and $\eta := g(\xi, .)$ are related by

$$\varphi^2 = -Id + \eta \otimes \xi;$$

3. $(\nabla_X\varphi)Y = g(X, Y)\xi - \eta(Y)\,X$, for all vectors X , Y .

Recall that a Riemannian manifold carries a Sasakian structure if and only if the cone over it is a Kähler manifold, as it is shown by a straightforward computation ([1]). From Corollary 4.1, Theorem 3.1 and Lemma 4.1 we thus obtain

Theorem 4.1 *A simply connected complete Spin^c manifold M carries a real Killing spinor if and only if it satisfies one of the two conditions below:*

1. the connection of the auxiliary bundle is flat and M is a simply connected spin manifold carrying real Killing spinors;

2. M is a Sasakian manifold.

The first case is not interesting for us, since the simply connected spin manifolds with real Killing spinors are already studied by C. Bär in [1]. We thus concentrate our efforts towards Sasakian manifolds, and start with the following

Proposition 4.2 *Every Sasakian manifold (M^{2k+1}, g, ξ) carries a canonical Spin^c structure. If M is Einstein, then the auxiliary bundle of the canonical Spin^c structure is flat, so if in addition M is simply connected, then it is spin.*

Proof. The first statement follows directly from the fact that the cone over M is Kähler, and thus carries a canonical Spin^c structure, whose restriction to M is the desired canonical Spin^c structure.

If M is Einstein, then its Einstein constant is $2k$ ([2], p.78), so $\bar{M}$ is Ricci flat by (13), (14). The auxiliary bundle of the canonical Spin^c structure of $\bar{M}$ which is just the canonical bundle $K = \Lambda^{k+1,0}\bar{M}$ is thus flat, and the same is true for its restriction to M. The last statement follows from Lemma 2.1.

Q.E.D.

One can actually construct the Spin^c structure more directly as follows: the frame bundle of every Sasakian manifold restricts to $U(k)$, by considering only adapted frames, i.e., orthonormal frames of the form $\{\xi, e_1, \varphi(e_1), ...e_k, \varphi(e_k)\}$. Then, just extend this bundle of adapted frames to a Spin^c_{2k} principal bundle using the canonical inclusion $U(k) \to \mathrm{Spin}^c_{2k}$ (cf. [12], p.392). We prefered the description which uses the cone over M since the computations are considerably simpler (e.g., for showing that, if M is Einstein and simply connected, it is spin).

The first description also has the advantage of directly showing (using Theorem 4.1) that the canonical Spin^c structure carries a Killing spinor, which is not obvious if one uses the second description.

Just as in the case of almost complex manifolds one can define an anti-canonical Spin^c structure for Sasakian manifolds, which has the same properties as the canonical one.

We recall that the parallel spinor of the canonical Spin^c structure of a Kähler manifold M^{2k+2} lies in $\Sigma_0 M$, so is always a positive half-spinor, and the parallel spinor of the anti-canonical Spin^c structure lies in $\Sigma_{k+1}M$, so it is positive (negative) for k odd (respectively even). Collecting these remarks together with Corollary 4.1 we obtain the following

Corollary 4.2 *The only simply connected Spin^c manifolds admitting real Killing spinors other than the spin manifolds are the non-Einstein Sasakian manifolds M^{2k+1} endowed with their canonical or anti-canonical Spin^c structure. For the canonical Spin^c structure, the dimension of the space of Killing spinors for the Killing constant $\frac{1}{2}$ is always equal to 1, and there is no Killing spinor for the constant $-\frac{1}{2}$. For the anti-canonical Spin^c structure, the dimensions of the spaces of Killing spinors for the Killing constants $\frac{1}{2}$ and $-\frac{1}{2}$ are 1 and 0 (0 and 1) for k odd (respectively even).*

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